Strong approximation of partial sums under dependence conditions with application to dynamical systems

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Abstract

In this paper, we obtain precise rates of convergence in the strong invariance principle for stationary sequences of real-valued random variables satisfying weak dependence conditions including strong mixing in the sense of Rosenblatt (1956) as a special case. Applications to unbounded functions of intermittent maps are given.

1 Introduction

The almost sure invariance principle is a powerful tool in both probability and statistics. It says that the partial sums of random variables can be approximated by those of independent Gaussian random variables, and that the approximation error between the trajectories of the two processes is negligible compared to their size. More precisely, when $(X_i)_{i\geq 1}$ is a sequence of i.i.d. centered real valued random variables with a finite second moment, a sequence $(Z_i)_{i\geq 1}$ of i.i.d. centered Gaussian variables may be constructed is such a way that

$$\sup_{1\leq k\leq n} \Big|\sum_{i=1}^{k}(X_i - Z_i)\Big| = o(a_n) \text{ almost surely,} \tag{1.1}$$

where $(a_n)_{n\geq 1}$ is a nondecreasing sequence of positive reals tending to infinity. The first result of this type is due to Strassen (1964) who obtained (1.1) with $a_n = (n \log\log n)^{1/2}$. To get smaller (a_n) additional information on the moments of X_1 is necessary. If $\mathbb{E}|X_1|^p < \infty$ for p in

$]2,4[$, by using the Skorohod embedding theorem, Breiman (1967) showed that (1.1) holds with $a_n = n^{1/p}(\log n)^{1/2}$. He also proved that $a_n = n^{1/p}$ cannot be improved under the p-th moment assumption for any $p > 2$. The Breiman paper highlights the fact that there is a gap between the direct result and its converse when using the Skorohod embedding. This gap was later filled by Komlós, Major and Tusnády (1976) for $p > 3$ and by Major (1976) for p in $]2,3]$: they obtained (1.1) with $a_n = n^{1/p}$ as soon as $\mathbb{E}|X_1|^p < \infty$ for any $p > 2$, using an explicit construction of the Gaussian random variables, based on quantile transformations.

There has been a great deal of work to extend these results to dependent sequences: see for instance Philipp and Stout (1975), Berkes and Philipp (1979), Dabrowski (1982), Bradley (1983), Shao (1993), Eberlein (1986), Wu (2007), Zhao and Woodroofe (2008), Gouëzel (2010), Berkes, Hörmann and Schauer (2010), and Cuny (2011), among others, for extensions of (1.1) under various dependence conditions.

In this paper, we are interested in the case of strictly stationary strongly mixing sequences. Recall that the strong mixing coefficient of Rosenblatt (1956) between two σ-algebras $\mathcal{F}$ and $\mathcal{G}$ is defined by

$$\alpha(\mathcal{F},\mathcal{G}) = \sup_{A\in\mathcal{F},B\in\mathcal{G}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| .$$

For a strictly stationary sequence $(X_i)_{i\in\mathbb{Z}}$ of real valued random variables, and the σ-algebra $\mathcal{F}_0 = \sigma(X_i, i \le 0)$ and $\mathcal{G}_n = \sigma(X_i, i \ge n)$, define then

$$\alpha(0) = 1 \text{ and } \alpha(n) = 2\alpha(\mathcal{F}_0,\mathcal{G}_n) \text{ for } n > 0 . \tag{1.2}$$

Concerning the extension of (1.1) in the strong mixing setting, Rio (1995-a) proved the following: assume that

$$\sum_{k=0}^{\infty} \int_0^{\alpha(k)} Q^2_{|X_0|}(u)du < \infty , \tag{1.3}$$

where $Q_{|X_0|}$ is given in Definition 2.1. Then the series $\mathbb{E}(X_0^2) + 2\sum_{k\ge1}\mathbb{E}(X_0X_k)$ is convergent to a nonnegative real σ^2 and one can construct a sequence $(Z_i)_{i\ge1}$ of zero mean i.i.d. Gaussian variables with variance σ^2 such that (1.1) holds true with $a_n = (n\log\log n)^{1/2}$. As shown in Theorem 3 of Rio (1995-a), the condition (1.3) cannot be improved. Recently Dedecker, Gouëzel and Merlevède (2010) proved that this result still holds if we replace the Rosenblatt strong mixing coefficients $\alpha(n)$ by the weaker coefficients defined in (2.1), provided that the underlying sequence is ergodic.

Still in the strong mixing setting, the best extension, up to our knowledge, of the Komlós, Major and Tusnády results is due to Shao and Lu (1987). Applying the Skorohod embedding, they obtained the following result (see also Corollary 9.3.1 in Lin and Lu (1996)): Let $p \in]2,4]$

and $r > p$. Assume that

$$\mathbb{E}(|X_0|^r) < \infty \quad \text{and} \quad \sum_{n\geq 1}(\alpha(n))^{(r-p)/(rp)} < \infty\,. \tag{1.4}$$

Then the series $\mathbb{E}(X_0^2) + 2\sum_{k\geq 1}\mathbb{E}(X_0X_k)$ is convergent to a nonnegative real σ^2 and one can construct a sequence $(Z_i)_{i\geq 1}$ of zero mean i.i.d. Gaussian variables with variance σ^2 such that (1.1) holds true with $a_n = n^{1/p}(\log n)^{1+(1+\lambda)/p}$, where $\lambda = (\log 2)/\log(r/(r-2))$.

Comparing (1.4) with (1.3) when p is close to 2, there appears to be a gap between the two above results. A reasonable conjecture is that Shao and Lu's result still holds under the weaker condition

$$\mathbb{E}(|X_0|^p) < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} k^{p-2}\int_0^{\alpha(k)} Q^p_{|X_0|}(u)du < \infty\,, \tag{1.5}$$

since the Rosenthal inequality of order p is true under (1.5) (see Theorem 6.3 in Rio (2000)) and may fail to hold if this condition is not satisfied (see Rio (2000), chapter 9). To compare (1.5) with (1.4), note that (1.5) is implied by: for $r > p$,

$$\sup_{x>0} x^r\mathbb{P}(|X_0| > x) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} n^{p-2}(\alpha(n))^{(r-p)/r} < \infty\,,$$

which is much weaker than (1.4). For example, in the case of bounded random variables ($r = \infty$), (1.4) needs $\alpha(n) = O(n^{-p})$, while (1.5) holds as soon as $\alpha(n) = O(n^{1-p}(\log n)^{-1-\varepsilon})$ for some positive ε.

Let us now give an outline of our results and methods of proofs. Our main result is Theorem 2.1, which ensures in particular that, for $p \in]2,3[$, (1.1) holds for $a_n = n^{1/p}(\log n)^{1/2-1/p}$ under (1.5). Furthermore the error in $\mathbb{L}^2$ is of the same order. The proof of our Theorem 2.1 is based on an explicit construction of the approximating sequence of i.i.d. Gaussian random variables with the help of conditional quantile transformations. The Gaussian random variables are constructed in such a way that the error of approximation in $\mathbb{L}^2$ between dyadic blocks of the initial sequence and the Gaussian one is exactly the expectation of the Wasserstein distance of order 2 between the corresponding conditional law of the initial sequence and the Gaussian one (see Definition 5.1 and the equality (4.4)). We then prove a conditional version of a functional inequality due to Rio (1998) (see our Lemma 5.1), allowing us to use the Lindeberg method to derive then suitable bounds for the $\mathbb{L}^2$-approximating error between blocks of the initial sequence and the Gaussian one (see our Proposition 5.1). This method allows us to get a smaller logarithmic factor than the extra factor $(\log n)^{1/2}$ induced by the Skorohod embedding. Moreover, it is possible to adapt it (by conditioning up to the future rather than to the past) to deal with the partial sums of non necessarily bounded functions f of iterates of expanding maps such as those considered in

Section 3. For such maps, Theorem 3.1 complements results obtained by Melbourne and Nicol (2005, 2009) when f is Hölder continuous. The rest of the paper is organized as follows: Section 4 is devoted to the proof of the main results whereas the technical tools are stated and proven in Appendix.

2 Definitions and main result

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Assume that there exists some strictly stationary sequence $(Y_i)_{i\in\mathbb{Z}}$ of real valued random variables on this probability space, and that the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is large enough to contain a sequence $(\delta_i)_{i\in\mathbb{Z}}$ of independent random variables with uniform distribution over $[0,1]$, independent of $(Y_i)_{i\in\mathbb{Z}}$. Define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = \sigma((Y_k, \delta_k) : k \le i)$. Let $\mathcal{F}_{-\infty} = \bigcap_{i\in\mathbb{Z}} \mathcal{F}_i$ and $\mathcal{F}_\infty = \bigvee_{i\in\mathbb{Z}} \mathcal{F}_i$. We shall denote by $\mathbb{E}_i$ the conditional expectation with respect to $\mathcal{F}_i$.

In this section we give rates of convergence in the almost sure and $\mathbb{L}^2$ invariance principle for functions of a stationary sequence $(Y_i)_{i\in\mathbb{Z}}$ satisfying weak dependence conditions that we specify below.

Definition 2.1. *For any nonnegative random variable X, define the "upper tail" quantile function Q_X by $Q_X(u) = \inf\{t \ge 0 : \mathbb{P}(X > t) \le u\}$.*

This function is defined on $[0,1]$, non-increasing, right continuous, and has the same distribution as X. This makes it very convenient to express the tail properties of X using Q_X. For instance, for $0 < \varepsilon < 1$, if the distribution of X has no atom at $Q_X(\varepsilon)$, then

$$\mathbb{E}(X \mathbb{1}_{X > Q_X(\varepsilon)}) = \sup_{\mathbb{P}(A)\le\varepsilon} \mathbb{E}(X\mathbb{1}_A) = \int_0^\varepsilon Q_X(u)du\,.$$

Definition 2.2. *Let μ be the probability distribution of a random variable X. If Q is an integrable quantile function, let $\widetilde{\mathrm{Mon}}(Q,\mu)$ be the set of functions g which are monotonic on some open interval of $\mathbb{R}$ and null elsewhere and such that $Q_{|g(X)|} \le Q$. Let $\widetilde{\mathcal{F}}(Q,\mu)$ be the closure in $\mathbb{L}^1(\mu)$ of the set of functions which can be written as $\sum_{\ell=1}^L a_\ell f_\ell$, where $\sum_{\ell=1}^L |a_\ell| \le 1$ and f_ℓ belongs to $\widetilde{\mathrm{Mon}}(Q,\mu)$.*

We now recall the definition of the dependent coefficients as considered in Dedecker, Gouëzel and Merlevède (2010).

Definition 2.3. *For any integrable random variable X, let us write $X^{(0)} = X - \mathbb{E}(X)$. For any random variable $Y = (Y_1, \cdots, Y_k)$ with values in $\mathbb{R}^k$ and any σ-algebra $\mathcal{F}$, let*

$$\alpha(\mathcal{F}, Y) = \sup_{(x_1,\ldots,x_k)\in\mathbb{R}^k} \Big\| \mathbb{E}\Big(\prod_{j=1}^k (\mathbb{1}_{Y_j \le x_j})^{(0)} \Big| \mathcal{F}\Big)^{(0)} \Big\|_1 .$$

For the sequence $\mathbf{Y} = (Y_i)_{i\in\mathbb{Z}}$, *let*

$$\alpha_{k,\mathbf{Y}}(0) = 1 \text{ and } \alpha_{k,\mathbf{Y}}(n) = \max_{1\le l\le k} \sup_{n\le i_1\le \ldots \le i_l} \alpha(\mathcal{F}_0, (Y_{i_1}, \ldots, Y_{i_l})) \text{ for } n > 0\,. \tag{2.1}$$

Remark 2.1. In the sequel, these coefficients will be considered for $k = 2$. In this case, for reader convenience notice that for any positive n,

$$\frac{1}{3}\tilde{\alpha}_{2,\mathbf{Y}}(n) \le \alpha_{2,\mathbf{Y}}(n) \le 3\tilde{\alpha}_{2,\mathbf{Y}}(n)\,,$$

where $\tilde{\alpha}_{2,\mathbf{Y}}(0) = 1$ and, for $n > 0$,

$$\tilde{\alpha}_{2,\mathbf{Y}}(n) = \sup_{i\ge j\ge n} \sup_{(x,y)\in\mathbb{R}^2} \|\mathbb{P}(Y_i \le x, Y_j \le y|\mathcal{F}_0) - \mathbb{P}(Y_i \le x, Y_j \le y)\|_1\,.$$

For any positive n, $\alpha_{k,\mathbf{Y}}(n) \le \alpha(n)$, where $\alpha(n)$ is defined by (1.2). We now introduce some quantities involving the rate of mixing and the quantile function Q. Define

$$\alpha_{2,\mathbf{Y}}^{-1}(x) = \min\{q \in \mathbb{N} : \alpha_{2,\mathbf{Y}}(q) \le x\} \quad \text{and} \quad R(x) = \alpha_{2,\mathbf{Y}}^{-1}(x)(Q(x) \vee 1) \tag{2.2}$$

(note that $\alpha_{2,\mathbf{Y}}^{-1}(x) \ge 1$ for $x < 1$). Set, for $p \ge 1$,

$$M_{p,\alpha}(Q) = \int_0^1 R^{p-1}(u)Q(u)du \quad \text{and} \quad \Lambda_{p,\alpha}(Q) = \sup_{u\in]0,1]} uR^{p-1}(u)Q(u)\,. \tag{2.3}$$

Note that, if $M_{p,\alpha}(Q) < \infty$ then $\Lambda_{p,\alpha}(Q) < \infty$, Also, if $\Lambda_{p,\alpha}(Q) < \infty$, then $M_{r,\alpha}(Q) < \infty$ for any $r < p$. Let us now state our main result.

Theorem 2.1. *Let* $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$ *where* f *belongs to* $\widetilde{\mathcal{F}}(Q, P_{Y_0})$ *(here* P_{Y_0} *denotes the law of* Y_0*). Assume that* $M_{2,\alpha}(Q) < \infty$. *Then the series* $\mathbb{E}(X_0^2) + 2\sum_{k\ge 1}\mathbb{E}(X_0X_k)$ *is convergent to a nonnegative real* σ^2. *Now let* $p \in]2, 3]$ *and suppose that* $\Lambda_{p,\alpha}(Q) < \infty$ *in the case* $p < 3$ *or* $M_{3,\alpha}(Q) < \infty$ *in the case* $p = 3$.

1. *Assume that* $\sigma^2 > 0$. *Then:*

   (a) *there exists a sequence* $(Z_i)_{i\ge 1}$ *of i.i.d. random variables with law* $N(0, \sigma^2)$ *such that, setting* $\Delta_k = \sum_{i=1}^k (X_i - Z_i)$,

   $$\sup_{k\le n} |\Delta_k| = O(n^{1/p}(\log n)^{1/2-1/p}) \text{ in } \mathbb{L}^2 \text{ and a.s. for } p < 3 \text{ if } M_{p,\alpha}(Q) < \infty.$$

   (b) *For any* $\varepsilon > 0$, *there exists a sequence* $(\widetilde{Z}_i)_{i\ge 1}$ *of i.i.d. random variables with law* $N(0, \sigma^2)$ *such that, setting* $\widetilde{\Delta}_k = \sum_{i=1}^k (X_i - \widetilde{Z}_i)$,

   $$\sup_{k\le n} |\widetilde{\Delta}_k| = O(n^{1/p}(\log n)^{1/2}(\log\log n)^{(1+\varepsilon)/p}) \text{ a.s.}$$

2. *Assume that* $\sigma^2 = 0$. *Let* $S_k = \sum_{i=1}^k X_i$. *Then*

   (a) $\sup_{k\le n} |S_k| = O(n^{1/p})$ *in* $\mathbb{L}^2$ *and* $\sup_{k\le n} |S_k| = O((n\log n)^{1/p}(\log\log n)^{(1+\varepsilon)/p})$ *a.s.*

   (b) *If* $p < 3$ *and* $M_{p,\alpha}(Q) < \infty$, *then* $\sup_{k\le n} |S_k| = o(n^{1/p})$ *a.s.*

Remark 2.2. The condition $M_{p,\alpha}(Q) < \infty$ can be rewritten in a complete equivalent way as

$$\sum_{k\ge 0}(1\vee k)^{p-2}\int_0^{\alpha_{2,\mathbf{Y}}(k)} Q^p(u)du < \infty\,. \tag{2.4}$$

(see Annexe C in Rio (2000)), which corresponds to (1.5) with $\alpha_{2,\mathbf{Y}}(k)$ instead of $\alpha(k)$.

