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On the Weak Invariance Principle for Stationary Sequences under Projective Criteria

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In this paper, we study the central limit theorem and its weak invariance principle for sums of a stationary sequence of random variables, via a martingale decomposition. Our conditions involve the conditional expectation of sums of random variables with respect to the distant past. The results contribute to the clarification of the central limit question for stationary sequences.

KEY WORDS: Central limit theorem; weak invariance principle; projective criteria; strong mixing sequences; martingale approximation.

MATHEMATICS SUBJECT CLASSIFICATIONS 1991: 60 F 05; 60 F 17.

1. INTRODUCTION

In the recent years, there has been a sustained effort towards a better understanding of the asymptotic behavior of stochastic processes.

For processes with short memory the theory of the weak invariance principle is very well fine tuned under various mixing condition (see the surveys by Peligrad,(27) Philipp,(30) Rio,(32) Dehling and Philipp,(11) and books by Bradley,(6,7)) among others. The only problem is that, in some

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situations, the mixing conditions are not verified. This is the reason why one of the new directions in modeling the dependence is to introduce new dependent structures defined by substantially reducing the classes of functions used in the definitions of mixing coefficients or by using innovative martingale-like conditions. In this way many new examples are included in the general structures and many general results can be established (see Ango Nzé and Doukhan[2] for such examples). Our paper is an effort in this direction. We obtain the central limit theorem and its invariance principles under summability conditions imposed to the conditional expectation of consecutive sums of random variables with respect to the distant past. For the sake of applications we shall apply our general results to obtain the invariance principle under conditions imposed to the moments of the conditional expectation of the individual summands. Our results will extend the best results known so far for strong mixing sequences and they will also provide some new insight into this class. Our method will also provide an alternative proof for Dedecker and Rio[8] result, and will give an extension of their invariance principle under a milder assumption and a more general normalization.

In this paper, we consider a strictly stationary sequence $(X_k)_{k\in Z}$ of centered random variables, with finite second moment and we shall address the central limit question and its invariance principle; namely we want to find a sequence of positive numbers $b_1, b_2, \dots$ with $b_n \to \infty$, and conditions ensuring that

$$\frac{S_n}{b_n} \xrightarrow{\mathcal{D}} N \sim \mathcal{N}(0,1), \quad \text{as } n \to \infty, \tag{1.1}$$

where $S_n = \sum_{i=1}^{n} X_i$, and for $t \in [0,1]$

$$\frac{S_{[nt]}}{b_n} \xrightarrow{\mathcal{D}} W(t), \quad \text{as } n \to \infty, \tag{1.2}$$

where W is the standard Brownian motion on [0, 1].
There is a vast literature on this subject. A certain restriction of the dependence structure is needed since, in general, a constant sequence obviously does not satisfy (1.1). Moreover also ergodicity or mixing in the ergodic sense are not sufficient. The classes of stochastic processes widely studied are martingales, uniformly mixing sequences, mixingales, associated sequences and so on.

Most of the results are in a certain sense of the form: Under certain dependence conditions, if

$$\lim_{n\to\infty} \frac{\mathrm{Var}(S_n)}{n} = \sigma^2 < \infty \tag{1.3}$$

and provided

$$\sigma^2 \neq 0, \tag{1.4}$$

then (for centered random variables)

$$\frac{S_n}{\sigma\sqrt{n}} \xrightarrow{\mathcal{D}} N \sim \mathcal{N}(0,1), \quad \text{as } n \to \infty. \tag{1.5}$$

There are several papers that contain important steps towards the elimination of Conditions (1.3) and (1.4). One way of doing that is to obtain CLT under different normalizations such as $b_n^2 = \mathrm{Var}(S_n)$ or $b_n = \sqrt{\frac{\pi}{2}}\mathbb{E}|S_n|$. This was contained in many papers involving dependent structures starting with a paper of Ibragimov[18] and followed by works of Dehling et al.,[12] Peligrad[28] or Wu and Woodroofe,[36] among others. Our results join the line of research that deals with the study of the central limit theorem under martingale like conditions when the variance of partial sums is not linear in n. As mentioned before this line of results goes back to Gordin[14] and was enriched by numerous authors including McLeish,[21,22] and more recently Dedecker and Rio,[8] Maxwell and Woodroofe,[20] Dedecker and Merlevède,[9] Wu and Woodroofe,[36] Peligrad and Utev,[29] among others. We considered both the normalization $b_n = \sqrt{\frac{\pi}{2}}\mathbb{E}|S_n|$ and $\sqrt{\mathrm{Var}(S_n)}$, and we obtained sharp results under conditions involving the conditional expectation of sums of random variables conditioned to the distant past.

Our paper is organized as follows. Section 2 deals with the results and several applications are given in Section 3. All the proofs are postponed in Section 4, where in particular some auxiliary results are stated and proved.

Throughout the paper, the notation $c_n \ll d_n$ means that $c_n = O(d_n)$, and the notation $c_n \sim d_n$ means that there exist two positive constants k_1 and k_2 such that $k_1 d_n \leqslant c_n \leqslant k_2 d_n$.

2. RESULTS

In this section, we present our main results that contain invariance principles for dependent structures under conditional moment conditions. The unifying element is a general theorem we present in the proofs section which is going to be exploited in two different directions to obtain sharp results under two types of normalizations. In order to formulate our results, we need further notations and definitions.

Notation 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T: \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. An element A of $\mathcal{A}$ is said to be invariant if $T(A) = A$. We denote by $\mathcal{I}$ the σ-algebra of all invariant sets. The probability $\mathbb{P}$ is ergodic if each element of $\mathcal{I}$ has measure 0 or 1. Let $\mathcal{M}_0$ be a σ-algebra of $\mathcal{A}$ satisfying

$\mathcal{M}_0 \subseteq T^{-1}(\mathcal{M}_0)$, and define the nondecreasing filtration $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ by $\mathcal{M}_i = T^{-i}(\mathcal{M}_0)$. Set $\mathcal{M}_{-\infty} = \bigcap_{n\geqslant 0} \mathcal{M}_{-n}$.

Definition 1. We define the process $\{W_n(t): t \in [0, 1]\}$ by

$$W_n(t) = \sum_{i=1}^{[nt]} X_i.$$

For each ω, $W_n(\cdot)$ is an element of the Skorohod space $D([0, 1])$ of all functions on $[0, 1]$ which have left-hand limits and continuous from the right. It is equipped with the Skorohod topology (see Billingsley,[3] Section 12). $W(t)$ denotes the standard Brownian motion on $[0, 1]$.

