The Annals of Probability
2000, Vol. 28, No. 3, 1336–1352

THE FUNCTIONAL CENTRAL LIMIT THEOREM UNDER THE STRONG MIXING CONDITION

BY FLORENCE MERLEVÈDE AND MAGDA PELIGRAD[1]

Université Paris VI and University of Cincinnati

We prove a central limit theorem for strongly mixing sequences under a sharp sufficient condition which combines the rate of the strong mixing coefficient with the quantile function. The result improves on all earlier central limit theorems for this type of dependence and answers a conjecture raised by Bradley in 1997.

Moreover, we derive the corresponding functional central limit theorem.

1. Introduction and results. Suppose $X := \{X_k, k \in \mathbb{Z}\}$ is a strictly stationary sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$. For $-\infty \le m \le p \le \infty$, let $\mathscr{F}_m^p$ denote the σ-field of events generated by the random variables $\{X_j, m \le j \le p\}$. For any two fields $\mathscr{A}$ and $\mathscr{B} \subset \mathscr{F}$, in 1956, Rosenblatt introduced the following measure of dependence:

$$\alpha(\mathscr{A}, \mathscr{B}) := \sup |P(A \cap B) - P(A)P(B)|, \qquad A \in \mathscr{A},\ B \in \mathscr{B}, \tag{1.1}$$

and the mixing coefficients:

$$\alpha_0 = 1/4 \text{ and } \alpha_n = \alpha(\mathscr{F}_{-\infty}^0, \mathscr{F}_n^\infty) \quad \text{for all } n \in \mathbb{N}^*. \tag{1.2}$$

If $\alpha_n \to 0$ as $n \to \infty$, we say that the sequence X is strongly mixing. Since 1956, a vaste body of work has been devoted to study the limiting behavior of strongly mixing sequences. This is a large class of random variables which contains both weakly dependent sequences and sequences with long range dependence. Examples include time series, Gaussian processes and Markov processes. These processes appear in other branches of mathematics, as well as statistics and mathematical physics, giving rise to a great deal of interest in their asymptotic properties.

The question concerning the central limit theorem in this setting is the following: under what assumptions, besides that of strong mixing, do there exist real numbers $a_1, a_2, a_3, \dots$ and positive numbers $b_1, b_2, b_3, \dots$ with $b_n \to \infty$ as $n \to \infty$, such that

$$\frac{S_n - a_n}{b_n} \xrightarrow{\mathscr{D}} N \sim \mathscr{N}(0,1) \quad \text{as } n \to \infty, \tag{1.3}$$

where $S_n = \sum_{i=1}^n X_i$.

A result of Ibragimov (1962) [see Theorem 18.1.1 in Ibragimov and Linnik (1971)] tells us that (under strict stationarity and strong mixing) if (1.3) holds

Received February 1999; revised March 2000.

[1]Supported in part by a Taft Research Grant.

AMS 1991 *subject classifications*. 60F05, 60F17.

Key words and phrases. Strong mixing, central limit theorem, weak invariance principle, quantiles.

with $b_n \to \infty$ as $n \to \infty$, then necessarily b_n has the representation $b_n^2 = nh(n)$ where $h(n)$ is a slowly varying function as $n \to \infty$.

An important problem is to determine subclasses of mixing sequences of random variables satisfying (1.3). Most of the research in this direction addresses the problem of a particular sequence of b_n's and little is known about (1.3) in its full generality. For the particular case of $b_n^2 = \operatorname{Var} S_n$ we would like to mention the characterization due to Denker (1986).

THEOREM 1.1. *Assume $\{X_k,\ k \in \mathbb{Z}\}$ is a strictly stationary and strongly mixing sequence such that*

$$\mathbb{E}X_0 = 0,\ \mathbb{E}X_0^2 < \infty \quad \textit{and} \quad \operatorname{Var}(S_n) \to \infty. \tag{1.4}$$

Then

$$\frac{S_n}{(\operatorname{Var}(S_n))^{1/2}} \xrightarrow{\mathscr{D}} N \sim \mathscr{N}(0,1) \quad \textit{as } n \to \infty \tag{1.5}$$

is equivalent to

$$\left\{ \frac{S_n^2}{\operatorname{Var}(S_n)} \right\}_{n \geq 1} \textit{ is an uniformly integrable family.}$$

The most useful results in this area point out classes of strong mixing sequences satisfying (1.4) under sharp sufficient conditions imposed on both the individual summands and the strong mixing coefficients. By the term "sharp sufficient conditions" we mean those conditions whose violation allows the construction of counterexamples. Some of the first sufficient conditions for central limit theorems in this setting are due to Ibragimov (1962). These conditions are in terms of moments of random variables and strong mixing rates. These results have a conclusion (1.3) for the particular choice $b_n = cn^{1/2}$.

More recently, Doukhan, Massart and Rio (1994) showed the particular role played by quantiles in deriving a central limit theorem. For any nonnegative random variable W, define the "upper tail" quantile function via

$$Q_W(u) = \inf\{t \geq 0\colon P(W > t) \leq u\}.$$

Doukhan, Massart and Rio (1994) proved the following.

THEOREM 1.2. *Suppose $\{X_k,\ k \in \mathbb{Z}\}$ is a strictly stationary, strongly mixing sequence of random variables such that $\mathbb{E}X_0 = 0$ and $\mathbb{E}X_0^2 < \infty$:*

(a) *If*

$$\sum_{n=1}^{\infty} \int_0^{\alpha_n} Q_{|X_0|}^2(u)\, du < \infty, \tag{1.6}$$

then $\sigma^2 = \mathbb{E}X_0^2 + 2\sum_{n=1}^{\infty} \mathbb{E}X_0 X_n$ exists in $[0,\infty)$, the sum being absolutely convergent.

(b) *If in addition,* $\sigma^2 > 0$, *then*

$$\frac{S_n}{n^{1/2}\sigma} \xrightarrow{\mathscr{D}} N \sim \mathscr{N}(0,1) \quad \textit{as } n \to \infty. \tag{1.7}$$

Part (a) is due to Rio [(1993) Theorem 1.2] and part (b) to Doukhan, Massart and Rio [(1994), Theorem 1]. This theorem sharpened earlier central limit theorems of Ibragimov (1962) and Herrndorf (1985).