Applications to geometric or arithmetic rates of mixing. Below we denote by H the cadlag inverse of the function Q. Assume first that, for some a in $]0,1[$, $\alpha_{2,\mathbf{Y}}(n) = O(a^n)$ as $n\to\infty$. Then $\alpha_{2,\mathbf{Y}}^{-1}(u) = O(|\log u|)$ as u decreases to 0. Hence $M_{p,\alpha}(Q) < \infty$ as soon as

$$\int_0^1 |\log u|^{p-1}Q^p(u)du < \infty\,.$$

This condition holds if $H(x) = O(\,(x\log x)^{-p}(\log\log x)^{-(1+\varepsilon)})$ as $x\to\infty$. In a similar way $\Lambda_{p,\alpha}(Q) < \infty$ if one of the following equivalent weaker conditions holds:

$$Q(u) = O(u^{-1/p}|\log u|^{-1+(1/p)}) \text{ as } u\downarrow 0\,,\;\; H(x) = O(\,x^{-p}(\log x)^{1-p}) \text{ as } x\uparrow\infty\,.$$

Suppose now that, for some real $q > 2$, $\alpha_{2,\mathbf{Y}}(n) = O(n^{1-q})$ as $n\to\infty$. Then $\alpha_{2,\mathbf{Y}}^{-1}(u) = O(u^{-1/(q-1)})$ as $u\to 0$. For p in $[2,q[$, we get that $M_{p,\alpha}(Q) < \infty$ as soon as

$$\int_0^1 |u|^{-1/(q-1)}Q^p(u)du < \infty\,.$$

This condition holds if $H(x) = O(\,(x^p\log(x)(\log\log x)^{1+\varepsilon})^{-(q-1)/(q-p)})$ as $x\to\infty$. In a similar way $\Lambda_{p,\alpha}(Q) < \infty$ if and only if $H(x) = O(\,x^{-p(q-1)/(q-p)})$ as $x\to\infty$. Note also that $\Lambda_{q,\alpha}(Q) < \infty$ if and only if Q is uniformly bounded over $]0,1]$.

3 Application to dynamical systems

In this section, we consider a class of piecewise expanding maps T of $[0,1]$ with a neutral fixed point, and their associated Markov chain Y_i whose transition kernel is the Perron-Frobenius operator of T with respect to the absolutely continuous invariant probability measure. Applying Theorem 2.1, we give a large class of unbounded functions f for which we can give rates of convergence close to optimal in the strong invariance principle of the partial sums of both $f \circ T^i$ and $f(Y_i)$.

For γ in $]0,1[$, we consider the intermittent map T_γ from $[0,1]$ to $[0,1]$, which is a modification of the Pomeau-Manneville map (1980):

$$T_\gamma(x) = \begin{cases} x(1+2^\gamma x^\gamma) & \text{if } x \in [0,1/2[\\ 2x-1 & \text{if } x \in [1/2,1]\,. \end{cases}$$

We denote by ν_γ the unique T_γ-invariant probability measure on $[0,1]$ which is absolutely continuous with respect to the Lebesgue measure. We denote by K_γ the Perron-Frobenius operator of T_γ with respect to ν_γ. Recall that for any bounded measurable functions f and g,

$$\nu_\gamma(f \cdot g \circ T_\gamma) = \nu_\gamma(K_\gamma(f)g)\,.$$

Let $(Y_i)_{i\geq 0}$ be a stationary Markov chain with invariant measure ν_γ and transition Kernel K_γ. It is well known (see for instance Lemma XI.3 in Hennion and Hervé (2001)) that on the probability space $([0,1],\nu_\gamma)$, the random vector $(T_\gamma, T_\gamma^2, \dots, T_\gamma^n)$ is distributed as $(Y_n, Y_{n-1}, \dots, Y_1)$.

To state our results for those intermittent maps, we need preliminary definitions.

Definition 3.1. A function H from $\mathbb{R}_+$ to $[0,1]$ is a tail function if it is non-increasing, right continuous, converges to zero at infinity, and $x \to xH(x)$ is integrable.

Definition 3.2. If μ is a probability measure on $\mathbb{R}$ and H is a tail function, let $\mathrm{Mon}(H,\mu)$ denote the set of functions $f : \mathbb{R} \to \mathbb{R}$ which are monotonic on some open interval and null elsewhere and such that $\mu(|f| > t) \leq H(t)$. Let $\mathcal{F}(H,\mu)$ be the closure in $\mathbb{L}^1(\mu)$ of the set of functions which can be written as $\sum_{\ell=1}^{L} a_\ell f_\ell$, where $\sum_{\ell=1}^{L} |a_\ell| \leq 1$ and $f_\ell \in \mathrm{Mon}(H,\mu)$.

Note that a function belonging to $\mathcal{F}(H,\mu)$ is allowed to explode at an infinite number of points. Note also that any function f with bounded variation (BV) such that $|f| \leq M_1$ and $\|df\| \leq M_2$ belongs to the class $\mathcal{F}(H,\mu)$ for any μ and the tail function $H = \mathbb{1}_{[0,M_1+2M_2)}$ (here and henceforth, $\|df\|$ denotes the variation norm of the signed measure df). In the unbounded case, if a function f is piecewise monotonic with N branches, then it belongs to $\mathcal{F}(H,\mu)$ for $H(t) = \mu(|f| > t/N)$. Finally, let us emphasize that there is no requirement on the modulus of continuity for functions in $\mathcal{F}(H,\mu)$.

Let Q denote the cadlag inverse of H. Then, for the random variable X defined by $X(\omega) = \omega$, $\mathrm{Mon}(H,\mu) = \widetilde{\mathrm{Mon}}(Q,\mu)$ and $\mathcal{F}(H,\mu) = \widetilde{\mathcal{F}}(Q,\mu)$. Furthermore Proposition 1.17 in Dedecker, Gouëzel and Merlevède (2010) states that there exists a positive constant C such that, for any $n > 0$, $\alpha_{2,\mathbf{Y}}(n) \leq Cn^{(\gamma-1)/\gamma}$. In addition, the computations page 817 in the same paper show that, for $p\gamma < 1$, the integrability conditions below are equivalent:

$$\int_0^1 R^{p-1}(u)Q(u)du < \infty \quad \text{and} \quad \int_0^\infty x^{p-1}(H(x))^{\frac{1-p\gamma}{1-\gamma}}dx < \infty\,. \tag{3.1}$$

Also, for p in $]2, 1/\gamma[$,

$$\Lambda_{p,\alpha}(Q) < \infty \text{ if and only if } H(x) = O(x^{-p(1-\gamma)/(1-p\gamma)}) \text{ as } x \to \infty \tag{3.2}$$

and, for $p = 1/\gamma$ and $H = \mathbb{1}_{[0,M)}$, $\Lambda_{p,\alpha}(Q) < \infty$ (see the previous section).

A modification of the proof of Theorem 2.1 leads to the result below for the Markov chain or the dynamical system associated to the transformation T_γ.

Theorem 3.1. *Let $\gamma < 1/2$. Let $f \in \mathcal{F}(H, \nu_\gamma)$ for some tail function H satisfying (3.1) with $p = 2$. Then the series*

$$\nu_\gamma((f - \nu_\gamma(f))^2) + 2\sum_{k>0} \nu_\gamma((f - \nu_\gamma(f))f \circ T_\gamma^k) \tag{3.3}$$

converges absolutely to a nonnegative number $\sigma^2(f)$. Let $p \in]2, 3]$ satisfying $p \le 1/\gamma$. Let Q denote the cadlag inverse of H. Suppose that $\Lambda_{p,\alpha}(Q) < \infty$ in the case $p < 3$ or $M_{3,\alpha}(Q) < \infty$ in the case $p = 3$.

1. *Let $(Y_i)_{i\ge1}$ be a stationary Markov chain with transition kernel K_γ and invariant measure ν_γ, and let $X_i = f(Y_i) - \nu_\gamma(f)$. The sequence $(X_i)_{i\ge0}$ satisfies the conclusions of Items 1 and 2 of Theorem 2.1 with $\sigma^2 = \sigma^2(f)$.*

2. *If $\sigma^2(f) = 0$, the sequence $(f \circ T_\gamma^i - \nu_\gamma(f))_{i\ge1}$ satisfies the conclusions of Item 2 of Theorem 2.1. If $\sigma^2(f) > 0$, enlarging the probability space $([0, 1], \nu_\gamma)$, there exist sequences $(Z_i^*)_{i\ge1}$ and $(\tilde{Z}_i^*)_{i\ge1}$ of i.i.d. random variables with law $N(0, \sigma^2(f))$ such that the random variables $\Delta_k = \sum_{i=1}^k (f \circ T_\gamma^i - \nu_\gamma(f) - Z_i^*)$ satisfy the conclusions of Item 1(a) of Theorem 2.1 and the random variables $\tilde{\Delta}_k = \sum_{i=1}^k (f \circ T_\gamma^i - \nu_\gamma(f) - \tilde{Z}_i^*)$ satisfy the conclusion of Item 1(b).*

Item 1 is direct by using Theorem 2.1 together with (3.1) and (3.2). Item 2 requires a proof that is given in Section 4.2.

Remark 3.1. Theorem 3.1 can be extended to generalized Pomeau-Manneville map (or GPM map) of parameter $\gamma \in (0, 1)$ as defined in Dedecker, Gouëzel and Merlevède (2010).

In the specific case of bounded variation functions, Theorem 3.1 provides the almost sure invariance principle below for the dynamical system associated to T_γ. Below we give the results in the case $\sigma^2(f) > 0$. The rates are slightly better in the case $\sigma^2(f) = 0$.

Corollary 3.1. *Let $\gamma \in]1/3, 1/2[$ and f be a function of bounded variation. Then the series in (3.3) converges absolutely to a nonnegative number $\sigma^2(f)$ and, for any $\varepsilon > 0$, there exists a sequence $(\widetilde{Z}_i^*)_{i\geq 1}$ of i.i.d. random variables with law $N(0, \sigma^2(f))$ such that*

$$\sup_{k\leq n} \Big|\sum_{i=1}^{k}(f\circ T_\gamma^i - \nu_\gamma(f) - \tilde{Z}_i^*)\Big| = O(n^\gamma(\log n)^{1/2}(\log\log n)^{(1+\varepsilon)\gamma}) \ a.s.$$

For the maps under consideration and Hölder continuous functions f, by using an approximation argument introduced by Berkes and Philipp (1979), Melbourne and Nicol (2009) obtained the following explicit error term in the almost sure invariance principle (see their Theorem 1.6 and their Remark 1.7): Let $p > 2$ and $0 < \gamma < 1/p$, then the error term in the almost sure invariance results is $O(n^{\beta+\varepsilon})$ where $\varepsilon > 0$ is arbitrarily small and $\beta = \frac{\gamma}{2} + \frac{1}{4}$ if γ belongs to $]1/4, 1/2[$ and $\beta = \frac{3}{8}$ if $\gamma \leq 1/4$. Consequently, for the modification of the Pomeau-Manneville map and functions f of bounded variation, Corollary 3.1 improves the error in the almost sure invariance principle obtained in Theorem 1.6 in Melbourne and Nicol (2009). Note also that, for $\gamma < 1/3$ and f of bounded variation, condition (3.1) is satisfied with $p = 3$, and Theorem 3.1 gives the error term $O(n^{1/3}(\log n)^{1/2}(\log\log n)^{(1+\varepsilon)/3})$ in the almost sure invariance principle.

4 Proofs

From now on, we denote by C a numerical constant which may vary from line to line. Throughout the proofs, to shorten the notations, we write $\alpha(n) = \alpha_{2,\mathbf{Y}}(n)$ and $\alpha^{-1}(u) = \alpha^{-1}_{2,\mathbf{Y}}(u)$. We also set, for $\lambda > 0$,

$$M_{3,\alpha}(Q,\lambda) = \int_0^1 Q(u)R(u)(R(u)\wedge\lambda)du\,. \tag{4.1}$$

We start by recalling some fact proved in Rio (1995-b), Lemma A.1.: for p in $]2, 3[$,

$$M_{3,\alpha}(Q,\lambda) = O(\lambda^{3-p}) \ \text{ as } \ \lambda \to +\infty \ \text{ if } \ \Lambda_{p,\alpha}(Q) < \infty\,. \tag{4.2}$$

4.1 Proof of Theorem 2.1.

Assume first that $\sigma^2 > 0$. For $L \in \mathbb{N}$, let $m(L) \in \mathbb{N}$ be such that $m(L) \leq L$. Let

$$I_{k,L} =]2^L + (k-1)2^{m(L)}, 2^L + k2^{m(L)}] \cap \mathbb{N} \text{ and } U_{k,L} = \sum_{i\in I_{k,L}} X_i\,,\ k \in \{1, \cdots, 2^{L-m(L)}\}\,.$$

For $k \in \{1, \cdots, 2^{L-m(L)}\}$, let $V_{k,L}$ be the $\mathcal{N}(0, \sigma^2 2^{m(L)})$-distributed random variable defined from $U_{k,L}$ via the conditional quantile transformation, that is

$$V_{k,L} = \sigma 2^{m(L)/2}\Phi^{-1}(\widetilde{F}_{k,L}(U_{k,L} - 0) + \delta_{2^L+k2^{m(L)}}(\widetilde{F}_{k,L}(U_{k,L}) - \widetilde{F}_{k,L}(U_{k,L} - 0)))\,, \tag{4.3}$$

where $\widetilde{F}_{k,L} := F_{U_{k,L}|\mathcal{F}_{2^L+(k-1)2^{m(L)}}}$ is the d.f. of $P_{U_{k,L}|\mathcal{F}_{2^L+(k-1)2^{m(L)}}}$ (the conditional law of $U_{k,L}$ given $\mathcal{F}_{2^L+(k-1)2^{m(L)}}$) and Φ^{-1} the inverse of the standard Gaussian distribution function Φ. Since $\delta_{2^L+k2^{m(L)}}$ is independent of $\mathcal{F}_{2^L+(k-1)2^{m(L)}}$, the random variable $V_{k,L}$ is independent of $\mathcal{F}_{2^L+(k-1)2^{m(L)}}$, and has the Gaussian distribution $N(0,\sigma^2 2^{m(L)})$. By induction on k, the random variables $(V_{k,L})_k$ are mutually independent and independent of $\mathcal{F}_{2^L}$. In addition

$$\begin{aligned}\mathbb{E}(U_{k,L}-V_{k,L})^2 &= \mathbb{E}\int_0^1 \big(F^{-1}_{U_{k,L}|\mathcal{F}_{2^L+(k-1)2^{m(L)}}}(u)-\sigma 2^{m(L)/2}\Phi^{-1}(u)\big)^2du \\ &:= \mathbb{E}\big(W_2^2(P_{U_{k,L}|\mathcal{F}_{2^L+(k-1)2^{m(L)}}}, G_{\sigma^2 2^{m(L)}})\big)\,, \end{aligned} \tag{4.4}$$

where $G_{\sigma^2 2^{m(L)}}$ is the Gaussian distribution $N(0,\sigma^2 2^{m(L)})$. Using Proposition 5.1 and stationarity, we then get that there exists a positive constant C such that

$$\mathbb{E}(U_{k,L}-V_{k,L})^2 \le C2^{m(L)/2}M_{3,\alpha}(Q,2^{m(L)/2})\,. \tag{4.5}$$

Now we construct a sequence $(Z'_i)_{i\ge 1}$ of i.i.d. Gaussian random variables with zero mean and variance σ^2 as follows. Let $Z'_1=\sigma\Phi^{-1}(\delta_1)$. For any $L\in\mathbb{N}$ and any $k\in\{1,\cdots,2^{L-m(L)}\}$ the random variables $(Z'_{2^L+(k-1)2^{m(L)}+1},\dots,Z'_{2^L+k2^{m(L)}})$ are defined in the following way. If $m(L)=0$, then $Z'_{2^L+k2^{m(L)}}=V_{k,L}$. If $m(L)>0$, then by the Skorohod lemma (1976), there exists a measurable function g from $\mathbb{R}\times[0,1]$ in $\mathbb{R}^{2^{m(L)}}$ such that, for any pair (V,δ) of independent random variables with respective laws $N(0,\sigma^2 2^{m(L)})$ and the uniform distribution over $[0,1]$, $g(V,\delta)=(N_1,\dots N_{2^{m(L)}})$ is a Gaussian random vector with i.i.d. components such that $V=N_1+\cdots+N_{2^{m(L)}}$. We then set

$$(Z'_{2^L+(k-1)2^{m(L)}+1},\dots,Z'_{2^L+k2^{m(L)}})=g(V_{k,L},\delta_{2^L+(k-1)2^{m(L)}+1})\,.$$

The so defined sequence (Z'_i) has the prescribed distribution.

