A Bernstein type inequality and moderate deviations for weakly dependent sequences

Florence Merlevède [a], Magda Peligrad [b 1] *and* Emmanuel Rio [c 2]

[a] Université Paris Est, Laboratoire de mathématiques, UMR 8050 CNRS, Bâtiment Copernic, 5 Boulevard Descartes, 77435 Champs-Sur-Marne, FRANCE. E-mail: florence.merlevede@univ-mlv.fr

[b] Department of Mathematical Sciences, University of Cincinnati, PO Box 210025, Cincinnati, Oh 45221-0025, USA. Email: magda.peligrad@uc.edu

[c] Université de Versailles, Laboratoire de mathématiques, UMR 8100 CNRS, Bâtiment Fermat, 45 Avenue des Etats-Unis, 78035 Versailles, FRANCE. E-mail: rio@math.uvsq.fr

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Abstract

In this paper we present a Bernstein-type tail inequality for the maximum of partial sums of a weakly dependent sequence of random variables that is not necessarily bounded. The class considered includes geometrically and subgeometrically strongly mixing sequences. The result is then used to derive asymptotic moderate deviation results. Applications are given for classes of Markov chains, iterated Lipschitz models and functions of linear processes with absolutely regular innovations.

1 Introduction and background results

In recent years there has been a great effort towards a better understanding of the structure and asymptotic behavior of stochastic processes. For processes with short memory one basic technique is approximation with independent random variables. In this approach, after a suitable

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blocking argument, the blocks are approximated by independent random vectors. For many examples including functionals of Gaussian processes, Harris recurrent Markov chains, time series, and for results including moment inequalities or central limit-type theorems with rate estimates, this method is very fruitful. Moreover, recently, the traditional measures of dependence that quantify the departure from independence have been fine tuned to include more examples than those covered by traditional mixing classes (see Dedecker and Prieur (2004)).

However, at this point, this method is not developed enough to handle Bernstein-type exponential inequalities. For instance, by the traditional blocking methods most of the exponential inequalities for tails of sums of weakly dependent random variables are known to hold only in an interval close to the central limit theorem range. Moreover the results are often restricted to bounded random variables, as for example in Adamczak (2008) (see the inequality (1.10)).

In this paper we will develop new methods to enlarge the interval on which the Bernstein inequality holds for some classes of weakly dependent random variables. The dependence coefficients used are weak enough to include sequences that are not necessarily strongly mixing in the traditional sense, so the results have a large applicability. Examples that can be treated this way include classes of Markov chains, iterated Lipschitz models and functions of linear processes with absolutely regular innovations.

Concerning the traditional large deviations principle, it is known from the paper by Bryc and Dembo (1996) that it is not satisfied by many classes of weakly dependent random variables. This is the reason why it is convenient to look at moderate deviations principles, which are intermediate results between central limit theorem and large deviations. We shall use the new developed Bernstein-type inequalities to obtain sharp moderate deviation asymptotic results for some classes of dependent random variables. The study will be made in the nonstationary setting.

We recall know some known results concerning the Bernstein-type inequalities. Let us consider a sequence $X_1, X_2, \ldots$ of centered real valued random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and set $S_n = X_1 + X_2 + \cdots + X_n$. We first recall the Bernstein inequality for random variables satisfying Condition (1.1) below. Suppose that the random variables $X_1, X_2, \ldots$ are independent and satisfy

$$\log \mathbb{E} \exp(tX_i) \leq \frac{\sigma_i^2 t^2}{2(1 - tB)} \quad \text{for positive constants} \quad \sigma_i \quad \text{and } B, \tag{1.1}$$

for any t in $[0, 1/B[$. Set $V_n = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$. Then

$$\mathbb{P}(S_n \geq \sqrt{2V_n x} + Bx) \leq \exp(-x).$$

When the random variables $X_1, X_2, \ldots$ are centered and uniformly bounded by M then (1.1) holds with $\sigma_i^2 = \mathrm{Var} X_i$, $B = M$ and the above inequality implies the usual Bernstein inequality

$$\mathbb{P}(S_n \geq y) \leq \exp\Big(-y^2(2V_n + 2yM)^{-1}\Big). \tag{1.2}$$

It is well known that (1.1) also holds true when the variables are centered and satisfy: there exist positive constants σ_i and B such that $\mathbb{E}|X_i|^k \leq k!\sigma_i^2 B^{k-2}/2$ for all $k \geq 2$, and that this last condition is satisfied by variables having exponential moments.
Assume now that the random variables $X_1, X_2, \ldots$ are independent, centered and satisfy the following weaker tail condition: for some $\delta > 0$, $\gamma \in (0,1)$ and $K > 0$,

$$\sup_i \mathbb{E}(\exp(\delta |X_i|^\gamma)) \leq K\,. \tag{1.3}$$

By the proof of Corollary 5.1 in Borovkov (2000-a) we infer that there exist two positive constants c_1 and c_2 depending only δ, γ and K such that

$$\mathbb{P}(S_n \geq y) \leq \exp\Big(-c_1 y^2/n\Big) + n\exp\Big(-c_2 y^\gamma\Big)\,. \tag{1.4}$$

More precise results for large deviations of sums of independent random variables with *semiexponential* tails (i.e. (1.3) is satisfied for $\gamma \in (0,1)$) may be found in Borovkov (2000-b).

Our interest is to extend the above inequalities to sequences of dependent random variables. Let us first assume that $X_1, X_2, \ldots$ is a strongly mixing sequence of real-valued and centered random variables (see (2.4) for the definition of the strong mixing coefficients $\alpha(n)$). Assume in addition that there exist two positive constants γ_1 and c such that the strong mixing coefficients of the sequence satisfy

$$\alpha(n) \leq \exp(-cn^{\gamma_1}) \quad \text{for any positive integer } n\,, \tag{1.5}$$

and there are constants $b \in]0, \infty[$ and γ_2 in $]0, +\infty]$ such that

$$\sup_{i>0} \mathbb{P}(|X_i| > t) \leq \exp(1 - (t/b)^{\gamma_2}) \quad \text{for any positive } t\,, \tag{1.6}$$

(when $\gamma_2 = +\infty$ (1.6) means that $\|X_i\|_\infty \leq b$ for any positive i).

Obtaining exponential bounds for this case is a challenging problem. To understand the difficulty of the problem we shall mention a possible approach. One of the available tools in the literature is Theorem 6.2 in Rio (2000) which is a Fuk-Nagaev type inequality (for a similar inequality, using the coupling coefficients τ recalled in Section 2 instead of the strong mixing ones, we refer to Theorem 2 in Dedecker and Prieur, 2004). This tail inequality is sharp for sequences of random variables with polynomial strongly mixing rates and finite moments up to a certain order, and one could be tempted to apply it to sequences with exponential or subexponential mixing rates. We shall argue that for these cases the inequality does not provide optimal results. To explain the situation, let γ be defined by $1/\gamma = (1/\gamma_1) + (1/\gamma_2)$. For any positive λ and any $r \geq 1$, Theorem 6.2 in Rio (2000) yields that there exists a positive constant C such that

$$\mathbb{P}(\sup_{1\leq k\leq n} |S_k| \geq 4\lambda) \leq 4\Big(1 + \frac{\lambda^2}{rns^2}\Big)^{-r/2} + 4Cn\lambda^{-1}\exp\Big(-c\frac{\lambda^\gamma}{b^\gamma r^\gamma}\Big), \tag{1.7}$$

where

$$s^2 = \sup_{i>0}\Big(\mathbb{E}(X_i^2) + 2\sum_{j>i}|\mathbb{E}(X_iX_j)|\Big).$$

Selecting in (1.7) $r = \lambda^{\gamma/(\gamma+1)}$ leads to

$$\mathbb{P}(\sup_{k\in[1,n]} |S_k| \geq 4\lambda) \leq 4\exp\Big(-\frac{\lambda^{\gamma/(\gamma+1)}\log 2}{2}\Big) + 4Cn\lambda^{-1}\exp\Big(-c\frac{\lambda^{\gamma/(\gamma+1)}}{b^\gamma}\Big) \tag{1.8}$$

for any $\lambda \geq 1 \vee (ns^2)^{(\gamma+1)/(\gamma+2)}$.
For stationary subgeometrically (absolutely regular) Markov chains $(Y_i)_{i\in\mathbb{Z}}$ having a petite set, and for bounded functions f (here $\gamma = \gamma_1$), this gives the following exponential deviation inequality for $S_n(f) = f(Y_1) + f(Y_2) + \cdots + f(Y_n)$. Under the centering condition $\mathbb{E}(f(Y_1)) = 0$, there exist positive constants K and L such that for any $\varepsilon > 0$ and $n > n_0(\varepsilon)$,

$$\mathbb{P}(|S_n(f)| \geq n\varepsilon) \leq K\exp\Big(-L(n\varepsilon)^{\gamma/(1+\gamma)}\Big), \tag{1.9}$$

which is also the inequality given in Theorem 10 of Douc, Guillin and Moulines (2008) for $\gamma \in (0,1]$ provided that their drift condition implies subexponential ergodicity (and then (1.5)) (see Section 2.1.1 for details on this drift condition). When $\gamma = 1$, this gives a power $\sqrt{n}$ in the exponential whereas in this case, Theorem 6 in Adamczak (2008) provides the following inequality: for any positive λ,

$$\mathbb{P}(|S_n(f)| \geq \lambda) \leq C\exp\Big(-\frac{1}{C}\min\Big(\frac{\lambda^2}{n\sigma^2}, \frac{\lambda}{\log n}\Big)\Big), \tag{1.10}$$

where $\sigma^2 = \lim_n n^{-1}\mathrm{Var} S_n(f)$ and C is a positive constant (here we take $m = 1$ in his condition (14) on the petite set). In addition, when $\gamma \in (0,1)$, the hope is to achieve the power n^γ in the exponential term of (1.9) instead of $n^{\gamma/(1+\gamma)}$, since the gap is filled asymptotically via the moderate deviations results (see Theorem 1 in Djellout and Guillin (2001) or Theorem 7 in Douc, Guillin and Fort (2008)).

In this paper, we extend the inequality (1.4) to dependent sequences allowing us to fill the gap above mentioned. To be more precise, we shall prove that, for α-mixing sequences satisfying (1.5) and (1.6) for $\gamma < 1$, there exists a positive η such that, for $n \geq 4$ and $\lambda \geq C(\log n)^\eta$

$$\mathbb{P}(\sup_{j\leq n} |S_j| \geq \lambda) \leq (n+1)\exp(-\lambda^\gamma/C_1) + \exp(-\lambda^2/(C_2 + C_2 nV)), \tag{1.11}$$

where C, C_1 and C_2 are positive constants depending on b, c, γ_1 and γ_2 and V is some constant, depending on the covariance properties of truncated random variables built from the initial sequence. In order to define precisely V we need to introduce truncation functions φ_M.

Notation 1. For any positive M let the function φ_M be defined by $\varphi_M(x) = (x \wedge M) \vee (-M)$.

With this notation, (1.11) holds with

$$V = \sup_{M>0} \sup_{i>0} \Big(\mathrm{Var}(\varphi_M(X_i)) + 2 \sum_{j>i} |\mathrm{Cov}(\varphi_M(X_i), \varphi_M(X_j))| \Big). \tag{1.12}$$

Let us mention that in the case $\gamma_1 = 1$ and $\gamma_2 = \infty$, a Bernstein type inequality very close to (1.10) (up to a logarithmic term) has been obtained by Merlevède, Peligrad and Rio (2009). The case $\gamma > 1$ will certainly give a different bound than (1.11) for the deviation inequality since it is the case for independent random variables (see for instance Theorems 3.1 and 3.2 in Liu and Watbled (2009)). This is outside the scope of the present paper to consider this case.

The main tool to prove (1.11) is our Proposition 2 allowing to derive a sharp upper bound for the Laplace transform of the partial sums. The proof of this proposition can be described as follows: the variables (suitably truncated) are partioned in blocks indexed by Cantor-type sets plus a remainder. The log-Laplace transform of each partial sum on the Cantor-type sets is then controlled with the help of our Proposition 1, and Lemma 3 of Appendix provides bounds for the log-Laplace transform of any sum of real-valued random variables. The proof of Proposition 1 is based on decorrelation arguments on the Cantor-type sets we consider together with adaptive truncations (at each decorrelation step, the variables get truncated at different levels, as in Bass (1985), to compensate for the diminishing block size). The assumption $\gamma < 1$ is crucial for the decorrelation steps. Proposition 2 alone does not lead directly to the inequality (1.11) and it has to be combined with coupling to reduce the number of variables considered in the partial sums. The strong mixing coefficients are thus not needed in their full generality: Theorem 1 gives (1.11) when (1.5) (up to a constant) is fulfilled by the dependence coefficients τ, as introduced by Dedecker and Prieur (2004), having exactly the *coupling property* in $\mathbb{L}^1$. In Section 2, we also prove the moderate deviations principle for the class of dependent sequences that we consider (see our Theorem 2). Section 2.1 is devoted to some applications and further comparisons with previous results are given (see Section 2.1.1). The proofs of Theorems 1 and 2 are postponed in Section 3. Technical results are given in Appendix.

2 Main results

We first define the dependence coefficients that we consider in this paper.

For any real random variable X in $\mathbb{L}^1$ and any σ-algebra $\mathcal{M}$ of $\mathcal{A}$, let $\mathbb{P}_{X|\mathcal{M}}$ be a conditional distribution of X given $\mathcal{M}$ and let $\mathbb{P}_X$ be the distribution of X. We consider the coefficient $\tau(\mathcal{M}, X)$ of weak dependence (Dedecker and Prieur, 2004) which is defined by

$$\tau(\mathcal{M}, X) = \Big\| \sup_{f \in \Lambda_1(\mathbb{R})} \Big| \int f(x) \mathbb{P}_{X|\mathcal{M}}(dx) - \int f(x) \mathbb{P}_X(dx) \Big| \Big\|_1 , \tag{2.1}$$

where $\Lambda_1(\mathbb{R})$ is the set of 1-Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$.

The coefficient τ has the following coupling property: If Ω is rich enough then the coefficient $\tau(\mathcal{M}, X)$ is the infimum of $\|X - X^*\|_1$ where X^* is independent of $\mathcal{M}$ and distributed as X (see Lemma 5 in Dedecker and Prieur (2004)). This coupling property allows to relate the coefficient τ to the strong mixing coefficient Rosenblatt (1956) defined by

$$\alpha(\mathcal{M}, \sigma(X)) = \sup_{A \in \mathcal{M}, B \in \sigma(X)} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| ,$$

(see Lemma 6 in Dedecker and Prieur (2004)).

If Y is a random variable with values in $\mathbb{R}^k$ equipped with the norm $|\cdot|_k$ defined by $|x - y|_k = \sum_{i=1}^k |x_i - y_i|$, the coupling coefficient τ is defined as follows: If $Y \in \mathbb{L}^1(\mathbb{R}^k)$,

$$\tau(\mathcal{M}, Y) = \sup\{\tau(\mathcal{M}, f(Y)), f \in \Lambda_1(\mathbb{R}^k)\} , \tag{2.2}$$

where $\Lambda_1(\mathbb{R}^k)$ is the set of 1-Lipschitz functions from $\mathbb{R}^k$ to $\mathbb{R}$.