Notice that Theorem 1.2 has two main conditions. One is (1.6) and the other imposed at the point (b) is the following:

$$\lim_{n\to\infty}\inf \frac{\mathbb{E}(S_n^2)}{n} > 0. \tag{1.8}$$

In a certain sense, Theorem 1.2 is of the form

$$(1.6) \quad \text{and} \quad (1.8) \Rightarrow (1.7).$$

Several papers address the problem of the minimality of conditions (1.6) and (1.8) for the central limit theorem in this setting. In a recent paper, Bradley (1997) constructed a class of strong mixing sequences which does not satisfy the central limit theorem. His sharp counterexample shows that, if (1.8) is assumed, then condition (1.6) is essentially sharp in Theorem 1.2, in the sense that if (1.6) is violated then (1.7) does not hold. For the case of mixing rates of the form $\alpha_n \sim n^{-\theta}$ as $n \to \infty$ for a given $\theta > 1$, this has already been shown by the construction of Doukhan, Massart and Rio (1994). When we say above that (1.7) no longer holds, we mean the central limit theorem does not hold under the normalization b_n^2 such that $\lim_{n\to\infty}(b_n^2/n) = \sigma^2$. It is possible that a central limit theorem might hold in this setting with a more general normalization. To see that more general normalizations are possible, examine the example given by Ibragimov and Rozanov [(1978), pages 179–180, Example 1], or the central limit theorems derived by Dehling, Denker and Philipp (1986), Denker (1986), Mori and Yoshihara (1986), Peligrad (1992) or Rosenblatt (1956). In all these papers $b_n^2 = nh(n)$ where $h(n)$ is slowly varying as $n \to \infty$. In view of the above, Bradley (1997) asked the following question: assuming (1.8), what conditions are "minimally" sufficient to insure that (1.3) holds with $b_n \to \infty$ as $n \to \infty$? In dealing with this question from the point of view of counterexamples, Bradley noticed that there remains a slight "gap" between the properties of his construction and the assumption (1.6) of Theorem 1.2 and he made the following conjecture.

CONJECTURE 1.1. *Assume that* $\{X_k,\ k \in \mathbb{Z}\}$ *is a strictly stationary, strongly mixing sequence of random variables such that* $\mathbb{E}X_0 = 0$, $\mathbb{E}X_0^2 < \infty$, (1.8) *holds and*

$$\int_0^{\alpha_n} Q^2_{|X_0|}(u)\,du = o\left(\frac{1}{n}\right) \quad \textit{as } n \to \infty. \tag{1.9}$$

Then there is a sequence $b_n \to \infty$ *as* $n \to \infty$, *such that*

$$\frac{S_n}{b_n} \xrightarrow{\mathscr{D}} N \sim \mathscr{N}(0,1) \quad \text{as } n \to \infty.$$

In this paper, we prove the truth of this conjecture. This result enlarges the class of strong mixing sequences known to satisfy a central limit theorem. Moreover, this result is as sharp as possible according to the second class of examples constructed in Bradley (1997). In addition, we identify b_n. Our main result is the following.

THEOREM 1.3. *Suppose that* $\{X_k,\ k \in \mathbb{Z}\}$ *is a strictly stationary, centered, strong mixing sequence with finite second moment. Assume that* (1.8) *and* (1.9) *hold. Then*

$$\frac{S_n}{\sqrt{(\pi/2)}\mathbb{E}|S_n|} \xrightarrow{\mathscr{D}} N \sim \mathscr{N}(0,1) \quad \text{as } n \to \infty. \tag{1.10}$$

For the sake of applications, we give the following corollary in term of conditions imposed to mixing rates and to moments of individual summands. It extends the corresponding results of Ibragimov (1962).

COROLLARY 1.1. *Suppose that* $X := \{X_k, k \in \mathbb{Z}\}$ *is a strictly stationary, centered, strong mixing sequence which satisfies* (1.8). *In addition if:*

(i) X *has moments of order* $2+\delta$ *finite, for a* $\delta > 0$ *and*

$$n\alpha_n^{\delta/(2+\delta)} \to 0 \quad \text{as } n \to \infty, \tag{1.11}$$

or if

(ii) X *is bounded and*

$$n\alpha_n \to 0 \quad \text{as } n \to \infty, \tag{1.12}$$

then (1.10) *holds.*

REMARK 1.1. By analyzing our proofs, we can easily see that under the additional assumption $\lim_{n\to\infty}(\mathbb{E}S_n^2/n) = \sigma^2 > 0$, all our results of type (1.10) hold with the normalization $\sigma\sqrt{n}$ instead of $\sqrt{(\pi/2)}\mathbb{E}|S_n|$.

The rest of the paper deals with the functional central limit.

We define the process $\{W_n(t): t \in [0,1]\}$ by

$$W_n(t) = \frac{\sum_{i=1}^{[nt]} X_i}{\sqrt{(\pi/2)}\mathbb{E}|S_n|},$$

square brackets designating here and throughout the paper the integer part, as usual. For each ω, $W_n(\cdot)$ is an element of the Skorohod space $D([0,1])$ of all functions on [0, 1] which have left-hand limits and are continuous from the right. It is equipped with the Skorohod topology [see Billingsley (1968), Section 14]. W denotes the standard Brownian motion on [0, 1].

THEOREM 1.4. *Assume that the conditions of Theorem* 1.3 *are satisfied. Then* W_n *converges in distribution to* W *in the Skorohod space* $D([0,1])$.

The results stated in this section are proved in Section 3. They are based on some preliminary material collected in Section 2.

Throughout the paper, the notation $c_n \ll d_n$ means that $c_n = O(d_n)$.

2. Preparatory material. In this section we collect some preliminary material.

Before stating the first lemma, let us recall the Rio's covariance inequality: For (say) square-integrable random variables Y and Z, Rio [(1993), Theorem 1.1] proved the covariance inequality

$$|\mathrm{Cov}(Y,Z)| \leq 4\int_0^{\alpha} Q_{|Y|}(u)Q_{|Z|}(u)\,du, \tag{2.1}$$

where $\alpha := \alpha(\sigma(Y),\sigma(Z))$. [Here the notation $\sigma(\cdots)$ means the σ-field generated by $(\cdots)$.] The inequality actually proved by Rio was slightly sharper (by up to a certain constant factor), but (2.1) is more convenient for what follows.

The following lemma is a consequence of (2.1) and of Theorem 6.3 and relation (C.4) in Annex C in Rio (2000).

LEMMA 2.1. *Suppose that* $\{X_k,\ k\in\mathbb{Z}\}$ *is a strictly stationary sequence having moment of order 2 finite. Then for all* $m\geq 1$,

$$\mathbb{E}\left(\sum_{i=1}^{m}\{X_i - EX_i\}\right)^2 \leq m\,\mathrm{Var}X_0 + 8m\sum_{i=1}^{m-1}\int_0^{\alpha_i} Q^2_{|X_0|}(u)\,du.$$

If in addition $EX_0^4<\infty$ *then we can find two universal constants* C_1 *and* C_2 *such that*

$$\mathbb{E}\left(\max_{1\leq j\leq m}\left|\sum_{i=1}^{j}\{X_i - EX_i\}\right|^4\right) \leq C_1 m^2\left(\sum_{i=0}^{m-1}\int_0^{\alpha_i} Q^2_{|X_0|}(u)\,du\right)^2 + C_2 m\sum_{i=0}^{m-1}(i+1)^2\int_0^{\alpha_i} Q^4_{|X_0|}(u)\,du$$

The next lemma refers to the structure of the variance of partial sums.