Set $S_j=\sum_{i=1}^j X_i$ and $T_j=\sum_{i=1}^j Z'_i$. Let

$$D_L := \sup_{\ell\le 2^L}\Big|\sum_{i=2^L+1}^{2^L+\ell}(X_i-Z'_i)\Big|\,.$$

Let $N\in\mathbb{N}^*$ and let $k\in]1,2^{N+1}]$. We first notice that $D_L\ge|(S_{2^{L+1}}-T_{2^{L+1}})-(S_{2^L}-T_{2^L})|$, so that, if K is the integer such that $2^K<k\le 2^{K+1}$, $|S_k-T_k|\le|X_1-Z'_1|+D_0+D_1+\cdots+D_K$. Consequently since $K\le N$,

$$\sup_{1\le k\le 2^{N+1}}|S_k-T_k|\le|X_1-Z'_1|+D_0+D_1+\cdots+D_N\,. \tag{4.6}$$

We first notice that the following decomposition is valid:

$$D_L\le D_{L,1}+D_{L,2}\,, \tag{4.7}$$

where

$$D_{L,1} := \sup_{k\le 2^{L-m(L)}} \Big| \sum_{\ell=1}^{k} (U_{\ell,L} - V_{\ell,L}) \Big| \text{ and } D_{L,2} := \sup_{k\le 2^{L-m(L)}} \sup_{\ell\in I_{k,L}} \Big| \sum_{i=\inf I_{k,L}}^{\ell} (X_i - Z_i) \Big| .$$

The main tools for proving Theorem 2.1 will be the two lemmas below. The first lemma allows us to control the fluctuation term $D_{L,2}$.

Lemma 4.1. *There exists positive constants c_1, $c_2 \ge 2$, c_3 and c_4 such that, for any positive λ,*

$$\mathbb{P}(D_{L,2} \ge 2\lambda) \le (c_1+2)2^L \exp\Big(-\frac{\lambda^2}{c_2\sigma^2 2^{m(L)}}\Big) + 2^L\lambda^{-3}\big(c_3 M_{3,\alpha}(Q,\lambda) + c_4\sigma^3\big) . \tag{4.8}$$

The second lemma gives a bound in $\mathbb{L}^2$ on the Gaussian approximation term $D_{L,1}$.

Lemma 4.2. *Let $p \in]2,3]$. Suppose that $\Lambda_{p,\alpha}(Q) < \infty$ in the case $p < 3$ and $M_{3,\alpha}(Q) < \infty$ in the case $p = 3$. Then*

$$\|D_{L,1}\|_2^2 \le C 2^L \Big(2^{(2-p)m(L)} + 2^{-m(L)/2} M_{3,\alpha}(Q, 2^{m(L)/2})\Big) . \tag{4.9}$$

Proof of Lemma 4.1. By the triangle inequality together with the stationarity of the sequences $(X_i)_i$ and $(Z_i)_i$, for any positive λ,

$$\mathbb{P}(D_{L,2} \ge 2\lambda) \le 2^{L-m(L)}\mathbb{P}\Big(\sup_{\ell\le 2^{m(L)}} |S_\ell| \ge \lambda\Big) + 2^{L-m(L)}\mathbb{P}\Big(\sup_{\ell\le 2^{m(L)}} |T_\ell| \ge \lambda\Big) . \tag{4.10}$$

By Lévy's inequality (see for instance Proposition 2.3 in Ledoux and Talagrand (1991)),

$$\mathbb{P}\Big(\sup_{\ell\le 2^{m(L)}} |T_\ell| \ge \lambda\Big) \le 2\exp\Big(-\frac{\lambda^2}{2\sigma^2 2^{m(L)}}\Big) . \tag{4.11}$$

On the other hand, applying Proposition 5.2, we get that

$$\mathbb{P}\Big(\sup_{\ell\le 2^{m(L)}} |S_\ell| \ge \lambda\Big) \le c_1 \exp\Big(-\frac{\lambda^2}{c_2\sigma^2 2^{m(L)}}\Big) + 2^{m(L)}\lambda^{-3}\big(c_3 M_{3,\alpha}(Q,\lambda) + c_4\sigma^3\big) .$$

Collecting the above inequalities, we then get Lemma 4.1. ⋄

Proof of Lemma 4.2. For any $\ell \in \{1, \cdots, 2^{L-m(L)}\}$, let $\widetilde{U}_{\ell,L} = U_{\ell,L} - \mathbb{E}_{2^L+(\ell-1)2^{m(L)}}(U_{\ell,L})$. Then $(\widetilde{U}_{\ell,L})_{\ell\ge 1}$ is a strictly stationary sequence of martingale differences adapted to the filtration $(\mathcal{F}_{2^L+\ell 2^{m(L)}})_{\ell\ge 1}$. Notice first that

$$\|D_{L,1}\|_2 \le \| \sup_{k\le 2^{L-m(L)}} |\sum_{\ell=1}^{k} (\widetilde{U}_{\ell,L} - V_{\ell,L})| \|_2 + \| \sup_{k\le 2^{L-m(L)}} |\sum_{\ell=1}^{k} (\widetilde{U}_{\ell,L} - U_{\ell,L})| \|_2 . \tag{4.12}$$

Let us deal with the first term on right hand. Since $V_{\ell,L}$ is independent of $\mathcal{F}_{2^L+(\ell-1)2^{m(L)}}$, the sequence $(\widetilde{U}_{\ell,L} - V_{\ell,L})_\ell$ is a martingale difference sequence with respect to the nondecreasing filtration $(\mathcal{F}_{2^L+\ell 2^{m(L)}})_\ell$. Hence, by the Doob-Kolmogorov maximal inequality, we get that

$$\begin{aligned}\| \sup_{k\le 2^{L-m(L)}} |\sum_{\ell=1}^{k}(\widetilde{U}_{\ell,L} - V_{\ell,L})|\|_2^2 &\le 4 \sum_{\ell=1}^{2^{L-m(L)}} \|\tilde{U}_{\ell,L} - V_{\ell,L}\|_2^2 \\ &\le 8 \sum_{\ell=1}^{2^{L-m(L)}} \|\widetilde{U}_{\ell,L} - U_{\ell,L}\|_2^2 + 8 \sum_{\ell=1}^{2^{L-m(L)}} \|U_{\ell,L} - V_{\ell,L}\|_2^2 .\end{aligned}$$

Since $V_{\ell,N}$ is independent of $\mathcal{F}_{2^L+(\ell-1)2^{m(L)}}$, $\mathbb{E}_{2^L+(\ell-1)2^{m(L)}}(V_{\ell,L}) = 0$. Consequently,

$$\|\tilde{U}_{\ell,L} - U_{\ell,L}\|_2^2 = \|\mathbb{E}_{2^L+(\ell-1)2^{m(L)}}(U_{\ell,L} - V_{\ell,L})\|_2^2 \le \|U_{\ell,L} - V_{\ell,L}\|_2^2 .$$

Using (4.5), it follows that

$$\| \sup_{k\le 2^{L-m(L)}} |\sum_{\ell=1}^{k}(\widetilde{U}_{\ell,L} - V_{\ell,L})|\|_2^2 \le C 2^{L-m(L)/2} M_{3,\alpha}(Q, 2^{m(L)/2}) . \tag{4.13}$$

We deal now with the second term in the right hand side of (4.12). According to Dedecker and Rio's maximal inequality (2000, Proposition 1), we obtain that

$$\begin{aligned}\| \sup_{k\le 2^{L-m(L)}} |\sum_{\ell=1}^{k}(\widetilde{U}_{\ell,L} - U_{\ell,L})|\|_2^2 &\le 4 \sum_{k=1}^{2^{L-m(L)}} \|\mathbb{E}_{2^L+(k-1)2^{m(L)}}(U_{k,L})\|_2^2 \\ &+8 \sum_{k=1}^{2^{L-m(L)}-1} \|\mathbb{E}_{2^L+(k-1)2^{m(L)}}(U_{k,L}) \Big(\sum_{i=k+1}^{2^{L-m(L)}} \mathbb{E}_{2^L+(k-1)2^{m(L)}}(U_{i,L})\Big)\|_1 .\end{aligned} \tag{4.14}$$

Stationarity leads to

$$\|\mathbb{E}_{2^L+(k-1)2^{m(L)}}(U_{k,L})\|_2^2 = \|\mathbb{E}_0(S_{2^{m(L)}})\|_2^2 \le 2 \sum_{i=1}^{2^{m(L)}} \sum_{j=1}^{i} \mathbb{E}|X_j \mathbb{E}_0(X_i)| . \tag{4.15}$$

Using Lemma 4 (page 679) in Merlevède and Peligrad (2006), we get that

$$\mathbb{E}|X_j \mathbb{E}_0(X_i)| \le 3 \int_0^{\|\mathbb{E}_0(X_i)\|_1} Q_{|X_0|} \circ G_{|X_0|}(u) du ,$$

where $G_{|X_0|}$ is the inverse of $L_{|X_0|}(x) = \int_0^x Q_{|X_0|}(u)du$. We will denote by L and G the same functions constructed from Q. Assume first that $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$ with $f = \sum_{\ell=1}^{L} a_\ell f_\ell$, where $f_\ell \in \widetilde{\mathrm{Mon}}(Q, P_{Y_0})$ and $\sum_{\ell=1}^{L} |a_\ell| \le 1$. According to Proposition 5.3,

$$\|\mathbb{E}_0(X_i)\|_1 \le 8 \int_0^{\alpha(i)} Q(u) du . \tag{4.16}$$

Since $Q_{|X_0|}(u) \le Q_{|f(Y_0)|}(u) + |\mathbb{E}(f(Y_0))|$, we see that $\int_0^x Q_{|X_0|}(u)du \le 2\int_0^x Q_{|f(Y_0)|}(u)du$. Since $f = \sum_{\ell=1}^L a_\ell f_\ell$, we get, according to Item (c) of Lemma 2.1 in Rio (2000),

$$\int_0^x Q_{|X_0|}(u)du \le 2\sum_{\ell=1}^L \int_0^x Q_{|a_\ell f_\ell(X_0)|}(u)du \le 2\sum_{\ell=1}^L |a_\ell| \int_0^x Q(u)du\,.$$

Since $\sum_{\ell=1}^L |a_\ell| \le 1$, it follows that $G(u/2) \le G_{|X_0|}(u)$. In particular, $G_{|X_0|}(u) \ge G(u/8)$. Using the fact that $Q_{|X_0|}$ is non-increasing and the change of variables $w = G(v)$,

$$\begin{aligned}\int_0^{\|\mathbb{E}_0(X_i)\|_1} Q_{|X_0|}\circ G_{|X_0|}(u)du &\le \int_0^{\|\mathbb{E}_0(X_i)\|_1} Q_{|X_0|}\circ G(u/8)du = 8\int_0^{\|\mathbb{E}_0(X_i)\|_1/8} Q_{|X_0|}\circ G(v)dv \\ &= 8\int_0^{G(\|\mathbb{E}_0(X_i)\|_1/8)} Q_{|X_0|}(w)Q(w)dw \le 8\int_0^{\alpha(i)} Q_{|X_0|}(w)Q(w)dw\,,\end{aligned}$$

where the last inequality follows from (4.16). Consequently, by Item (c) of Lemma 2.1 in Rio (2000),

$$\mathbb{E}|X_j\mathbb{E}_0(X_i)| \le 48\sum_{\ell=1}^L |a_\ell| \int_0^{\alpha(i)} Q_{|f_\ell(Y_0)|}(u)Q(u)du \le 48\int_0^{\alpha(i)} Q^2(u)du\,, \tag{4.17}$$

and the same inequality holds if $f \in \widetilde{\mathcal{F}}(Q, P_{Y_0})$ by applying Fatou's lemma. Consequently starting from (4.15), we derive that

$$\sum_{k=1}^{2^{L-m(L)}} \|\mathbb{E}_{2^L+(k-1)2^{m(L)}}(U_{k,L})\|_2^2 \le 96\times 2^{L-m(L)}\sum_{i=1}^{2^{m(L)}} i\int_0^{\alpha(i)} Q^2(u)du\,. \tag{4.18}$$

We now bound up the second term in the right hand side of (4.14). Stationarity yields that

$$\|\mathbb{E}_{2^L+(k-1)2^{m(L)}}(U_{k,L})\Big(\sum_{i=k+1}^{2^{L-m(L)}} \mathbb{E}_{2^L+(k-1)2^{m(L)}}(U_{i,L})\Big)\|_1 \le \sum_{j=1}^{2^{m(L)}}\sum_{i=2^{m(L)}+1}^{2^L-(k-1)2^{m(L)}} \mathbb{E}|X_j\mathbb{E}_0(X_i)|\,.$$

Using Inequality (4.17), we then derive that

$$\sum_{k=1}^{2^{L-m(L)}-1} \|\mathbb{E}_{2^L+(k-1)2^{m(L)}}(U_{k,L})\Big(\sum_{i=k+1}^{2^{L-m(L)}} \mathbb{E}_{2^L+(k-1)2^{m(L)}}(U_{i,L})\Big)\|_1 \le 48\times 2^L \sum_{i=2^{m(L)}+1}^{2^L}\int_0^{\alpha(i)} Q^2(u)du\,. \tag{4.19}$$

Starting from (4.14) and considering the bounds (4.18) and (4.19), we get that

$$\begin{aligned}\|\sup_{k\le 2^{L-m(L)}} |\sum_{\ell=1}^k (\widetilde{U}_{\ell,L} - U_{\ell,L})|\|_2^2 &\le C2^{L-m(L)}\int_0^1 Q(u)R(u)(\alpha^{-1}(u)\wedge 2^{m(L)})du \\ &\le C2^{L-m(L)}M_{3,\alpha}(Q, 2^{m(L)})\,,\end{aligned} \tag{4.20}$$

since $R(u) \geq \alpha^{-1}(u)$. Starting from (4.12) and considering the bounds (4.13), (4.20) and (4.2) in the case $p < 3$, we then get (4.9), which ends the proof of Lemma 4.2. $\diamond$

Proof of Item 1(a) of Theorem 2.1. We choose $Z_i = Z_i'$ with

$$m(L) = \Big[\frac{2L}{p} - \frac{2}{p}\log_2 L\Big]\,, \text{ so that } \frac{1}{2}\Big(\frac{2^L}{L}\Big)^{2/p} \leq 2^{m(L)} \leq \Big(\frac{2^L}{L}\Big)^{2/p}\,, \tag{4.21}$$

square brackets designating as usual the integer part and $\log_2(x) = (\log x)/(\log 2)$. Starting from (4.8), we now prove that

$$D_{L,2} = O(2^{L/p} L^{1/2-1/p}) \text{ in } \mathbb{L}^2 \text{ for } p \leq 3 \text{ and a.s. for } p < 3 \text{ if } M_{p,\alpha}(Q) < \infty\,. \tag{4.22}$$