Definition 2. Following Definition 0.15 in Bradley,[6] we will say that a sequence $(h(n), n = 1, 2, 3, \ldots)$ of positive numbers is slowly varying in the strong sense if there exists a continuous function $f: (0, \infty) \to (0, \infty)$ such that $f(n) = h(n)$ for all $n \in \mathbb{N}$, and $f(x)$ is slowly varying as $x \to \infty$.

Our first result is in the spirit of Theorem 1 in Dedecker and Merlevède.[9] The main difference is basically that one of the conditions is imposed to the conditional expectation with respect to the distant past, condition that is easier to verify for many dependent structures:

Theorem 1. Let X_0 be a $\mathcal{M}_0$-measurable, centered real random variable such that $\mathbb{E}X_0^2 < \infty$. Define the strictly stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$. Set $S_n := \sum_{k=1}^n X_k$ and $\sigma_n^2 = \mathrm{Var}(S_n)$. Assume that

$$\sigma_n^2 = nh(n) \text{ where } h(n) \text{ is slowly varying in the strong sense,} \tag{2.1}$$

$$\frac{S_n^2}{\sigma_n^2} \text{ is uniformly integrable,} \tag{2.2}$$

$$\|\mathbb{E}(S_n | \mathcal{M}_{-n})\|_1 = o(\sigma_n) \text{ as } n \to \infty, \tag{2.3}$$

and that there exists a positive $\mathcal{M}_{-\infty}$ random variable η such that

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{S_n^2}{\sigma_n^2} | \mathcal{M}_{-n}\right) = \eta \text{ in } \mathbb{L}^1. \tag{2.4}$$

Then the random variable η satisfies $\eta = \eta \circ T$ almost surely and $\sigma_n^{-1} S_n$ converges in distribution to $\sqrt{\eta} N$, where N is a standard Gaussian *r.v.* independent of $\mathcal{I}$.

Remark 1. Notice that the random variable η defined in Theorem 1 is both the limit in $\mathbb{L}^1$ of $\sigma_n^{-2}\mathbb{E}(S_n^2|\mathcal{M}_{-\infty})$ and of $\sigma_n^{-2}\mathbb{E}(S_n^2|\mathcal{I})$.

Remark 2. If $\sigma_n^2 \to \infty$, as $n \to \infty$, the conditions (2.3) as well as (2.1) can be both replaced by

$$\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2 = o(\sigma_n) \text{ as } n \to \infty \tag{2.5}$$

(see Theorem 8.13 in Bradley[6] or also Lemma 1 in Wu and Woodroofe[36]).
In addition, under (2.2), (2.5) is also implied by: $\|\mathbb{E}(S_n|\mathcal{M}_0)\|_1 = o(\sigma_n)$ as $n \to \infty$.
The following functional version of Theorem 1 also holds:

Theorem 2. Assume that the conditions of Theorem 1 are satisfied and that in addition

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} \lambda^2 \mathbb{P}\left(\max_{1\leqslant i\leqslant n} \frac{|S_i|}{\sigma_n} \geqslant \lambda \right) = 0. \tag{2.6}$$

Then $\sigma_n^{-1} W_n$ converges in distribution to $\sqrt{\eta}\, W$ in the Skorohod space $D([0,1])$, where W is a standard Brownian motion independent of $\mathcal{I}$.

Remark 3. Clearly Conditions (2.2) and (2.6) are satisfied if

$$\frac{\max_{1\leqslant i\leqslant n} S_i^2}{\sigma_n^2} \text{ is uniformly integrable.} \tag{2.7}$$

Our next Corollary contains sufficient conditions for the invariance principle and it is obtained by verifying the conditions of Theorem 2. As we shall see, it extends the corresponding result by Dedecker and Rio[8] providing at the same time another proof of their result.

Corollary 1. Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ and $(X_i)_{i\in\mathbb{Z}}$ be as in Theorem 1. In addition assume that $\sigma_n^2 = nh(n)$ where $h(n)$ is slowly varying in the strong sense and satisfies $\lim_{n\to\infty} h(n) = c$, where c is a strictly positive constant possibly infinite, and that

$$\frac{n}{\sigma_n^2}\mathbb{E}(X_0 S_n|\mathcal{M}_0) \text{ is convergent in } \mathbb{L}^1 \text{ to a random variable } \mu \text{ as } n \to \infty. \tag{2.8}$$

Then the invariance principle of Proposition 2 holds with $\eta = c^{-1}\mathbb{E}(X_0^2|\mathcal{I}) + 2\mathbb{E}(\mu|\mathcal{I})$.

Remark 4. We point out that the condition (2.8) is not needed in its full strength and can be replaced by the couple of weaker conditions

$$\frac{n}{\sigma_n^2}\mathbb{E}(X_0 S_n|\mathcal{M}_0) \text{ is uniformly integrable,}$$

and

$$\frac{n}{\sigma_n^2}\mathbb{E}(X_0 S_n|\mathcal{M}_{-n}) \text{ is convergent in } \mathbb{L}^1.$$

Remark 5. Notice that this corollary contains the case treated in the paper of Dedecker and Rio[8] when

$$\mathbb{E}(X_0 S_n|\mathcal{M}_0) \text{ is convergent in } \mathbb{L}^1 \text{ to a random variable } \mu \text{ as } n\to\infty. \tag{2.9}$$

Under this condition it is easy to see that $\frac{\sigma_n^2}{n}$ is convergent to a non-negative number c, and according to Corollary 1, if c is strictly larger than 0, the invariance principle holds. Our corollary also contains the interesting case when $\lim_{n\to\infty}\mathbb{E}|X_0\mathbb{E}(S_n|\mathcal{M}_0)|=\infty$, and then when necessarily by (2.8), $\frac{\sigma_n^2}{n}$ is convergent to infinity.

We would like to point out that the condition of uniform integrability in Theorem 1 as well as the condition imposed to the maximum of partial sums in Theorem 2 are both sufficient conditions for the results in these two theorems under the normalization $b_n:=\sigma_n$. These conditions are not needed to be verified if we change the normalization and the remaining conditions can be imposed to the individual summands. Next theorem deals with the invariance principle under the normalization $b_n:=\sqrt{(\pi/2)}\mathbb{E}|S_n|$. In order to formulate this result, we need further definitions and notations.

Definition 3. For any integrable random variable Y, define the "upper tail" quantile function Q_Y by $Q_Y(u)=\inf\{t\geqslant 0:\mathbb{P}(|Y|>t)\leqslant u\}$. Note that, on the set $[0,\ P(|Y|>0)]$, the function $H_Y: x\to\int_0^x Q_Y(u)du$ is an absolutely continuous and increasing function with values in $[0,\ E|Y|]$. Denote by G_Y the inverse of H_Y.