LEMMA 2.2. *Assume that* $\{X_k,\ k\in\mathbb{Z}\}$ *is a strictly stationary sequence of random variables such that* $\mathbb{E}X_0=0$, $\mathbb{E}X_0^2<\infty$, (1.8) *holds and*

$$\lim_{n\to\infty} n\mathbb{E}(X_0X_n) = 0. \tag{2.2}$$

Then we have the representation

$$\sigma_n^2 = nh(n), \tag{2.3}$$

where $\sigma_n^2 = \mathbb{E}S_n^2$ *and* $h(n)$ *is a slowly varying function of* n. *Moreover,* $h(n)$ *has an extension to the whole real line which is slowly varying.*

PROOF. To prove (2.3), we first need to show that for every $j \geq 1$,

$$\lim_{n\to\infty} \frac{\sigma_{nj}^2}{\sigma_n^2} = j. \tag{2.4}$$

We shall prove this by induction.

First, it is obvious that (2.4) is true for $j = 1$; then assume that (2.4) is true for any $j < k$. To prove (2.4) for the rank k, we notice that by stationarity,

$$\sigma_{nk}^2 = \sigma_{n(k-1)}^2 + \sigma_n^2 + 2\mathbb{E}\left(\sum_{i=1}^{n} X_i \sum_{l=n+1}^{nk} X_i\right).$$

Now by using stationarity, easy computations lead to

$$\left|\mathbb{E}\left(\sum_{i=1}^{n} X_i \sum_{l=n+1}^{nk} X_l\right)\right| \leq \sum_{i=1}^{nk-1} i|\mathbb{E}(X_0 X_i)|,$$

which combined with (1.8) entails that

$$\left|\frac{\sigma_{nk}^2}{\sigma_n^2} - \frac{\sigma_{n(k-1)}^2}{\sigma_n^2} - 1\right| \ll \frac{\sum_{i=1}^{nk-1} i|\mathbb{E}(X_0 X_i)|}{n}.$$

This in turn together with (2.2) implies that the left-hand side of the above inequality is a $o(1)$ as $n \to \infty$.

We now finish the proof of (2.4) by using the recurrence assumption. It remains to show that $h(n)$ admits an extension to a slowly varying function of a continuous variable. We do not give the proof here since it is the same as the one in Ibragimov and Linnik [(1971), pages 327–328], with just trivial modifications, for example, involving the use of (1.8) and (2.2). $\square$

REMARK 2.1. By using (2.1), it is easy to see that (1.9) implies (2.2). It follows that the conclusions of Lemma 2.2 hold under the assumptions of Conjecture 1.1.

Before stating the next two lemmas which are technical in nature and needed for the proof of Conjecture 1.1, let us first give the following extension of the strong mixing coefficients α_n: let the sequence $(\alpha_n,\ n \in \mathbb{N})$ of dependence coefficients be extended to a continuous, nonincreasing function $\alpha(\cdot)$: $[0, \infty) \to (0, 1/4]$ with $\alpha(0) = 1/4$ and $\alpha(n) = \alpha_n$ for $n = 1, 2, 3, \ldots$. Moreover for each $n = 0, 1, 2, \ldots$, let $\alpha(x)$ be linear on $[n, n+1]$.

LEMMA 2.3. *For any nonincreasing on* $(0, 1)$ *positive function* $Q(\cdot)$ *satisfying* (1.9) *there exists a nonincreasing and continuous on* $(0, 1)$ *function* Q_* *which satisfies*

$$x \int_0^{\alpha(x)} Q_*^2(u)\, du \to 0 \quad \textit{as } x \to \infty \tag{2.5}$$

and such that

$$Q(u) \leq Q_*(u) \leq Q(u/2) \quad \textit{for all } u \in (0, 1). \tag{2.6}$$

PROOF. Let define the function Q_* as follows: for each $u \in (0, 1)$,

$$Q_*(u) := \frac{2}{u} \int_{u/2}^{u} Q(v)\, dv.$$

Since the function $Q(\cdot)$ is positive and nonincreasing on (0, 1), one has by elementary arguments that Q_* is nonincreasing and continuous on (0, 1) and satisfies (2.6). Then for each $n = 1, 2, 3, \ldots$,

$$\begin{aligned}\int_0^{\alpha(n)} Q_*^2(u)\, du &= 2 \int_0^{(1/2)\alpha(n)} Q_*^2(2u)\, du \\ &\le 2 \int_0^{(1/2)\alpha(n)} Q^2(u)\, du \\ &\le 2 \int_0^{\alpha(n)} Q^2(u)\, du,\end{aligned}$$

which shows that $n \int_0^{\alpha(n)} Q_*^2(u)\, du \to 0$ as $n \to \infty$ since $Q(\cdot)$ satisfies (1.9). Consequently, since we also have $(n+1) \int_0^{\alpha(n)} Q_*^2(u)\, du \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned}x \int_0^{\alpha(x)} Q_*^2(u)\, du &\le ([x]+1) \int_0^{\alpha([x])} Q_*^2(u)\, du \\ &\to 0 \quad \text{as } x \to \infty,\end{aligned}$$

which completes the proof of this lemma. □

From now on, we will always take $Q(\cdot) = Q_{|X_0|}(\cdot)$.

LEMMA 2.4. *Assume* (1.9) *holds. Then there exists a continuous, nondecreasing function* $a(\cdot)$: $[0, \infty) \to [1, \infty)$ *with* $a(x) \to \infty$ *and* $a(x+1)/a(x) \to 1$ *as* $x \to \infty$*, such that the following two relations are satisfied:*

$$\lim_{x \to \infty} a(x) x \int_0^{\alpha(x)} Q_*^2(u)\, du = 0 \tag{2.7}$$

and

$$\lim_{x \to \infty} a(x) x \alpha(x) Q_*^2(\alpha(x)) = 0, \tag{2.8}$$

where Q_* *is defined as in Lemma 2.3.*

PROOF. Notice first that because of Lemma 2.3, (1.9) implies (2.5) which in turn entails that $\lim_{x\to\infty} ax \int_0^{\alpha(x)} Q_*^2(u)\, du = 0$, for all $a > 0$. Then, we can construct a continuous, nondecreasing function $a(\cdot)$: $[0, \infty) \to [1, \infty)$ with $a(x) \to \infty$ and $a(x+1)/a(x) \to 1$ as $x \to \infty$ and such that (2.7) is satisfied.