To prove the almost sure part in (4.22), take

$$\lambda = \lambda_L = K 2^{m(L)/2}\sqrt{L} \text{ with } K = \sqrt{2c_2\sigma^2 \log 2}\,. \tag{4.23}$$

Then, on one hand,

$$\sum_{L>0} 2^L \exp\Big(-\frac{\lambda_L^2}{c_2\sigma^2 2^{m(L)}}\Big) = \sum_{L\geq 0} 2^{L-2L} < \infty \text{ and } \sum_{L>0} 2^L \lambda_L^{-3} < \infty\,,$$

for $p < 3$. On the other hand, since $M_{3,\alpha}(Q, a\lambda) \leq a M_{3,\alpha}(Q,\lambda)$ for any $a \geq 1$,

$$2^L \lambda_L^{-3} M_{3,\alpha}(Q,\lambda_L) \leq 2^{L-3m(L)/2} L^{-1} M_{3,\alpha}(Q, K2^{m(L)/2})\,.$$

Therefore, from the choice of $m(L)$ made in (4.21),

$$\sum_{L>0} 2^L \lambda_L^{-3} M_{3,\alpha}(Q,\lambda_L) \leq C \sum_{L>0} (2^L/L)^{(p-3)/p} M_{3,\alpha}(Q, (2^L/L)^{1/p})\,.$$

Next, for $p \in]2,3[$,

$$\sum_{L\,:\,\frac{2^L}{L} \geq R^p(x)} \Big(\frac{2^L}{L}\Big)^{1-3/p} \leq C R^{p-3}(x) \text{ and } \sum_{L\,:\,\frac{2^L}{L} \leq R^p(x)} \Big(\frac{2^L}{L}\Big)^{1-2/p} \leq C R^{p-2}(x)\,,$$

which ensures that

$$\sum_{L>0} 2^L \lambda_L^{-3} M_{3,\alpha}(Q,\lambda_L) \leq C M_{p,\alpha}(Q)\,. \tag{4.24}$$

Consequently under (2.4), we derive that $\sum_{L>0} \mathbb{P}(D_{L,2} \geq 2\lambda_L) < \infty$ implying the almost sure part of (4.22) via the Borel-Cantelli lemma.

We now prove the $\mathbb{L}^2$ part of (4.22). Clearly

$$\mathbb{E}(D_{L,2}^2) = 8\int_0^\infty \lambda \mathbb{P}(D_{L,2} \geq 2\lambda) d\lambda \leq 4\lambda_L^2 + 8\int_{\lambda_L}^\infty \lambda \mathbb{P}(D_{L,2} \geq 2\lambda) d\lambda\,. \tag{4.25}$$

We now apply (4.8). First, from (4.23),

$$\int_{\lambda_L}^{\infty} \lambda \exp\Big(-\frac{\lambda^2}{c_2\sigma^2 2^{m(L)}}\Big) d\lambda = c_2\sigma^2 2^{m(L)-L} \text{ and } 2^L \int_{\lambda_L}^{\infty} \frac{c_4\sigma^3}{\lambda^2} d\lambda = c_4\sigma^3 \frac{2^L}{\lambda_L}\,.$$

In the case $p < 3$ and $\Lambda_{p,\alpha}(Q) < \infty$, from (4.2), there exists a positive constant C depending on p and $\Lambda_{p,\alpha}(Q)$ such that

$$\int_{\lambda_L}^{\infty} \frac{c_3 2^L}{\lambda^2} M_{3,\alpha}(Q,\lambda) d\lambda \le C \int_{\lambda_L}^{\infty} \lambda^{1-p} d\lambda \le \frac{C 2^L}{(p-2)\lambda_L^{p-2}}\,. \tag{4.26}$$

Now, by (4.23) again, $(K/2)2^{L/p}L^{1/2-1/p} \le \lambda_L \le K 2^{L/p}L^{1/2-1/p}$, and consequently, collecting the above estimates, we get that $\mathbb{E}(D_{L,2}^2) = O(\lambda_L^2)$, which implies the $\mathbb{L}^2$ part of (4.22).

We now deal with $D_{L,1}$. We will prove that

$$D_{L,1} = O(2^{L/p}L^{1/2-1/p}) \text{ in } \mathbb{L}^2 \text{ for } p \le 3 \text{ and a.s. for } p < 3 \text{ if } M_{p,\alpha}(Q) < \infty\,. \tag{4.27}$$

We first derive from Lemma 4.2 that $\|D_{L,1}\|_2^2 \le C 2^{L-m(L)(p-2)/2}$ (applying (4.2) in the case $p < 3$), which implies the $\mathbb{L}^2$ part of (4.27).

Next, from (4.9) together with the Markov inequality,

$$\sum_{L>0} \mathbb{P}(D_{L,1} \ge \lambda_L) \le C \sum_{L>0} 2^{L+(1-p)m(L)} + C \sum_{L>0} \frac{2^L}{L 2^{3m(L)/2}} M_{3,\alpha}(Q, 2^{m(L)/2})\,,$$

where λ_L is defined by (4.23). Repeating exactly the same arguments as in the proof of (4.24), we get that the second series on right hand in the above inequality is convergent for $p < 3$. Now $2^{L+(1-p)m(L)} \le 2^{p-1}2^{L(2-p)/p}L^{2(p-1)/p}$, which ensures the convergence of the first series on right hand. Hence, by the Borel-Cantelli lemma $D_{L,1} = O(\lambda_L)$ almost surely, which completes the proof of (4.27). Finally Item 1(a) of Theorem 2.1 follows from both (4.27), (4.22) and (4.6) and (4.7). ⋄

Proof of Item 1(b) of Theorem 2.1. We choose $\widetilde{Z}_i = Z_i'$ with $m(L) = [(2L/p) + (2(1+\varepsilon)/p)\log_2(1 \vee \log L)]$. Following the proof of Item 1(a) with this selection of $m(L)$, Item 1(b) follows. ⋄

Proof of Item 2 of Theorem 2.1. Starting from the decomposition (4.6), we just have to bound the random variables $D_L := \sup_{\ell \le 2^L} |S_{2^L+\ell} - S_{2^L}|$ both almost surely and in $\mathbb{L}^2$. Applying Proposition 5.2 in case where $\sigma^2 = 0$, we get that for any positive λ,

$$\mathbb{P}(D_L \ge \lambda) \le c 2^L \lambda^{-3} M_{3,\alpha}(Q,\lambda)\,, \tag{4.28}$$

where c is a positive constant. Using computations as in (4.25) and (4.26), we then get that for any positive λ_L, $\|D_L\|_2^2 \leq 4\lambda_L^2 + C2^L\lambda_L^{2-p}$. Choosing $\lambda_L = 2^{L/p}$ gives the $\mathbb{L}^2$ part of Item 2 (a). To prove the almost sure parts, we start from (4.28) and choose, for $\delta > 0$ arbitrarily small,

$$\lambda = 2^{L/p}L^{1/p}(1 \vee \log L)^{(1+\varepsilon)/p} \text{ and } \lambda = \delta 2^{L/p} \text{ if } p \in]2,3[\text{ and } M_{p,\alpha}(Q) < \infty\,.$$

The Borel-Cantelli lemma then implies that

$$D_L = O(2^{L/p}L^{1/p}(1 \vee \log L)^{(1+\varepsilon)/p}) \text{ a.s. and } D_L = o(2^{L/p}) \text{ a.s. if } p \in]2,3[\text{ and } M_{p,\alpha}(Q) < \infty\,.$$

This ends the proof of the almost sure part of Item 2 and then of the theorem. ⋄

4.2 Proof of Item 2 of Theorem 3.1.

If $\sigma^2(f) > 0$, similarly as for the proof of Theorem 2.1, we start by constructing a sequence $(Z_i'^*)_{i\geq 1}$ of i.i.d. gaussian random variables with mean zero and variance $\sigma^2(f)$ depending on the sequence $(m(L))_{L\geq 0}$ defined either as in (4.21) or as in the proof of Item 1(b). Define for any $k \in \{1, \cdots, 2^{L-m(L)}\}$,

$$I_{k,L} =]2^L + (k-1)2^{m(L)}, 2^L + k2^{m(L)}] \cap \mathbb{N} \text{ and } U_{k,L}^* = \sum_{i \in I_{k,L}} (f \circ T_\gamma^i - \nu_\gamma(f))\,.$$

For $k \in \{1, \cdots, 2^{L-m(L)}\}$, let $V_{k,L}^*$ be the $\mathcal{N}(0, \sigma^2 2^{m(L)})$-distributed random variable defined from $U_{k,L}^*$ via the conditional quantile transformation, that is

$$V_{k,L}^* = \sigma(f)2^{m(L)/2}\Phi^{-1}(F_{k,L}^*(U_{k,L}^* - 0) + \delta_{2^L+k2^{m(L)}}(F_{k,L}^*(U_{k,L}^*) - F_{k,L}^*(U_{k,L}^* - 0)))\,, \qquad (4.29)$$

where $F_{k,L}^* := F_{U_{k,L}^*|\tilde{\mathcal{G}}_{2^L+k2^{m(L)}+1}}$ is the d.f. of the conditional law of $U_{k,L}^*$ given $\tilde{\mathcal{G}}_{2^L+k2^{m(L)}+1}$, where $\tilde{\mathcal{G}}_m = \sigma(T_\gamma^m, (\delta_i)_{i\geq m})$ and Φ^{-1} the inverse of the standard Gaussian distribution function Φ. Since $\delta_{2^L+k2^{m(L)}}$ is independent of $\tilde{\mathcal{G}}_{2^L+k2^{m(L)}+1}$, the random variable $V_{k,L}^*$ is independent of $\tilde{\mathcal{G}}_{2^L+k2^{m(L)}+1}$, and has the Gaussian distribution $N(0, \sigma^2(f)2^{m(L)})$. By induction on k, the random variables $(V_{k,L}^*)_k$ are mutually independent and independent of $\tilde{\mathcal{G}}_{2^{L+1}+1}$. Let us construct now the sequence $(Z_i'^*)_{i\geq 1}$ as follows. Let $Z_1'^* = \sigma(f)\Phi^{-1}(\delta_1)$. For any $L \in \mathbb{N}$ and any $k \in \{1, \cdots, 2^{L-m(L)}\}$, the random variables $(Z'^*_{2^L+(k-1)2^{m(L)}+1}, \ldots, Z'^*_{2^L+k2^{m(L)}})$ are defined in the following way. If $m(L) = 0$, then $Z'^*_{2^L+k2^{m(L)}} = V_{k,L}^*$. If $m(L) > 0$, then there exists a measurable function g from $\mathbb{R} \times [0,1]$ in $\mathbb{R}^{2^{m(L)}}$ such that, for any pair (V, δ) of independent random variables with respective laws $N(0, \sigma^2(f)2^{m(L)})$ and the uniform distribution over $[0,1]$, $g(V, \delta) = (N_1, \ldots N_{2^{m(L)}})$ is a Gaussian random vector with i.i.d. components such that $V = N_1 + \cdots + N_{2^{m(L)}}$. We then set

$$(Z'^*_{2^L+(k-1)2^{m(L)}+1}, \ldots, Z^*_{2^L+k2^{m(L)}}) = g(V_{k,L}^*, \delta_{2^L+(k-1)2^{m(L)}+1})\,.$$

The so defined sequence $(Z_i'^*)$ has the prescribed distribution.

Set now $S_j^* = \sum_{i=1}^j (f \circ T_\gamma^i - \nu_\gamma(f))$, $T_j^* = \sum_{i=1}^j Z_i'^*$ if $\sigma^2(f) > 0$ and $T_j^* = 0$ otherwise, and let

$$D_L^* := \sup_{0 \le \ell \le 2^L} |(S^*_{2^L+\ell} - T^*_{2^L+\ell}) - (S^*_{2^{L+1}} - T^*_{2^{L+1}})| \,.$$

Similarly as in the proof of (4.6), we get that

$$\sup_{1 \le k \le 2^{N+1}} |S_k^* - T_k^*| \le |S_1^* - T_1^*| + 2D_0^* + 2D_1^* + \cdots + 2D_N^* \,. \tag{4.30}$$

For any $L \in \mathbb{N}$, on the probability space $([0,1], \nu_\gamma)$, the random variable $(T_\gamma^{2^L+1}, T_\gamma^{2^L+2}, \dots, T_\gamma^{2^{L+1}})$ is distributed as $(Y_{2^{L+1}}, Y_{2^{L+1}-1}, \dots, Y_{2^L+1})$, where $(Y_i)_{i\ge 1}$ is a stationary Markov chain with transition kernel K_γ and invariant measure ν_γ. From our construction of the random variables $Z_i'^*$, for any $L \in \mathbb{N}$,

$$(T_\gamma^{2^L+1}, \dots, T_\gamma^{2^{L+1}}, Z'^*_{2^L+1}, \dots, Z'^*_{2^{L+1}}) =^{\mathcal{D}} (Y_{2^{L+1}}, \dots, Y_{2^L+1}, Z'_{2^{L+1}}, \dots, Z'_{2^L+1}) \,,$$

where the sequence $(Z_i')_{2^L+1 \le i \le 2^{L+1}}$ is defined from $(Y_i, \delta_i)_{2^L < i \le 2^{L+1}}$ as in the proof of Theorem 2.1. It follows that

$$D_L^* =^{\mathcal{D}} D_L \text{ where } D_L := \sup_{0 \le \ell \le 2^L} |(S_{2^L+\ell} - T_{2^L+\ell}) - (S_{2^L} - T_{2^L})|$$

and, for any $j \ge 1$, $T_j = \sum_{i=1}^j Z_i'$ if $\sigma^2(f) > 0$ and $T_j = 0$ otherwise. Hence we have, for any positive λ, $\mathbb{P}(D_L^* \ge \lambda) = \mathbb{P}(D_L \ge \lambda)$. Proceeding as in the proof of Theorem 2.1, Item 2 follows. $\diamond$

5 Appendix

Next lemma is a parametrized version of Theorem 1 of Rio (1998). We first need the following definitions.

Definition 5.1. For P and Q two probability laws on the real line with respective distribution functions F and G, the Wasserstein distance of order 2 is defined by

$$W_2^2(P,Q) = \int_0^1 (F^{-1}(u) - G^{-1}(u))^2 du \,.$$

Definition 5.2. Λ_2 is the class of real functions f which are continuously differentiable and such that $|f'(x) - f'(y)| \le |x - y|$ for any $(x,y) \in \mathbb{R} \times \mathbb{R}$.

Lemma 5.1. *Let Z be a random variable with values in a purely non atomic Lebesgue space $(E, \mathcal{L}(E), m)$ and $\mathcal{F} = \sigma(Z)$. For real random variables U and V, let $P_{U|\mathcal{F}}$ be the law of U given $\mathcal{F}$ and P_V be the law of V. Assume that V is independent of $\mathcal{F}$. Let $\sigma^2 > 0$ and N be a $\mathcal{N}(0, \sigma^2)$-distributed random variable independent of $\sigma(Z, U, V)$. Then*

$$\mathbb{E}\big(W_2^2(P_{U|\mathcal{F}}, P_V)\big) \le 16 \sup_{f \in \Lambda_2(E)} \mathbb{E}\big(f(U+N, Z) - f(V+N, Z)\big) + 8\sigma^2 \,,$$

where $\Lambda_2(E)$ denotes the set of measurable functions $f : \mathbb{R} \times E \to \mathbb{R}$ wrt the σ-fields $\mathcal{L}(\mathbb{R} \times E)$ and $\mathcal{B}(\mathbb{R})$, such that $f(\cdot, z) \in \Lambda_2$ and $f(0, z) = f'(0, z) = 0$ for any $z \in E$.