The τ-mixing coefficients $\tau_X(i) = \tau(i)$ of a sequence $(X_i)_{i \in \mathbb{Z}}$ of real-valued random variables are then defined by

$$\tau_k(i) = \max_{1 \le \ell \le k} \frac{1}{\ell} \sup \Big\{ \tau(\mathcal{M}_p, (X_{j_1}, \cdots, X_{j_\ell})),\ p + i \le j_1 < \cdots < j_\ell \Big\} \text{ and } \tau(i) = \sup_{k \ge 0} \tau_k(i) , \tag{2.3}$$

where $\mathcal{M}_p = \sigma(X_j, j \le p)$ and the above supremum is taken over p and $(j_1, \dots j_\ell)$. Recall that the strong mixing coefficients $\alpha(i)$ are defined by:

$$\alpha(i) = \sup_{p \in \mathbb{Z}} \alpha(\mathcal{M}_p, \sigma(X_j, j \ge i + p)) . \tag{2.4}$$

Define now the function $Q_{|Y|}$ by $Q_{|Y|}(u) = \inf\{t > 0, \mathbb{P}(|Y| > t) \le u\}$ for u in $]0, 1]$. To compare the τ-mixing coefficients with the strong mixing ones, let us mention that, by Lemma 7 in Dedecker and Prieur (2004),

$$\tau(i) \le 2 \int_0^{2\alpha(i)} Q(u) du , \text{ where } Q = \sup_{k \in \mathbb{Z}} Q_{|X_k|}. \tag{2.5}$$

Let $(X_j)_{j\in\mathbb{Z}}$ be a sequence of centered real valued random variables and let $\tau(i)$ be defined by (2.3). Let $\tau(x) = \tau([x])$ (square brackets denoting the integer part). Throughout, we assume that there exist positive constants γ_1, a and c such that

$$\tau(x) \leq a\exp(-cx^{\gamma_1}) := \tau^*(x) \quad \text{for any } x \geq 1\,, \tag{2.6}$$

and that, for some constants γ_2 in $]0,+\infty]$ and b in $]0,+\infty[$, the following tail condition is satisfied: for any positive t,

$$\sup_{k>0} \mathbb{P}(|X_k| > t) \leq \exp(1-(t/b)^{\gamma_2}) := H(t)\,. \tag{2.7}$$

Suppose furthermore that

$$\gamma < 1 \text{ where } \gamma \text{ is defined by } 1/\gamma = 1/\gamma_1 + 1/\gamma_2\,. \tag{2.8}$$

Theorem 1. *Let $(X_j)_{j\in\mathbb{Z}}$ be a sequence of centered real valued random variables and let V be defined by (1.12). Assume that (2.6), (2.7) and (2.8) are satisfied. Then V is finite and, for any $n \geq 4$, there exist positive constants C_1, C_2, C_3 and C_4 depending only on a, b, c, γ and γ_1 such that, for any positive x,*

$$\mathbb{P}\Big(\sup_{j\leq n}|S_j| \geq x\Big) \leq n\exp\Big(-\frac{x^\gamma}{C_1}\Big) + \exp\Big(-\frac{x^2}{C_2(1+nV)}\Big) + \exp\Big(-\frac{x^2}{C_3 n}\exp\Big(\frac{x^{\gamma(1-\gamma)}}{C_4(\log x)^\gamma}\Big)\Big)\,.$$

Remark 1. *Let us mention that if the sequence $(X_j)_{j\in\mathbb{Z}}$ satisfies (2.7) and is strongly mixing with strong mixing coefficients satisfying (1.5), then, from (2.5), (2.6) is satisfied (with an other constant), and Theorem 1 applies.*

Remark 2. *If there exists $\delta > 0$ such that $\mathbb{E}\exp(\delta|X_i|^{\gamma_2})) \leq K$ for any positive i, then setting $C = 1\vee\log K$, Markov inequality yields that the process $(X_i)_{i\in\mathbb{Z}}$ satisfies (2.7) with $b = (C/\delta)^{1/\gamma_2}$.*

Remark 3. *If $(X_i)_{i\in\mathbb{Z}}$ satisfies (2.6) and (2.7), then*

$$\begin{aligned} V &\leq \sup_{i>0}\Big(\mathbb{E}(X_i^2) + 4\sum_{k>0}\int_0^{\tau(k)/2} Q_{|X_i|}(G(v))dv\Big) \\ &= \sup_{i>0}\Big(\mathbb{E}(X_i^2) + 4\sum_{k>0}\int_0^{G(\tau(k)/2)} Q_{|X_i|}(u)Q(u)du\Big), \end{aligned}$$

where G is the inverse function of $x \mapsto \int_0^x Q(u)du$ (see Section 3.3 for a proof). Here the random variables do not need to be centered. Note also that, in the strong mixing case, using (2.5), we have $G(\tau(k)/2) \leq 2\alpha(k)$.

This result is one of the main tools to derive the moderate deviations principle (MDP) given in Theorem 2 below. In our terminology the MDP stays for the following type of asymptotic behavior:

Definition 1. We say that the MDP holds for a sequence $(T_n)_n$ of random variables with speed $a_n \to 0$ and good rate function $I(\cdot)$, if the level sets $\{x, I(x) \le \lambda\}$ are compact for all $\lambda < \infty$, and for each Borel set A,

$$\begin{aligned} -\inf_{t \in A^o} I(t) &\le \liminf_n a_n \log \mathbb{P}(\sqrt{a_n} T_n \in A) \\ &\le \limsup_n a_n \log \mathbb{P}(\sqrt{a_n} T_n \in A) \le -\inf_{t \in \bar{A}} I(t) \,, \end{aligned} \tag{2.9}$$

where $\bar{A}$ denotes the closure of A and A^o the interior of A.

Theorem 2. *Let $(X_i)_{i \in \mathbb{Z}}$ be a sequence of random variables as in Theorem 1 and let $S_n = \sum_{i=1}^n X_i$ and $\sigma_n^2 = \mathrm{Var} S_n$. Assume in addition that $\liminf_{n\to\infty} \sigma_n^2/n > 0$. Then for all positive sequences a_n with $a_n \to 0$ and $a_n n^{\gamma/(2-\gamma)} \to \infty$, $\{\sigma_n^{-1} S_n\}$ satisfies (2.9) with rate function $I(t) = t^2/2$.*

If we impose a stronger degree of stationarity we obtain the following corollary.

Corollary 1. *Let $(X_i)_{i \in \mathbb{Z}}$ be a second order stationary sequence of centered real valued random variables. Assume that (2.6), (2.7) and (2.8) are satisfied. Let $S_n = \sum_{i=1}^n X_i$ and $\sigma_n^2 = \mathrm{Var} S_n$. Assume in addition that $\sigma_n^2 \to \infty$. Then $\lim_{n\to\infty} \sigma_n^2/n = \sigma^2 > 0$, and for all positive sequences a_n with $a_n \to 0$ and $a_n n^{\gamma/(2-\gamma)} \to \infty$, $\{n^{-1/2} S_n\}$ satisfies (2.9) with rate function $I(t) = t^2/(2\sigma^2)$.*

The proof is direct by using the fact that (2.6) and (2.7) imply that $\sum_{k>0} k |\mathrm{Cov}(X_0, X_k)| < \infty$, and then $\lim_{n\to\infty} \sigma_n^2/n = \sigma^2 > 0$ since $\sigma_n^2 \to \infty$ (see Lemma 1 in Bradley (1997)).

2.1 Applications

2.1.1 Instantaneous functions of absolutely regular processes

Let $(Y_j)_{j \in \mathbb{Z}}$ be a strictly stationary sequence of random variables with values in a Polish space E, and let f be a measurable function from E to $\mathbb{R}$. Set $X_j = f(Y_j)$. Consider now the case where the sequence $(Y_k)_{k \in \mathbb{Z}}$ is absolutely regular (or β-mixing) in the sense of Rozanov and Volkonskii (1959). Setting $\mathcal{F}_0 = \sigma(Y_i, i \le 0)$ and $\mathcal{G}_k = \sigma(Y_i, i \ge k)$, this means that

$$\beta(k) = \beta(\mathcal{F}_0, \mathcal{G}_k) \to 0 \,, \text{ as } k \to \infty \,,$$

with $\beta(\mathcal{A},\mathcal{B}) = \frac{1}{2}\sup\{\sum_{i\in I}\sum_{j\in J}|\mathbb{P}(A_i\cap B_j)-\mathbb{P}(A_i)\mathbb{P}(B_j)|\}$, the maximum being taken over all finite partitions $(A_i)_{i\in I}$ and $(B_i)_{i\in J}$ of Ω respectively with elements in $\mathcal{A}$ and $\mathcal{B}$. If we assume that

$$\beta(n) \leq 2\exp(-cn^{\gamma_1}) \text{ for any positive } n, \tag{2.10}$$

where $c > 0$ and $\gamma_1 > 0$, and that the random variables X_j are centered and satisfy (2.7) for some positive γ_2 such that $1/\gamma = 1/\gamma_1 + 1/\gamma_2 > 1$, then Theorem 1 and Corollary 1 apply to the sequence $(X_j)_{j\in\mathbb{Z}}$ (recall that $\alpha(n) \leq \beta(n)/2$). Furthermore, as shown in Viennet (1997), by Delyon's (1990) covariance inequality,

$$V \leq \mathbb{E}(f^2(Y_0)) + 4\sum_{k>0}\mathbb{E}(B_k f^2(Y_0)),$$

for some sequence $(B_k)_{k>0}$ of random variables with values in $[0,1]$ satisfying $\mathbb{E}(B_k) \leq \beta(k)$ (see Rio (2000, Section 1.6) for more details).

We now give an example where $(Y_j)_{j\in\mathbb{Z}}$ satisfies (2.10). Let $(Y_j)_{j\geq 0}$ be an E-valued irreducible ergodic and stationary Markov chain with a transition probability P having a unique invariant probability measure π (by Kolmogorov extension Theorem one can complete $(Y_j)_{j\geq 0}$ to a sequence $(Y_j)_{j\in\mathbb{Z}}$). To simplify the exposition, we assume in the rest of the section that the chain has an atom, that is there exists $A \subset E$ with $\pi(A) > 0$ and ν a probability measure such that $P(x,\cdot) = \nu(\cdot)$ for any x in A. If

$$\text{there exists } \delta > 0 \text{ and } \gamma_1 > 0 \text{ such that } \mathbb{E}_\nu(\exp(\delta\tau^{\gamma_1})) < \infty\,, \tag{2.11}$$

where $\tau = \inf\{n \geq 0;\, Y_n \in A\}$, then the β-mixing coefficients of the sequence $(Y_j)_{j\geq 0}$ satisfy (2.10) with the same γ_1 (see Proposition 9.6 and Corollary 9.1 in Rio (2000) for more details). Consequently the following corollary holds :

Corollary 2. *Suppose that $\pi(f) = 0$ and that there exist $b \in]0,\infty[$ and $\gamma_2 \in [0,\infty]$ such that*

$$\pi(|f| > t) \leq \exp(1-(t/b)^{\gamma_2}) \quad \textit{for any positive } t\,. \tag{2.12}$$

If (2.11) holds and $1/\gamma_1 + 1/\gamma_2 > 1$, then $(f(Y_i), i \in \mathbb{Z}\}$ satisfies both the conclusions of Theorem 1 and of Corollary 1 with rate function $I_f(t) = t^2/(2\sigma_f^2)$ (here $\sigma_f^2 = \lim_{n\to\infty} n^{-1}\mathbb{E}\big(\sum_{i=1}^n f(Y_i)\big)^2$).

Concerning the MDP, notice that Lemmas 5 and 7 in Djellout and Guillin (2001) imply that under (2.11) with $\gamma_1 < 1$, the MDP holds for $\{n^{-1/2}\sum_{i=1}^n f(Y_i)\}$ with rate function $I_f(t) = t^2/(2\sigma_f^2)$ and speed a_n satisfying $a_n \searrow 0$, and $a_n n^{\gamma_1/(2-\gamma_1)} \to \infty$ as soon as

$$\limsup_{n\to\infty} a_n \log\Big(n\mathbb{P}\Big(\sum_{k=0}^{\tau}|f(Y_k)| \geq \sqrt{\frac{n}{a_n}}\Big)\Big) = -\infty\,. \tag{2.13}$$

Their result extends, in the context of chains with an atom, works by de Acosta (1997) and Chen and de Acosta (1998) for bounded functionals of geometrically ergodic Markov chains.

Condition (2.13) comes from the use of the regeneration method constructed via the splitting technique on return times to the atom and links the rate of ergodicity of the chain with the growth of f. If the function f is bounded then (2.11) implies (2.13). When f is an unbounded functional, conditions ensuring that (2.13) holds are given in Douc, Guillin and Moulines (2008). To describe them, introduce the following "subgeometric drift" condition due to Douc, Fort, Moulines and Soulier (2004):

Assumption SGD. *There exist a concave, non decreasing, differentiable function* $\varphi : [1, +\infty) \to \mathbb{R}^+$*, a mesurable function* $V : E \to [1, \infty)$ *and a positive constant* r *satisfying* $\varphi(1) > 0$*,* $\lim_{x\to\infty} \varphi(x) = \infty$*,* $\lim_{x\to\infty} \varphi'(x) = \infty$*,* $\sup_{x\in A} V(x) < \infty$ *and* $PV \le V - \varphi \circ V + r\mathbb{1}_A$*.*

By Proposition 2.2 in Douc, Fort, Moulines and Soulier (2004) (and their computations on page 1365), if Assumption SGD holds with $\varphi(x) = c(x+d)\big(\log(x+d)\big)^{(1-\gamma_1)/\gamma_1}$ for $c > 0$, $\gamma_1 \in (0,1)$ and sufficiently large d, then (2.11) is satisfied. In addition if $\pi(V) < \infty$, according to Theorems 4 and 9 in Douc, Guillin and Moulines (2008), (2.13) is satisfied for all functions f such that

$$\sup_{x\in E} |f(x)|/\psi \circ V(x) < \infty, \text{ where } \psi(x) = \big(\log(1+x)\big)^{1/\gamma_2} \text{ with } \gamma_2 \text{ a positive constant}, \quad (2.14)$$

as soon as $a_n \to 0$ and $a_n n^{\gamma/(2-\gamma)} \to \infty$, where $\gamma^{-1} = \gamma_1^{-1} + \gamma_2^{-1}$. Since $\pi(V) < \infty$, notice that (2.14) implies (2.12). It follows that for the MDP, our Corollary 2 gives alternative conditions to the ones imposed in Theorem 9 of Douc, Guillin and Moulines (2008). In addition as in their paper, Corollary 2 can be extended to the one petite set case.

Let us now make some comments about the exponential deviation inequality obtained in Corollary 2. Assume that Assumption SGD holds with V such that $\pi(V) < \infty$ and $\varphi(x) = c(x+d)\big(\log(x+d)\big)^{(1-\gamma_1)/\gamma_1}$ for $c > 0$, $\gamma_1 \in (0,1)$ and sufficiently large d. If f is a bounded and centered function w.r.t. π, then Theorem 10 in Douc, Guillin and Moulines (2008) states that (1.9) holds with $\gamma = \gamma_1$, whereas Corollary 2 gives a better rate (the power is γ_1 instead of $\gamma_1/(1+\gamma_1)$). If we relax now the assumption that f is bounded by assuming that (2.14) holds then Theorem 1 in Bertail and Clémençon (2009) combined with Theorem 4 in Douc, Guillin and Moulines (2008) implies (1.9) with γ such that $\gamma^{-1} = \gamma_1^{-1} + \gamma_2^{-1}$ whereas Corollary 2 again gives a better rate.

2.1.2 Iterative Lipschitz models

In this section, we give an example of iterative Lipschitz model, which fails to be irreducible, to which our results apply. For sake of simplicity, we do not take the iterative Lipschitz models in their full generality, as defined in Diaconis and Freedman (1999) and Duflo (1996).