Now since $Q_*(\cdot)$ is a nonincreasing function, we have

$$Q_*^2(\alpha(x)) \le Q_*^2(u) \quad \text{for all } 0 \le u \le \alpha(x),$$

which yields

$$a(x)x\alpha(x)Q_*^2(\alpha(x)) \le a(x)x\int_0^{\alpha(x)} Q_*^2(u)\,du.$$

This last inequality shows that (2.7) implies (2.8) and ends the proof of this lemma. $\square$

The next lemma is a consequence of Theorem 4 in Rio (1995).

LEMMA 2.5. *Let $\{X_n, n \ge 1\}$ be a sequence of real random variables such that, for each $n \ge 1$, $P(a_n \le X_n \le b_n) = 1$ where $a_n \le b_n$ are real numbers. Denote by $\mathscr{F}_1^n = \sigma(X_1, \dots, X_n)$. Then, we can redefine $\{X_n, n \ge 1\}$ onto a richer probability space on which there exists a sequence $\{X_n^*, n \ge 1\}$ of independent random variables such that, for each $n \ge 1$, X_n and X_n^* have the same distribution and*

$$\mathbb{E}(|X_n - X_n^*|) \le 2(b_n - a_n)\alpha\big(\mathscr{F}_1^{n-1}, \sigma(X_n)\big).$$

Moreover, for every $n > 1$, X_n^* and $(X_1, \dots, X_{n-1})$ are independent r.v.'s.

PROOF. Let $\{\delta_n\}_{n\ge1}$ be a sequence of independent random variables uniformly distributed on $[0, 1]$, independent of $\{X_n\}_{n\ge1}$. We redefine $\{X_n\}_{n\ge1}$ on a new probability space which supports $\{\delta_n\}_{n\ge1}$. Let $X_1^* = X_1$ and suppose that $X_1^*, \dots, X_{n-1}^*$ are defined. Then, we apply Theorem 4 in Rio (1995) with $\mathscr{A} = \mathscr{A}_{n-1} = \sigma(X_1, \dots, X_{n-1}, \delta_1, \dots, \delta_{n-1})$ and $X = X_n$. Thus there exists a random variable X_n^*, measurable with respect to the σ-field $\mathscr{A}_{n-1} \vee \sigma(X_n) \vee \sigma(\delta_n)$, independent of $\mathscr{A}_{n-1}$ and distributed as X_n, such that

$$\mathbb{E}(|X_n - X_n^*|) \le 2(b_n - a_n)\alpha\big(\mathscr{A}_{n-1}, \sigma(X_n)\big).$$

To finish the proof, we notice that by independence,

$$\alpha(\mathscr{A}_{n-1}, \sigma(X_n)) = \alpha\big(\mathscr{F}_1^{n-1}, \sigma(X_n)\big). \qquad \square$$

3. Proofs.

3.1. *Proof of Conjecture* 1.1 *and Theorem* 1.3. We shall start the proof by taking a continuous, nondecreasing function $a(\cdot)$: $[0, \infty) \to [1, \infty)$ with $a(x) \to \infty$ more slowly than $\ln(x)$ and $a(x+1)/a(x) \to 1$ as $x \to \infty$ and which in addition satisfies (2.7) and (2.8).

Now define $Q_*(\cdot)$ as in Lemma 2.3 and note that for $x \in [0, \infty)$, the quantity

$$a^2(x)x^2Q_*^2(\alpha(x))$$

is continuous, nonnegative and nondecreasing. In addition, once it becomes positive, it is strictly increasing since x^2 is strictly increasing and $a^2(x)$ and $Q_*^2(\alpha(x))$ are each nondecreasing. We also have $a^2(x)x^2Q_*^2(\alpha(x)) \to \infty$ as $x \to \infty$. Consequently for each $n = 1, 2, 3, \dots$, there exists a unique positive number θ_n such that

$$a^2(\theta_n)\theta_n^2Q_*^2(\alpha(\theta_n)) = n. \tag{3.1}$$

Notice that the number θ_n is not necessarily an integer. Note also that $\theta_n \to \infty$ as $n \to \infty$. Now let $q_n := [\theta_n] + 1$ and $p_n := [a(\theta_n)\theta_n] + 1$.

By construction, it is clear that p_n and q_n converge to infinity and that $q_n = o(p_n)$. We now divide the variables in big blocks of size p_n and small blocks of size q_n in the following way: let us set $k_n = [n/(p_n + q_n)]$. For a given positive integer n, the set $1, 2, \ldots, n$ is being partitioned into blocks of consecutive integers, the blocks being $I_1, J_1, \ldots, I_{k_n}, J_{k_n}, J_{k_n+1}$, such that for each $1 \leq j \leq k_n$, I_j contains p_n integers and J_j contains q_n integers, while j_{k_n+1} contains at most $(p_n + q_n - 1)$ integers.

Denote by $Y_j := \sum_{i \in I_j} X_i$ and $Z_j := \sum_{i \in J_j} X_i$ for $1 \leq j \leq k_n$ and $Z_{k_n+1} := \sum_{i \in J_{k_n+1}} X_i$ and let us truncate the variables X_i in the following way: Set $T_n = Q_{|X_0|}(\alpha(\theta_n))$ and

$$X'_i = \begin{cases} X_i I(|X_i| \leq T_n) - \mathbb{E} X_i I(|X_i| \leq T_n), & \text{if } X_i \text{ is an unbounded variable,} \\ X_i, & \text{if } \operatorname{ess\,sup} X_i = A \text{ a.s.} \end{cases}$$

(3.2) and

$$X''_i = \begin{cases} X_i I(|X_i| > T_n) - \mathbb{E} X_i I(|X_i| > T_n), & \text{if } X_i \text{ is an unbounded variable,} \\ 0, & \text{if } \operatorname{ess\,sup} X_i = A \quad \text{a.s.} \end{cases}$$

For $j = 1, 2, \ldots, k_n$, set $Y'_j := \sum_{i \in I_j} X'_i$, $Y''_j := \sum_{i \in I_j} X''_i$ and $Z'_j := \sum_{i \in J_j} X'_i$, $Z''_j := \sum_{i \in J_j} X''_i$ for $j = 1, 2, \ldots, k_n + 1$.