Proof of Lemma 5.1. Notice first that

$$\mathbb{E}\big(W_2^2(P_{U|\mathcal{F}}, P_V)\big) \le 2\mathbb{E}\big(W_2^2(P_{U+N|\mathcal{F}}, P_{V+N})\big) + 8\sigma^2 \,. \tag{5.1}$$

Let G be the d.f. of P_{V+N}. Since E is a Lebesgue space, there exists a regular version of the conditional distribution function of $U+N$ given Z, that is, a function $(x, z) \to F_z(x)$ from $\mathbb{R} \times E$ in $\mathbb{R}$ such that, for any real x, $F_Z(x) = \mathbb{E}(\mathbb{1}_{U+N \le x}|Z)$ almost surely.

Notice in addition that, for any z in E, F_z is a C^∞ increasing distribution function. Let now $H_z(x) = F_z(x) - G(x)$, $A_z = \{y \in \mathbb{R} : H_z(y) = 0\}$, and for any $(x, z) \in \mathbb{R} \times E$, let

$$h(x, z) = d(x, A_z \cup \{0\}) \operatorname{sign} H_z(x) \text{ and } f(x, z) = \int_0^x h(y, z) dy \,, \tag{5.2}$$

where $d(x, A_z \cup \{0\})$ is the distance of x to the random set $A_z \cup \{0\}$ and $\operatorname{sign} y = 1$ for $y > 0$, 0 for $y = 0$ and -1 for $y < 0$.

For z fixed, $f(0, z) = f'(0, z) = 0$ and it is shown in Rio (1998, Inequality (7)) that $f(\cdot, z)$ belongs to Λ_2, and that for any $u \in]0, 1[$,

$$f(F_z^{-1}(u), z) - f(G^{-1}(u), z) \ge \frac{1}{8}\big(F_z^{-1}(u) - G^{-1}(u)\big)^2 \,.$$

Hence, for any $z \in E$,

$$\begin{aligned} W_2^2(P_{U+N|Z=z}, P_V) &= \int_0^1 \big(F_z^{-1}(u) - G^{-1}(u)\big)^2 du \\ &\le 8\Big(\int_{\mathbb{R}} f(x, z) dP_{U+N|Z=z} - \int_{\mathbb{R}} f(x, z) dP_{V+N}\Big) \,. \end{aligned} \tag{5.3}$$

We prove now that the function f defined by (5.2) is $\mathcal{L}(\mathbb{R} \times E) - \mathcal{B}(\mathbb{R})$ measurable. Notice first that since for any fixed z, $x \mapsto h(x, z)$ is continuous we get that

$$f(x, z) = \lim_{n \to \infty} \frac{x}{n} \sum_{i=1}^{n} h(itn^{-1}, z) \,.$$

Therefore the mesurability of f will follow from the mesurability of h. With this aim, it is enough to prove the mesurability of the restriction h_n of h to $[-n,n]\times E$ for any positive integer n.

Let $\varphi : [-n,n] \to [0,1]$ be the one to one bicontinuous map defined by $\varphi(x) = (n-x)/(2n)$. We then define

$$\begin{aligned} g : [0,1]\times E &\to \mathbb{R} \\ (x,z) &\mapsto h(\varphi^{-1}(x),z)\,. \end{aligned} \tag{5.4}$$

The mesurability of h_n will then follow from the mesurability of g. Since E is purely non atomic, $(E,\mathcal{L}(E),m)$ is isomorphic to $([0,1],\mathcal{L}([0,1]),\lambda_{[0,1]})$ where $\mathcal{L}([0,1])$ and $\lambda_{[0,1]}$ are respectively the Lebesgue σ-algebra and the Lebesgue measure on $[0,1]$ (see for instance Theorem 4.3 in De La Rue (1993)). Hence the following theorem due to Lipiński (1972) which is recalled in Grande (1976) also holds in $[0,1]\times E$.

Theorem 5.1. *(Lipiński, 1972) Let g be a bounded function from $[0,1]\times E$ into $\mathbb{R}$ such that*

1. *the cross sections $g_x(t) = g(x,t)$ and $g^z(t) = g(t,z)$ are respectively $\mathcal{L}(E)$ and $\mathcal{L}([0,1])$-measurable,*
2. *for all $t\in[0,1]$, $k_t(z) = \int_0^t g(x,z)dx$ is $\mathcal{L}(E)$-measurable,*
3. *for all $z\in E$, the cross section g^z is a derivative.*

Then g is measurable wrt the σ-fields $\mathcal{L}([0,1]\times E)$ and $\mathcal{B}(\mathbb{R})$.

We now apply Theorem 5.1 to the function g defined by (5.4). Items 2 and 3 as well as the second part of Item 1 follows directly from the fact that if z is fixed, then the function $x\to g(x,z)$ is continuous (recall that $h(\cdot,z)$ and φ^{-1} are continuous). It remains to show that for all $x\in[0,1]$ the cross section g_x is Lebesgue-measurable. Let us then prove that for any $x\in[-n,n]$ and any $\delta>0$,

$$\{z\in E : g(x,z)\geq\delta\}\in\mathcal{L}(E) \text{ and } \{z\in E : g(x,z)\leq-\delta\}\in\mathcal{L}(E) \tag{5.5}$$

which will end the proof of the mesurability of g and then of f. For any $x\in[-n,n]$ and any $\delta>0$, we notice that

$$\{z\in E : g(x,z)\geq\delta\} = \begin{cases} \{z\in E : H_z(x)>0\}\cap\{z\in E : d(x,A_z)\geq\delta\} & \text{if } |x|\geq\delta \\ \emptyset & \text{if } |x|<\delta. \end{cases}$$

If $|x| \geq \delta$,

$$\begin{aligned}
&\{z \in E \,:\, H_z(x) > 0\} \cap \{z \in E \,:\, d(x, A_z) \geq \delta\} \\
&\quad = \{z \in E \,:\, H_z(x) > 0\} \cap \{z \in E \,:\,]x-\delta, x+\delta[\cap A_z = \emptyset\} \\
&\quad = \{z \in E \,:\, H_z(y) > 0\,,\ \forall y \in]x-\delta, x+\delta[\}\,.
\end{aligned}$$

Using the fact that the function $H_z(\cdot)$ is continuous, we get that if $|x| \geq \delta$,

$$\begin{aligned}
&\{z \in E \,:\, H_z(x) > 0\} \cap \{z \in E \,:\, d(x, A_z) \geq \delta\} \\
&\quad = \bigcup_{p \in \mathbb{N}^*} \{z \in E \,:\, H_z(y) \geq \frac{1}{p}\,,\ \forall y \in]x-\delta, x+\delta[\cap \mathbb{Q}\}\,,
\end{aligned}$$

which proves the first part of (5.5) since $\{z \in E \,:\, H_z(a) \geq p^{-1}\}$ belongs to $\mathcal{L}(E)$ for any $a \in \mathbb{Q}$ and any $p \in \mathbb{N}^*$. The second part of (5.5) follows from the same arguments by changing the sign. This ends the proof of the $\mathcal{L}(\mathbb{R} \times E) - \mathcal{B}(\mathbb{R})$ measurability of f defined by (5.2).

Next $P_{(U+N,Z)}$ and $P_{(V+N,Z)}$ are absolutely continuous wrt $\lambda \otimes P_Z$. Hence, starting from (5.1) and using (5.3), the lemma follows. $\diamond$

Proposition 5.1. *Let $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$, where f belongs to $\widetilde{\mathcal{F}}(Q, P_{Y_0})$. Assume that $M_{2,\alpha}(Q) < \infty$. Then the series $\mathbb{E}(X_0^2) + 2\sum_{k\geq 1} \mathbb{E}(X_0 X_k)$ is convergent to a nonnegative real σ^2. If $\sigma^2 > 0$, then there exists a positive constant C depending on σ^2 such, that for any $n > 0$,*

$$\mathbb{E}\big(W_2^2(P_{S_n|\mathcal{F}_0}, G_{n\sigma^2})\big) \leq C n^{1/2} M_{3,\alpha}(Q, n^{1/2})\,, \tag{5.6}$$

where $M_{3,\alpha}(Q, n^{1/2})$ is defined in (4.1).

Proof of Proposition 5.1. Let $(N_i)_{i\in\mathbb{Z}}$ be a sequence of independent random variables with normal distribution $\mathcal{N}(0, \sigma^2)$. Suppose furthermore that the sequence $(N_i)_{i\in\mathbb{Z}}$ is independent of $\mathcal{F}_\infty$. Let N be a $\mathcal{N}(0, \sigma^2)$-distributed random variable, independent of $\mathcal{F}_\infty \vee \sigma(N_i, i \in \mathbb{Z})$. Set $T_n = N_1 + N_2 + \cdots + N_n$. Let $Z = ((Y_i, \delta_i) \,:\, i \leq 0)$ and $E = (\mathbb{R} \times [0,1])^{\mathbb{Z}^-}$. Notice that $(E, \mathcal{L}(E), P_Z)$ is a purely non atomic Lebesgue space. From Lemma 5.1, we have to bound

$$\Delta(\varphi) = \mathbb{E}(\varphi(S_n + N, Z) - \varphi(T_n + N, Z))\,, \tag{5.7}$$

for any function φ in $\Lambda_2(E)$. With this aim, we apply the Lindeberg method.

Notation 5.1. Let

$$\varphi_k(x, Z) = \int_{\mathbb{R}} \varphi(t, Z) \phi_{\sigma\sqrt{n-k+1}}(x - t)\, dt\,.$$

Let $S_0 = 0$, and, for $k > 0$, let $\Delta_k = \varphi_k(S_{k-1} + X_k, Z) - \varphi_k(S_{k-1} + N_k, Z)$.

Since the sequence $(N_i)_{i\in\mathbb{Z}}$ is independent of the sequence $(X_i)_{i\in\mathbb{Z}}$,

$$\mathbb{E}(\varphi(S_n+N,Z)-\varphi(T_n+N,Z))=\sum_{k=1}^{n}\mathbb{E}(\Delta_k)\,. \tag{5.8}$$

We first show that for any real $u\in[0,1]$,

$$|\mathbb{E}(\Delta_k)|\le C\big((n-k+1)^{-1/2}+D_k(u)\big)\,, \tag{5.9}$$

where

$$\begin{aligned}D_k(u)&=(n-k+1)^{1/2}\int_0^{\alpha(k)}Q(x)dx+\sum_{i>[k/2]}\int_0^{\alpha(i)}Q^2(x)dx\\&+\int_0^{u}Q(x)R(x)dx+(n-k+1)^{-1/2}\int_0^{1}Q(x)R(x)R(x\vee u)dx\,.\end{aligned} \tag{5.10}$$

We now prove (5.9). For the sake of brevity, write $\varphi_k(x,Z)=\varphi_k(x)$ and $\varphi(x,Z)=\varphi(x)$ (the derivatives are taken wrt x). By the Taylor formula at order 3,

$$\Big|\mathbb{E}\big(\varphi_k(S_{k-1}+N_k)-\varphi_k(S_{k-1})-\frac{\sigma^2}{2}\varphi_k''(S_{k-1})\big)\Big|\le\frac{\|\varphi_k^{(3)}\|_\infty}{6}\mathbb{E}|N|^3\,.$$

Now Lemma 6.1 in Dedecker, Merlevède and Rio (2009) gives that, almost surely,

$$\|\varphi_k^{(i)}\|_\infty\le c_i\sigma^{2-i}(n-k+1)^{(2-i)/2}\text{ for any integer }i\ge 2 \tag{5.11}$$

where the c_i's are universal constants. Therefore

$$\Big|\mathbb{E}\big(\varphi_k(S_{k-1}+N_k)-\varphi_k(S_{k-1})-\frac{\sigma^2}{2}\varphi_k''(S_{k-1})\big)\Big|\le C(n-k+1)^{-1/2}\,.$$

Hence to prove (5.9), it remains to show that

$$\Big|\mathbb{E}\big(\varphi_k(S_{k-1}+X_k)-\varphi_k(S_{k-1})-\frac{\sigma^2}{2}\varphi_k''(S_{k-1})\big)\Big|\le CD_k(u)\,, \tag{5.12}$$

where $D_k(u)$ is defined by (5.10). To prove (5.12), we follow the lines of the proof of Proposition 2(a) of Rio (1995-b) with $b_2=\|\varphi_k^{(2)}\|_\infty$, $b_3=\|\varphi_k^{(3)}\|_\infty$ and the modifications below. Since f belongs to $\widetilde{\mathcal{F}}(Q,P_{Y_0})$, we can write

$$X_i=\lim_{N\to\infty}\mathbb{L}^1\sum_{\ell=1}^{N}a_{\ell,N}\big(f_{\ell,N}(Y_i)-\mathbb{E}(f_{\ell,N}(Y_i))\big)\,,$$

with $f_{\ell,N}$ belonging to $\widetilde{\text{Mon}}(Q, P_{Y_0})$ and $\sum_{\ell=1}^N |a_{\ell,N}| \leq 1$. For $u \in [0,1]$, let the function g_u be defined by $g_u(x) = (x \wedge Q(u)) \vee (-Q(u))$. Since there exists a subsequence $m(N)$ tending to infinity such that $\sum_{\ell=1}^{m(N)} a_{\ell,m(N)} g_u \circ f_{\ell,m(N)}(Y_0)$ is convergent in $\mathbb{L}^1$, for any $i \geq 0$, we define

$$\bar{X}_i = \lim_{N\to\infty} \mathbb{L}^1 \sum_{\ell=1}^{m(N)} a_{\ell,m(N)} \big(g_u \circ f_{\ell,m(N)}(Y_i) - \mathbb{E}(g_u \circ f_{\ell,m(N)}(Y_i))\big) \quad \text{and} \quad \tilde{X}_i = X_i - \bar{X}_i \,.$$

Let also

$$Q_u(x) := Q(x) \mathbb{I}_{x \leq u} \text{ and } \bar{Q}_u(x) := Q(x \vee u)\,.$$

Since $Q_{|g_u \circ f_{\ell,m(N)}(Y_i)|} \leq \bar{Q}_u$, this means that $\bar{X}_i = r(Y_i) - \mathbb{E}(r(Y_i))$ where r belongs to $\widetilde{\mathcal{F}}(\bar{Q}_u, P_{Y_0})$.