Notation 2. In order to simplify the presentation of the results, we will denote by Q, G and H the respective functions Q_{X_0}, G_{X_0} and H_{X_0}, when no confusion is possible.

Theorem 3. Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ and $(X_i)_{i\in\mathbb{Z}}$ be as in Theorem 1. In addition assume that (2.1) and (2.4) holds and that

$$n^2 \int_0^{n^{-1}\|\mathbb{E}(S_n|\mathcal{M}_{-n})\|_1} Q\circ G(u)du = o\left(\sigma_n^2\right) \text{ as } n\to\infty. \tag{2.10}$$

Then $B_n^{-1}W_n$ converges in distribution to $\sqrt{\eta}\,W$, as $n\to\infty$, in the Skorohod topology, where $B_n := \sqrt{(\pi/2)}\mathbb{E}|S_n|$ and W is a standard Brownian motion independent of $\mathcal{I}$.

Reasoning as in Herrndorf[16] on page 99, let us mention that, as a consequence of the weak invariance principle stated in Theorem 3, $\mathbb{E}|S_n|$ has the representation $\sqrt{n}h'(n)$, where $(h'(n), n=1,2,3,\ldots)$ is a sequence of positive numbers which is slowly varying in the strong sense.

Condition (2.10) combines the tail of the distribution of X with the size of $\|\mathbb{E}(S_n|\mathcal{M}_{-n})\|_1$. To have a better insight into the significance of this condition, we give the following easy applications:

Corollary 2. Let $(\mathcal{M}_i)_{i\in Z}$ and $(X_i)_{i\in Z}$ be as in Theorem 1. In addition assume that (2.1) and (2.4) hold. Moreover suppose that

(i) $\mathbb{P}(|X_0|\leqslant T)=1$ for a positive number T and $\lim_{n\to\infty} n\sigma_n^{-2}\|\mathbb{E}(S_n|\mathcal{M}_{-n})\|_1=0$, or that
(ii) there exist $r>2$ and $c>0$ such that $P(|X_0|>x)\leqslant(c/x)^r$ and $\lim_{n\to\infty} n^{r/(r-1)}\sigma_n^{-2}\left(\|\mathbb{E}(S_n|\mathcal{M}_{-n})\|_1\right)^{(r-2)/(r-1)}=0$,

then the conclusion of Theorem 3 holds.

The proof is straightforward by using the definitions of the quantile function and of the function $G(\cdot)$ and is therefore omitted.

Remark 6. If X_0 is a bounded real random variable and $\liminf_{n\to\infty}\dfrac{\sigma_n^2}{n}>0$, then

$$\|\mathbb{E}(S_n|\mathcal{M}_{-n})\|_1 = o(1) \text{ as } n\to\infty$$

entails (2.10).

As another application of Theorem 3, we also have the following result:

Corollary 3. Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ and $(X_i)_{i\in\mathbb{Z}}$ be as in Theorem 1. In addition assume that $\sigma_n^2\to\infty$, (2.4) holds and that

$$\sum_{i=1}^{n} i\int_0^{\|\mathbb{E}(X_i|\mathcal{M}_0)\|_1} Q\circ G(u)du = o\left(\sigma_n^2\right) \text{ as } n\to\infty. \tag{2.11}$$

Then the conclusion of Theorem 3 holds.

In the same spirit as Corollary 2, we give the following application of Corollary 3:

Corollary 4. Let $(\mathcal{M}_i)_{i\in Z}$ and $(X_i)_{i\in Z}$ be as in Theorem 1. In addition assume that $\sigma_n^2 \to \infty$, (2.4) holds. Moreover suppose that for a positive number T

(i) $\mathbb{P}(|X_0| \leqslant T) = 1$ and $\sum_{i=1}^{n} i \,\|\mathbb{E}(X_i|\mathcal{M}_0)\|_1 = o\left(\sigma_n^2\right)$ as $n \to \infty$, or that

(ii) there exist $r > 2$ and $c > 0$ such that $P(|X_0| > x) \leqslant (c/x)^r$ and $\sum_{i=1}^{n} i \,\|\mathbb{E}(X_i|\mathcal{M}_0)\|_1^{(r-2)/(r-1)} = o\left(\sigma_n^2\right)$ as $n \to \infty$,

then the conclusion of Theorem 3 holds.

3. EXAMPLES

3.1. Strong Mixing Sequences

We shall apply the results of the previous section to the case of strongly mixing sequences. We first need some definitions.

Definition 4. For two σ-algebras $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{A}$, the strong mixing coefficient of Rosenblatt[33] is defined by $\alpha(\mathcal{U},\mathcal{V}) = \sup\{|\mathbb{P}(U \cap V) - \mathbb{P}(U)\mathbb{P}(V)| : U \in \mathcal{U},\ V \in \mathcal{V}\}$.
We say that a strictly stationary sequence, $(X_i)_{i\in\mathbb{Z}}$, is strongly mixing if

$$\lim_{n\to\infty} \alpha(\mathcal{M}_0, \sigma(X_k, k \geqslant n)) = 0. \tag{3.1}$$

An equivalent definition of the strong mixing coefficients, based on the conditional expectation, is given in Bradley,[6] Theorem 4.4, item (a2): $\alpha(\mathcal{U},\mathcal{V}) = \frac{1}{4}\sup\{\|\mathbb{E}(Y|\mathcal{U})\|_1/\|Y\|_\infty :$ where Y is $\mathcal{V}$-measurable with $\mathbb{E}(Y) = 0\}$.

As a corollary of our Theorem 1, we shall formulate first a result of Denker,[13] that is: *A strictly stationary, centered and strongly mixing sequence with $\sigma_n \to \infty$ satisfies the central limit theorem under the normalization σ_n if and only if $\left(\frac{S_n^2}{\sigma_n^2}\right)$ is an uniformly integrable family*. Indeed, if we assume that $\sigma_n \to \infty$, (3.1) and (2.2) then σ_n^2 has the representation $\sigma_n^2 = nh(n)$ where $h(n)$ is a slowly varying function in the strong sense (see Denker,[13] page 272). In addition, by a standard truncation argument along with the equivalent definition of the strong mixing coefficients, we infer that (2.2) entails both (2.3) and (2.4) with $\eta = 1$. Then the central limit theorem follows now by our theorem 1. For the necessity, a result of Ibragimov[17] (see Theorem 18.1.1 in Ibragimov and Linnik[19]), tells us that, if a strictly stationary strongly mixing sequence satisfies the central limit theorem with the normalization σ_n with $\sigma_n \to \infty$, then necessarily σ_n^2 has the representation $\sigma_n^2 = nh(n)$ where $h(n)$ is a slowly varying function in the strong sense, and (2.2) is satisfied.