Now, let us consider sequences $\{Y'^*_j\}_{1 \leq j \leq k_n}$ and $\{Z'^*_j\}_{1 \leq j \leq k_n}$ of independent real random variables each distributed as Y'_j and Z'_j, respectively, and defined as in Lemma 2.5. Since

$$S_n = \sum_{j=1}^{k_n} Y'_j + \sum_{j=1}^{k_n+1} Z'_j + \sum_{i=1}^{n} X''_i,$$

it is easy to see that we have

$$\left| \frac{S_n}{b_n} - \frac{\sum_{j=1}^{k_n} Y'^*_j}{b_n} \right| \leq \frac{\left| \sum_{j=1}^{k_n} (Y'_j - Y'^*_j) \right|}{b_n} + \frac{\left| \sum_{j=1}^{k_n} (Z'_j - Z'^*_j) \right|}{b_n} \tag{3.3}$$
$$+ \frac{\sum_{i=1}^{n} |X''_i|}{b_n} + \frac{\left| \sum_{j=1}^{k_n} Z'^*_j \right|}{b_n} + \frac{\left| Z'_{k_n+1} \right|}{b_n},$$

where $b_n^2 = k_n \sigma_{p_n}^2$ with $\sigma_n^2 := \mathbb{E} S_n^2$.

We treat each term from the right-hand side of the above relation separately, to show that

$$\lim_{n \to \infty} \mathbb{E} \left| \frac{S_n}{b_n} - \frac{\sum_{j=1}^{k_n} Y'^*_j}{b_n} \right| = 0. \tag{3.4}$$

First, since for all $1 \le j \le k_n$, $|Y'_j| \le 2p_n Q_{|X_0|}(\alpha(\theta_n))$ and $|Z'_j| \le 2q_n Q_{|X_0|} \times (\alpha(\theta_n))$, Lemma 2.5 yields

$$\frac{\sum_{j=1}^{k_n} \mathbb{E}|Y'_j - Y'^*_j|}{b_n} \le \frac{8p_n k_n \alpha_{q_n} Q_{|X_0|}(\alpha(\theta_n))}{b_n}$$

and

$$\frac{\sum_{j=1}^{k_n} \mathbb{E}|Z'_j - Z'^*_j|}{b_n} \le \frac{8q_n k_n \alpha_{p_n} Q_{|X_0|}(\alpha(\theta_n))}{b_n}.$$

Now by using (1.8), (2.6), the definition of q_n and (3.1), we derive

$$\begin{aligned}\frac{p_n k_n \alpha_{q_n} Q_{|X_0|}(\alpha(\theta_n))}{b_n} &\ll \sqrt{n}\alpha(\theta_n) Q_*(\alpha(\theta_n)) \\ &\ll a(\theta_n)\theta_n \alpha(\theta_n) Q_*^2(\alpha(\theta_n)),\end{aligned}$$

which converges to 0 by (2.8).

Moreover, since $q_n \le p_n$ and (α_n) is decreasing, we also have

$$\frac{q_n k_n \alpha_{p_n} Q_{|X_0|}(\alpha(\theta_n))}{b_n} \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$\frac{\mathbb{E}\sum_{j=1}^{k_n} |Y'_j - Y'^*_j|}{b_n} \to 0 \quad \text{as } n \to \infty \tag{3.5}$$

and

$$\frac{\mathbb{E}\sum_{j=1}^{k_n} |Z'_j - Z'^*_j|}{b_n} \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

We now treat the last two terms in the extreme right-hand side of (3.3). First notice that for $r_n \ll p_n$, according to Lemma 2.1, we have

$$\begin{aligned}\operatorname{Var}\left(\frac{S''_{r_n}}{\sigma_{p_n}}\right) &\le \frac{r_n \operatorname{Var}(X''_1)}{\sigma^2_{p_n}} + \frac{8r_n \sum_{i=1}^{r_n} \int_0^{\alpha_i} Q^2_{|X_0|I(|X_0|>T_n)}(u)\,du}{\sigma^2_{p_n}} \\ &=: I_1 + I_2,\end{aligned} \tag{3.7}$$

where $S''_n := \sum_{i=1}^n X''_i$. First, by using (1.8), we derive that

$$I_1 \ll \frac{r_n}{p_n}\operatorname{Var}(X''_1) \ll \mathbb{E}(X_0^2 I(|X_0| > T_n)),$$

which converges to 0 as $n \to \infty$ since $\mathbb{E}X_0^2 < \infty$ and $T_n \to \infty$ in the unbounded case.

On the other hand, in order to analyze the second term in (3.7), notice that

$$Q_{|X_0|I(|X_0|>T_n)}(u) = \begin{cases} Q_{|X_0|}(u), & \text{if } u < \alpha(\theta_n), \\ 0, & \text{if } u \ge \alpha(\theta_n), \end{cases} \tag{3.8}$$

whence

$$\sum_{i=1}^{r_n}\int_0^{\alpha_i} Q^2_{|X_0|I(|X_0|>T_n)}(u)\,du \le r_n\int_0^{\alpha(\theta_n)} Q^2_{|X_0|}(u)\,du.$$

By using (1.8), the definition of p_n and (2.6), we find that

$$\begin{aligned} I_2 &\ll p_n\int_0^{\alpha(\theta_n)} Q^2_{|X_0|}(u)\,du\\ &\ll \alpha(\theta_n)\theta_n\int_0^{\alpha(\theta_n)} Q^2_*(u)\,du, \end{aligned}$$

which converges to 0 by (2.7).

Finally, by the above considerations, for $r_n \ll p_n$ we get

$$\operatorname{Var}\left(\frac{S''_{r_n}}{\sigma_{p_n}}\right) \to 0 \quad \text{as } n\to\infty. \tag{3.9}$$

Then, if we denote by $\sigma''^2_n := \mathbb{E}(S''_n)^2$, we have particularly shown both

$$\frac{\sigma''_{p_n}}{\sigma_{p_n}} \to 0 \quad \text{as } n\to\infty \tag{3.10}$$

and

$$\frac{\sigma''_{q_n}}{\sigma_{p_n}} \to 0 \quad \text{as } n\to\infty. \tag{3.11}$$

Moreover, (3.10) also implies that

$$\frac{\sigma'_{p_n}}{\sigma_{p_n}} \to 1 \quad \text{as } n\to\infty, \tag{3.12}$$

where $\sigma'^2_n := \mathbb{E}(S'_n)^2$. Notice now that

$$I := \mathbb{E}\left(\frac{\sum_{j=1}^{k_n} Z'^*_j}{b_n}\right)^2 = \frac{k_n\sigma'^2_{q_n}}{b_n^2} = \frac{\sigma'^2_{q_n}}{\sigma^2_{p_n}}.$$

However, by (2.3) and from the standard properties of $h(n)$, we obtain

$$\frac{\sigma^2_{q_n}}{\sigma^2_{p_n}} = \frac{q_n h(q_n)}{p_n h(p_n)} \ll \left(\frac{q_n}{p_n}\right)^{1-\varepsilon} \quad \text{for every } \varepsilon>0. \tag{3.13}$$