By the Taylor integral formula,

$$\begin{aligned} \varphi_k(S_k) - \varphi_k(S_{k-1}) - \varphi_k'(S_{k-1})X_k &= X_k \int_0^1 (\varphi_k'(S_{k-1} + vX_k) - \varphi_k'(S_{k-1}))dv \\ &= X_k \int_0^1 (\varphi_k'(S_{k-1} + vX_k) - \varphi_k'(S_{k-1} + v\bar{X}_k))dv \\ &+ X_k \bar{X}_k \int_0^1\int_0^1 v\varphi_k''(S_{k-1} + vv'\bar{X}_k)dvdv' \,. \end{aligned} \tag{5.13}$$

The first term on right hand side is bounded up by $b_2|X_k(X_k - \bar{X}_k)|/2$. Moreover

$$\Big| \int_0^1\int_0^1 v\varphi_k''(S_{k-1} + vv'\bar{X}_k)dvdv' - \frac{1}{2}\varphi_k''(S_{k-1})\Big| \leq \frac{b_3}{6}|\bar{X}_k|\,.$$

Setting $h_u(x) = x - g_u(x)$, we get that for any f belonging to $\widetilde{\text{Mon}}(Q, P_{Y_0})$,

$$\begin{aligned} &\mathbb{E}\big|(f(Y_k) - \mathbb{E}(f(Y_k)))(h_u \circ f_\ell(Y_k) - \mathbb{E}(h_u \circ f(Y_k)))\big| \\ &\qquad \leq \mathbb{E}|f(Y_k)h_u(f(Y_k))| + 3\mathbb{E}|f(Y_k)|\mathbb{E}|h_u(f(Y_k))|\,. \end{aligned}$$

Since $Q_{|f(Y_k)|} \leq Q$ and $Q_{|h_u(f(Y_k))|} \leq (Q - Q(u))_+ \leq Q_u$, we derive that

$$\begin{aligned} &\mathbb{E}\big|(f(Y_k) - \mathbb{E}(f(Y_k)))(h_u \circ f_\ell(Y_k) - \mathbb{E}(h_u \circ f(Y_k)))\big| \\ &\qquad \leq \int_0^u Q^2(x)dx + 3\Big(\int_0^1 Q(x)dx\Big)\Big(\int_0^u Q(x)dx\Big) \leq 4\int_0^u Q^2(x)dx\,, \end{aligned}$$

by using Lemma 2.1(a) in Rio (2000). Now, by Fatou lemma,

$$\begin{aligned} &\mathbb{E}|X_k(X_k - \bar{X}_k)| \leq \liminf_{N\to\infty} \sum_{\ell=1}^{m(N)} \sum_{j=1}^{m(N)} |a_{\ell,m(N)}||a_{j,m(N)}| \\ &\times \mathbb{E}\big|(f_{\ell,m(N)}(Y_k) - \mathbb{E}(f_{\ell,m(N)}(Y_k)))(h_u \circ f_{j,m(N)}(Y_k) - \mathbb{E}(h_u \circ f_{j,m(N)}(Y_k)))\big|\,, \end{aligned}$$

whence

$$\mathbb{E}|X_k(X_k - \bar{X}_k)| \le 4\int_0^u Q^2(x)dx\,. \tag{5.14}$$

Similarly, by using Lemma 2.1 in Rio (2000) and the fact that $Q_{|g_u \circ f(Y_k)|} \le \bar{Q}_u$ for any f belonging to $\widetilde{\mathrm{Mon}}(Q, P_{Y_0})$, we derive that

$$\mathbb{E}|X_k(\bar{X}_k)^2| \le 8\int_0^1 Q^2(x)Q(x\vee u)dx\,. \tag{5.15}$$

It follows that

$$\begin{aligned}&\big|\mathbb{E}(\varphi_k(S_k) - \varphi_k(S_{k-1}) - \varphi_k'(S_{k-1})X_k - \frac{1}{2}\varphi_k''(S_{k-1})X_k\bar{X}_k)\big| \\ &\le 2b_2\int_0^u Q^2(x)dx + \frac{4b_3}{3}\int_0^1 Q^2(x)Q(x\vee u)dx\,. \end{aligned}\tag{5.16}$$

Now we control the second order term. Let

$$\Gamma_k(k,i) = \varphi_k''(S_{k-i}) - \varphi_k''(S_{k-i-1})\,, \tag{5.17}$$

and

$$r = \alpha^{-1}(u)\,. \tag{5.18}$$

Clearly

$$\varphi_k''(S_{k-1})X_k\bar{X}_k = \sum_{i=1}^{(r\wedge k)-1}\Gamma_k(k,i)X_k\bar{X}_k + \varphi_k''(S_{k-(r\wedge k)})X_k\bar{X}_k\,,$$

Since $|\Gamma_k(k,i)| \le b_3|X_{k-i}|$, by stationarity we get that for any $i \le (r\wedge k)-1$,

$$\big|\operatorname{Cov}(\Gamma_k(k,i), X_k\bar{X}_k)\big| \le b_3\|X_0\big(\mathbb{E}_0(X_i\bar{X}_i) - \mathbb{E}(X_k\bar{X}_k)\big)\|_1\,.$$

Applying Proposition 5.3 with $m=1$, $q=2$, $k_1=0$, $k_2=k_3=i$, $f_{j_1}=f_{j_2}=f$ and $f_{j_3}\in\widetilde{\mathcal{F}}(\bar{Q}_u, P_{Y_0})$, we derive that

$$\big|\operatorname{Cov}(\Gamma_k(k,i), X_k\bar{X}_k)\big| \le 32b_3\int_0^{\alpha(i)} Q^2(x)Q(x\vee u)dx\,.$$

Since $|\varphi_k''(S_{k-(r\wedge k)})| \le b_2$ a.s., we also get by stationarity that

$$\big|\operatorname{Cov}(\varphi_k''(S_{k-(r\wedge k)}), X_k\bar{X}_k))\big| \le b_2\|\mathbb{E}_0(X_{r\wedge k}\bar{X}_{r\wedge k}) - \mathbb{E}(X_{r\wedge k}\bar{X}_{r\wedge k})\|_1\,.$$

Applying Proposition 5.3 with $m=0$, $q=2$, $k_1=k_2=r\wedge k$, $f_{j_1}=f$ and $f_{j_2}\in\widetilde{\mathcal{F}}(\bar{Q}_u, P_{Y_0})$, and noting that $\alpha(r)\le u$, we also get that

$$\big|\operatorname{Cov}(\varphi_k''(S_{k-(r\wedge k)}), X_k\bar{X}_k))\big| \;\le\; 16b_2\Big(\int_0^u Q(x)Q(u)dx\,\mathbb{1}_{r\le k} + \int_0^{\alpha(k)} Q(x)Q(x\vee u)dx\,\mathbb{1}_{k<r}\Big)\,.$$

Hence

$$\frac{1}{2}|\operatorname{Cov}(\varphi_k''(S_{k-1}), X_k\bar{X}_k)| \leq 8b_2\int_0^u Q(x)Q(u)dx\,\mathbb{1}_{r\leq k} + 8b_2\int_0^{\alpha(k)} Q(x)Q(x\vee u)dx\,\mathbb{1}_{k<r} + 16b_3\int_0^1 Q^2(x)R(x\vee u)dx\,,$$

which together with (5.16) and (5.14) implies that

$$\Big|\mathbb{E}(\varphi_k(S_k)-\varphi_k(S_{k-1})-\varphi_k'(S_{k-1})X_k)-\frac{1}{2}\mathbb{E}(\varphi_k''(S_{k-1}))\mathbb{E}(X_k^2)\Big| \leq 12b_2\int_0^u Q^2(x)dx + 8b_2\int_0^{\alpha(k)} Q(x)Q(x\vee u)dx\,\mathbb{1}_{k<r} + \frac{52}{3}b_3\int_0^1 Q^2(x)R(x\vee u)dx\,. \quad (5.19)$$

To give now an estimate of the expectation of $\varphi_k'(S_{k-1})X_k$, we write

$$\varphi_k'(S_{k-1}) = \varphi_k'(0) + \sum_{i=1}^{k-1}(\varphi_k'(S_{k-i})-\varphi_k'(S_{k-i-1}))\,.$$

Hence

$$\mathbb{E}(\varphi_k'(S_{k-1})X_k) = \sum_{i=1}^{k-1}\operatorname{Cov}\big(\varphi_k'(S_{k-i})-\varphi_k'(S_{k-i-1}), X_k\big) + \mathbb{E}(\varphi_k'(0)X_k)\,. \quad (5.20)$$

Now $\varphi_k'(0)$ is a $\mathcal{F}_0$-measurable random variable, and since $\varphi'(0)=0$ and φ' is 1-Lipschitz wrt x,

$$|\varphi_k'(0)| = \Big|\int(\varphi'(u)-\varphi'(0))\phi_{\sigma\sqrt{n-k+1}}(-u)du\Big| \leq \sigma\sqrt{n-k+1} \text{ a.s.}$$

Applying Proposition 5.3 with $m=0$, $q=1$, $k_1=k$ and $f_{j_1}=f$, it follows that

$$\mathbb{E}(\varphi_k'(0)X_k) \leq \sigma\sqrt{n-k+1}\|\mathbb{E}_0(X_k)\|_1 \leq 8\sigma\sqrt{n-k+1}\int_0^{\alpha(k)/2} Q(x)dx\,. \quad (5.21)$$

We give now an estimate of $\sum_{i=1}^{k-1}\operatorname{Cov}\big(\varphi_k'(S_{k-i})-\varphi_k'(S_{k-i-1}), X_k\big)$. Using the stationarity and noting that $|\varphi_k'(S_{k-i})-\varphi_k'(S_{k-i-1})| \leq b_2|X_{k-i}|$, we have

$$|\operatorname{Cov}(\varphi_k'(S_{k-i})-\varphi_k'(S_{k-i-1}), X_k)| \leq b_2\|X_0\mathbb{E}_0(X_i)\|_1\,.$$

Now, for any $i\geq r$, $\alpha(i)\leq u$. So applying Proposition 5.3 with $m=1$, $q=1$, $k_1=0$, $k_2=i$, $f_{j_1}=f_{j_2}=f$, we get, for any $k\geq i\geq r$, that

$$|\operatorname{Cov}(\varphi_k'(S_{k-i})-\varphi_k'(S_{k-i-1}), X_k)| \leq 16b_2\int_0^u Q^2(x)\mathbb{1}_{x<\alpha(i)}dx\,. \quad (5.22)$$

From now on, we assume that $i < r \wedge k$. Let us replace X_k by $\bar{X}_k$. Since by stationarity,

$$| \operatorname{Cov}(\varphi_k'(S_{k-i}) - \varphi_k'(S_{k-i-1}), X_k - \bar{X}_k)| \le b_2 \|X_0 \mathbb{E}_0(X_i - \bar{X}_i)\|_1 ,$$

we can apply Proposition 5.3 with $m = 1$, $q = 1$, $k_1 = 0$, $k_2 = i$, $f_{j_1} = f$ and $f_{j_2} \in \widetilde{\mathcal{F}}(\bar{Q}_u, P_{Y_0})$. It follows that

$$| \operatorname{Cov}(\varphi_k'(S_{k-i}) - \varphi_k'(S_{k-i-1}), X_k - \bar{X}_k)| \le 16 b_2 \int_0^u Q^2(x) \mathbb{1}_{x < \alpha(i)} dx . \tag{5.23}$$

Now

$$\varphi_k'(S_{k-i}) - \varphi_k'(S_{k-i-1}) - \varphi_k''(S_{k-i-1}) X_{k-i} = R_{k,i} ,$$

where $R_{k,i}$ is $\mathcal{F}_{k-i}$-measurable and $|R_{k,i}| \le b_3 X_{k-i}^2/2$. Hence, by stationarity,

$$| \operatorname{Cov}(R_{k,i}, \bar{X}_k)| \le b_3 \|X_0^2 \mathbb{E}_0(\bar{X}_i)\|_1 / 2 .$$

Applying Proposition 5.3 with $m = 2$, $q = 1$, $k_1 = k_2 = 0$, $k_3 = i$, $f_{j_1} = f_{j_2} = f$ and $f_{j_3} \in \widetilde{\mathcal{F}}(\bar{Q}_u, P_{Y_0})$, we get that

$$| \operatorname{Cov}(R_{k,i}, \bar{X}_k)| \le 32 b_3 \int_0^{\alpha(i)} Q^2(x) Q(x \vee u) dx . \tag{5.24}$$

In order to estimate the term $\operatorname{Cov}(\varphi_k''(S_{k-i-1}) X_{k-i}, \bar{X}_k)$, we introduce the decomposition below:

$$\varphi_k''(S_{k-i-1}) = \sum_{l=1}^{(i-1)\wedge(k-i-1)} (\varphi_k''(S_{k-i-l}) - \varphi_k''(S_{k-i-l-1})) + \varphi_k''(S_{(k-2i)\vee 0}) .$$

For any $l \in \{1, \cdots, (i-1) \wedge (k-i-1)\}$, by using the notation (5.17) and stationarity, we get that

$$| \operatorname{Cov}(\Gamma_k(k, l+i) X_{k-i}, \bar{X}_k)| \le b_3 \|X_{-l} X_0 \mathbb{E}_0(\bar{X}_i)\|_1 .$$

Applying Proposition 5.3 with $m = 2$, $q = 1$, $k_1 = -\ell$, $k_2 = 0$, $k_3 = i$, $f_{j_1} = f_{j_2} = f$ and $f_{j_3} \in \widetilde{\mathcal{F}}(\bar{Q}_u, P_{Y_0})$, we then derive that

$$| \operatorname{Cov}(\Gamma_k(k, l+i) X_{k-i}, \bar{X}_k)| \le 64 b_3 \int_0^{\alpha(i)} Q^2(x) Q(x \vee u) dx . \tag{5.25}$$

As a second step, we bound up $| \operatorname{Cov}(\varphi_k''(S_{(k-2i)\vee 0}), X_{k-i} \bar{X}_k)|$. Assume first that $i \le [k/2]$. Clearly, using the notation (5.17),

$$\varphi_k''(S_{k-2i}) = \sum_{l=i}^{(r-1)\wedge(k-i-1)} \Gamma_k(k, l+i) + \varphi''(S_{(k-i-r)\vee 0}) .$$

Now for any $l \in \{i, \cdots, (r-1) \wedge (k-i-1)\}$, by stationarity,

$$|\operatorname{Cov}(\Gamma_k(k, l+i), X_{k-i}\bar{X}_k)| \leq b_3 \|X_{-l}\big(\mathbb{E}_{-l}(X_0\bar{X}_i) - \mathbb{E}(X_0\bar{X}_i)\big)\|_1 .$$

Hence applying Proposition 5.3 with $m = 1$, $q = 2$, $k_1 = -l$, $k_2 = 0$, $k_3 = i$, $f_{j_1} = f_{j_2} = f$ and $f_{j_3} \in \widetilde{\mathcal{F}}(\bar{Q}_u, P_{Y_0})$, we derive that

$$|\operatorname{Cov}(\Gamma_k(k, l+i), X_{k-i}\bar{X}_k)| \leq 32 b_3 \int_0^{\alpha(l)} Q^2(x) Q(x \vee u) dx . \tag{5.26}$$

If $i \leq k - r$, then stationarity implies that

$$|\operatorname{Cov}(\varphi_k''(S_{k-i-r}), X_{k-i}\bar{X}_k)| \leq b_2 \|\mathbb{E}_0(X_r\bar{X}_{i+r}) - \mathbb{E}(X_r\bar{X}_{i+r})\|_1 .$$

Noting that $\alpha(r) \leq u < \alpha(i)$ and applying Proposition 5.3 with $m = 0$, $q = 2$, $k_0 = 0$, $k_1 = r$, $k_2 = i + r$, $f_{j_1} = f$ and $f_{j_2} \in \widetilde{\mathcal{F}}(\bar{Q}_u, P_{Y_0})$, we also get that

$$|\operatorname{Cov}(\varphi_k''(S_{k-i-r}), X_{k-i}\bar{X}_k)| \leq 16 b_2 \int_0^u \mathbb{1}_{x<\alpha(i)} Q(x) Q(u) dx . \tag{5.27}$$

Now if $i > k - r$, then we write that

$$|\operatorname{Cov}(\varphi_k''(0), X_{k-i}\bar{X}_k)| \leq b_2 \|\mathbb{E}_0(X_{k-i}\bar{X}_k) - \mathbb{E}(X_{k-i}\bar{X}_k)\|_1 .$$

Applying Proposition 5.3 with $m = 0$, $q = 2$, $k_0 = 0$, $k_1 = k - i$, $k_2 = k$, $f_{j_1} = f$ and $f_{j_2} \in \widetilde{\mathcal{F}}(\bar{Q}_u, P_{Y_0})$, and noting that for $i \leq [k/2]$, $\alpha(k-i) \leq \alpha([k/2])$, we obtain that

$$|\operatorname{Cov}(\varphi_k''(0), X_{k-i}\bar{X}_k)| \leq 16 b_2 \int_0^{\alpha([k/2])} Q(x) Q(x \vee u) dx . \tag{5.28}$$

Assume now that $i \geq [k/2] + 1$. For any $i \leq k$, the stationarity entails that

$$|\mathbb{E}(\varphi_k''(0) X_{k-i}\bar{X}_k)| \leq b_2 \|X_0 \mathbb{E}_0(\bar{X}_i)\|_1 .$$