Now if the coefficient of weak dependence (3.1) is weakened into the following one:

$$\alpha_{2,\infty}(n) := \sup_{k \geqslant 0} \alpha(\mathcal{M}_0, \sigma(X_n, X_{k+n})),$$

then the following result holds:

Theorem 4. Let $(\mathcal{M}_i)_{i \in Z}$ and $(X_i)_{i \in Z}$ be as in Theorem 3. Assume that $\sigma_n^2 \to \infty$ and that

$$\lim_{n \to \infty} \frac{1}{\sigma_n^2} \sum_{i=1}^{n} i \int_0^{\alpha_{2,\infty}(i)} Q^2(u) du = 0, \tag{3.2}$$

then the conclusion of Theorem 3 holds with $\eta = 1$.

It is clear that (3.2) is implied by

$$\liminf_{n \to \infty} \frac{\sigma_n^2}{n} > 0 \text{ and } \lim_{n \to \infty} n \int_0^{\alpha_{2,\infty}(n)} Q^2(u) du = 0.$$

It follows that Theorem 4 extends Theorem 1.4 of Merlevède and Peligrad[24] and also Theorem 2.2 of Merlevède[23] to the context when we do not have necessarily that $\liminf_{n \to \infty} \frac{\sigma_n^2}{n} > 0$. Moreover it is also sharper even when this last condition is imposed.

Theorem 4 is optimal in the sense that according to Theorem 2 of Bradley,[5] if we only assume that $\sigma_n^2 \to \infty$, then a rate of convergence to infinity of the series $\sum_{i=1}^{n} i \int_0^{\alpha_{2,\infty}(i)} Q^2(u) du$, is needed to obtain the central limit theorem. Moreover, by a combination of the construction from Theorem 2 in Bradley[5] with our Theorem 4, we infer that the following result holds.

Theorem 5. Suppose $\lambda(1), \lambda(2), \lambda(3), \ldots$ is a nonincreasing sequence of numbers in $(0, 1]$ such that $\lambda(n) \to 0$ as $n \to \infty$. Suppose $q : (0, 1) \to [0, \infty)$ is a nonincreasing, right-continuous function such that $\int_0^1 q^2(u) du < \infty$. Then there exists a strictly stationary sequence $(X_i)_{i \in \mathbb{Z}}$ of centered *r.v.*'s in $\mathbb{L}^2$, with $\sigma_n^2 = \operatorname{Var} S_n \to \infty$, satisfying the following properties:

(a) The quantile function of $|X_0|$, that is Q_{X_0}, coincides with q on an interval $[0,\delta)$, $\delta>0$.
(b) The strong mixing coefficients $\alpha_n := \alpha(\mathcal{M}_0, \sigma(X_k, k \geqslant n))$ of $(X_i)_{i\in Z}$ satisfy for all $n\in\mathbb{N}$: $\alpha_n \leqslant \lambda(n)$.
(c) $\limsup_{n\to\infty} \frac{1}{\sigma_n^2}\sum_{i=1}^{n} i\int_0^{\lambda(i)} Q^2(u)du > 0.$

In addition, the sequence $b_n^{-1}S_n$ cannot converge in distribution to a standard Gaussian variable, regardless of any choice of the normalizing constants, b_n.

Proof of Theorem 5. The construction of this example was given in Bradley[(5)] (see the proof of his Theorem 2). In his example the following property holds:

$$\sum_{i=1}^{\infty} i\int_0^{\lambda(i)} Q^2(u)du = \infty.$$

Suppose now by contradiction that the example satisfies

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2}\sum_{i=1}^{n} i\int_0^{\lambda(i)} Q^2(u)du = 0.$$

Then the sequence $\{X_i, i\in\mathbb{Z}\}$ from the Bradley's example has to satisfy the condition of our Theorem 4 and it should satisfy then the central limit theorem. This leads to a contradiction, and as a consequence Bradley's example has to satisfy Item (c), even with $\lambda(i)$ replaced by α_i. □

We notice that our Theorem 4 gives an additional insight on Bradley's example, and at the same time feels a portion of the small gap between the class of strong mixing sequences known to satisfy the central limit theorem and the conditions which allow the construction of counterexamples.

3.2. A Class of Linear Processes

Let $(\xi_i, i\in\mathbb{Z})$ be a strictly stationary sequence of martingale differences with finite second moment σ^2. In addition, let $(a_i, i\geqslant 0)$ be a sequence of real numbers such that $\sum_{i\geqslant 0} a_i^2 < \infty$. Then we consider the causal linear process defined by

$$X_k = \sum_{i\geqslant 0} a_i \xi_{k-i}. \tag{3.3}$$

Letting $b_n = a_0 + \cdots + a_n$, for this class of processes, the following result holds

Proposition 1. Assume $(X_k)_{k\in Z}$ is a linear process defined as above. Assume that

$$\sum_{k=0}^{n-1} b_k^2 \to \infty, \text{ as } n\to\infty, \tag{3.4}$$

and that

$$\sum_{j=0}^{\infty}(b_{n+j}-b_j)^2 = o\left(\sum_{k=0}^{n-1} b_k^2\right). \tag{3.5}$$

Then σ_n^2 satisfies (2.1) and $\sigma_n^{-1}\sum_{k=1}^n X_k$ converges in distribution to $\sqrt{\sigma^{-2}\mathbb{E}(\xi_0^2|\mathcal{I})}\,N$, where N is a standard Gaussian *r.v.* independent of $\mathcal{I}$.

Remark 7. Notice that Wu and Woodroofe[36] pointed out that (3.5) is a necessary and sufficient condition in order for (2.5) to hold for the sequence $(X_k)_{k\in Z}$.

The next example shows that the conditions (3.4) and (3.5) are not sufficient to ensure that linear processes $(X_k)_{k\in Z}$ as defined in this paper satisfy the weak invariance principle.

Example 1. There is a linear process $(X_k)_{k\in Z}$ satisfying the conditions of Proposition 1 and such that the weak invariance principle does not hold.

In order to derive the weak invariance principle for the class of linear processes studied in this section, we impose an additional condition on the moment of the random variables $(\xi_i, i\in\mathbb{Z})$.