Since $q_n = o(p_n)$, it follows that

$$\frac{\sigma^2_{q_n}}{\sigma^2_{p_n}} \to 0 \quad \text{as } n\to\infty,$$

which combined with (3.11) shows that

$$I = o(1) \quad \text{as } n\to\infty. \tag{3.14}$$

Moreover, if we denote by l_n the number of terms in Z_{k_n+1}, using once again (2.3), the standard properties of $h(n)$ and the fact that $l_n \ll p_n$, we derive

$$\frac{\sigma_{l_n}^2}{k_n \sigma_{p_n}^2} = \frac{l_n h(l_n)}{k_n p_n h(p_n)} \ll \left(\frac{l_n}{p_n}\right)^{1-\varepsilon} \frac{1}{k_n} \quad \text{for every } \varepsilon > 0 \tag{3.15}$$
$$= o(1) \quad \text{as } n \to \infty.$$

This relation, together with the fact that, by (3.9), $\lim_{n\to\infty}(\sigma_{l_n}''^2/\sigma_{p_n}^2) = 0$ yields

$$\lim_{n\to\infty} \frac{\mathbb{E}Z_{k_n+1}'^2}{b_n^2} = 0. \tag{3.16}$$

It remains to look at $(\mathbb{E}\sum_{i=1}^n |X_i''|/b_n)$. By using stationarity, the definition of b_n and (1.8), we derive that

$$\frac{\mathbb{E}\sum_{i=1}^n |X_i''|}{b_n} \le \frac{2n\mathbb{E}|X_0|I(|X_0| > T_n)}{b_n} \ll \sqrt{n}\mathbb{E}|X_0|I(|X_0| > T_n).$$

However, it is well known that if U is a random variable uniformly distributed on the interval [0, 1] and if W is a nonnegative random variable, then the r.v. $Q_W(U)$ has the same distribution as the r.v. W, and then

$$\mathbb{E}(W) = \int_0^1 Q_W(u)\,du.$$

This last equality applied to the r.v. $|X_0|I(|X_0| > T_n)$ together with (3.8) yield

$$\mathbb{E}|X_0|I(|X_0| > T_n) = \int_0^{\alpha(\theta_n)} Q_{|X_0|}(u)\,du.$$

Then the Cauchy–Schwarz inequality entails that

$$\frac{\mathbb{E}\sum_{i=1}^n |X_i''|}{b_n} \ll \sqrt{n\alpha(\theta_n)\int_0^{\alpha(\theta_n)} Q_{|X_0|}^2(u)\,du}.$$

By using now (3.1) combined with (2.6), (2.7) and (2.8), we find that

$$\lim_{n\to\infty} \frac{\mathbb{E}\sum_{i=1}^n |X_i''|}{b_n} = 0. \tag{3.17}$$

Finally (3.4) follows by combining (3.5) with (3.6), (3.17), (3.14) and (3.16).

Let us concentrate now on the limiting behavior of $\{\sum_{j=1}^{k_n} Y_j'^*\}_{n\ge1}$. As a consequence of stationarity and of (3.12), it follows that

$$\lim_{n\to\infty} \frac{\operatorname{Var}(\sum_{j=1}^{k_n} Y_j'^*)}{b_n^2} = 1. \tag{3.18}$$

We just have to check the Liapunov condition; that is,

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{k_n} \mathbb{E}(Y_j'^*)^4}{b_n^4} = 0. \tag{3.19}$$

Because of stationarity, in order to verify (3.19), we shall apply Lemma 2.1 which gives

$$\mathbb{E}\biggl(\sum_{i=1}^{p_n} X_i'\biggr)^4 \ll p_n^2\biggl(\sum_{i=0}^{p_n}\int_0^{\alpha_i} Q^2_{|X_0|I(|X_0|\le T_n)}(u)\,du\biggr)^2 + p_n\sum_{i=0}^{p_n}(i+1)^2\int_0^{\alpha_i} Q^4_{|X_0|I(|X_0|\le T_n)}(u)\,du.$$

Next, by using the facts that $Q_{|X_0|I(|X_0|\le T_n)}(u) \le Q_{|X_0|}(u)$, $Q^4_{|X_0|I(|X_0|\le T_n)}(u) \le T_n^2 Q^2_{|X_0|}(u)$, $\int_0^{\alpha_i} Q^2_{|X_0|}(u)\,du = o(1/i)$ and since $\ln p_n = o(\sqrt{p_n}Q_{|X_0|}(\alpha_{q_n}))$ we obtain

$$\mathbb{E}\biggl(\sum_{i=1}^{p_n} X_i'\biggr)^4 \ll p_n^2(\ln p_n)^2 + p_n T_n^2\sum_{i=0}^{p_n}(i+1)^2\int_0^{\alpha_i} Q^2_{|X_0|}(u)\,du = o\bigl(p_n^3T_n^2\bigr). \tag{3.20}$$

Now by the definition of b_n^2, of p_n and by using (1.8), (2.6) and (3.1), we derive

$$\frac{k_n\mathbb{E}\bigl(\sum_{i=1}^{p_n}X_i'\bigr)^4}{b_n^4} = o\biggl(\frac{p_n^2Q_*^2(\alpha(\theta_n))}{n}\biggr) = o(1). \tag{3.21}$$

Then the classical Liapunov's theorem [see, e.g., Theorem 7.3 in Billingsley (1968)] yields

$$\frac{\sum_{j=1}^{k_n}Y_j'^*}{b_n} \xrightarrow{\mathscr{D}} N \sim \mathscr{N}(0,1) \quad \text{as } n\to\infty, \tag{3.22}$$

which combined with (3.4) ends the proof of the conjecture.

Now we prove that

$$\lim_{n\to\infty}\frac{b_n}{\sqrt{\pi/2}\,\mathbb{E}\bigl|\sum_{j=1}^{k_n}Y_j'^*\bigr|} = 1. \tag{3.23}$$

To do this, it is enough to show that $\bigl\{\sum_{j=1}^{k_n}Y_j'^*/b_n\bigr\}_{n\ge1}$ is a uniformly integrable family. Indeed recall that by Theorem 5.4 in Billingsley (1968), if $\sum_{j=1}^{k_n}Y_j'^*/b_n \to^{\mathscr{D}} N$ as $n\to\infty$ and if $\bigl\{\sum_{j=1}^{k_n}Y_j'^*/b_n\bigr\}_{n\ge1}$ is a uniformly integrable family, then $\mathbb{E}\bigl|\sum_{j=1}^{k_n}Y_j'^*\bigr| \to \mathbb{E}|N|$ as $n\to\infty$, and in our case $\mathbb{E}|N| = \sqrt{2/\pi}$. Then notice that (3.23) holds since (3.18) entails that $\bigl\{\sum_{j=1}^{k_n}Y_j'^*/b_n\bigr\}_{n\ge1}$ is a uniformly integrable family.