Hence applying Proposition 5.3 with $m = 1$, $q = 1$, $k_0 = 0$, $k_1 = i$, $f_{j_1} = f$ and $f_{j_2} \in \widetilde{\mathcal{F}}(\bar{Q}_u, P_{Y_0})$, and noting that for $i \geq [k/2] + 1$, $\alpha(i) \leq \alpha([k/2])$, we obtain that

$$|\mathbb{E}(\varphi_k''(0) X_{k-i}\bar{X}_k)| \leq 16 b_2 \int_0^{\alpha([k/2])} Q(x) Q(x \vee u) dx . \tag{5.29}$$

Adding the inequalities (5.21)-(5.29), summing on i and l, and using the fact that

$$\sum_{i=1}^{k-1} \mathbb{1}_{x<\alpha(i)} \leq \alpha^{-1}(x) , \sum_{i=1}^{r} \mathbb{1}_{x<\alpha(i)} \leq \alpha^{-1}(x \vee u) \text{ and } \sum_{i=1}^{r} i \mathbb{1}_{x<\alpha(i)} \leq (\alpha^{-1}(x \vee u))^2 ,$$

we then get:

$$|\mathbb{E}(\varphi'(S_{k-1})X_k) - \sum_{i=1}^{r-1}\mathbb{E}(\varphi''(S_{k-2i}))\mathbb{E}(X_{k-i}\bar{X}_k)\mathbb{1}_{i\leq[k/2]}| \leq C(n-k+1)^{1/2}\int_0^{\alpha(k)} Q(x)dx +$$
$$48b_2\int_0^u Q(x)R(x)dx + 24kb_2\int_0^{\alpha([k/2])} Q(x)Q(x\vee u)dx$$
$$+128b_3\int_0^1 Q(x)R(x)R(x\vee u)dx\,. \qquad (5.30)$$

It remains to bound up

$$A_k := \sum_{i=1}^{r-1}\mathbb{E}(\varphi_k''(S_{k-2i}))\mathbb{E}(X_{k-i}\bar{X}_k)\mathbb{1}_{i\leq[k/2]} - \sum_{i=1}^{\infty}\mathbb{E}(\varphi_k''(S_{k-1}))\mathbb{E}(X_{k-i}X_k)\,.$$

We first note that by stationarity,

$$\sum_{i\geq r}|\mathbb{E}(\varphi_k''(S_{k-1}))\mathbb{E}(X_{k-i}X_k)| \leq b_2\sum_{i\geq r}|\mathbb{E}(f(Y_0)\mathbb{E}_0(X_i))|\,.$$

Applying Proposition 5.3 and noting that $\alpha(i)\leq u$ for $i\geq r$, we get that

$$\sum_{i\geq r}|\mathbb{E}(\varphi_k''(S_{k-1}))\mathbb{E}(X_{k-i}X_k)| \leq 8b_2\sum_{i\geq r}\int_0^{\alpha(i)} Q^2(x)dx \leq 8b_2\int_0^u Q(x)R(x)dx\,. \qquad (5.31)$$

By stationarity we also have

$$\sum_{i=1}^{r-1}|\mathbb{E}(\varphi_k''(S_{k-1}))\mathbb{E}(X_{k-i}(X_k-\bar{X}_k))| \leq b_2\sum_{i=1}^{r-1}|\mathbb{E}(f(Y_0)\mathbb{E}_0(X_i-\bar{X}_i))|\,.$$

Next, noting that $u<\alpha(i)$ for all $i<r$ and applying Proposition 5.3, we get that

$$\begin{aligned}\sum_{i=1}^{r-1}|\mathbb{E}(\varphi_k''(S_{k-1}))\mathbb{E}(X_{k-i}(X_k-\bar{X}_k))| &\leq 8b_2\int_0^u Q^2(x)\sum_{i=1}^{r-1}\mathbb{1}_{x<\alpha(i)}dx\\ &\leq 8b_2\int_0^u Q^2(x)\alpha^{-1}(x)dx\,. \qquad (5.32)\end{aligned}$$

In addition, another application of Proposition 5.3 gives

$$\sum_{i=1+[k/2]}^{r-1}|\mathbb{E}(\varphi_k''(S_{k-1}))\mathbb{E}(X_{k-i}\bar{X}_k)| \leq 8b_2\sum_{i>[k/2]}\int_0^{\alpha(i)} Q^2(x)dx\,. \qquad (5.33)$$

In order to bound up the last term, we still write

$$\mathbb{E}(\varphi_k''(S_{k-1})-\varphi_k''(S_{k-2i}))\mathbb{E}(X_{k-i}\bar{X}_k)\mathbb{1}_{i\leq[k/2]} = \sum_{l=1}^{2i-1}\mathbb{E}(\Gamma_k(k,l))\mathbb{E}(f(Y_0)\mathbb{E}_0(\bar{X}_i))\mathbb{1}_{i\leq[k/2]}\,.$$

This decomposition, Proposition 5.3 and Lemma 2.1 in Rio (2000) then yield:

$$\sum_{i=1}^{r-1} |\mathbb{E}(\varphi_k''(S_{k-1}) - \varphi_k''(S_{k-2i}))\mathbb{E}(X_{k-i}\bar{X}_k)|\mathbb{1}_{i\le[k/2]} \le 8b_3 \sum_{i=1}^{r-1} i \int_0^{\alpha(i)} Q^2(x)Q(x\vee u)dx$$
$$\le 8b_3 \int_0^1 Q(x)R(x)R(x\vee u)dx\,. \tag{5.34}$$

Hence (5.31), (5.32), (5.33) and (5.34) together entail that

$$|A_k| \le 16b_2 \int_0^u Q(x)R(x)dx + 8b_2 \sum_{i>[k/2]} \int_0^{\alpha(i)} Q^2(x)dx + 8b_3 \int_0^1 Q(x)R(x)R(x\vee u)dx\,. \tag{5.35}$$

The inequalities (5.35), (5.30), (5.19) together with (5.11) then yield (5.9).

Notice now that

$$\sum_{k=1}^{n} \sqrt{n-k+1} \int_0^{\alpha(k)} Q(x)dx \le n^{1/2} \int_0^1 (\alpha^{-1}(x)\wedge n)Q(x)dx\,,$$

and that

$$\sum_{k=1}^{n} \sum_{i>[k/2]} \int_0^{\alpha(i)} Q^2(x)dx \le 2\sum_{i\ge 1}(i\wedge n) \int_0^{\alpha(i)} Q^2(x)dx$$
$$\le 2\int_0^1 Q(x)R(x)(\alpha^{-1}(x)\wedge n)dx \le 2n^{1/2}\int_0^1 Q(x)R(x)(R(x)\wedge n^{1/2})dx\,.$$

Moreover

$$n^{1/2}\int_0^1 (\alpha^{-1}(x)\wedge n)Q(x)dx \le n^{1/2}\int_0^1 Q(x)R(x)(R(x)\wedge n^{1/2})dx\,.$$

Hence to prove Proposition 5.1, it remains to select $u = u_k$ in such a way that

$$\sum_{k=1}^{n} \int_0^{u_k} Q(x)R(x)dx + \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \int_0^1 Q(x)R(x)R(x\vee u_k)dx \le Cn^{1/2}M_{3,\alpha}(Q, n^{1/2})\,. \tag{5.36}$$

Let $R^{-1}(y) = \inf\{v\in[0,1] : R(v)\le y\}$ be the right continuous inverse of R. Since R is right continuous, $x < R^{-1}(y)$ if and only if $R(x) > y$. We now choose $u_k = R^{-1}(k^{1/2})$, so that

$$R(u_k) \le k^{1/2} \text{ and } R(x) > k^{1/2} \text{ for any } x < u_k\,. \tag{5.37}$$

With this choice of u_k, on one hand,

$$\sum_{k=1}^{n} \int_0^{u_k} Q(x)R(x)dx = \int_0^1 Q(x)R(x) \sum_{k=1}^{n} \mathbb{1}_{R(x)>\sqrt{k}}dx \le \int_0^1 Q(x)R(x)(R^2(x)\wedge n)dx$$
$$\le n^{1/2}\int_0^1 Q(x)R(x)(R(x)\wedge n^{1/2})dx\,. \tag{5.38}$$

On the other hand, by using (5.37), we obtain

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\int_0^1 Q(x)R(x)R(x\vee u_k)dx \le \sum_{k=1}^{n}\frac{1}{\sqrt{k}}\int_{u_k}^1 Q(x)R^2(x)dx + \sum_{k=1}^{n}\int_0^{u_k} Q(x)R(x)dx\,. \quad (5.39)$$

Next

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\int_{u_k}^1 Q(x)R^2(x)dx \le \sum_{k=1}^{n}\frac{1}{\sqrt{k}}\int_{u_n}^1 Q(x)R^2(x)dx \le 2n^{1/2}M_{3,\alpha}(Q,n^{1/2})\,. \quad (5.40)$$

Combining (5.39) with (5.40) and (5.38), we then get (5.36) ending the proof of the proposition. $\diamond$

Proposition 5.2. *For f in $\widetilde{\mathcal{F}}(Q,P_{Y_0})$, let $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$. Set $S_n^* = \max_{1\le k\le n}|S_k|$. Assume that $M_{2,\alpha}(Q)<\infty$. Then the series $\mathbb{E}(X_0^2)+2\sum_{k\ge 1}\mathbb{E}(X_0X_k)$ is convergent to a non-negative real σ^2 and for any positive real λ,*

$$\mathbb{P}(S_n^*\ge 5\lambda) \;\le\; c_1\exp\Big(-\frac{\lambda^2}{c_2 n\sigma^2}\Big) + c_3 n\lambda^{-3}M_{3,\alpha}(Q,\lambda) + c_4 n\sigma^3\lambda^{-3}\,,$$

where $M_{3,\alpha}(Q,n^{1/2})$ is defined in (4.1) and c_1, c_2, c_3 and c_4 are positive constants not depending on σ^2, so that the first term vanishes if $\sigma^2=0$.

Proof of Proposition 5.2. Assume first that $X_i=\sum_{\ell=1}^L a_\ell f_\ell(Y_i) - \sum_{\ell=1}^L a_\ell\mathbb{E}(f_\ell(Y_i))$, with f_ℓ belonging to $\widetilde{\mathrm{Mon}}(Q,P_{Y_0})$ and $\sum_{\ell=1}^L |a_\ell|\le 1$. Let $M>0$ and $g_M(x)=(x\wedge M)\vee(-M)$. For any $i\ge 0$, we first define

$$X_i' = \sum_{\ell=1}^{L} a_\ell\big(g_M\circ f_\ell(Y_i) - \mathbb{E}(g_M\circ f_\ell(Y_i))\big) \quad\text{and}\quad X_i''=X_i-X_i'\,.$$

Let q be a positive integer such that $q\le n$. Let us first show that

$$\max_{1\le k\le n}|S_k| \le \max_{1\le k\le n}|\mathbb{E}_k(S_n)| + 2qM + \max_{1\le k\le n}\mathbb{E}_k\Big(\sum_{i=1}^n |X_i''|\Big) + \max_{1\le k\le n}\mathbb{E}_k\Big(\sum_{i=1}^n |\mathbb{E}_{i-q}(X_i')|\Big)\,. \quad (5.41)$$

Notice that

$$S_k = \mathbb{E}_k(S_n) - \sum_{i=k+1}^n \mathbb{E}_k(X_i'') - \sum_{i=k+1}^n \mathbb{E}_k(X_i')\,.$$

Now

$$\sum_{i=k+1}^n \mathbb{E}_k(X_i') = \sum_{i=k+1}^n \mathbb{E}_k(X_i'-\mathbb{E}_{i-q}(X_i')) - \sum_{i=k+1}^n \mathbb{E}_k(\mathbb{E}_{i-q}(X_i'))\,.$$

The inequality (5.41) follows by noticing that

$$\sum_{i=k+1}^{n} \mathbb{E}_k(X'_i - \mathbb{E}_{i-q}(X'_i)) = \sum_{i=k+1}^{q+k} (\mathbb{E}_k(X'_i) - \mathbb{E}_{i-q}(X'_i)) \le 2qM\,.$$

Notice now that $(\mathbb{E}_k(S_n))_{k\ge1}$, $\Big(\mathbb{E}_k\big(\sum_{i=1}^n |X''_i|\big)\Big)_{k\ge1}$ and $\Big(\mathbb{E}_k\big(\sum_{i=1}^n |\mathbb{E}_{i-q}(X'_i)|\big)\Big)_{k\ge1}$ are martingales with respect to the filtration $(\mathcal{F}_k)_{k\ge1}$. Therefore from (5.41) and the Doob maximal inequality, we infer that for any nondecreasing, non negative, convex and even function φ and if $qM \le \lambda$,

$$\mathbb{P}(S_n^* \ge 5\lambda) \;\le\; \frac{\mathbb{E}(\varphi(S_n))}{\varphi(\lambda)} + \lambda^{-1}\sum_{i=1}^n \mathbb{E}|X''_i| + \lambda^{-1}\sum_{i=1}^n \|\mathbb{E}_{i-q}(X'_i)\|_1\,. \tag{5.42}$$

Choose $u = R^{-1}(\lambda)$, $q = \alpha^{-1}(u)\wedge n$ and $M = Q(u)$. Since R is right continuous, we have $R(u)\le\lambda$, hence $qM \le R(u)\le \lambda$. Note also that

$$\sum_{k=1}^n \mathbb{E}(|X''_k|) \le 2n\int_0^u Q(x)dx \le 2n\int_0^1 Q(x)\mathbb{1}_{R(x)>\lambda}dx\,. \tag{5.43}$$

In addition using Proposition 5.3, we get that

$$\|\mathbb{E}_{i-q}(X'_i)\|_1 \le 8\int_0^{\alpha(q)/2} Q(x)dx\,. \tag{5.44}$$

Since $\alpha(q)/2 \le u$,

$$\sum_{i=1}^n \|\mathbb{E}_{i-q}(X'_i)\|_1 \le 8n\int_0^1 Q(x)\mathbb{1}_{R(x)>\lambda}dx\,.$$

It follows that

$$\begin{aligned}\lambda^{-1}\Big(\sum_{i=1}^n \mathbb{E}|X''_i| + \sum_{i=1}^n \|\mathbb{E}_{i-q}(X'_i)\|_1\Big) \;&\le\; 10n\lambda^{-1}\int_0^1 Q(x)\mathbb{1}_{R(x)>\lambda}dx\\ &\le\; 10n\lambda^{-2}\int_0^1 Q(x)R(x)\mathbb{1}_{R(x)>\lambda}dx\,. \end{aligned}\tag{5.45}$$

To control now the first term in the inequality (5.42), we choose the even convex function φ such that

$$\varphi(t) = \begin{cases} 0 & \text{if } 0\le t\le \lambda/2\\ \frac{1}{6}(t-\frac{\lambda}{2})^3 & \text{if } \lambda/2\le t\le\lambda\\ \frac{\lambda^3}{48} + \frac{\lambda}{4}(t-\lambda)^2 + \frac{\lambda^2}{8}(t-\lambda) & \text{if } t\ge\lambda\,. \end{cases}$$

Clearly $\|\varphi^{(2)}\|_\infty \le \lambda/2$ and $\|\varphi^{(3)}\|_\infty \le 1$. Let $(N_i)_{i\in\mathbb{Z}}$ be a sequence of independent random variables with normal distribution $\mathcal{N}(0,\sigma^2)$. Suppose furthermore that the sequence $(N_i)_{i\in\mathbb{Z}}$ is independent of $(X_i)_{i\in\mathbb{N}}$. Set $T_n = N_1 + N_2 + \cdots + N_n$ and $\varphi_k(x) = \mathbb{E}(\varphi(x + T_n - T_k))$. With this notation and setting $S_0 = 0$,

$$\mathbb{E}(\varphi(S_n) - \varphi(T_n)) = \sum_{k=1}^{n} \mathbb{E}(\varphi_k(S_{k-1} + X_k) - \varphi_k(S_{k-1} + Y_k))\,.$$