Example: Autoregressive Lipschitz model. For δ in $[0,1[$ and C in $]0,1]$, let $\mathcal{L}(C,\delta)$ be the class of 1-Lipschitz functions f which satisfy

$$f(0)=0 \quad \text{and} \quad |f'(t)| \le 1 - C(1+|t|)^{-\delta} \quad \text{almost everywhere.}$$

Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a sequence of i.i.d. real-valued random variables. For $\eta \in]0,1]$, let $ARL(C,\delta,\eta)$ be the class of Markov chains on $\mathbb{R}$ defined by

$$Y_n = f(Y_{n-1}) + \varepsilon_n \quad \text{with} \quad f \in \mathcal{L}(C,\delta) \quad \text{and} \quad \mathbb{E}(\exp(\lambda|\varepsilon_0|^\eta)) < \infty \text{ for some } \lambda > 0. \tag{2.15}$$

For this model, there exists an unique invariant probability measure μ (see Proposition 2 of Dedecker and Rio (2000)) and the following result holds (see Section 4.0.2 for the proof) :

Corollary 3. *Assume that $(Y_i)_{i\in\mathbb{Z}}$ belongs to $ARL(C,\delta,\eta)$. Let g be a 1-Lipschitz function. Assume furthermore that, for some ζ in $[0,1]$ and some positive constant c, $|g(x)| \le c(1+|x|^\zeta)$ for any real x. If $\delta+\zeta>0$, then $(g(Y_i)-\mathbb{E}(g(Y_i)))_{i\in\mathbb{Z}}$ satisfies both the conclusions of Theorem 1 and of Corollary 1 with $\gamma_2 = \eta(1-\delta)/\zeta$ and $\gamma_1 = \eta(1-\delta)(\eta(1-\delta)+\delta)^{-1}$.*

Note that $\gamma = \eta(1-\delta)(\eta(1-\delta)+\delta+\zeta)^{-1}$. An element of $ARL(C,\delta,\eta)$ may fail to be irreducible and then strongly mixing in the general case. However, if the common distribution of the ε_i's has an absolutely continuous component which is bounded away from 0 in a neighborhood of the origin, then the chain is irreducible and fits in the example of Tuominen and Tweedie (1994), Section 5.2. In this case, the rate of ergodicity can be derived from Theorem 2.1 in Tuominen and Tweedie (1994) (cf. Ango-Nzé (1994) for exact rates of ergodicity).

2.1.3 Functions of linear processes with absolutely regular innovations

Let f be a 1-Lipshitz function. We consider here the case where

$$X_n = f\Big(\sum_{j\ge 0} a_j Y_{n-j}\Big) - \mathbb{E} f\Big(\sum_{j\ge 0} a_j Y_{n-j}\Big),$$

where $A = \sum_{j\ge 0}|a_j| < \infty$ and $(Y_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence of real-valued random variables which is absolutely regular in the sense of Rozanov and Volkonskii; namely, $\beta(k)\to 0$, as $k\to\infty$ (see Section 2.1.1 for the definitions).

According to Section 3.1 in Dedecker and Merlevède (2006), if the innovations $(Y_i)_{i\in\mathbb{Z}}$ are in $\mathbb{L}^2$, the following bound holds for the τ-mixing coefficient associated to the sequence $(X_i)_{i\in\mathbb{Z}}$:

$$\tau(i) \le 2\|Y_0\|_1 \sum_{j\ge i}|a_j| + 4\|Y_0\|_2 \sum_{j=0}^{i-1} |a_j|\, \beta^{1/2}(i-j)\,.$$

Assume that there exists $\gamma_1 > 0$ and $c' > 0$ such that, for any positive integer k,

$$a_k \leq \exp(-c' k^{\gamma_1}) \text{ and } \beta(k) \leq \exp(-c' k^{\gamma_1}) .$$

Then the τ-mixing coefficients of $(X_j)_{j\in\mathbb{Z}}$ satisfy (2.6). Let us now focus on the tails of the random variables X_i. Assume that $(Y_i)_{i\in\mathbb{Z}}$ satisfies (2.7). Define the convex functions ψ_η for $\eta > 0$ in the following way: $\psi_\eta(-x) = \psi_\eta(x)$, and for any $x \geq 0$,

$$\psi_\eta(x) = \exp(x^\eta) - 1 \text{ for } \eta \geq 1 \quad \text{and} \quad \psi_\eta(x) = \int_0^x \exp(u^\eta) du \text{ for } \eta \in]0, 1].$$

Let $\| \,.\, \|_{\psi_\eta}$ be the usual corresponding Orlicz norm. Since the function f is 1-Lipshitz, we get that $\|X_0\|_{\psi_{\gamma_2}} \leq 2A \|Y_0\|_{\psi_{\gamma_2}}$. Next, if $(Y_i)_{i\in\mathbb{Z}}$ satisfies (2.7), then $\|Y_0\|_{\psi_{\gamma_2}} < \infty$. Furthermore, it can easily be proven that, if $\|Z\|_{\psi_\eta} \leq 1$, then $\mathbb{P}(|Z| > t) \leq \exp(1 - t^\eta)$ for any positive t. Hence, $(X_i)_{i\in\mathbb{Z}}$ satisfies (2.7) with the same parameter γ_2, and therefore the conclusions of Theorem 1 and Corollary 1 hold with γ defined by $1/\gamma = 1/\gamma_1 + 1/\gamma_2$, provided that $\gamma < 1$.

3 Proofs

3.1 Some auxiliary results

For any positive real M, let $\varphi_M(x) = (x \wedge M) \vee (-M)$. The aim of this section is essentially to give suitable bounds of the Laplace transform of the truncated sums

$$\bar{S}_M(K) = \sum_{i\in K} \overline{X}_M(i) \text{ where } \bar{X}_M(i) = \varphi_M(X_i) - \mathbb{E}(\varphi_M(X_i)) \ , \tag{3.1}$$

and K is a finite set of integers.

We first define some constants, depending only on b, γ, γ_1 that are needed in the following. Let

$$c_0 = (2(2^{1/\gamma} - 1))^{-1}(2^{(1-\gamma)/\gamma} - 1) \,, \ c_1 = \min(c^{1/\gamma_1} c_0/4, 2^{-1/\gamma}) \,, \tag{3.2}$$

$$c_2 = 2^{-(1+2\gamma_1/\gamma)} c_1^{\gamma_1} b^{-1} \ , \ c_3 = 2^{-\gamma_1/\gamma} b^{-1} \,, \text{ and } \kappa = \min\big(c_2, c_3\big) \,. \tag{3.3}$$

The following proposition, based on decorrelation arguments on Cantor-type sets, is the key tool to derive Proposition 2, that provides upper bounds for the log-Laplace transform of the partial sums of the random variables $\overline{X}_M(i)$ for a suitable M.

Proposition 1. *Let $(X_j)_{j\geq 1}$ be a sequence of centered and real valued random variables satisfying (2.6), (2.7) and (2.8). Let A and ℓ be two positive integers such that $A2^{-\ell} \geq (1 \vee 2c_0^{-1})$. Let*

$M = H^{-1}(a^{-1}\tau^*(c^{-1/\gamma_1}A))$. *Then there exists a subset* $K_A^{(\ell)}$ *of* $\{1,\ldots,A\}$ *with* $\mathrm{Card}(K_A^{(\ell)}) \geq A/2$, *such that for any positive* $t \leq \kappa\big(A^{\gamma-1} \wedge (2^\ell/A)\big)^{\gamma_1/\gamma}$, *where* κ *is defined by (3.3),*

$$\log \mathbb{E}\exp\big(t\bar{S}_M(K_A^{(\ell)})\big) \leq t^2 v^2 A + t^2\big(ab\ell(2A)^{1+\frac{\gamma_1}{\gamma}} + 4b^2A^\gamma(2A)^{\frac{2\gamma_1}{\gamma}}\big)\exp\Big(-\frac{1}{2}\Big(\frac{c_1A}{2^\ell}\Big)^{\gamma_1}\Big), \tag{3.4}$$

with $\bar{S}_M(K_A^{(\ell)})$ *defined by (3.1) and*

$$v^2 = \sup_{T>0}\sup_{K\subset\mathbb{N}^*}\frac{1}{\mathrm{Card}K}\mathrm{Var}\sum_{i\in K}\varphi_T(X_i) \tag{3.5}$$

(the maximum being taken over all nonempty finite sets K *of integers).*

Remark 4. *Notice that* $v^2 \leq V$ *(the proof is immediate).*

Proof of Proposition 1. The proof is divided in several steps.

Step 1. The construction of $K_A^{(\ell)}$. Let c_0 be defined by (3.2) and $n_0 = A$. $K_A^{(\ell)}$ will be a finite union of 2^ℓ disjoint sets of consecutive integers with same cardinal spaced according to a recursive "Cantor"-like construction. We first define an integer d_0 as follows:

$$d_0 = \begin{cases} \sup\{m \in 2\mathbb{N}\,,\ m \leq c_0 n_0\} & \text{if } n_0 \text{ is even} \\ \sup\{m \in 2\mathbb{N}+1\,,\ m \leq c_0 n_0\} & \text{if } n_0 \text{ is odd.} \end{cases}$$

It follows that $n_0 - d_0$ is even. Let $n_1 = (n_0 - d_0)/2$, and define two sets of integers of cardinal n_1 separated by a gap of d_0 integers as follows

$$I_{1,1} = \{1,\ldots,n_1\},\ I_{1,2} = \{n_1 + d_0 + 1,\ldots,n_0\}\,.$$

We define now the integer d_1 by

$$d_1 = \begin{cases} \sup\{m \in 2\mathbb{N}\,,\ m \leq c_0 2^{-(\ell\wedge\frac{1}{\gamma})} n_0\} & \text{if } n_1 \text{ is even} \\ \sup\{m \in 2\mathbb{N}+1\,,\ m \leq c_0 2^{-(\ell\wedge\frac{1}{\gamma})} n_0\} & \text{if } n_1 \text{ is odd.} \end{cases}$$

Noticing that $n_1 - d_1$ is even, we set $n_2 = (n_1 - d_1)/2$, and define four sets of integers of cardinal n_2 by

$$I_{2,1} = \{1,\ldots,n_2\},\ I_{2,2} = \{n_2 + d_1 + 1,\ldots,n_1\},\ I_{2,i+2} = (n_1 + d_0) + I_{2,i} \text{ for } i = 1,2\,.$$

Iterating this procedure j times (for $1 \leq j \leq \ell$), we then get a finite union of 2^j sets, $(I_{j,k})_{1\leq k\leq 2^j}$, of consecutive integers, with same cardinal, constructed by induction from $(I_{j-1,k})_{1\leq k\leq 2^{j-1}}$ as follows: First, for $1 \leq k \leq 2^{j-1}$, we have $I_{j-1,k} = \{a_{j-1,k},\ldots,b_{j-1,k}\}$, where $1 + b_{j-1,k} - a_{j-1,k} = n_{j-1}$ and

$$1 = a_{j-1,1} < b_{j-1,1} < a_{j-1,2} < b_{j-1,2} < \cdots < a_{j-1,2^{j-1}} < b_{j-1,2^{j-1}} = n_0.$$

Let $n_j = 2^{-1}(n_{j-1} - d_{j-1})$ and

$$d_j = \begin{cases} \sup\{m \in 2\mathbb{N}\,,\ m \le c_0 2^{-(\ell \wedge \frac{j}{\gamma})} n_0\} & \text{if } n_j \text{ is even} \\ \sup\{m \in 2\mathbb{N}+1\,,\ m \le c_0 2^{-(\ell \wedge \frac{j}{\gamma})} n_0\} & \text{if } n_j \text{ is odd.} \end{cases}$$

Then $I_{j,k} = \{a_{j,k}, a_{j,k}+1, \ldots, b_{j,k}\}$, where the double indexed sequences $(a_{j,k})$ and $(b_{j,k})$ are defined as follows:

$$a_{j,2k-1} = a_{j-1,k}\,,\ b_{j,2k} = b_{j-1,k}\,,\ b_{j,2k} - a_{j,2k} + 1 = n_j \text{ and } b_{j,2k-1} - a_{j,2k-1} + 1 = n_j\,.$$

With this selection, we then get that there is exactly d_{j-1} integers between $I_{j,2k-1}$ and $I_{j,2k}$ for any $1 \le k \le 2^{j-1}$.

Finally we get

$$K_A^{(\ell)} = \bigcup_{k=1}^{2^\ell} I_{\ell,k}\,.$$

Since $\mathrm{Card}(I_{\ell,k}) = n_\ell$, for any $1 \le k \le 2^\ell$, we get that $\mathrm{Card}(K_A^{(\ell)}) = 2^\ell n_\ell$. Now notice that

$$A - \mathrm{Card}(K_A^{(\ell)}) = \sum_{j=0}^{\ell-1} 2^j d_j \le A c_0 \Big(\sum_{j\ge 0} 2^{j(1-1/\gamma)} + \sum_{j \ge 1} 2^{-j} \Big) \le A/2\,.$$

Consequently

$$A \ge \mathrm{Card}(K_A^{(\ell)}) \ge A/2 \text{ and } n_\ell \le A2^{-\ell}\,.$$

The following notation will be useful for the rest of the proof: For any k in $\{0,1,\ldots,\ell\}$ and any j in $\{1,\ldots,2^\ell\}$, we set

$$K_{A,k,j}^{(\ell)} = \bigcup_{i=(j-1)2^{\ell-k}+1}^{j2^{\ell-k}} I_{\ell,i}\,. \tag{3.6}$$

Notice that $K_A^{(\ell)} = K_{A,0,1}^{(\ell)}$ and that for any k in $\{0,1,\ldots,\ell\}$

$$K_A^{(\ell)} = \bigcup_{j=1}^{2^k} K_{A,k,j}^{(\ell)}\,, \tag{3.7}$$

where the union is disjoint.

In what follows we shall also use the following notation: for any integer j in $[0,\ell]$, we set

$$M_j = H^{-1}\big(a^{-1}\tau^*(c^{-1/\gamma_1} A 2^{-(\ell\wedge\frac{j}{\gamma})})\big)\,. \tag{3.8}$$

Since $H^{-1}(y) = b\big(\log(e/y)\big)^{1/\gamma_2}$ for any $y \le e$, we get that for any $x \ge 1$,

$$H^{-1}(a^{-1}\tau^*(c^{-1/\gamma_1}x)) = b\big(1 + x^{\gamma_1}\big)^{1/\gamma_2} \le b(2x)^{\gamma_1/\gamma_2}\,. \tag{3.9}$$

Consequently since for any j in $[0,\ell]$, $A2^{-(\ell\wedge\frac{j}{\gamma})}\geq 1$, the following bound is valid:

$$M_j \leq b(2A2^{-(\ell\wedge\frac{j}{\gamma})})^{\gamma_1/\gamma_2}\,. \tag{3.10}$$

Step 2. Proof of Inequality (3.4) with $K_A^{(\ell)}$ defined in step 1.