Proposition 2. Let $(\xi_i, i\in\mathbb{Z})$ be a strictly stationary sequence of martingale differences such that $\mathbb{E}|\xi_0|^{2+\delta}<\infty$, with $\delta>0$. Let $(a_i, i\geqslant 0)$ be a sequence of real numbers such that $\sum_{i\geqslant 0} a_i^2<\infty$ and satisfying (3.4) and (3.5). Let $(\xi_i, i\in\mathbb{Z})$ be the linear process defined by (3.3). Then $\sigma_n^{-1}W_n$ converges in distribution to $\sqrt{\sigma^{-2}\mathbb{E}(\xi_0^2|\mathcal{I})}\,W$ in the Skorohod space $D([0,1])$, where W is a standard Brownian motion independent of $\mathcal{I}$.

4. PROOFS

We start this section by two general technical results that will be a building block in the proof of all the results of this paper.

4.1. Some General Results

In the spirit of Theorem 18.4.1 in Ibragimov and Linnik,[19] we first give the following result:

Proposition 3. Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ and $(X_i)_{i\in\mathbb{Z}}$ be as in Theorem 1. Assume that (2.1) holds and that there exists a sequence of integers q_n which converges to infinity, satisfying $q_n = o(n)$, as $n\to\infty$,and such that (4.1)

(a) $\sqrt{\frac{n}{q_n}}\frac{\mathbb{E}|\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})|}{\sigma_{q_n}}\to 0,\ \text{as } n\to\infty,$

(b) for any $\varepsilon>0$, $\sigma_{q_n}^{-2}\mathbb{E}\left(S_{q_n}^2\mathbb{I}\big(|S_{q_n}|\geqslant \varepsilon\sigma_{q_n}\sqrt{n/q_n}\big)\right)\to 0,\ \text{as } n\to\infty,$

(c) there exists a positive $\mathcal{M}_{-\infty}$ random variable η such that $\|\mathbb{E}(\sigma_{q_n}^{-2}S_{q_n}^2|\mathcal{M}_{-q_n})-\eta\|_1\to 0$ as $n\to\infty$.

(d) $\|\mathbb{E}(\sigma_{q_n}^{-1}S_{q_n}|\mathcal{M}_0)\|_2\to 0$ as $n\to\infty$.

Then the random variable η satisfies $\eta=\eta\circ T$ almost surely and $B_n^{-1}S_n$ converges in distribution to $\sqrt{\eta}N$ where $B_n=\sqrt{(n/q_n)}\sigma_{q_n}$ or $\sqrt{(\pi/2)}\mathbb{E}|S_n|$ and N is a standard Gaussian random variable independent of $\mathcal{I}$.

We can also derive the functional version of Proposition 3. With the notation of Proposition 3, the following result holds.

Proposition 4. Assume that the conditions of Proposition 3 are satisfied and that in addition, for all positive ε,

$$\lim_{n\to\infty}\frac{n}{q_n}\mathbb{P}\left(\max_{1\leqslant i\leqslant q_n}\frac{|\sum_{j=1}^{i}X_j|}{\sigma_{q_n}}\geqslant\varepsilon\sqrt{\frac{n}{q_n}}\right)=0. \tag{4.2}$$

Then $B_n^{-1}W_n$ converges in distribution to $\sqrt{\eta}W$ in the Skorohod space $D([0,1])$, where $B_n=\sqrt{(n/q_n)}\sigma_{q_n}$ or $\sqrt{(\pi/2)}\mathbb{E}|S_n|$ and W is a standard Brownian motion independent of $\mathcal{I}$.

According to the proof of Propositions 3 and 4, let us mention that the same conclusions also hold for the stable convergence replacing the convergence in distribution. This concept (more precise than convergence in distribution) was first introduced by Rényi,[31] and the equivalence between stability and weak- L^1 convergence of some functions of the variables was made clear by Aldous and Eagleson.[1]

Remark 8. Item (b) of Proposition 3 and Condition (4.2) are obviously both satisfied if the following condition holds: for any $\varepsilon>0$,

$$\sigma_{q_n}^{-2}\mathbb{E}\Big(\max_{1\leqslant i\leqslant q_n}S_i^2\mathbb{I}\big(\sigma_{q_n}^{-1}\max_{1\leqslant i\leqslant q_n}|S_i|\geqslant\varepsilon\sqrt{n/q_n}\big)\Big)\to 0 \text{ as } n\to\infty. \tag{4.3}$$

Also, if $\sigma_n^2 \to \infty$, as $n \to \infty$, (2.1) as well as Item (d) of (4.1) can be both replaced by (2.5).

4.1.1. Proofs of Proposition 3 and 4

In this section, we prove Proposition 4. Indeed Proposition 4 states the functional version of the result given in Proposition 3. In this sense, it is a more general result. For this reason, the proof of Proposition 3 is left to the reader.
Let m be a fixed integer and q_n as in Proposition 3. Set $p_n = m\,q_n$.

As it is suggested for instance page 243 in Philipp[30] (and exploited more recently by Shao[34] or by Merlevède,[23]) the proof is mainly based on the Bernstein-type blocking arguments, namely the index set $\{1, 2, 3, \ldots, [nt]\}$ is partitioned into an alternating sequence of "big" blocks of size p_n and "small" blocks of size q_n $\Big($in fact we will obtain k_{nt} "big" blocks and k_{nt} "small" blocks, where $k_{nt} = \left[\frac{[nt]}{(m+1)q_n}\right]\Big)$, and on martingale approximations of the "big blocks" and "small blocks" random variables. For the sake of clarity, we have divided the proof in several steps. We start with the proof of the invariance of η.

Invariance of η. We show that if (c) of (4.1) holds, the random variable η satisfies $\eta = \eta \circ T$ almost surely (or equivalently that η is measurable with respect to the $\mathbb{P}$-completion of $\mathcal{I}$). From (c) of (4.1) and both the facts that $(X_i)_{i\in\mathbb{Z}}$ is strictly stationary and $\mathcal{M}_{-q_n} \subseteq \mathcal{M}_{-q_n+1}$, we have

$$\lim_{n\to\infty} \left\|\mathbb{E}\Big(\eta \circ T - \frac{S_{q_n}^2 \circ T}{\sigma_{q_n}^2}\Big|\mathcal{M}_{-q_n}\Big)\right\|_1 = 0. \tag{4.4}$$

On the other hand

$$\begin{aligned}\left\|\mathbb{E}\Big(\frac{S_{q_n}^2 - S_{q_n}^2 \circ T}{\sigma_{q_n}^2}\Big|\mathcal{M}_{-q_n}\Big)\right\|_1 &\leqslant \frac{1}{\sigma_{q_n}^2}\|(X_1 - X_{q_n+1})(2S_{q_n} - X_1 + X_{q_n+1})\|_1 \\ &\leqslant \frac{4}{\sigma_{q_n}^2}\|X_0\|_2(\|S_{q_n}\|_2 + \|X_0\|_2).\end{aligned}$$