We finish the proof of Theorem 1.3 by using (3.4) together with (3.23) which yields

$$\lim_{n\to\infty}\frac{b_n}{\sqrt{\pi/2}\,\mathbb{E}|S_n|} = 1$$

and consequently the desired result. □

3.2. *Proof of Theorem* 1.4. In order to derive the functional form of the central limit theorem, we notice that under strong mixing, $W_n(t)$ has asymptotically independent increments [Lemma 1.1 in Peligrad (1986) could be used in order to see this]. Then the central limit theorem will imply that the finite-dimensional distributions of $W_n(t)$ will converge to the corresponding ones of the Brownian motion. By Prohorov's theorem [see Billingsley (1968), Theorem 6.1], the functional form of the central limit theorem will result by proving the tightness of $W_n(t)$. We shall make use of the same notation from the proof of Conjecture 1.1.

Denote by $W'_n(t) = \sum_{i=1}^{[nt]} X'_i/b_n$ and by $W''_n(t) = \sum_{i=1}^{[nt]} X''_i/b_n$, where X'_i and X''_i designate the truncated variables as described in (3.2).

Let us prove first that $W''_n(t)$ is negligible for the weak convergence: for all $\varepsilon > 0$, we have

$$P\left(\sup_{0\le t\le 1}\left|W''_n(t)\right| \ge \varepsilon\right) \le P\left(\frac{\sum_{i=1}^n \left|X''_i\right|}{b_n} \ge \varepsilon\right) \le \frac{1}{\varepsilon}\frac{\mathbb{E}\sum_{i=1}^n \left|X''_i\right|}{b_n}, \tag{3.24}$$

which is convergent to 0 by (3.17).

Now take p_n and q_n as defined in the beginning of the proof of Conjecture 1.1. Set $k_{nt} = \left[[nt]/(p_n + q_n)\right]$ and divide the sequence of random variables $\{X'_n\}$ in big and small blocks as in the proof of Conjecture 1.1. We obviously have

$$\frac{\sum_{i=1}^{[nt]} X'_i}{b_n} = \frac{\sum_{j=1}^{k_{nt}} Y'_j}{b_n} + \frac{\sum_{j=1}^{k_{nt}} Z'_j}{b_n} + \frac{R'_{n,t}}{b_n}, \tag{3.25}$$

where

$$R'_{n,t} := \sum_{i=1}^{[nt]} X'_i - \left(\sum_{j=1}^{k_{nt}} Y'_j + \sum_{j=1}^{k_{nt}} Z'_j\right).$$

Note that $R'_{n,t}$ is a sum of at most $p_n + q_n$ consecutive X'_i's.

Let us consider now the independent variables $\{Y'^*_j\}_{i\le j\le k_{nt}}$ (respectively, $\{Z'^*_j\}_{1\le j\le k_{nt}}$), each distributed as Y'_j (respectively, Z'_j) and defined as in Lemma 2.5. Notice that for all $\varepsilon > 0$, Markov's inequality yields

$$P\left(\sup_{0\le t\le 1}\frac{\left|\sum_{j=1}^{k_{nt}} Y'_j - \sum_{j=1}^{k_{nt}} Y'^*_j\right|}{b_n} \ge \varepsilon\right) \le P\left(\frac{\sum_{j=1}^{k_{nt}} |Y'_j - Y'^*_j|}{b_n} \ge \varepsilon\right) \le \frac{\mathbb{E}(\sum_{j=1}^{k_n} |Y'_j - Y'^*_j|)}{\varepsilon b_n} \tag{3.26}$$

which converges to 0 by (3.5).

Similarly to (3.6), for all $\varepsilon > 0$ we derive that

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le 1} \frac{|\sum_{j=1}^{k_{nt}} Z'_j - \sum_{j=1}^{k_{nt}} Z'^*_j|}{b_n} \ge \varepsilon\right) = 0. \tag{3.27}$$

Moreover, by stationarity and Markov's inequality, for all $\varepsilon > 0$ we have

$$\begin{aligned} P\left(\sup_{0\le t\le 1} \frac{|R'_{n,t}|}{b_n} \ge \varepsilon\right) &\le (k_n+1) P\left(\max_{1\le i\le p_n+q_n} \frac{|\sum_{j=1}^{i} X'_j|}{b_n} \ge \varepsilon\right) \\ &\le \frac{(k_n+1)}{\varepsilon^4 b_n^4} \mathbb{E}\left(\max_{1\le i\le p_n+q_n}\left|\sum_{j=1}^{i} X'_j\right|^4\right) \end{aligned}$$

By Lemma 2.1 and since $q_n \le p_n$, we obtain

$$\begin{aligned} \frac{k_n}{b_n^4}\mathbb{E}\left(\max_{1\le i\le p_n+q_n}\left|\sum_{j=1}^{i} X'_j\right|^4\right) \ll \frac{k_n}{b_n^4}\Biggl\{ & p_n^2\left(\sum_{i=0}^{2p_n}\int_0^{\alpha_i} Q^2_{|X_0|I(|X_0|\le T_n)}(u)\,du\right)^2 \\ & + p_n\sum_{i=0}^{2p_n}(i+1)^2\int_0^{\alpha_i} Q^4_{|X_0|I(|X_0|\le T_n)}(u)\,du\Biggr\}, \end{aligned}$$

which converges to zero as $n \to \infty$ by involving the arguments used to prove (3.21). This shows that for all $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le 1} \frac{|R'_{n,t}|}{b_n} \ge \varepsilon\right) = 0. \tag{3.28}$$

Combining (3.25), (3.26), (3.27) and (3.28), we reduced the problem to prove the tightness of the random elements $\{\sum_{j=1}^{k_{nt}} Y'^*_j/b_n\}$ and $\{\sum_{j=1}^{k_{nt}} Z'^*_j/b_n\}$ which are based on sums of independent random variables.