Consider the decomposition (3.7), and notice that for any $i=1,2$, $\mathrm{Card}(K_{A,1,i}^{(\ell)})\leq A/2$ and

$$\tau\big(\sigma(X_i\,:\,i\in K_{A,1,1}^{(\ell)}),\bar S_{M_0}(K_{A,1,2}^{(\ell)})\big)\leq A\tau(d_0)/2\,.$$

Since $\overline{X}_{M_0}(j)\leq 2M_0$, we get that $|\bar S_{M_0}(K_{A,1,i}^{(\ell)})|\leq AM_0$. Consequently, by using Lemma 2 from Appendix, we derive that for any positive t,

$$|\mathbb{E}\exp\big(t\bar S_{M_0}(K_A^{(\ell)})\big)-\prod_{i=1}^{2}\mathbb{E}\exp\big(t\bar S_{M_0}(K_{A,1,i}^{(\ell)})\big)|\leq \frac{At}{2}\tau(d_0)\exp(2tAM_0)\,.$$

Since the random variables $\bar S_{M_0}(K_A^{(\ell)})$ and $\bar S_{M_0}(K_{A,1,i}^{(\ell)})$ are centered, their Laplace transform are greater than one. Hence applying the elementary inequality

$$|\log x-\log y|\leq |x-y| \text{ for } x\geq 1 \text{ and } y\geq 1, \tag{3.11}$$

we get that, for any positive t,

$$|\log\mathbb{E}\exp\big(t\bar S_{M_0}(K_A^{(\ell)})\big)-\sum_{i=1}^{2}\log\mathbb{E}\exp\big(t\bar S_{M_0}(K_{A,1,i}^{(\ell)})\big)|\leq \frac{At}{2}\tau(d_0)\exp(2tAM_0)\,.$$

The next step is to compare $\mathbb{E}\exp\big(t\bar S_{M_0}(K_{A,1,i}^{(\ell)})\big)$ with $\mathbb{E}\exp\big(t\bar S_{M_1}(K_{A,1,i}^{(\ell)})\big)$ for $i=1,2$. The random variables $\bar S_{M_0}(K_{A,1,i}^{(\ell)})$ and $\bar S_{M_1}(K_{A,1,i}^{(\ell)})$ have values in $[-AM_0,AM_0]$, hence applying the inequality

$$|e^{tx}-e^{ty}|\leq |t||x-y|(e^{|tx|}\vee e^{|ty|})\,, \tag{3.12}$$

we obtain that, for any positive t,

$$\big|\mathbb{E}\exp\big(t\bar S_{M_0}(K_{A,1,i}^{(\ell)})\big)-\mathbb{E}\exp\big(t\bar S_{M_1}(K_{A,1,i}^{(\ell)})\big)\big|\leq te^{tAM_0}\mathbb{E}\big|\bar S_{M_0}(K_{A,1,i}^{(\ell)})-\bar S_{M_1}(K_{A,1,i}^{(\ell)})\big|\,.$$

Notice that

$$\mathbb{E}\big|\bar S_{M_0}(K_{A,1,i}^{(\ell)})-\bar S_{M_1}(K_{A,1,i}^{(\ell)})\big|\leq 2\sum_{j\in K_{A,1,i}^{(\ell)}}\mathbb{E}|(\varphi_{M_0}-\varphi_{M_1})(X_j)|\,.$$

Since for all $x\in\mathbb{R}$, $|(\varphi_{M_0}-\varphi_{M_1})(x)|\leq M_0\ \mathbb{1}_{|x|>M_1}$, we get that

$$\mathbb{E}|(\varphi_{M_0}-\varphi_{M_1})(X_j)|\leq M_0\mathbb{P}(|X_j|>M_1)\leq a^{-1}M_0\tau^*(c^{-\frac{1}{\gamma_1}}A2^{-(\ell\wedge\frac{1}{\gamma})})\,.$$

Consequently, since $\mathrm{Card}(K^{(\ell)}_{A,1,i}) \le A/2$, for any $i = 1, 2$ and any positive t,

$$\big|\mathbb{E}\exp\big(t\bar{S}_{M_0}(K^{(\ell)}_{A,1,i})\big) - \mathbb{E}\exp\big(t\bar{S}_{M_1}(K^{(\ell)}_{A,1,i})\big)\big| \le tAa^{-1}M_0 e^{tAM_0}\tau^*(c^{-\frac{1}{\gamma_1}}A2^{-(\ell\wedge\frac{1}{\gamma})})\,.$$

Using again the fact that the variables are centered and taking into account the inequality (3.11), we derive that for any $i = 1, 2$ and any positive t,

$$\big|\log\mathbb{E}\exp\big(t\bar{S}_{M_0}(K^{(\ell)}_{A,1,i})\big) - \log\mathbb{E}\exp\big(t\bar{S}_{M_1}(K^{(\ell)}_{A,1,i})\big)\big| \le a^{-1}e^{2tAM_0}\tau^*(c^{-\frac{1}{\gamma_1}}A2^{-(\ell\wedge\frac{1}{\gamma})})\,. \qquad (3.13)$$

Now for any $k = 1, \dots, \ell$ and any $i = 1, \dots, 2^k$, $\mathrm{Card}(K^{(\ell)}_{A,k,i}) \le 2^{-k}A$. By iterating the above procedure, we then get for any $k = 1, \dots, \ell$, and any positive t,

$$\begin{aligned}|\sum_{i=1}^{2^{k-1}}\log\mathbb{E}\exp\big(t\bar{S}_{M_{k-1}}(K^{(\ell)}_{A,k-1,i})\big) \ &- \ \sum_{i=1}^{2^k}\log\mathbb{E}\exp\big(t\bar{S}_{M_{k-1}}(K^{(\ell)}_{A,k,i})\big)| \\ &\le \ 2^{k-1}\frac{tA}{2^k}\tau(d_{k-1})\exp\big(\frac{2tAM_{k-1}}{2^{k-1}}\big)\,,\end{aligned}$$

and for any $i = 1, \dots, 2^k$,

$$|\log\mathbb{E}\exp\big(t\bar{S}_{M_{k-1}}(K^{(\ell)}_{A,k,i})\big) - \log\mathbb{E}\exp\big(t\bar{S}_{M_k}(K^{(\ell)}_{A,k,i})\big)| \le a^{-1}\tau^*(c^{-\frac{1}{\gamma_1}}A2^{-(\ell\wedge\frac{k}{\gamma})})\exp\big(\frac{2tAM_{k-1}}{2^{k-1}}\big)\,.$$

Hence finally, we get that for any $j = 1, \dots, \ell$, and any positive t,

$$\begin{aligned}&|\sum_{i=1}^{2^{j-1}}\log\mathbb{E}\exp\big(t\bar{S}_{M_{j-1}}(K^{(\ell)}_{A,j-1,i})\big) - \sum_{i=1}^{2^j}\log\mathbb{E}\exp\big(t\bar{S}_{M_j}(K^{(\ell)}_{A,j,i})\big)| \\ &\quad\le \frac{tA}{2}\tau(d_{j-1})\exp(2tAM_{j-1}2^{1-j}) + 2^j a^{-1}\tau^*(c^{-\frac{1}{\gamma_1}}A2^{-(\ell\wedge\frac{j}{\gamma})})\exp(2tAM_{j-1}2^{1-j})\,.\end{aligned}$$

Set $k_\ell = \sup\{j \in \mathbb{N}\,,\ j/\gamma < \ell\}$, and notice that $0 \le k_\ell \le \ell - 1$. Since $K^{(\ell)}_A = K^{(\ell)}_{A,0,1}$, we then derive that for any positive t,

$$\begin{aligned}&|\log\mathbb{E}\exp\big(t\bar{S}_{M_0}(K^{(\ell)}_A)\big) - \sum_{i=1}^{2^{k_\ell+1}}\log\mathbb{E}\exp\big(t\bar{S}_{M_{k_\ell+1}}(K^{(\ell)}_{A,k_\ell+1,i})\big)| \\ &\quad\le \frac{tA}{2}\sum_{j=0}^{k_\ell}\tau(d_j)\exp\big(\frac{2tAM_j}{2^j}\big) + 2a^{-1}\sum_{j=0}^{k_\ell-1}2^j\tau^*(2^{-1/\gamma}c^{-1/\gamma_1}A2^{-j/\gamma})\exp\big(\frac{2tAM_j}{2^j}\big) \\ &\qquad+2^{k_\ell+1}a^{-1}\tau^*(c^{-1/\gamma_1}A2^{-\ell})\exp(2tAM_{k_\ell}2^{-k_\ell})\,. \qquad (3.14)\end{aligned}$$

Notice now that for any $i = 1, \dots, 2^{k_\ell+1}$, $S_{M_{k_\ell+1}}(K^{(\ell)}_{A,k_\ell+1,i})$ is a sum of $2^{\ell-k_\ell-1}$ blocks, each of size n_ℓ and bounded by $2M_{k_\ell+1}n_\ell$. In addition the blocks are equidistant and there is a gap of size

$d_{k_\ell+1}$ between two blocks. Consequently, by using Lemma 2 along with Inequality (3.11) and the fact that the variables are centered, we get that

$$
\begin{aligned}
&|\log \mathbb{E}\exp\big(t\bar{S}_{M_{k_\ell+1}}(K^{(\ell)}_{A,k_\ell+1,i})\big) - \sum_{j=(i-1)2^{\ell-k_\ell-1}+1}^{i2^{\ell-k_\ell-1}} \log \mathbb{E}\exp\big(t\bar{S}_{M_{k_\ell+1}}(I_{\ell,j})\big)| \\
&\quad \leq tn_\ell 2^\ell 2^{-k_\ell-1}\tau(d_{k_\ell+1})\exp(2tM_{k_\ell+1}n_\ell 2^{\ell-k_\ell-1})\,. \qquad (3.15)
\end{aligned}
$$

Starting from (3.14) and using (3.15) together with the fact that $n_\ell \leq A2^{-\ell}$, we obtain:

$$
\begin{aligned}
&|\log \mathbb{E}\exp\big(t\bar{S}_{M_0}(K^{(\ell)}_{A})\big) - \sum_{j=1}^{2^\ell} \log \mathbb{E}\exp\big(t\bar{S}_{M_{k_\ell+1}}(I_{\ell,j})\big)| \\
&\quad \leq \frac{tA}{2}\sum_{j=0}^{k_\ell}\tau(d_j)\exp\big(\frac{2tAM_j}{2^j}\big) + 2a^{-1}\sum_{j=0}^{k_\ell-1}2^j\tau^*(2^{-1/\gamma}c^{-1/\gamma_1}A2^{-j/\gamma})\exp\big(\frac{2tAM_j}{2^j}\big) \\
&\quad +2^{k_\ell+1}a^{-1}\tau^*(c^{-1/\gamma_1}A2^{-\ell})\exp\big(\frac{2tAM_{k_\ell}}{2^{k_\ell}}\big) + tA\tau(d_{k_\ell+1})\exp(tM_{k_\ell+1}A2^{-k_\ell})\,. \qquad (3.16)
\end{aligned}
$$

Notice that for any $j = 0,\dots,\ell-1$, we have $d_j + 1 \geq [c_0A2^{-(\ell\wedge\frac{j}{\gamma})}]$ and $c_0A2^{-(\ell\wedge\frac{j}{\gamma})} \geq 2$. Whence

$$
d_j \geq (d_j+1)/2 \geq c_0A2^{-(\ell\wedge\frac{j}{\gamma})-2}\,.
$$

Consequently setting $c_1 = \min(\frac{1}{4}c^{1/\gamma_1}c_0, 2^{-1/\gamma})$ and using (2.6), we derive that for any positive t,

$$
\begin{aligned}
&|\log \mathbb{E}\exp\big(t\bar{S}_{M_0}(K^{(\ell)}_{A})\big) - \sum_{j=1}^{2^\ell} \log \mathbb{E}\exp\big(t\bar{S}_{M_{k_\ell+1}}(I_{\ell,j})\big)| \\
\leq\ & \frac{tAa}{2}\sum_{j=0}^{k_\ell}\exp\Big(-\big(c_1A2^{-j/\gamma}\big)^{\gamma_1} + \frac{2tAM_j}{2^j}\Big) + 2\sum_{j=0}^{k_\ell-1}2^j\exp\Big(-\big(c_1A2^{-j/\gamma}\big)^{\gamma_1} + \frac{2tAM_j}{2^j}\Big) \\
&\quad +2^{k_\ell+1}\exp\Big(-\big(A2^{-\ell}\big)^{\gamma_1} + \frac{2tAM_{k_\ell}}{2^{k_\ell}}\Big) + tAa\exp\Big(-\big(c_1A2^{-\ell}\big)^{\gamma_1} + tM_{k_\ell+1}A2^{-k_\ell}\Big)\,.
\end{aligned}
$$

By (3.10), we get that $2AM_j2^{-j} \leq b2^{\gamma_1/\gamma}(2^{-j}A)^{\gamma_1/\gamma}$ for any $0 \leq j \leq k_\ell$. Also, since $k_\ell + 1 \geq \gamma\ell$ and $\gamma < 1$, we have that $M_{k_\ell+1} \leq b(2A2^{-\ell})^{\gamma_1/\gamma_2} \leq b(2A2^{-\gamma\ell})^{\gamma_1/\gamma_2}$. Whence

$$
M_{k_\ell+1}A2^{-k_\ell} = 2M_{k_\ell+1}A2^{-(k_\ell+1)} \leq b2^{\gamma_1/\gamma}A^{\gamma_1/\gamma}2^{-\gamma_1\ell}\,.
$$

In addition,

$$
2AM_{k_\ell}2^{-k_\ell} \leq b2^{2\gamma_1/\gamma}(A2^{-k_\ell-1})^{\gamma_1/\gamma} \leq b2^{2\gamma_1/\gamma}A^{\gamma_1/\gamma}2^{-\gamma_1\ell}\,.
$$

Hence, if $t \le c_2 A^{\gamma_1(\gamma-1)/\gamma}$ where $c_2 = 2^{-(1+2\gamma_1/\gamma)} c_1^{\gamma_1} b^{-1}$, we derive that

$$
\begin{aligned}
&|\log \mathbb{E} \exp\big(t\bar S_{M_0}(K_A^{(\ell)})\big) - \sum_{j=1}^{2^\ell} \log \mathbb{E}\exp\big(t\bar S_{M_{k_\ell+1}}(I_{\ell,j})\big)| \\
\le\ & \frac{tAa}{2}\sum_{j=0}^{k_\ell} \exp\Big(-\frac12 \big(c_1 A 2^{-j/\gamma}\big)^{\gamma_1}\Big) + 2\sum_{j=0}^{k_\ell-1} 2^j \exp\Big(-\frac12\big(c_1 A 2^{-j/\gamma}\big)^{\gamma_1}\Big) \\
& + (2^{k_\ell+1} + tAa)\exp\big(-(c_1 A 2^{-\ell})^{\gamma_1}/2\big)\,.
\end{aligned}
$$

Since $2^{k_\ell} \le 2^{\ell\gamma} \le A^\gamma$, it follows that for any $t \le c_2 A^{\gamma_1(\gamma-1)/\gamma}$,

$$
|\log \mathbb{E} \exp\big(t\bar S_{M_0}(K_A^{(\ell)})\big) - \sum_{j=1}^{2^\ell} \log \mathbb{E}\exp\big(t\bar S_{M_{k_\ell+1}}(I_{\ell,j})\big)| \le (2a\ell t A + 4A^\gamma)\exp\Big(-\frac12\Big(\frac{c_1 A}{2^\ell}\Big)^{\gamma_1}\Big)\,. \tag{3.17}
$$

We bound up now the log Laplace transform of each $\bar S_{M_{k_\ell+1}}(I_{\ell,j})$ using the following elementary fact. Let $g(x) = x^{-2}(e^x - x - 1)$: for any centered random variable U such that $\|U\|_\infty \le M$, and any positive t,

$$
\mathbb{E}\exp(tU) \le 1 + t^2 g(tM)\mathbb{E}(U^2)\,. \tag{3.18}
$$

Notice that $\|\bar S_{M_{k_\ell+1}}(I_{\ell,j})\|_\infty \le 2M_{k_\ell+1} n_\ell \le b2^{\gamma_1/\gamma}(A2^{-\ell})^{\gamma_1/\gamma}$. Since $t \le b^{-1}2^{-\gamma_1/\gamma}(2^\ell/A)^{\gamma_1/\gamma}$, by using (3.5), we then get that $\log \mathbb{E}\exp\big(t\bar S_{M_{k_\ell+1}}(I_{\ell,j})\big) \le t^2 v^2 n_\ell$. Consequently, for any $t \le \kappa\big(A^{\gamma_1(\gamma-1)/\gamma} \wedge (2^\ell/A)^{\gamma_1/\gamma}\big)$, the following inequality holds:

$$
\log \mathbb{E}\exp\big(t\bar S_{M_0}(K_A^{(\ell)})\big) \le t^2 v^2 A + (2a\ell t A + 4A^\gamma)\exp\big(-\big(c_1 A 2^{-\ell}\big)^{\gamma_1}/2\big)\,. \tag{3.19}
$$

Notice now that $\|\bar S_{M_0}(K_A^{(\ell)})\|_\infty \le 2M_0 A \le b2^{\gamma_1/\gamma}A^{\gamma_1/\gamma}$. Hence if $t \le b^{-1}2^{-\gamma_1/\gamma}A^{-\gamma_1/\gamma}$, by using (3.18) together with (3.5), we derive that

$$
\log \mathbb{E}\exp\big(t\bar S_{M_0}(K_A^{(\ell)})\big) \le t^2 v^2 A\,, \tag{3.20}
$$

which proves (3.4) in this case.