Using the fact that $\sigma_{q_n} \to \infty$, we derive that

$$\lim_{n\to\infty} \left\|\mathbb{E}\Big(\frac{S_{q_n}^2 - S_{q_n}^2 \circ T}{\sigma_{q_n}^2}\Big|\mathcal{M}_{-q_n}\Big)\right\|_1 = 0,$$

which together with (c) of (4.1) imply that

$$\lim_{n\to\infty} \|\mathbb{E}\Big(\eta - \frac{S_{q_n}^2 \circ T}{\sigma_{q_n}^2}\Big|\mathcal{M}_{-q_n}\Big)\|_1 = 0. \tag{4.5}$$

Combining (4.4) and (4.5), it follows that $\lim_{n\to\infty} \|\mathbb{E}\Big(\eta - \eta \circ T|\mathcal{M}_{-q_n}\Big)\|_1 = 0$, which by the reverse martingale theorem in $\mathbb{L}^1$ implies that

$$\|\mathbb{E}\Big(\eta - \eta \circ T|\mathcal{M}_{-\infty}\Big)\|_1 = 0. \tag{4.6}$$

According to (c) of (4.1), the random variable η is $\mathcal{M}_{-\infty}$-measurable. Therefore (4.6) implies that $\mathbb{E}\Big(\eta \circ T|\mathcal{M}_{-\infty}\Big) = \eta$. The fact that $\eta = \eta \circ T$ almost surely follows from this last consideration and the fact that since $\mathbb{E}\Big(\eta \circ T|\mathcal{M}_{-\infty}\Big) \overset{\mathcal{L}}{\sim} \eta \circ T$ then $\mathbb{E}\Big(\eta \circ T|\mathcal{M}_{-\infty}\Big) = \eta \circ T$ (see Lemma 3 in Dedecker and Merlevède(9)).

Step 1. We divide the variables $\{X_i\}$ in big blocks of size $p_n = m\, q_n$ and small blocks of size q_n, in the following way: Let us set for all $1 \leqslant j \leqslant k_n$,

$$Y_{j,n} = \sum_{i=(j-1)(p_n+q_n)+1}^{(j-1)(p_n+q_n)+p_n} X_i \text{ and } Z_{j,n} = \sum_{i=(j-1)(p_n+q_n)+p_n+1}^{j\,(p_n+q_n)} X_i.$$

Then we have the following decomposition:

$$S_{[nt]} = \sum_{j=1}^{k_{nt}} Y_{j,n} + \sum_{j=1}^{k_{nt}} Z_{j,n} + R_n(t),$$

where $R_n(t) := S_{[nt]} - \Big(\sum_{j=1}^{k_{nt}} Y_{j,n} + \sum_{j=1}^{k_{nt}} Z_{j,n}\Big)$.

Step 2. The martingale decomposition.
First for all $j \geqslant 1$, set $\mathcal{F}_{j,n}^Z = \mathcal{M}_{j\,(p_n+q_n)}$ and $\mathcal{F}_{j,n}^Y = \mathcal{M}_{(j-1)(p_n+q_n)+p_n}$. Define now the following martingales:

$$M_n'(t) := \sum_{j=1}^{k_{nt}} \Big\{Y_{j,n} - \mathbb{E}\Big(Y_{j,n}|\mathcal{F}_{j-1,n}^Y\Big)\Big\} \quad \text{and}$$

$$M_n''(t) := \sum_{j=1}^{k_{nt}} \Big\{Z_{j,n} - \mathbb{E}\Big(Z_{j,n}|\mathcal{F}_{j-1,n}^Z\Big)\Big\}.$$

Write now

$$S_{[nt]} = M'_n(t) + M''_n(t) + \sum_{j=1}^{k_{nt}} \mathbb{E}(Y_{j,n}|\mathcal{F}^Y_{j-1,n}) + \sum_{j=1}^{k_{nt}} \mathbb{E}(Z_{j,n}|\mathcal{F}^Z_{j-1,n}) + R_n(t), \tag{4.7}$$

and set $b^2_{n,m} = k_n \sigma^2_{p_n}$.

The idea of the proof is to show that: for each $m \geqslant 1$, there exists a nonnegative and $\mathcal{M}_{-\infty}$-measurable random variable η such that

$$\frac{M'_n(\cdot)}{b_{n,m}} \overset{\mathcal{D}}{\to} \sqrt{\eta}\, W, \text{ as } n \to \infty, \text{ in the Skorohod topology,} \tag{4.8}$$

and that for each positive ε,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{t\in[0,1]} \Big| \frac{S_{[nt]} - M'_n(t)}{b_{n,m}} \Big| \geqslant \varepsilon \Big) = 0, \tag{4.9}$$

and finally that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{t\in[0,1]} \Big| \frac{S_{[nt]}}{B_n} - \frac{S_{[nt]}}{b_{n,m}} \Big| \geqslant \varepsilon \Big) = 0, \tag{4.10}$$

where $B_n = \sqrt{(n/q_n)}\sigma_{q_n}$ or $\sqrt{(\pi/2)}\mathbb{E}|S_n|$.
Indeed, according to Theorem 3.2 in Billingsley,[(3)] (4.8) together with (4.9) and (4.10) will give the desired result.

Step 3. At this step, we prove Relation (4.9) by taking into account Decomposition (4.7).