To bound up $\mathbb{E}(\varphi_k(S_{k-1} + X_k) - \varphi_k(S_{k-1} + Y_k))$, we proceed as in the proof of Proposition 5.1 with the following modifications: the φ_k's are deterministic, $b_2 = \|\varphi_k^{(2)}\|_\infty \le \lambda/2$ and $b_3 = \|\varphi_k^{(3)}\|_\infty \le 1$. Here $\mathbb{E}(\varphi_k'(0)X_k) = 0$ and $\varphi_k''(S_\ell)$ is $\mathcal{F}_\ell$-measurable for any $\ell \in \mathbb{Z}$. We then infer that the following bound is valid: for any $k = 1, \dots, n$,

$$\mathbb{E}(\varphi_k(S_{k-1} + X_k) - \varphi_k(S_{k-1} + Y_k)) \le \sigma^3 + C\lambda \int_0^u Q(x)R(x)dx + C\int_0^1 Q(x)R(x)R(x \vee u)dx\,,$$

where C is a positive constant not depending on σ^2. Choosing $u = R^{-1}(\lambda)$, we get that

$$\int_0^u Q(x)R(x)dx = \int_0^1 Q(x)R(x)\mathbb{1}_{R(x)>\lambda}dx\,,$$

and

$$\int_0^1 Q(x)R(x)R(x \vee u)dx \le \int_0^1 Q(x)R(x)(R(x) \wedge \lambda)dx\,.$$

It follows that

$$\mathbb{E}(\varphi(S_n) - \varphi(T_n)) \le n\sigma^3 + 2CnM_{3,\alpha}(Q,\lambda)\,. \tag{5.46}$$

It remains to compute $\mathbb{E}(\varphi(T_n))$. We have that $6\mathbb{E}(\varphi(T_n)) \le \mathbb{E}\big(T_n - \lambda/2\big)_+^3$. Hence, using the fact that $t^2 = \lambda^2/4 + (t - \lambda/2)^2 + \lambda(t - \lambda/2)$, we obtain:

$$\mathbb{E}(\varphi(T_n)) \le \frac{e^{-\lambda^2/(8n\sigma^2)}}{6}\int_0^\infty e^{-\lambda x/(2n\sigma^2)}\frac{x^3}{\sigma\sqrt{2n\pi}}dx\,.$$

Using the change of variables $y = \lambda x/(2n\sigma^2)$, we derive that

$$\mathbb{E}(\varphi(T_n)) \le \frac{\lambda^3}{\sqrt{2\pi}}\Big(\frac{(2n\sigma^2)}{\lambda^2}\Big)^{7/2} e^{-\lambda^2/(8n\sigma^2)}\,. \tag{5.47}$$

Starting from (5.42) and collecting the bounds (5.45), (5.46) and (5.47), the proposition is proved for any variable $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$ with $f = \sum_{\ell=1}^{L} a_\ell f_\ell$ and $f_\ell \in \widetilde{\mathrm{Mon}}(Q, P_{Y_0})$, $\sum |a_\ell| \le 1$. Since these functions are dense in $\widetilde{\mathcal{F}}(Q, P_{Y_0})$ by definition, the result follows by applying Fatou's lemma. $\diamond$

Next proposition deals with general covariance inequalities for α-dependent random variables.

Proposition 5.3. *Let m and q be two nonnegative integers. For any $(m+q)$-tuple of integers $(j_\ell)_{1\le\ell\le m+q}$, let $X_i^{(j_\ell)} = f_{j_\ell}(Y_i) - \mathbb{E}(f_{j_\ell}(Y_i))$, where f_{j_ℓ} belongs to $\widetilde{\mathcal{F}}(Q_{j_\ell}, P_{Y_0})$ for $1 \le \ell \le m+q$. Suppose that $Q_{j_\ell}^q$ is integrable for $\ell \ge m+1$. Define the coefficients $\alpha_{k,\mathbf{Y}}(n)$ as in (2.1). Then for any integers $(j_\ell)_{1\le\ell\le m+q}$ and any integers $(k_\ell)_{0\le\ell\le m+q}$ such that $k_0 \le k_1 \le \cdots \le k_{m+q}$ and $k_{m+1} - k_m = \ell$,*

$$\Big\| \prod_{i=1}^{m} X_{k_i}^{(j_i)} \Big(\mathbb{E}_{k_m}\big(\prod_{i=m+1}^{m+q} X_{k_i}^{(j_i)} \big) - \mathbb{E}\big(\prod_{i=m+1}^{m+q} X_{k_i}^{(j_i)} \big) \Big) \Big\|_1 \le 2^{m+q+2} \int_0^{2^{q-2}\alpha_{q,\mathbf{Y}}(\ell)} \prod_{i=1}^{m+q} Q_{j_i}(x)dx \,,$$

and

$$\Big\| \prod_{i=1}^{m} f_{j_i}(Y_{k_i}) \Big(\mathbb{E}_{k_m}\big(\prod_{i=m+1}^{m+q} X_{k_i}^{(j_i)} \big) - \mathbb{E}\big(\prod_{i=m+1}^{m+q} X_{k_i}^{(j_i)} \big) \Big) \Big\|_1 \le 2^{q+2} \int_0^{2^{q-2}\alpha_{q,\mathbf{Y}}(\ell)} \prod_{i=1}^{m+q} Q_{j_i}(x)dx \,,$$

with the convention that $\prod_{i=1}^{0} = \prod_{i=m+1}^{m} = 1$.

Proof of proposition 5.3. Assume first that $f_{j_\ell} = \sum_{r=1}^{N} a_r g_{j_\ell,r}$ where $\sum_{r=1}^{N} |a_r| \le 1$ and $g_{j_\ell,r}$ belongs to $\widetilde{\mathrm{Mon}}(Q_{j_\ell}, P_{Y_0})$ for $1 \le \ell \le m+q$. To soothe the notation, let also

$$X_{i,r}^{(j_\ell)} = g_{j_\ell,r}(Y_i) - \mathbb{E}(g_{j_\ell,r}(Y_i)) \,. \tag{5.48}$$

We then have that

$$\begin{aligned} &\Big\| \prod_{i=1}^{m} X_{k_i}^{(j_i)} \Big(\mathbb{E}_{k_m}\big(\prod_{i=m+1}^{m+q} X_{k_i}^{(j_i)} \big) - \mathbb{E}\big(\prod_{i=m+1}^{m+q} X_{k_i}^{(j_i)} \big) \Big) \Big\|_1 \\ &\le \prod_{p=1}^{m+q} \big(\sum_{r_p=1}^{N} |a_{r_p}| \big) \Big\| \prod_{i=1}^{m} X_{k_i,r_i}^{(j_i)} \Big(\mathbb{E}_{k_m}\big(\prod_{i=m+1}^{m+q} X_{k_i,r_i}^{(j_i)} \big) - \mathbb{E}\big(\prod_{i=m+1}^{m+q} X_{k_i,r_i}^{(j_i)} \big) \Big) \Big\|_1 \,. \end{aligned}$$

Now setting

$$A := \Big| \prod_{i=1}^{m} X_{k_i,r_i}^{(j_i)} \Big| \operatorname{sign} \Big\{ \mathbb{E}_{k_m}\big(\prod_{i=m+1}^{m+q} X_{k_i,r_i}^{(j_i)} \big) - \mathbb{E}\big(\prod_{i=m+1}^{m+q} X_{k_i,r_i}^{(j_i)} \big) \Big\} \,,$$

we get that

$$\begin{aligned} &\Big\| \prod_{i=1}^{m} X_{k_i,r_i}^{(j_i)} \Big(\mathbb{E}_{k_m}\big(\prod_{i=m+1}^{m+q} X_{k_i,r_i}^{(j_i)} \big) - \mathbb{E}\big(\prod_{i=m+1}^{m+q} X_{k_i,r_i}^{(j_i)} \big) \Big) \Big\|_1 \\ &= \mathbb{E} \left(A \Big(\mathbb{E}_{k_m}\big(\prod_{i=m+1}^{m+q} X_{k_i,r_i}^{(j_i)} \big) - \mathbb{E}\big(\prod_{i=m+1}^{m+q} X_{k_i,r_i}^{(j_i)} \big) \Big) \right) = \mathbb{E} \left((A - \mathbb{E}(A)) \prod_{i=m+1}^{m+q} X_{k_i,r_i}^{(j_i)} \right) . \end{aligned}$$

From Proposition A.1 and Lemma A.1 in Dedecker and Rio (2008), we have that

$$\mathbb{E} \left((A - \mathbb{E}(A)) \prod_{i=m+1}^{m+q} X_{k_i,r_i}^{(j_i)} \right) \le 2^{q+2} \int_0^{\bar{\alpha}/2} Q_{|A|}(x) \prod_{i=m+1}^{m+q} Q_{j_i}(x)dx \,,$$

where

$$\bar\alpha = \sup_{(t_1,\dots,t_{q+1})\in\mathbf{R}^{q+1}} \Big|\mathbb{E}\big((\mathbb{1}_{A\le t_1} - \mathbb{P}(A\le t_1))\prod_{i=m+1}^{m+q}(\mathbb{1}_{g_{j_i,r_i}(Y_{k_i})\le t_{i-m+1}} - \mathbb{P}(g_{j_i,r_i}(Y_{k_i})\le t_{i-m+1}))\big)\Big|\,.$$

By monotonocity of the functions g_{j_i,r_i}, we then get that

$$\begin{aligned}\bar\alpha &\le 2^q \sup_{(t_1,\dots,t_{q+1})\in\mathbf{R}^{q+1}} \Big|\mathbb{E}\big((\mathbb{1}_{A\le t_1} - \mathbb{P}(A\le t_1))\prod_{i=m+1}^{m+q}(\mathbb{1}_{Y_{k_i}\le t_{i-m+1}} - \mathbb{P}(Y_{k_i}\le t_{i-m+1}))\big)\Big| \\ &\le 2^{q-1}\alpha_{q,\mathbf{Y}}(\ell)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}\left((A-\mathbb{E}(A))\prod_{i=m+1}^{m+q}X_{k_i,r_i}^{(j_i)}\right) \le 2^{q+2}\int_0^{2^{q-2}\alpha_{q,\mathbf{Y}}(\ell)} Q_{|A|}(x)\prod_{i=m+1}^{m+q}Q_{j_i}(x)dx \\ &\le 2^{q+2}\int_0^{2^{q-2}\alpha_{q,\mathbf{Y}}(\ell)}\prod_{i=1}^{m}\Big(Q_{j_i}(x)+\int_0^1 Q_{j_i}(x)dx\Big)\prod_{i=m+1}^{m+q}Q_{j_i}(x)dx\,.\end{aligned}$$

Hence taking into account that $\prod_{i=1}^{m+q}\big(\sum_{r_i=1}^N|a_{r_i}|\big)\le 1$ and using Lemma 2.1 in Rio (2000), the inequality is proved for functions $f_{j_\ell}=\sum_{r=1}^N a_r g_{j_\ell,r}$ where $\sum_{r=1}^N|a_r|\le 1$ and $g_{j_\ell,r}$ belongs to $\widetilde{\mathrm{Mon}}(Q_{j_\ell},P_{Y_0})$ for $1\le\ell\le m+q$.

It remains to prove that the inequality remains valid for f_{j_ℓ} belonging to $\widetilde{\mathcal{F}}(Q_{j_\ell},P_{Y_0})$ for $1\le\ell\le m+q$. By definition,

$$X_i^{(j_\ell)} = \lim_{N\to\infty}\mathbb{L}^1\sum_{r=1}^N a_{r,N}X_{i,r,N}^{(j_\ell)}\,,$$

where $\sum_{r=1}^N|a_{r,N}|\le 1$ and $X_{i,r,N}^{(j_\ell)} = g_{j_\ell,r,N}(Y_i)-\mathbb{E}(g_{j_\ell,r,N}(Y_i))$ with the $g_{j_\ell,r,N}$ belonging to $\widetilde{\mathrm{Mon}}(Q_{j_\ell},P_{Y_0})$ for $1\le\ell\le m+q$. Hence, by Fatou lemma the proposition will hold if we can prove that the following inequality holds almost surely

$$\begin{aligned}&\mathbb{E}_{k_m}\Big(\prod_{i=m+1}^{m+q}X_{k_i}^{(j_i)}\Big)-\mathbb{E}\Big(\prod_{i=m+1}^{m+q}X_{k_i}^{(j_i)}\Big) \\ &\quad= \lim_{N\to\infty}\prod_{i=m+1}^{m+q}\Big(\sum_{r_i=1}^N a_{r_i,N}\Big)\Big(\mathbb{E}_{k_m}\Big(\prod_{i=m+1}^{m+q}X_{k_i,r_i,N}^{(j_i)}\Big)-\mathbb{E}\Big(\prod_{i=m+1}^{m+q}X_{k_i,r_i,N}^{(j_i)}\Big)\Big)\,. \qquad (5.49)\end{aligned}$$

With this aim, notice that for any $m+1\le\ell\le m+q$,

$$X_i^{(j_\ell)} = \sum_{r=1}^N a_{r,N}X_{i,r,N}^{(j_\ell)}+\epsilon_{i,N}^{(j_\ell)}\,,$$

with $\lim_{N\to\infty}\|\epsilon^{(j_\ell)}_{i,N}\|_1=0$. In addition, since for $m+1\le \ell\le m+q$, $Q^q_{j_\ell}$ is integrable and $g_{j_\ell,r,N}$ belongs to $\widetilde{\text{Mon}}(Q_{j_\ell},P_{Y_0})$, it follows that $\|X^{(j_\ell)}_{i,r,N}\|_q\le 2\|Q_{j_\ell}\|_q$ and next $\|X^{(j_\ell)}_i\|_q\le 2\|Q_{j_\ell}\|_q$ by an application of Fatou lemma. Consequently the $\epsilon^{(j_\ell)}_{i,N}$'s are in $\mathbb{L}^q$ and satisfy $\|\epsilon^{(j_\ell)}_{i,N}\|_q\le 4\|Q_{j_\ell}\|_q$. Now

$$\begin{aligned}\|\epsilon^{(j_{m+1})}_{k_{m+1},N}\prod_{i=m+2}^{m+q}X^{(j_i)}_{k_i}\|_1 &\le 2^{q-1}\int_0^1 Q_{|\epsilon^{(j_{m+1})}_{k_{m+1},N}|}(x)\prod_{i=m+2}^{m+q}Q_{j_i}(x)dx\\ &\le 2^{q-1}\int_0^1 Q_{|\epsilon^{(j_{m+1})}_{k_{m+1},N}|}(x)Q_*^{q-1}(x)dx\,,\end{aligned}$$

where $Q_*=\max_{m+2\le i\le m+q}Q_{j_i}$. Now for any positive M, $Q_*^{q-1}\le M^{q-1}+Q_*^{q-1}\mathbb{1}_{Q_*>M}$. Hence,

$$\begin{aligned}2^{1-q}\|\epsilon^{(j_{m+1})}_{k_{m+1},N}\prod_{i=m+2}^{m+q}X^{(j_i)}_{k_i}\|_1 &\le M^{q-1}\|\epsilon^{(j_{m+1})}_{k_{m+1},N}\|_1+\|\epsilon^{(j_{m+1})}_{k_{m+1},N}\|_q\|Q_*\mathbb{1}_{Q_*>M}\|_q^{q-1}\\ &\le M^{q-1}\|\epsilon^{(j_{m+1})}_{k_{m+1},N}\|_1+4\|Q_{j_{m+1}}\|_q\|Q_*\mathbb{1}_{Q_*>M}\|_q^{q-1}\,,\end{aligned}$$

which tends to zero by letting first N tends to infinity and after M. Similarly, we can show that for any $\ell\in\{1,\dots,q-1\}$,

$$\lim_{N\to\infty}\Big\|\Big(\prod_{i=1}^{\ell}X^{(j_{m+i})}_{k_{m+i},r,N}\Big)\epsilon^{(j_{m+\ell+1})}_{k_{m+\ell+1},N}\prod_{i=m+\ell+2}^{m+q}X^{(j_i)}_{k_i}\Big\|_1=0\,.$$

This ends the proof of (5.49) and then of the proposition. ⋄

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