Now if $b^{-1}2^{-\gamma_1/\gamma}A^{-\gamma_1/\gamma} \le t \le \kappa\big(A^{\gamma_1(\gamma-1)/\gamma} \wedge (2^\ell/A)^{\gamma_1/\gamma}\big)$, by using (3.19), we derive that (3.4) holds, which completes the proof of Proposition 1. ⋄

We now bound up the Laplace transform of the sum of truncated random variables on $[1, A]$. Let

$$
\mu = \big(2(2 \vee 4c_0^{-1})/(1-\gamma)\big)^{\frac{2}{1-\gamma}} \text{ and } c_4 = b2^{\gamma_1/\gamma}3^{\gamma_1/\gamma_2}c_0^{-\gamma_1/\gamma_2}\,, \tag{3.21}
$$

where c_0 is defined in (3.2). Define also

$$
\nu = \big(c_4\big(3 - 2^{(\gamma-1)\frac{\gamma_1}{\gamma}}\big) + \kappa^{-1}\big)^{-1}\big(1 - 2^{(\gamma-1)\frac{\gamma_1}{\gamma}}\big)\,, \tag{3.22}
$$

where κ is defined by (3.3).

Proposition 2. *Let $(X_j)_{j\geq 1}$ be a sequence of centered real valued random variables satisfying (2.6), (2.7) and (2.8). Let A be an integer. Let $M = H^{-1}(a^{-1}\tau^*(c^{-1/\gamma_1}A))$ and $\bar{X}_M(k)$ be defined by (3.1) for any positive k. Then, if $A \geq \mu$ with μ defined by (3.21), for any positive $t < \nu A^{\gamma_1(\gamma-1)/\gamma}$, where ν is defined by (3.22), we get that*

$$\log \mathbb{E}\Big(\exp(t \textstyle\sum_{k=1}^{A} \bar{X}_M(k))\Big) \leq \frac{AV(A)t^2}{1 - t\nu^{-1}A^{\gamma_1(1-\gamma)/\gamma}}\,, \tag{3.23}$$

where $V(A) = 50v^2 + \nu_1 \exp(-\nu_2 A^{\gamma_1(1-\gamma)}(\log A)^{-\gamma})$ and ν_1, ν_2 are positive constants depending only on a, b, c, γ and γ_1, and v^2 is defined by (3.5).

Proof of Proposition 2. Let $A_0 = A$ and $X^{(0)}(k) = X_k$ for any $k = 1, \ldots, A_0$. Let ℓ be a fixed positive integer, to be chosen later, which satisfies

$$A_0 2^{-\ell} \geq (2 \vee 4c_0^{-1})\,. \tag{3.24}$$

Let $K_{A_0}^{(\ell)}$ be the discrete Cantor type set as defined from $\{1, \ldots, A\}$ in Step 1 of the proof of Proposition 1. Let $A_1 = A_0 - \mathrm{Card}K_{A_0}^{(\ell)}$ and define for any $k = 1, \ldots, A_1$,

$$X^{(1)}(k) = X_{i_k} \text{ where } \{i_1, \ldots, i_{A_1}\} = \{1, \ldots, A\} \setminus K_A\,.$$

Now for $i \geq 1$, let $K_{A_i}^{(\ell_i)}$ be defined from $\{1, \ldots, A_i\}$ exactly as $K_A^{(\ell)}$ is defined from $\{1, \ldots, A\}$. Here we impose the following selection of ℓ_i:

$$\ell_i = \inf\{j \in \mathbb{N}\,,\, A_i 2^{-j} \leq A_0 2^{-\ell}\,\}\,. \tag{3.25}$$

Set $A_{i+1} = A_i - \mathrm{Card}K_{A_i}^{(\ell_i)}$ and $\{j_1, \ldots, j_{A_{i+1}}\} = \{1, \ldots, A_{i+1}\} \setminus K_{A_{i+1}}^{(\ell_{i+1})}$. Define now

$$X^{(i+1)}(k) = X^{(i)}(j_k) \text{ for } k = 1, \ldots, A_{i+1}\,.$$

Let

$$m(A) = \inf\{m \in \mathbb{N}\,,\, A_m \leq A2^{-\ell}\}\,. \tag{3.26}$$

Note that $m(A) \geq 1$, since $A_0 > A2^{-\ell}$ ($\ell \geq 1$). In addition, $m(A) \leq \ell$ since for all $i \geq 1$, $A_i \leq A2^{-i}$.

Obviously, for any $i = 0, \ldots, m(A) - 1$, the sequences $(X^{(i+1)}(k))$ satisfy (2.6), (2.7) and (3.5) with the same constants. Now we set $T_0 = M = H^{-1}(a^{-1}\tau^*(c^{-1/\gamma_1}A_0))$, and for any integer $j = 0, \ldots, m(A)$, $T_j = H^{-1}(a^{-1}\tau^*(c^{-1/\gamma_1}A_j))$. With this definition, we then define for all integers i and j,

$$X_{T_j}^{(i)}(k) = \varphi_{T_j}\big(X^{(i)}(k)\big) - \mathbb{E}\varphi_{T_j}\big(X^{(i)}(k)\big)\,.$$

According to (3.9), for any integer $j \geq 0$,

$$T_j \leq b(2A_j)^{\gamma_1/\gamma_2}\,. \tag{3.27}$$

For any $j = 1, \dots, m(A)$ and $i < j$, define

$$Y_i = \sum_{k \in K_{A_i}^{(\ell_i)}} X_{T_i}^{(i)}(k)\,,\ Z_i = \sum_{k=1}^{A_i} (X_{T_{i-1}}^{(i)}(k) - X_{T_i}^{(i)}(k)) \text{ for } i > 0, \text{ and } R_j = \sum_{k=1}^{A_j} X_{T_{j-1}}^{(j)}(k)\,.$$

The following decomposition holds:

$$\sum_{k=1}^{A_0} X_{T_0}^{(0)}(k) = \sum_{i=0}^{m(A)-1} Y_i + \sum_{i=1}^{m(A)-1} Z_i + R_{m(A)}\,. \tag{3.28}$$

To control the terms in the decomposition (3.28), we need the following elementary lemma.

Lemma 1. *For any $j = 0, \dots, m(A) - 1$, $A_{j+1} \geq \frac{1}{3} c_0 A_j$.*

Proof of Lemma 1. Notice that for any i in $[0, m(A)[$, we have $A_{i+1} \geq [c_0 A_i] - 1$. Since $c_0 A_i \geq 2$, we derive that $[c_0 A_i] - 1 \geq ([c_0 A_i] + 1)/3 \geq c_0 A_i/3$, which completes the proof. ⋄

Using (3.27), a useful consequence of Lemma 1 is that for any $j = 1, \dots, m(A)$

$$2 A_j T_{j-1} \leq c_4 A_j^{\gamma_1/\gamma} \tag{3.29}$$

where c_4 is defined by (3.21)

A bound for the Laplace transform of $R_{m(A)}$.

The random variable $|R_{m(A)}|$ is a.s. bounded by $2A_{m(A)} T_{m(A)-1}$. By using (3.29) and (3.26), we then derive that

$$\|R_{m(A)}\|_\infty \leq c_4 (A_{m(A)})^{\gamma_1/\gamma} \leq c_4 \big(A 2^{-\ell}\big)^{\gamma_1/\gamma}\,. \tag{3.30}$$

Hence, if $t \leq c_4^{-1} (2^\ell / A)^{\gamma_1/\gamma}$, by using (3.18) together with (3.5), we obtain

$$\log \mathbb{E}\big(\exp(t R_{m(A)})\big) \leq t^2 v^2 A 2^{-\ell} \leq t^2 (v\sqrt{A})^2 := t^2 \sigma_1^2\,. \tag{3.31}$$

A bound for the Laplace transform of the Y_i's.

Notice that for any $0 \leq i \leq m(A) - 1$, by the definition of ℓ_i and (3.24), we get that

$$2^{-\ell_i} A_i = 2^{1-\ell_i} (A_i/2) > 2^{-\ell} (A/2) \geq (1 \vee 2 c_0^{-1})\,.$$

Now, by Proposition 1, we get that for any $i \in [0, m(A)[$ and any $t \leq \kappa \big(A_i^{\gamma-1} \wedge (2^{\ell_i}/A_i)\big)^{\gamma_1/\gamma}$ with κ defined by (3.3),

$$\log \mathbb{E}\big(e^{tY_i}\big) \leq t^2 \Big(v\sqrt{A_i} + \big(\sqrt{ab\ell_i} (2A_i)^{\frac{1}{2} + \frac{\gamma_1}{2\gamma}} + 2b A_i^{\gamma/2} (2A_i)^{\gamma_1/\gamma}\big) \exp\big(-\frac{1}{4} \big(c_1 A_i 2^{-\ell_i}\big)^{\gamma_1}\big)\Big)^2\,.$$

Notice now that $\ell_i \le \ell \le A$, $A_i \le A2^{-i}$ and $2^{-\ell-1}A \le 2^{-\ell_i}A_i \le 2^{-\ell}A$. Taking into account these bounds and the fact that $\gamma < 1$, we then get that for any i in $[0, m(A)[$ and any $t \le \kappa\big((2^i/A)^{1-\gamma} \wedge (2^\ell/A)\big)^{\gamma_1/\gamma}$,

$$\log \mathbb{E}\big(e^{tY_i}\big) \le t^2\Big(v\frac{A^{1/2}}{2^{i/2}} + \Big(2^{1+\frac{\gamma_1}{\gamma}}\sqrt{b}(\sqrt{a}+\sqrt{b})\frac{A^{1+\frac{\gamma_1}{\gamma}}}{(2^i)^{\frac{\gamma}{2}+\frac{\gamma_1}{2\gamma}}}\Big)\exp\Big(-\frac{c_1^{\gamma_1}}{2^{2+\gamma_1}}\Big(\frac{A}{2^\ell}\Big)^{\gamma_1}\Big)\Big)^2 := t^2\sigma_{2,i}^2 \,. \quad (3.32)$$

A bound for the Laplace transform of the Z_i's.

Notice first that for any $1 \le i \le m(A)-1$, Z_i is a centered random variable, such that

$$|Z_i| \le \sum_{k=1}^{A_i}\Big(\big|(\varphi_{T_{i-1}} - \varphi_{T_i})(X^{(i)}(k))\big| + \mathbb{E}\big|(\varphi_{T_{i-1}} - \varphi_{T_i})(X^{(i)}(k))\big|\Big)\,.$$

Consequently, using (3.29) we then get that $\|Z_i\|_\infty \le 2A_iT_{i-1} \le c_4A_i^{\gamma_1/\gamma}$. In addition, since $|(\varphi_{T_{i-1}} - \varphi_{T_i})(x)| \le (T_{i-1} - T_i)\,\mathbb{I}_{x>T_i}$, and the random variables $(X^{(i)}(k))$ satisfy (2.7), by the definition of T_i, we get that

$$\mathbb{E}|Z_i|^2 \le (2A_iT_{i-1})^2a^{-1}\tau^*(c^{-1/\gamma_1}A_i) \le c_4^2A_i^{2\gamma_1/\gamma}\exp(-A_i^{\gamma_1})\,.$$

Hence applying (3.18) to the random variable Z_i, we get for any positive t,

$$\mathbb{E}\exp(tZ_i) \le 1 + t^2g(c_4tA_i^{\gamma_1/\gamma})c_4^2A_i^{2\gamma_1/\gamma}\exp(-A_i^{\gamma_1})\,.$$

Now, since $A_i \le A2^{-i}$, for any positive real t satisfying $t \le (2c_4)^{-1}(2^i/A)^{\gamma_1(1-\gamma)/\gamma}$, we have that $c_4tA_i^{\gamma_1/\gamma} \le A_i^{\gamma_1}/2$. Since $g(x) \le e^x$ for $x \ge 0$, we infer that for $t \le (2c_4)^{-1}(2^i/A)^{\gamma_1(1-\gamma)/\gamma}$,

$$\log \mathbb{E}\exp(tZ_i) \le c_4^2t^2(2^{-i}A)^{2\gamma_1/\gamma}\exp(-A_i^{\gamma_1}/2)\,.$$

By taking into account that for any $1 \le i \le m(A)-1$, $A_i \ge A_{m(A)-1} > A2^{-\ell}$ (by definition of $m(A)$), it follows that for any i in $[1, m(A)[$ and any positive t satisfying $t \le (2c_4)^{-1}(2^i/A)^{\gamma_1(1-\gamma)/\gamma}$,

$$\log \mathbb{E}\exp(tZ_i) \le t^2\big(c_4(2^{-i}A)^{\gamma_1/\gamma}\exp(-(A2^{-\ell})^{\gamma_1}/4)\big)^2 := t^2\sigma_{3,i}^2\,. \quad (3.33)$$

End of the proof. Let

$$C = c_4\Big(\frac{A}{2^\ell}\Big)^{\gamma_1/\gamma} + \frac{1}{\kappa}\sum_{i=0}^{m(A)-1}\Big(\Big(\frac{A}{2^i}\Big)^{1-\gamma} \vee \frac{A}{2^\ell}\Big)^{\gamma_1/\gamma} + 2c_4\sum_{i=1}^{m(A)-1}\Big(\frac{A}{2^i}\Big)^{\gamma_1(1-\gamma)/\gamma}\,,$$

and

$$\sigma = \sigma_1 + \sum_{i=0}^{m(A)-1}\sigma_{2,i} + \sum_{i=1}^{m(A)-1}\sigma_{3,i}\,,$$

where σ_1, $\sigma_{2,i}$ and $\sigma_{3,i}$ are respectively defined in (3.31), (3.32) and (3.33).

Notice that $m(A) \leq \ell \leq 2\log A/\log 2$. We choose now $\ell = \inf\{j \in \mathbb{N} : 2^j \geq A^\gamma (\log A)^{\gamma/\gamma_1}\}$. This selection is compatible with (3.24) if

$$(2 \vee 4c_0^{-1})(\log A)^{\gamma/\gamma_1} \leq A^{1-\gamma}\,. \tag{3.34}$$

Now we use the fact that for any positive δ and any positive u, $\delta \log u \leq u^\delta$. Hence if $A \geq 3$,

$$(2 \vee 4c_0^{-1})(\log A)^{\gamma/\gamma_1} \leq (2 \vee 4c_0^{-1}) \log A \leq 2(1-\gamma)^{-1}(2 \vee 4c_0^{-1})A^{(1-\gamma)/2}\,,$$

which implies that (3.34) holds as soon as $A \geq \mu$ where μ is defined by (3.21). It follows that

$$C \leq \nu^{-1} A^{\gamma_1(1-\gamma)/\gamma}\,. \tag{3.35}$$

In addition

$$\sigma \leq 5v\sqrt{A} + 5\sqrt{b}(\sqrt{a}+\sqrt{b})2^{2\gamma_1/\gamma}A^{1+\gamma_1/\gamma} \exp\Big(-\frac{c_1^{\gamma_1}}{2^{2+\gamma_1}}(A2^{-\ell})^{\gamma_1}\Big) + c_4 A^{\gamma_1/\gamma}\exp\Big(-\frac{1}{4}(A2^{-\ell})^{\gamma_1}\Big)\,.$$

Consequently, since $A2^{-\ell} \geq \frac{1}{2}A^{1-\gamma}(\log A)^{-\gamma/\gamma_1}$, there exists positive constants ν_1 and ν_2 depending only on a, b, c, γ and γ_1 such that

$$\sigma^2 \leq A\big(50v^2 + \nu_1 \exp(-\nu_2 A^{\gamma_1(1-\gamma)}(\log A)^{-\gamma})\big) = AV(A)\,. \tag{3.36}$$

Starting from the decomposition (3.28) and the bounds (3.31), (3.32) and (3.33), we aggregate the contributions of the terms by using Lemma 3 given in the appendix. Then, by taking into account the bounds (3.35) and (3.36), Proposition 2 follows. ⋄

3.2 Proof of Theorem 1

Let us give the idea of the proof. We first use the classical Bernstein-type blocking arguments on partial sums of suitably truncated variables at a level M; namely, the index set $\{1,\dots,n\}$ is partitioned into blocks of size A. Then the blocks of even indexes are approximated by independent blocks with same marginals using the exact coupling coefficient in $\mathbb{L}^1$ (see Lemma 5 in Dedecker and Prieur (2004)); and similarly for the blocks of odd indexes. We then bound the log-Laplace transforms of each block of size A with the help of Proposition 2. The end of the proof consists in choosing optimally both A and M.