First, since for all $t \in [0, 1]$, $\{M''_n(t)\}$ is a martingale with respect to the filtration $\{\mathcal{F}^Z_{j,n}\}_{j\geqslant 1}$, Markov's inequality combined with Doob's inequality yields for all $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} \frac{|M''_n(t)|}{b_{n,m}} \geqslant \varepsilon \right) \leqslant \frac{\mathbb{E}\Big(\sup_{0\leqslant t\leqslant 1} |M''_n(t)|^2 \Big)}{\varepsilon^2 b^2_{n,m}} \leqslant 4 \frac{\mathbb{E}\big(M''_n(1)\big)^2}{\varepsilon^2 b^2_{n,m}}$$
$$\leqslant 4 \frac{\sum_{j=1}^{k_n} \mathbb{E}(Z_{j,n})^2}{\varepsilon^2 b^2_{n,m}} = \frac{4}{\varepsilon^2} \frac{\sigma^2_{q_n}}{\sigma^2_{m\,q_n}}. \tag{4.11}$$

According to (2.1) and to the definition of the slowly varying functions, it follows that

$$\lim_{n\to\infty} \frac{\sigma^2_{q_n}}{\sigma^2_{m\,q_n}} = \frac{1}{m}. \tag{4.12}$$

Then Inequality (4.11) combined with (4.12) yields that, for each positive ε,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{|M_n''(t)|}{b_{n,m}}\geqslant\varepsilon\right)=0. \tag{4.13}$$

Now we show that for each positive ε,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{|\sum_{k=1}^{k_{nt}}\mathbb{E}\left(Z_{k,n}|\mathcal{F}^Z_{k-1,n}\right)|}{b_{n,m}}\geqslant\varepsilon\right)=0. \tag{4.14}$$

To this aim we first observe that Markov's inequality yields that

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{\left|\sum_{j=1}^{k_{nt}}\mathbb{E}\left(Z_{j,n}|\mathcal{F}^Z_{j-1,n}\right)\right|}{b_{n,m}}\geqslant\varepsilon\right)&\leqslant\mathbb{P}\left(\frac{\sum_{j=1}^{k_n}\left|\mathbb{E}\left(Z_{j,n}|\mathcal{F}^Z_{j-1,n}\right)\right|}{b_{n,m}}\geqslant\varepsilon\right)\\&\leqslant\frac{\mathbb{E}\left(\sum_{j=1}^{k_n}\left|\mathbb{E}\left(Z_{j,n}|\mathcal{F}^Z_{j-1,n}\right)\right|\right)}{\varepsilon b_{n,m}}.\end{aligned} \tag{4.15}$$

Next by stationarity, we derive that

$$\sum_{k=1}^{k_n}\mathbb{E}\left|\mathbb{E}\left(Z_{k,n}|\mathcal{F}^Z_{k-1,n}\right)\right|=k_n\mathbb{E}|\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})|.$$

It follows that

$$b_{n,m}^{-1}\sum_{k=1}^{k_n}\mathbb{E}\left|\mathbb{E}\left(Z_{k,n}|\mathcal{F}^Z_{k-1,n}\right)\right|\leqslant\sqrt{\frac{n}{m\,q_n}}\,\frac{\sigma_{q_n}}{\sigma_{m\,q_n}}\,\frac{\mathbb{E}|\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})|}{\sigma_{q_n}},$$

Hence by using (4.12) together with Item (a) of (4.1), we get that

$$\lim_{n\to\infty}b_{n,m}^{-1}\sum_{k=1}^{k_n}\mathbb{E}\left|\mathbb{E}\left(Z_{k,n}|\mathcal{F}^Z_{k-1,n}\right)\right|=0, \tag{4.16}$$

which combined with (4.15) ends the proof of (4.14).

Our task now is to prove that for each positive ε,

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{\left|\sum_{j=1}^{k_{nt}}\mathbb{E}\left(Y_{j,n}|\mathcal{F}^Y_{j-1,n}\right)\right|}{b_{n,m}}\geqslant\varepsilon\right)=0. \tag{4.17}$$

Using again Markov's inequality together with stationarity, we derive that

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{\left|\sum_{j=1}^{k_{nt}}\mathbb{E}\left(Y_{j,n}|\mathcal{F}_{j-1,n}^{Y}\right)\right|}{b_{n,m}}\geqslant\varepsilon\right)\leqslant k_n\frac{\mathbb{E}|\mathbb{E}(S_{p_n}|\mathcal{M}_{-q_n})|}{\varepsilon b_{n,m}}. \tag{4.18}$$

For all $k\geqslant 1$, define

$$U_{k,n}=S_{kq_n}-S_{(k-1)q_n}, \tag{4.19}$$

(where $S_0=0$), and since $p_n=m\,q_n$, we write

$$S_{p_n}=\sum_{k=1}^{m}U_{k,n}. \tag{4.20}$$

Using the decomposition (4.20), we derive that

$$\mathbb{E}|\mathbb{E}(S_{p_n}|\mathcal{M}_{-q_n})|\leqslant\sum_{j=1}^{m}\mathbb{E}\left|\mathbb{E}\left(U_{j,n}|\mathcal{M}_{-q_n}\right)\right|\leqslant m\mathbb{E}|\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})|.$$

Consequently

$$k_n\frac{\mathbb{E}|\mathbb{E}(S_{p_n}|\mathcal{M}_{-q_n})|}{b_{n,m}}\leqslant m^{1/2}\sqrt{\frac{n}{q_n}}\,\frac{\sigma_{q_n}}{\sigma_{m\,q_n}}\,\frac{\mathbb{E}|\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})|}{\sigma_{q_n}}.$$

By using (4.12) together with Item (a) of (4.1), it follows that

$$\lim_{n\to\infty}k_n\frac{\mathbb{E}|\mathbb{E}(S_{p_n}|\mathcal{M}_{-q_n})|}{b_{n,m}}=0. \tag{4.21}$$

Starting from Inequality (4.18) and using (4.21), we end the proof of (4.17).

Now we want to show that for each positive ε,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{0\leqslant t\leqslant 1}\frac{|R_n(t)|}{b_{n,m}}\geqslant\varepsilon\Big)=0. \tag{4.22}$$

Using stationarity, notice first that, for all $\varepsilon>0$

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\leqslant t\leqslant 1}\frac{|R_n(t)|}{b_{n,m}}\geqslant 2\varepsilon\Big)&\leqslant(k_n+1)\mathbb{P}\Big(\max_{1\leqslant j\leqslant m\,q_n}\frac{|S_j|}{b_{n,m}}\geqslant 2\varepsilon\Big)\\&\leqslant(k_n+1)\left\{\mathbb{P}\Big(\max_{1\leqslant j\leqslant m}\frac{|S_{jq_n}|}{b_{n,m}}\geqslant\varepsilon\Big)+m\mathbb{P}\Big(\max_{1\leqslant j\leqslant q_n}\frac{|S_j|}{b_{n,m}}\geqslant\varepsilon\Big)\right\}.\end{aligned} \tag{4.23}$$

To treat the first term in the right-hand side, we use the notation (4.19) and notice that by stationarity

$$\mathbb{P}\Big(\max_{1\leqslant j\leqslant m}\frac{|S_{jq_n}|}{b_{n,m}}\geqslant\varepsilon\Big)=\mathbb{P}\Big(\max_{1\leqslant j\leqslant m}\frac{|\sum_{i=1}^{j}U_{i,n}|}{b_{n,m}}\geqslant\varepsilon\Big)\leqslant\mathbb{P}\Big(m\max_{1\leqslant j\leqslant m}\frac{|U_{j,n}|}{b_{n,m}}\geqslant\varepsilon\Big)$$
$$\leqslant m\mathbb{P}\Big(\frac{|S_{q_n}|}{b_{n,m}}\geqslant\frac{\varepsilon}{m}\Big). \tag{4.24}$$

Then starting from (4.23) and using (4.24), we clearly derive that

$$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant 1}\frac{|R_n(t)|}{b_{n,m}}\geqslant 2\varepsilon\Big)\leqslant 2m(k_n+1)\mathbb{P}\Big(\max_{1\leqslant j\leqslant q_n}\frac{|S_j|}{b_{n,m}}\geqslant\frac{\varepsilon}{m}\Big),$$

which entails that (4.22) holds by using (4.12) and provided that Condition (4.2) is satisfied.