According to Theorem 8.3 in Billingsley (1968) formulated for random elements in $D([0,1])$ as in Billingsley [(1968), page 137], by using the stationarity it is enough to prove that for all $\varepsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{\delta}P\left(\sup_{0\le t\le \delta}\frac{|\sum_{j=1}^{k_{nt}} Y'^*_j|}{b_n} \ge \varepsilon\right) = 0 \tag{3.29}$$

and

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{\delta}P\left(\sup_{0\le t\le \delta}\frac{|\sum_{j=1}^{k_{nt}} Z'^*_j|}{b_n} \ge \varepsilon\right) = 0. \tag{3.30}$$

Notice first that for all $\varepsilon > 0$, Markov's inequality leads to

$$\begin{aligned} P\left(\sup_{0\le t\le \delta}\frac{|\sum_{j=1}^{k_{nt}} Y'^*_j|}{b_n} \ge \varepsilon\right) &= P\left(\max_{1\le i\le k_{n\delta}}\frac{|\sum_{j=1}^{i} Y'^*_j|}{b_n} > \varepsilon\right) \\ &\le \frac{\mathbb{E}(\max_{1\le i\le k_{n\delta}}|\sum_{j=1}^{i} Y'^*_j|^4)}{\varepsilon^4 b_n^4}. \end{aligned} \tag{3.31}$$

Now since the $\{Y_j'^*\}$ are independent, the upper bound of Rosenthal's inequality [see, e.g., Theorem 2.11 in Hall and Heyde (1980)] yields

$$\mathbb{E}\left(\max_{1\le i\le k_{n\delta}}\left|\sum_{j=1}^{i} Y_j'^*\right|^4\right) \ll \left(\mathbb{E}\left(\sum_{j=1}^{k_{n\delta}} Y_j'^*\right)^2\right)^2 + \mathbb{E}\left(\max_{1\le i\le k_{n\delta}} |Y_i'^*|^4\right)$$
$$\le k_{n\delta}^2\left(\mathbb{E}\left(\sum_{i=1}^{p_n} X_i'\right)^2\right)^2 + k_{n\delta}\mathbb{E}\left(\sum_{i=1}^{p_n} X_i'\right)^4.$$

Then the definition of b_n and of $k_{n\delta}$ combined with (3.21) and (3.12) entails that

$$\frac{\mathbb{E}(\max_{1\le i\le k_{n\delta}}|\sum_{j=1}^{i} Y_j'^*|^4)}{b_n^4} \ll \frac{k_{n\delta}^2\big(\mathbb{E}(S_{p_n})^2\big)^2}{k_n^2\big(\mathbb{E}(S_{p_n})^2\big)^2} + \frac{k_{n\delta}\mathbb{E}\Big(\sum_{i=1}^{p_n} X_i'\Big)^4}{b_n^4}$$
$$\ll \delta^2 + \delta\frac{k_n\mathbb{E}\big(\sum_{i=1}^{p_n} X_i'\big)^4}{b_n^4}$$
$$= \delta(\delta + o(1)) \quad \text{as } n\to\infty.$$

This last result combined with (3.31) leads to (3.29). Similarly, (3.30) holds and the tightness is proved. The only difference in the proof is that

$$\frac{\mathbb{E}\big(\max_{1\le i\le k_{n\delta}}|\sum_{j=1}^{i} Z_j'^*|^4\big)}{b_n^4} \ll \delta\left(\delta\frac{\big(\mathbb{E}\big(S'_{q_n}\big)^2\big)^2}{\big(\mathbb{E}\big(S_{p_n}\big)^2\big)^2} + o(1)\right) \quad \text{as } n\to\infty$$

and then we use (3.14) to prove the convergence to zero of the extreme right-hand side of the above inequality.

Finally, we end the proof by using the fact that $\lim_{n\to\infty}(b_n/\sqrt{\pi/2}\mathbb{E}|S_n|)=1$. $\square$

Acknowledgments. The authors thank Richard Bradley for useful discussions and the referees not only for carefully reading the manuscript but also for having pointed out a gap in a previous version of the paper and giving various suggestions to overcome this difficulty.

REFERENCES

BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

BRADLEY, R. C. (1997). On quantiles and the central limit theorem question for strongly mixing sequences. *J. Theoret. Probab.* **10** 507–555.

DEHLING, H., DENKER, M. and PHILIPP, W. (1986). Central limit theorems for mixing sequences of random variables under minimal conditions. *Ann. Probab.* **14** 1359–1370.

DENKER, M. (1986). Uniform integrability and the central limit theorem for strongly mixing processes. In *Dependence in Probability and Statistics. A Survey of Recent Results* (E. Eberlein and M. S. Taqqu, eds.). Birkhäuser, Boston.

DOUKHAN, P., MASSART, P. and RIO, E. (1994). The functional central limit theorem for strongly mixing processes. *Ann. Inst. H. Poincaré Probab. Statist.* **30** 63–82.

HALL, P. and HEYDE, C. C. (1980). *Martingale Limit Theory and Its Application*. Academic Press, New York.

HERRNDORF, N. (1985). A functional central limit theorem for strongly mixing sequences of random variables. *Z. Wahrsch. Verw. Gebiete* **69** 541–550.

IBRAGIMOV, I. A. (1962). Some limit theorems for stationary processes. *Teor. Verojatnost. i Primenen.* **7** 361–392.

IBRAGIMOV, I. A. and LINNIK, YU. V. (1971). *Independent and Stationary Sequences of Random Variables*. Wolters-Noorhoff, Groningen.

IBRAGIMOV, I. A. and ROZANOV, IU. A. (1978). *Gaussian Random Processes.* Springer, Berlin.

MORI, T. and YOSHIHARA, K. (1986). A note on the central limit theorem for stationary strong-mixing sequences. *Yokohama Math J.* **35** 143–146.

PELIGRAD, M. (1986). Recent advances in the central limit theorem and its weak invariance principles for mixing sequences of random variables. In *Dependence in Probability and statistics. A Survey of Recent Results* (E. Eberlein and M. S. Taqqu, eds.). Birkhäuser, Boston.

PELIGRAD, M. (1992). On the central limit theorem for decomposed strong mixing sequences. *Stochastic Process Appl.* **42** 181–193.

RIO, E. (1993). Covariance inequalities for strongly mixing processes. *Ann. Inst. H. Poincaré Probab. Statist.* **29** 587–597.

RIO, E. (1995). The functional law of the iterated logarithm for stationary strongly mixing sequences. *Ann. Probab.* **23** 1188–1203.

RIO, E. (2000). Théorie asymptotique des processus aléatoires faiblement dépendants. In *Mathématiques et Applications de la SMAI.* **31**. Springer, Berlin.

ROSENBLATT, M. (1956). A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci. U.S.A.* **42** 43–47.

LSTA, UNIVERSITÉ PARIS VI
TOUR 45-55, 4 PLACE JUSSIEU
75252 PARIS CEDEX 5
FRANCE
E-MAIL: merleve@ccr.jussieu.fr

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF CINCINNATI
P.O. BOX 210025
CINCINNATI, OHIO 45221-0025
E-MAIL: peligrm@math.uc.edu