• If $\lambda \geq bn^{\gamma_1/\gamma}$, note that $\sup(|S_1|,|S_2|,\dots,|S_n|) \leq |X_1|+|X_2|+\cdots+|X_n|$. Hence

$$\mathbb{P}\big(\sup_{j\leq n}|S_j| \geq \lambda\big) \leq \sum_{i=1}^n \mathbb{P}(|X_i| \geq \lambda/n) \leq ne\exp\Big(-\frac{\lambda^{\gamma_2}}{(bn)^{\gamma_2}}\Big) \wedge 1 \leq \sqrt{ne}\exp\Big(-\frac{\lambda^{\gamma_2}}{2(bn)^{\gamma_2}}\Big).$$

Now, if $\lambda \geq bn^{\gamma_1/\gamma}$, then $(\lambda/(nb))^{\gamma_2} \geq (\lambda/b)^{\gamma}$. Hence, for any $n \geq 3$,

$$\mathbb{P}\Big(\sup_{j\leq n} |S_j| \geq \lambda\Big) \leq n \exp\Big(-\frac{\lambda^\gamma}{2b^\gamma}\Big) .$$

• Let $\zeta = \mu \vee (2/\gamma_2)^{1/\gamma_1}$ where μ is defined by (3.21). Assume that $a \vee b(4\zeta)^{\gamma_1/\gamma} \leq \lambda \leq bn^{\gamma_1/\gamma}$. Let p be a real in $[1, \frac{n}{2}]$, to be chosen later on. Let

$$A = \Big[\frac{n}{2p}\Big], \ k = \Big[\frac{n}{2A}\Big] \text{ and } M = H^{-1}(a^{-1}\tau^*(c^{-\frac{1}{\gamma_1}}A)) .$$

For any set of natural numbers K, denote

$$\bar{S}_M(K) = \sum_{i\in K} \bar{X}_M(i) \text{ where } \bar{X}_M(k) \text{ is defined by (3.1).}$$

For i integer in $[1, 2k]$, let $I_i = \{1 + (i-1)A, \dots, iA\}$. Let also $I_{2k+1} = \{1 + 2kA, \dots, n\}$. Set

$$\bar{S}_1(j) = \sum_{i=1}^{j} \bar{S}_M(I_{2i-1}) \ \text{ and } \bar{S}_2(j) = \sum_{i=1}^{j} \bar{S}_M(I_{2i}).$$

We then get the following inequality

$$\sup_{j\leq n} |S_j| \leq \sup_{j\leq k+1} |\bar{S}_1(j)| + \sup_{j\leq k} |\bar{S}_2(j)| + 2AM + \sum_{i=1}^{n} |X_i - \overline{X}_M(i)| . \tag{3.37}$$

Now

$$\mathbb{P}\Big(\sum_{i=1}^{n} |X_i - \overline{X}_M(i)| \geq \lambda\Big) \leq \frac{1}{\lambda}\sum_{i=1}^{n} \mathbb{E}|X_i - \overline{X}_M(i)| \leq \frac{2n}{\lambda}\int_M^\infty H(x)dx.$$

Now recall that $\log H(x) = 1 - (x/b)^{\gamma_2}$. It follows that the function $x \to \log(x^2 H(x))$ is nonincreasing as soon as $x \geq b(2/\gamma_2)^{1/\gamma_2}$. Hence, for $M \geq b(2/\gamma_2)^{1/\gamma_2}$,

$$\int_M^\infty H(x)dx \leq M^2 H(M) \int_M^\infty \frac{dx}{x^2} = MH(M).$$

Whence

$$\mathbb{P}\Big(\sum_{i=1}^{n} |X_i - \overline{X}_M(i)| \geq \lambda\Big) \leq 2n\lambda^{-1} MH(M) \ \text{ for } \ M \geq b(2/\gamma_2)^{1/\gamma_2}. \tag{3.38}$$

Using (3.38) together with our selection of M, we get for all positive λ that

$$\mathbb{P}\Big(\sum_{i=1}^{n} |X_i - \overline{X}_M(i)| \geq \lambda\Big) \leq 2n\lambda^{-1} M \exp(-A^{\gamma_1}) \ \text{ for } \ A \geq (2/\gamma_2)^{1/\gamma_1} .$$

By using Lemma 5 in Dedecker and Prieur (2004), we get the existence of independent random variables $(\bar S^*_M(I_{2i}))_{1\le i\le k}$ with the same distribution as the random variables $\bar S_M(I_{2i})$ such that

$$\mathbb{E}|\bar S_M(I_{2i}) - \bar S^*_M(I_{2i})| \le A\tau(A) \le aA\exp\big(-cA^{\gamma_1}\big)\,. \tag{3.39}$$

The same is true for the sequence $(\bar S_M(I_{2i-1}))_{1\le i\le k+1}$. Hence for any positive λ such that $\lambda \ge 2AM$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{j\le n}|S_j| \ge 6\lambda\Big) &\le a\lambda^{-1}A(2k+1)\exp\big(-cA^{\gamma_1}\big) + 2n\lambda^{-1}M\exp(-A^{\gamma_1})\\ &\quad +\mathbb{P}\Big(\max_{j\le k+1}\Big|\sum_{i=1}^{j}\bar S^*_M(I_{2i-1})\Big| \ge \lambda\Big) + \mathbb{P}\Big(\max_{j\le k}\Big|\sum_{i=1}^{j}\bar S^*_M(I_{2i})\Big| \ge \lambda\Big)\,.\end{aligned}$$

For any positive t, due to the independence and since the variables are centered, $(\exp(t\bar S_M(I_{2i})))_i$ is a submartingale. Hence Doob's maximal inequality entails that for any positive t,

$$\mathbb{P}\Big(\max_{j\le k}\sum_{i=1}^{j}\bar S^*_M(I_{2i}) \ge \lambda\Big) \le e^{-\lambda t}\prod_{i=1}^{k}\mathbb{E}\Big(\exp(t\bar S_M(I_{2i}))\Big)\,.$$

To bound the Laplace transform of each random variable $\bar S_M(I_{2i})$, we apply Proposition 2 to the sequences $(X_{i+s})_{i\in\mathbb{Z}}$ for suitable values of s. Hence we derive that, if $A \ge \mu$ then for any positive t such that $t < \nu A^{\gamma_1(\gamma-1)/\gamma}$ (where ν is defined by (3.22)),

$$\sum_{i=1}^{k}\log\mathbb{E}\Big(\exp(t\bar S_M(I_{2i}))\Big) \le Akt^2\frac{V(A)}{1-t\nu^{-1}A^{\gamma_1(1-\gamma)/\gamma}}\,. \tag{3.40}$$

Obviously the same inequalities hold true for the sums associated to $(-X_i)_{i\in\mathbb{Z}}$. Now the usual Chernoff computations for the optimization of t in (3.40) lead to

$$\mathbb{P}\Big(\max_{j\le k}\Big|\sum_{i=1}^{j}\bar S^*_M(I_{2i})\Big| \ge \lambda\Big) \le 2\exp\Big(-\frac{\lambda^2}{4AkV(A)+2\lambda\nu^{-1}A^{\gamma_1(1-\gamma)/\gamma}}\Big)\,.$$

Similarly, we obtain that

$$\mathbb{P}\Big(\max_{j\le k+1}\Big|\sum_{i=1}^{j}\bar S^*_M(I_{2i-1})\Big| \ge \lambda\Big) \le 2\exp\Big(-\frac{\lambda^2}{4A(k+1)V(A)+2\lambda\nu^{-1}A^{\gamma_1(1-\gamma)/\gamma}}\Big)\,.$$

Choose now $p = n(b^{-1}\lambda)^{-\gamma/\gamma_1}$. It follows that $2A \le (\lambda/b)^{\gamma/\gamma_1}$ and, since $M \le b(2A)^{\gamma_1/\gamma_2}$, we obtain that $2AM \le b(2A)^{\gamma_1/\gamma} \le \lambda$. Also, since $\lambda/b \ge (4\zeta)^{\gamma_1/\gamma}$, $p \le n/4$ and consequently $A \ge n/(4p) \ge (b^{-1}\lambda)^{\gamma/\gamma_1}/4 \ge \zeta \ge \mu$. Hence we get that for $a \vee b(4\zeta)^{\gamma_1/\gamma} \le \lambda \le bn^{\gamma_1/\gamma}$,

$$\mathbb{P}\Big(\sup_{j\le n}|S_j| \ge 6\lambda\Big) \le 1\wedge\Big(4n\exp\Big(-(c\wedge 1)\frac{\lambda^\gamma}{4^{\gamma_1}b^\gamma}\Big) + 4\exp\Big(-\frac{\lambda^2}{4nB(\lambda)+2\lambda^{2-\gamma}\nu^{-1}b^{\gamma-1}}\Big)\Big)\,,$$

with $B(\lambda) = 50v^2 + \nu_1 \exp(-\tilde{\nu}_2 \lambda^{\gamma(1-\gamma)} (\log \lambda)^{-\gamma})$, where $\tilde{\nu}_2$ is a positive constant depending on a, b, c, γ and γ_1. The result follows from the previous bound.

• To end the proof, we mention that if $\lambda \le a \vee b(4\zeta)^{\gamma_1/\gamma}$, then

$$\mathbb{P}\Big(\sup_{j\le n} |S_j| \ge \lambda\Big) \le 1 \le e \exp\Big(-\frac{\lambda^\gamma}{a^\gamma \vee b^\gamma (4\zeta)^{\gamma_1}}\Big) ,$$

which is less than $n \exp(-\lambda^\gamma / C_1)$ as soon as $n \ge 3$ and $C_1 \ge a^\gamma \vee b^\gamma (4\zeta)^{\gamma_1}$. $\diamond$

3.3 Proof of Remark 3

Setting $W_i = \varphi_M(X_i)$ we first bound $\text{Cov}(W_i, W_{i+k})$). Applying (4.2) of Proposition 1 in Dedecker and Doukhan (2003), we derive that, for any positive k,

$$|\text{Cov}(W_i, W_{i+k})| \le 2 \int_0^{\gamma(\mathcal{M}_i, W_{i+k})/2} Q_{|W_i|} \circ G_{|W_{i+k}|}(u) du$$

where

$$\gamma(\mathcal{M}_i, W_{i+k}) = \|\mathbb{E}(W_{i+k}|\mathcal{M}_i) - \mathbb{E}(W_{i+k})\|_1 \le \tau(k) ,$$

since $x \mapsto \varphi_M(x)$ is 1-Lipschitz. Now for any j, $Q_{|W_j|} \le Q_{|X_j|} \le Q$, implying that $G_{|W_j|} \ge G$, where G is the inverse function of $u \to \int_0^u Q(v) dv$. Taking $j = i$ and $j = i + k$, we get that

$$|\text{Cov}(W_i, W_{i+k})| \le 2 \int_0^{\tau(k)/2} Q_{|X_i|} \circ G(u) du .$$

Making the change-of-variables $u = G(v)$ we also have

$$|\text{Cov}(W_i, W_{i+k})| \le 2 \int_0^{G(\tau(k)/2)} Q_{X_i}(u) Q(u) du , \tag{3.41}$$

proving the remark.

3.4 Proof of Theorem 2

We first use Theorem 1 allowing essentially us to reduce the proof of the MDP to the one of bounded random variables. For any $n \ge 1$, let $T = T_n$ where (T_n) is a sequence of real numbers greater than 1 such that $\lim_{n\to\infty} T_n = \infty$, that will be specified later. We truncate the variables at the level T_n. So we consider

$$X_i' = \varphi_{T_n}(X_i) - \mathbb{E}\varphi_{T_n}(X_i) , \; W_i' = X_i - \varphi_{T_n}(X_i) \text{ and } X_i'' = X_i - X_i' ,$$

where we recall that $\varphi_T(x) = (x \wedge T) \vee (-T)$. Let $S'_n = \sum_{i=1}^n X'_i$ and $S''_n = \sum_{i=1}^n X''_i$. To prove the result, by exponentially equivalence lemma in Dembo and Zeitouni (1998, Theorem 4.2.13. p130), it suffices to prove that for any $\eta > 0$,

$$\limsup_{n\to\infty} a_n \log \mathbb{P}\big(\frac{\sqrt{a_n}}{\sigma_n}|S''_n| \geq \eta\big) = -\infty\,, \tag{3.42}$$

and

$$\{\frac{1}{\sigma_n}S'_n\} \text{ satisfies (2.9) with the rate function } I(t) = \frac{t^2}{2}\,. \tag{3.43}$$

To prove (3.42), we first notice that $|x - \varphi_T(x)| = (|x| - T)_+$. Hence $Q_{|W'_i|} \leq (Q - T)_+$ and, denoting by V''_{T_n} the upper bound for the variance of S''_n (corresponding to V for the variance of S_n) we have, by Remark 3 ,

$$V''_{T_n} \leq \int_0^1 (Q(u) - T_n)_+^2 du + 4\sum_{k>0}\int_0^{\tau_{W'}(k)/2} (Q(G_{T_n}(v)) - T_n)_+ dv\,.$$

where G_T is the inverse function of $x \to \int_0^x (Q(u) - T)_+ du$ and the coefficients $\tau_{W'}(k)$ are the τ-mixing coefficients associated to $(W'_i)_i$. Next, since $x \to x - \varphi_T(x)$ is 1-Lipschitz, we have that $\tau_{W'}(k) \leq \tau_X(k) = \tau(k)$. Moreover, $G_T \geq G$, because $(Q - T)_+ \leq Q$. Since Q is nonincreasing, it follows that

$$V''_{T_n} \leq \int_0^1 (Q(u) - T_n)_+^2 du + 4\sum_{k>0}\int_0^{\tau(k)/2} (Q(G(v)) - T_n)_+ dv.$$

Hence

$$\lim_{n\to+\infty} V''_{T_n} = 0. \tag{3.44}$$

The sequence (X''_i) satisfies (2.6) and we now prove that it satisfies also (2.7) for n large enough. With this aim, we first notice that

$$\big|\mathbb{E}(\varphi_{T_n}(X_i))\big| \leq \mathbb{E}|W'_i| \leq \int_{T_n}^\infty H(x)dx < b \text{ for } n \text{ large enough.}$$

Hence for n large enough, $|X''_i| \leq (|X_i| - T_n)_+ + b \leq b \vee |X_i|$ provided that $T_n \geq b$. Then for n large enough, the sequence (X''_i) satisfies (2.7) and we can apply Theorem 1 to the sequence (X''_i): for any $\eta > 0$, and n large enough

$$\mathbb{P}\Big(\sqrt{\frac{a_n}{\sigma_n^2}}|S''_n| \geq \eta\Big) \leq n\exp\Big(-\frac{\eta^\gamma\sigma_n^\gamma}{C_1 a_n^{\frac{\gamma}{2}}}\Big) + \exp\Big(-\frac{\eta^2\sigma_n^2}{C_2 a_n(1 + nV''_{T_n})}\Big) + \exp\Big(-\frac{\eta^2\sigma_n^2}{C_3 n a_n}\exp\Big(\frac{\eta^\delta\sigma_n^\delta}{C_4 a_n^{\frac{\delta}{2}}}\Big)\Big)\,,$$

where $\delta = \gamma(1-\gamma)/2$. This proves (3.42), since $a_n \to 0$, $a_n n^{\gamma/(2-\gamma)} \to \infty$, $\lim_{n\to\infty} V''_{T_n} = 0$ and $\liminf_{n\to\infty} \sigma_n^2/n > 0$.

We turn now to the proof of (3.43). Let $p_n = [n^{1/(2-\gamma)}]$ and $q_n = \delta_n p_n$ where δ_n is a sequence of integers tending to zero and such that $\delta_n^{\gamma_1} n^{\gamma_1/(2-\gamma)}/\log n \to \infty$ and $\delta_n^{\gamma_1} a_n n^{\gamma_1/(2-\gamma)} \to \infty$ (this is always possible since $\gamma_1 \geq \gamma$ and by assumption $a_n n^{\gamma/(2-\gamma)} \to \infty$). Let now $m_n = [n/(p_n + q_n)]$. We divide the variables $\{X_i'\}$ in big blocks of size p_n and small blocks of size q_n, in the following way: Let us set for all $1 \leq j \leq m_n$,

$$Y_{j,n} = \sum_{i=(j-1)(p_n+q_n)+1}^{(j-1)(p_n+q_n)+p_n} X_i' \quad \text{and} \quad Z_{j,n} = \sum_{i=(j-1)(p_n+q_n)+p_n+1}^{j\,(p_n+q_n)} X_i' \,.$$

Then we have the following decomposition:

$$S_n' = \sum_{j=1}^{m_n} Y_{j,n} + \sum_{j=1}^{m_n} Z_{j,n} + \sum_{i=m_n(p_n+q_n)+1}^{n} X_i' \,. \tag{3.45}$$

For any $j = 1, \dots, m_n$, let now $I(n,j) = \{(j-1)(p_n+q_n)+1, \dots, (j-1)(p_n+q_n)+p_n\}$. These intervals are of cardinal p_n. Let $\ell_n = \inf\{k \in \mathbb{N}^*, 2^k \geq \varepsilon_n^{-1} p_n^{\gamma/2} a_n^{-1/2}\}$, where ε_n a sequence of positive numbers tending to zero and satisfying

$$\varepsilon_n^2 a_n n^{\gamma/(2-\gamma)} \to \infty. \tag{3.46}$$

For each $j \in \{1, \dots, m_n\}$, we construct discrete Cantor sets, $K_{I(n,j)}^{(\ell_n)}$, as described in the proof of Proposition 1 with $A = p_n$, $\ell = \ell_n$, and the following selection of c_0,

$$c_0 = \frac{\varepsilon_n}{1+\varepsilon_n} \frac{2^{(1-\gamma)/\gamma} - 1}{2^{1/\gamma} - 1} \,.$$

Notice that clearly with the selections of p_n and ℓ_n, $p_n 2^{-\ell_n} \to \infty$. In addition with the selection of c_0 we get that for any $1 \leq j \leq m_n$,

$$\mathrm{Card}(K_{I(n,j)}^{(\ell_n)})^c \leq \frac{\varepsilon_n p_n}{1+\varepsilon_n}$$

and

$$K_{I(n,j)}^{(\ell_n)} = \bigcup_{i=1}^{2^{\ell_n}} I_{\ell_n,i}(p_n, j) \,,$$

where the $I_{\ell_n,i}(p_n, j)$ are disjoint sets of consecutive integers, each of same cardinal such that

$$\frac{p_n}{2^{\ell_n}(1+\varepsilon_n)} \leq \mathrm{Card} I_{\ell_n,i}(p_n, j) \leq \frac{p_n}{2^{\ell_n}} \,. \tag{3.47}$$

With this notation, we derive that

$$\sum_{j=1}^{m_n} Y_{j,n} = \sum_{j=1}^{m_n} S'\big(K_{I(n,j)}^{(\ell_n)}\big) + \sum_{j=1}^{m_n} S'\big((K_{I(n,j)}^{(\ell_n)})^c\big) \,. \tag{3.48}$$

Combining (3.45) with (3.48), we can rewrite S'_n as follows

$$S'_n = \sum_{j=1}^{m_n} S'\big(K^{(\ell_n)}_{I(n,j)}\big) + \sum_{k=1}^{r_n} \tilde{X}_i\,, \tag{3.49}$$

where $r_n = n - m_n \mathrm{Card} K^{(\ell_n)}_{I(n,1)}$ and the $\tilde{X}_i$ are obtained by renumbering the X'_i and satisfy (2.6) and (2.7) with the same constants. Since $r_n = o(n)$, applying Theorem 1 and using the fact that $\liminf_{n\to\infty} \sigma_n^2/n > 0$, we get that for any $\eta > 0$,

$$\limsup_{n\to\infty} a_n \log \mathbb{P}\Big(\frac{\sqrt{a_n}}{\sigma_n} \sum_{k=1}^{r_n} \tilde{X}_i \geq \eta\Big) = -\infty\,. \tag{3.50}$$

Hence to prove (3.43), it suffices to prove that

$$\{\sigma_n^{-1} \sum_{j=1}^{m_n} S'\big(K^{(\ell_n)}_{I(n,j)}\big)\} \text{ satisfies (2.9) with the rate function } I(t) = t^2/2. \tag{3.51}$$

With this aim, we choose now $T_n = \varepsilon_n^{-1/2}$ where ε_n is defined by (3.46).

By using Lemma 5 in Dedecker and Prieur (2004), we get the existence of independent random variables $\big(S^*(K^{(\ell_n)}_{I(n,j)})\big)_{1\leq j\leq m_n}$ with the same distribution as the random variables $S'(K^{(\ell_n)}_{I(n,j)})$ such that

$$\sum_{j=1}^{m_n} \mathbb{E}|S'(K^{(\ell_n)}_{I(n,j)}) - S^*(K^{(\ell_n)}_{I(n,j)})| \leq \tau(q_n) \sum_{j=1}^{m_n} \mathrm{Card} K^{(\ell_n)}_{I(n,j)}\,.$$

Consequently, since $\sum_{j=1}^{m_n} \mathrm{Card} K^{(\ell_n)}_{I(n,j)} \leq n$, we derive that for any $\eta > 0$ and any $n \geq 2^{2-\gamma}$,

$$a_n \log \mathbb{P}\Big(\frac{\sqrt{a_n}}{\sigma_n} \Big|\sum_{j=1}^{m_n} (S'(K^{(\ell_n)}_{I(n,j)}) - S^*(K^{(\ell_n)}_{I(n,j)}))\Big| \geq \eta\Big) \leq a_n \log\Big(\frac{an\sqrt{a_n}}{\eta\sigma_n} \exp\Big(-\frac{c\delta_n^{\gamma_1} n^{\gamma_1/(2-\gamma)}}{2^{\gamma_1}}\Big)\Big)\,,$$

which tends to $-\infty$ by the fact that $\liminf_n \sigma_n^2/n > 0$ and the selection of δ_n. Hence the proof of the MDP for $\{\sigma_n^{-1}\sum_{j=1}^{m_n} S'\big(K^{(\ell_n)}_{I(n,j)}\big)\}$ is reduced to proving the MDP for $\{\sigma_n^{-1}\sum_{j=1}^{m_n} S^*\big(K^{(\ell_n)}_{I(n,j)}\big)\}$. By Ellis Theorem, to prove (3.51) it remains then to show that, for any real t,

$$a_n \sum_{j=1}^{m_n} \log \mathbb{E} \exp\Big(tS'\big(K^{(\ell_n)}_{I(n,j)}\big)/\sqrt{a_n\sigma_n^2}\Big) \to \frac{t^2}{2} \text{ as } n \to \infty\,. \tag{3.52}$$

As in the proof of Proposition 1, we decorrelate step by step. Using Lemma 2 and taking into account the fact that the variables are centered together with the inequality (3.11), we obtain, proceeding as in the proof of Proposition 1, that for any real t,

$$\Big|\sum_{j=1}^{m_n} \log \mathbb{E} \exp\Big(tS'\big(K^{(\ell_n)}_{I(n,j)}\big)/\sqrt{a_n\sigma_n^2}\Big) - \sum_{j=1}^{m_n}\sum_{i=1}^{2^{\ell_n}} \log \mathbb{E}\exp\Big(tS'\big(I_{\ell_n,i}(p_n,j)\big)/\sqrt{a_n\sigma_n^2}\Big)\Big| \leq$$

$$\frac{a|t|m_n p_n}{\sqrt{a_n\sigma_n^2}}\Big(\exp\Big(-c\frac{c_0^{\gamma_1}}{4}\frac{p_n^{\gamma_1}}{2^{\ell_n\gamma_1}} + 2\frac{|t|}{\sqrt{\varepsilon_n a_n\sigma_n^2}}\frac{p_n}{2^{\gamma\ell_n}}\Big) + \sum_{j=0}^{k_{\ell_n}} \exp\Big(-c\frac{c_0^{\gamma_1}}{4}\frac{p_n^{\gamma_1}}{2^{j\gamma_1/\gamma}} + 2\frac{|t|}{\sqrt{\varepsilon_n a_n\sigma_n^2}}\frac{p_n}{2^j}\Big)\Big)\,,$$

where $k_{\ell_n} = \sup\{j \in \mathbb{N}\,,\, j/\gamma < \ell_n\}$. By the selection of p_n and ℓ_n, since $\liminf_{n\to\infty} \sigma_n^2/n > 0$ and $\varepsilon_n^2 a_n n^{\gamma/(2-\gamma)} \to \infty$, we get that $2^{-\gamma_1 \ell_n} 2^{\gamma \ell_n} p_n^{\gamma_1 - 1} \sqrt{\varepsilon_n a_n \sigma_n^2}$ tends to ∞ as n goes to ∞. Consequently for n large enough, there exist positive constants K_1 and K_2 depending on c, γ and γ_1 such that

$$a_n \Big| \sum_{j=1}^{m_n} \log \mathbb{E} \exp \Big(tS'\big(K_{I(n,j)}^{(\ell_n)}\big)/\sqrt{a_n \sigma_n^2} \Big) - \sum_{j=1}^{m_n} \sum_{i=1}^{2^{\ell_n}} \log \mathbb{E} \exp \Big(tS'\big(I_{\ell_n,i}(p_n,j)\big)/\sqrt{a_n\sigma_n^2}\Big) \Big|$$
$$\leq aK_1 |t| \sqrt{a_n n} \log(n) \exp\Big(-K_2 \big(\varepsilon_n^2 a_n n^{\gamma/(2-\gamma)}\big)^{\gamma/2} n^{\gamma(1-\gamma)/(2-\gamma)} \Big)\,, \tag{3.53}$$

which converges to zero by the selection of ε_n.

Hence (3.52) will hold if we prove that for any real t

$$a_n \sum_{j=1}^{m_n} \sum_{k=1}^{2^{\ell_n}} \log \mathbb{E} \exp \Big(tS'\big(I_{\ell_n,i}(p_n,j)/\sqrt{a_n \sigma_n^2}\Big) \to \frac{t^2}{2} \text{ as } n \to \infty\,. \tag{3.54}$$

With this aim, we first notice that, by the selection of ℓ_n and the fact that $\varepsilon_n \to 0$,

$$\|S'\big(I_{\ell_n,i}(p_n,j)\|_\infty \leq 2T_n 2^{-\ell_n} p_n = o(\sqrt{na_n}) = o(\sqrt{\sigma_n^2 a_n})\,. \tag{3.55}$$

In addition, since $\lim_n V''_{T_n} = 0$ and the fact that $\liminf_n \sigma_n^2/n > 0$, we have $\lim_n \sigma_n^{-2} \mathrm{Var} S'_n = 1$. Notice that by (3.49) and the fact that $r_n = o(n)$,

$$\mathrm{Var} S'_n = \mathbb{E}\Big(\sum_{j=1}^{m_n} \sum_{i=1}^{2^{\ell_n}} S'(I_{\ell_n,i}(p_n,j)) \Big)^2 + o(n) \text{ as } n \to \infty\,.$$

Also, straightforward computations as in the proof of Remark 3 show that under (2.6) and (2.7),

$$\mathbb{E}\Big(\sum_{j=1}^{m_n} \sum_{i=1}^{2^{\ell_n}} \big(S'\big(I_{\ell_n,i}(p_n,j)\big)\Big)^2 = \sum_{j=1}^{m_n} \sum_{i=1}^{2^{\ell_n}} \mathbb{E}\big(S'^2\big(I_{\ell_n,i}(p_n,j)\big)\big) + o(n) \text{ as } n \to \infty\,.$$

Hence

$$\lim_{n\to\infty} (\sigma_n)^{-2} \sum_{j=1}^{m_n} \sum_{i=1}^{2^{\ell_n}} \mathbb{E}\big(S'^2\big(I_{\ell_n,i}(p_n,j)\big)\big) = 1\,. \tag{3.56}$$

Consequently (3.54) holds by taking into account (3.55) and (3.56) and by using Lemma 2.3 in Arcones (2003). ⋄

4 Appendix

4.0.1 Technical lemmas

We first give the following decoupling inequality.

Lemma 2. *Let Y_1, ..., Y_p be real-valued random variables each a.s. bounded by M. For every $i \in [1, p]$, let $\mathcal{M}_i = \sigma(Y_1, ..., Y_i)$ and for $i \geq 2$, let Y_i^* be a random variable independent of $\mathcal{M}_{i-1}$ and distributed as Y_i. Then for any real t,*

$$|\mathbb{E}\exp\Big(t\sum_{i=1}^{p} Y_i\Big) - \prod_{i=1}^{p}\mathbb{E}\exp(tY_i)| \leq |t|\exp(|t|Mp)\sum_{i=2}^{p}\mathbb{E}|Y_i - Y_i^*|\,.$$

In particular, we have for any real t,

$$|\mathbb{E}\exp\Big(t\sum_{i=1}^{p} Y_i\Big) - \prod_{i=1}^{p}\mathbb{E}\exp(tY_i)| \leq |t|\exp(|t|Mp)\sum_{i=2}^{p}\tau(\sigma(Y_1, \dots, Y_{i-1}), Y_i)\,,$$

where τ is defined by (2.2).

Proof of Lemma 2. Set $U_k = Y_1 + Y_2 + \cdots + Y_k$. We first notice that

$$\mathbb{E}\big(e^{tU_p}\big) - \prod_{i=1}^{p}\mathbb{E}\big(e^{tY_i}\big) = \sum_{k=2}^{p}\Big(\mathbb{E}\big(e^{tU_k}\big) - \mathbb{E}\big(e^{tU_{k-1}}\big)\mathbb{E}\big(e^{tY_k}\big)\Big)\prod_{i=k+1}^{p}\mathbb{E}\big(e^{tY_i}\big) \tag{4.1}$$

with the convention that the product from $p+1$ to p has value 1. Now

$$|\mathbb{E}\exp(tU_k) - \mathbb{E}\exp(tU_{k-1})\mathbb{E}\exp(tY_k)| \leq \|\exp(tU_{k-1})\|_\infty \|\mathbb{E}\big(e^{tY_k} - e^{tY_k^*}|\mathcal{M}_{k-1}\big)\|_1\,.$$

Using (3.12) we then derive that

$$|\mathbb{E}\exp(tU_k) - \mathbb{E}\exp(tU_{k-1})\mathbb{E}\exp(tY_k)| \leq |t|\exp(|t|kM)\|Y_k - Y_k^*\|_1\,. \tag{4.2}$$

Since the variables are bounded by M, starting from (4.1) and using (4.2), the result follows.$\diamond$

One of the tools we use repeatedly is the technical lemma below, which provides bounds for the log-Laplace transform of any sum of real-valued random variables.

Lemma 3. *Let $Z_0, Z_1, \dots$ be a sequence of real valued random variables. Assume that there exist positive constants $\sigma_0, \sigma_1, \dots$ and $c_0, c_1, \dots$ such that, for any positive i and any t in $[0, 1/c_i[$,*

$$\log\mathbb{E}\exp(tZ_i) \leq (\sigma_i t)^2/(1 - c_i t)\,.$$

Then, for any positive n and any t in $[0, 1/(c_0 + c_1 + \cdots + c_n)[$,

$$\log\mathbb{E}\exp(t(Z_0 + Z_1 + \cdots + Z_n)) \leq (\sigma t)^2/(1 - Ct),$$

where $\sigma = \sigma_0 + \sigma_1 + \cdots + \sigma_n$ and $C = c_0 + c_1 + \cdots + c_n$.

Proof of Lemma 3. Lemma 3 follows from the case $n = 1$ by induction on n. Let L be the log-Laplace of $Z_0 + Z_1$. Define the functions γ_i by

$$\gamma_i(t) = (\sigma_i t)^2/(1 - c_i t) \text{ for } t \in [0, 1/c_i[\text{ and } \gamma_i(t) = +\infty \text{ for } t \geq 1/c_i.$$

For u in $]0, 1[$, let $\gamma_u(t) = u\gamma_1(t/u) + (1 - u)\gamma_0(t/(1 - u))$. From the Hölder inequality applied with $p = 1/u$ and $q = 1/(1 - u)$, we get that $L(t) \leq \gamma_u(t)$ for any nonnegative t. Now, for t in $[0, 1/C[$, choose $u = (\sigma_1/\sigma)(1 - Ct) + c_1 t$ (here $C = c_0 + c_1$ and $\sigma = \sigma_0 + \sigma_1$). With this choice $1 - u = (\sigma_0/\sigma)(1 - Ct) + c_0 t$, so that u belongs to $]0, 1[$ and $L(t) \leq \gamma_u(t) = (\sigma t)^2/(1 - Ct)$, which completes the proof of Lemma 3. $\diamond$

4.0.2 Proof of Corollary 3.

The proof of Corollary 3 is based on the following proposition together with Remark 5.

Proposition 3. *Assume that $(Y_i)_{i\in\mathbb{Z}}$ belongs to $ARL(C, \delta, \eta)$. Then the probability μ satisfies*

$$\int \exp(\omega|x|^{\eta(1-\delta)})\mu(dx) < +\infty\,, \tag{4.3}$$

for any positive $\omega < \lambda(8(1 - \delta))^{-\eta}C^\eta$.
Furthermore there exist positive constants a and c such that the following bound holds for the τ-mixing coefficient associated to the stationary Markov chain $(Y_i)_{i\in\mathbb{Z}}$:

$$\tau(n) \leq a \exp(-cn^{\eta(1-\delta)/(\eta(1-\delta)+\delta)}). \tag{4.4}$$

Remark 5. *Due to the definition of τ, if h is a 1-Lipschitz function, then the τ-mixing coefficients $(\tau_{h(Y)}(n))_{n\geq 1}$ associated to the sequence $(h(Y_i))_{i\in\mathbb{Z}}$ satisfy also (4.4). On an other hand, according to Remark 2, (4.3) entails that if $|g(x)| \leq c(1 + |x|^\zeta)$ the process $(g(Y_i) - \mathbb{E}(g(Y_i)))_{i\in\mathbb{Z}}$ satisfies (2.7) for some $b > 0$ and $\gamma_2 = \eta(1 - \delta)/\zeta$.*

Proof of Proposition 3.

We leave to the reader the case $\delta = 0$, which is easy to treat. Throughout $\delta > 0$. We start by proving (4.3). Let K be the transition kernel of the stationary Markov chain $(Y_i)_{i\in\mathbb{Z}}$ belonging to $ARL(C, \delta, \eta)$. For $n > 0$, we write $K^n g$ for the function $\int g(y)K^n(x, dy)$.

Let $k_0 = \inf\{k \in \mathbb{N}^* : k\eta(1 - \delta) \geq 1\}$. To prove (4.3) it suffices to show that

$$\sum_{k\geq k_0} \frac{\omega^k}{k!} \int |x|^{k\eta(1-\delta)}\mu(dx) < +\infty\,. \tag{4.5}$$

Let $S = k\eta(1-\delta)+\delta$ and $V(x) = |x|^S$. Arguing as in the proof of Proposition 2 of Dedecker and Rio (2000), we get that

$$\frac{[KV(x)]^{1/S}}{|x|} \leq 1 + \frac{1}{|x|^\delta}\left(\frac{C}{(1-\delta)}\left[\frac{1}{|x|^{1-\delta}} - \left(1+\frac{1}{|x|}\right)^{1-\delta} + \frac{(1-\delta)\|\varepsilon_0\|_S}{C|x|^{1-\delta}}\right]\right).$$

Then for $|x| \geq R_S = \big(2C^{-1}(1-\delta)\|\varepsilon_0\|_S + 2\big)^{1/(1-\delta)}$,

$$|x|^{-1}[KV(x)]^{1/S} \leq 1 - C(2-2\delta)^{-1}|x|^{-\delta}.$$

On the other hand, for $x \in [-R_S, R_S]$, $[KV(x)]^{1/S} \leq |f(x)| + \|\varepsilon_0\|_S \leq R_S + \|\varepsilon_0\|_S$. Furthermore, $\|\varepsilon_0\|_S \leq (\delta + (1-\delta)\|\varepsilon_0\|_S)^{1/(1-\delta)} \leq R_S$. Therefore, for any $S \geq 1$,

$$KV(x) \leq V(x) - C(2-2\delta)^{-1}|x|^{S-\delta} + 2^S R_S^S.$$

Integrating w.r.t. μ this inequality gives for $S \geq 1$,

$$\mu(KV) \leq \mu(V) - C(2-2\delta)^{-1}\int |x|^{S-\delta}\mu(dx) + 2^S R_S^S.$$

Since $\mu(KV) = \mu(V)$, it follows that for $S \geq 1$,

$$\frac{C}{2(1-\delta)}\int |x|^{S-\delta}\mu(dx) \leq 2^S R_S^S \leq 2^{S/(1-\delta)} R_S^S.$$

Now

$$R_S^S \leq 4^{S/(1-\delta)}\left(1 + \Big(\frac{4(1-\delta)}{C}\|\varepsilon_0\|_S\Big)^{S/(1-\delta)}\right) \leq 4^{S/(1-\delta)} + \Big(\frac{4(1-\delta)}{C}\Big)^{S/(1-\delta)}\mathbb{E}(|\varepsilon_0|^{S/(1-\delta)}).$$

For $k \geq k_0$ and $S = k\eta(1-\delta)+\delta$, we get that

$$\frac{C}{2(1-\delta)}\int |x|^{S-\delta}\mu(dx) \leq 8^{k\eta+\beta} + \Big(\frac{8(1-\delta)}{C}\Big)^{k\eta+\beta}\mathbb{E}(|\varepsilon_0|^{k\eta+\beta}),$$

where $\beta = \delta/(1-\delta)$. Hence

$$\frac{C}{2(1-\delta)}\sum_{k\geq k_0}\frac{\omega^k}{k!}\int |x|^{k\eta(1-\delta)}\mu(dx)$$
$$\leq 8^\beta \exp(\omega 8^\eta) + (8C^{-1}(1-\delta))^\beta\mathbb{E}\Big(|\varepsilon_0|^\beta \exp\big(\omega(8C^{-1}(1-\delta))^\eta|\varepsilon_0|^\eta\big)\Big) < \infty,$$

provided that $\omega < \lambda(8(1-\delta))^{-\eta}C^\eta$. Consequently (4.5) holds and so does (4.3).

We turn now to the proof of (4.4). We denote by $(Y_n^x)_{n\geq 0}$ the chain starting from $Y_0 = x$. According to Inequality (3.5) in Dedecker and Prieur (2004), for $j_k > \cdots > j_1 > 0$,

$$\tau(\sigma(Y_0),(Y_{j_1},\cdots,Y_{j_k})) \leq \sum_{l=1}^k \int\int \|Y_{j_l}^x - Y_{j_l}^y\|_1\mu(dx)\mu(dy). \tag{4.6}$$

Now we use the same scheme of proof than the one of Proposition 3 in Dedecker and Rio (2000). Since $|Y_n^x - Y_n^y| = |f(Y_{n-1}^x) - f(Y_{n-1}^y)|$, we have

$$|Y_n^x - Y_n^y| \le \Big(1 - \frac{C}{(1+\max(|Y_{n-1}^x|, |Y_{n-1}^y|))^\delta}\Big) |Y_{n-1}^x - Y_{n-1}^y| \,. \tag{4.7}$$

Set $\alpha(t) = 1 - C(1+t)^{-\delta}$ and $\Sigma_k = |\varepsilon_1| + \cdots + |\varepsilon_k|$. Noting that $\max(|Y_{n-1}^x|, |Y_{n-1}^y|) \le |x| + |y| + \Sigma_{n-1}$, and iterating (4.7) n times, we get

$$|Y_n^x - Y_n^y| \le \alpha^n(|x| + |y| + \Sigma_{n-1})|x-y|.$$

Then setting $I_n(x,y) := \mathbb{E}(\alpha^n(|x|+|y|+\Sigma_{n-1}))$, we get

$$\tau(\sigma(Y_0), (Y_{j_1}, \cdots, Y_{j_k})) \le \sum_{l=1}^k \int\int I_{j_l}(x,y)\mu(dx)\mu(dy) \,. \tag{4.8}$$

Setting $\Gamma_{n-1} = \Sigma_{n-1} - (n-1)\mathbb{E}|\varepsilon_0|$, we first write:

$$I_n(x,y) = n\int_0^1 \mathbb{P}\Big(\Gamma_{n-1} > [C/u]^{1/\delta} - [1 + |x| + |y| + (n-1)\mathbb{E}|\varepsilon_0|]\Big)(1-u)^{n-1}du.$$

Set $A_n(x,y) = C[2(1+|x|+|y|+(n-1)\mathbb{E}|\varepsilon_0|)]^{-\delta}$. We have

$$\begin{aligned} I_n(x,y) &\le (1 - A_n(x,y))^n + n\int_0^{A_n(x,y)} \mathbb{P}\left(2\Gamma_{n-1} > [C/u]^{1/\delta}\right)(1-u)^{n-1}du \\ &:= I_n^{(1)}(x,y) + I_n^{(2)}(x,y) \end{aligned} \tag{4.9}$$

We first control $\int\int I_n^{(1)}(x,y)|x-y|\mu(dx)\mu(dy)$. Set $B_n(x) = C[4(1+|x|+(n-1)\mathbb{E}|\varepsilon_0|)]^{-\delta}$. Then we have

$$\begin{aligned} \int I_n^{(1)}(x,y)|x-y|\mu(dy) &\le (1 - B_n(x))^n \int |x-y|\mu(dy) \\ &\quad + n\int_0^{B_n(x)} \Big[\int |x-y| \mathbb{1}_{(4|y|>[C/u]^{1/\delta})}\mu(dy)\Big](1-u)^{n-1}du. \end{aligned}$$

Let $0 < \omega < \lambda(8(1-\delta))^{-\eta}C^\eta$. Since by (4.3), $x \to \exp(\omega|x|^{\eta(1-\delta)})$ belongs to $\mathbb{L}^1(\mu)$, we have the finite upper bound

$$\int |x-y|\mathbb{1}_{(4|y|>[C/u]^{1/\delta})}\mu(dy) \le \exp\Big(-\frac{\omega}{2}\big(\frac{C}{4^\delta u}\big)^{\eta(1-\delta)/\delta}\Big)\int |x-y|\exp\Big(\frac{\omega}{2}|y|^{\eta(1-\delta)}\Big)\mu(dy).$$

Consequently there exists $c > 0$ such that, for any positive n,

$$\begin{aligned} \int I_n^{(1)}(x,y)|x-y|\mu(dy) &\le (1 - B_n(x))^n \int |x-y|\mu(dy) \\ &\quad + \exp(-cn^{\eta(1-\delta)/(\eta(1-\delta)+\delta)})\int |x-y|\exp\Big(\frac{\omega}{2}|y|^{\eta(1-\delta)}\Big)\mu(dy). \end{aligned} \tag{4.10}$$

Notice that that $(1 - B_n(x))^n \le \exp(-nB_n(x))$. If $|x| \le n\|\varepsilon_0\|_1$,

$$\exp(-nB_n(x)) \le \exp(-Cn(4 + 8n\|\varepsilon_0\|_1)^{-\delta}).$$

Therefore there exists $c > 0$, such that

$$\int_0^{n\|\varepsilon_0\|_1} (1 - B_n(x))^n \int |x - y|\mu(dy)\mu(dx) \le \exp(-cn^{(1-\delta)}) \int |x - y|\mu(dy)\mu(dx)\,. \qquad (4.11)$$

On the other hand, using again the fact that $x \to \exp(\omega|x|^{\eta(1-\delta)})$ belongs to $\mathbb{L}^1(\mu)$, we have the finite upper bound

$$\begin{aligned}
&\int_{|x|>n\|\varepsilon_0\|_1} |x|(1 - B_n(x))^n\mu(dx) \\
&\le \int_{n\|\varepsilon_0\|_1}^{\infty} |x| \exp\Big(\frac{\omega}{2}|x|^{\eta(1-\delta)}\Big) \exp\Big(-\frac{\omega}{2}|x|^{\eta(1-\delta)} - Cn(1 + 2|x|)^{-\delta}\Big)\mu(dx) \\
&\le 2\exp(-cn^{\eta(1-\delta)/(\eta(1-\delta)+\delta)}) \int_0^{\infty} |x| \exp\Big(\frac{\omega}{2}|x|^{\eta(1-\delta)}\Big)\mu(dx) \qquad (4.12)
\end{aligned}$$

where c is a positive constant. Starting from (4.10) and using (4.11) and (4.12), we derive that there exist positive constants c and K such that

$$\int\int I_n^{(1)}(x, y)|x - y|\mu(dy)\mu(dx) \le K \exp(-cn^{\eta(1-\delta)/(\eta(1-\delta)+\delta)}) \qquad (4.13)$$

We control now $\int\int I_n^{(2)}(x, y)|x - y|\mu(dx)\mu(dy)$. Using Borovkov inequality (1.4), there exists positive constants c_1 and c_2 depending on C, δ and η such that

$$\mathbb{P}\left(2\Gamma_{n-1} > [C/u]^{1/\delta}\right) \le \exp\Big(-c_1/(nu^{2/\delta})\Big) + n\exp\Big(-c_2(1/u)^{\eta/\delta}\Big)\,.$$

Consequently, there exists a positive constant $c_3 > 0$ such that

$$\int_0^1 \mathbb{P}\left(2\Gamma_{n-1} > [C/u]^{1/\delta}\right)(1 - u)^{n-1}du \le 2\exp(-c_3 n^{\min\left(\frac{\eta}{\eta+\delta}, \frac{2-\delta}{2+\delta}\right)})\,.$$

Since $\eta \in [0, 1]$ and $\delta > 0$, $\min\left(\frac{\eta}{\eta+\delta}, \frac{2-\delta}{2+\delta}\right) \ge \eta(1 - \delta)/(\eta(1 - \delta) + \delta)$. Then there exists $c > 0$ such that

$$\int\int I_n^{(2)}(x, y)|x - y|\mu(dy)\mu(dx) \le \exp(-cn^{\eta(1-\delta)/(\eta(1-\delta)+\delta)}) \int |x - y|\mu(dy)\mu(dx)\,. \qquad (4.14)$$

Starting from (4.8) and combining (4.9) and (4.13) and (4.14), we get that there exist positive constants a and c such that

$$\tau(\sigma(Y_0), (Y_{j_1}, \cdots, Y_{j_k})) \le ak\exp(-cj_1^{\eta(1-\delta)/(\eta(1-\delta)+\delta)})\,,$$

which combined with the definition (2.3) ends the proof of (4.4). $\diamond$

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