Finally, starting from Decomposition (4.7) and gathering (4.13), (4.14), (4.17) and (4.22), we infer that (4.9) holds.

Step 4. We turn now to the proof of (4.8). Since $\{M_n'(t)\}_{n\geqslant 1}$ is a sequence of martingale with respect to the filtration $\{\mathcal{F}_{k,n}^Y\}$, we have only to apply the classical weak invariance principle for triangular arrays of martingales. To this aim, we shall apply Theorem 18.2 in Billingsley[3] (see also Hall and Heyde[15]). Then, setting for all $j\geqslant 1$,

$$W_{j,n}=Y_{j,n}-\mathbb{E}(Y_{j,n}|\mathcal{F}_{j-1,n}^Y),$$

it suffices to prove that for every $t\in[0,1]$ and each positive ε,

$$\lim_{n\to\infty}\frac{1}{b_{n,m}^2}\sum_{j=1}^{k_{nt}}\mathbb{E}\left(W_{j,n}^2\mathbb{I}(|W_{j,n}|\geqslant\varepsilon b_{n,m})\right)=0, \tag{4.25}$$

and that there exists a nonnegative and $\mathcal{I}$-measurable random variable η such that,

$$\lim_{n\to\infty}\sum_{j=1}^{k_{nt}}\mathbb{E}\left(\frac{|W_{j,n}|^2}{b_{n,m}^2}\Big|\mathcal{F}_{j-1,n}^Y\right)=t\,\eta \text{ in probability.} \tag{4.26}$$

We first prove that (4.25) holds. To this aim notice first that by stationarity,

$$
\begin{aligned}
& b_{n,m}^{-2}\sum_{k=1}^{k_{nt}}\mathbb{E}\left(W_{j,n}^2\mathbb{I}(|W_{j,n}|\geqslant 4\varepsilon b_{n,m})\right) \\
&\leqslant \frac{1}{\sigma_{m\,q_n}^2}\mathbb{E}\left(\left(S_{m\,q_n}-\mathbb{E}(S_{m\,q_n}|\mathcal{M}_{-q_n})\right)^2\mathbb{I}\left(|S_{m\,q_n}-\mathbb{E}(S_{m\,q_n}|\mathcal{M}_{-q_n})|\geqslant 4\varepsilon b_{n,m}\right)\right) \\
&\leqslant \frac{2}{\sigma_{m\,q_n}^2}\mathbb{E}\left(S_{m\,q_n}^2\mathbb{I}\left(|S_{m\,q_n}|\geqslant 2\varepsilon b_{n,m}\right)\right)+\frac{2}{\sigma_{m\,q_n}^2}\mathbb{E}\left|\mathbb{E}(S_{m\,q_n}|\mathcal{M}_{-q_n})\right|^2. \qquad (4.27)
\end{aligned}
$$

To treat the first term in the right-hand side, we first notice that $x^2\mathbb{I}(|x|>2\varepsilon)\leqslant 4(|x|-\varepsilon)_+^2$. Then using the convexity of the function $(|x|-\varepsilon)_+^2$ followed by the fact that $(|x|-\varepsilon)_+^2\leqslant x^2\mathbb{I}(|x|>\varepsilon)$, we derive that

$$
\left(\frac{1}{m}\sum_{i=1}^m x_i\right)^2\mathbb{I}\left(\left|\sum_{i=1}^m x_i\right|>2m\varepsilon\right)\leqslant\frac{4}{m}\sum_{i=1}^m x_i^2\mathbb{I}\left(|x_i|>\varepsilon\right). \qquad (4.28)
$$

Starting from this inequality and using Decomposition (4.20) together with stationarity, we infer that

$$
\frac{1}{\sigma_{m\,q_n}^2}\mathbb{E}\left(S_{m\,q_n}^2\mathbb{I}\left(|S_{m\,q_n}|\geqslant 2\varepsilon b_{n,m}\right)\right)\leqslant\frac{4m^2}{\sigma_{m\,q_n}^2}\mathbb{E}\left(S_{q_n}^2\mathbb{I}\left(|S_{q_n}|\geqslant\frac{\varepsilon b_{n,m}}{m}\right)\right),
$$

which converges to zero as $n\to\infty$, by using Item (b) of (4.1) combined with (4.12). Hence,

$$
\lim_{n\to\infty}\frac{1}{\sigma_{m\,q_n}^2}\mathbb{E}\left(S_{m\,q_n}^2\mathbb{I}\left(|S_{m\,q_n}|\geqslant 2\varepsilon b_{n,m}\right)\right)=0. \qquad (4.29)
$$

Now we show that

$$
\lim_{n\to\infty}\sigma_{m\,q_n}^{-2}\mathbb{E}\left(\mathbb{E}(S_{m\,q_n}|\mathcal{M}_{-q_n})\right)^2=0. \qquad (4.30)
$$

Using again Decomposition (4.20), we clearly get

$$
\mathbb{E}\left(\mathbb{E}(S_{m\,q_n}|\mathcal{M}_{-q_n})\right)^2\leqslant m^2\mathbb{E}\left(\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})\right)^2. \qquad (4.31)
$$

Now, notice that, Item (d) of (4.1) implies that

$$
\lim_{n\to\infty}\frac{\|\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})\|_2}{\sigma_{q_n}}=0. \qquad (4.32)
$$

Then (4.30) follows by using (4.12) together with (4.31) and (4.32).

By combining (4.29) and (4.30) with (4.27), we derive (4.25).

We turn now to the proof of (4.26), and we shall prove that the convergence holds in $\mathbb{L}^1$. For this task, we first notice that

$$\mathbb{E}\left(W_{j,n}^2|\mathcal{F}_{j-1,n}^Y\right)=\mathbb{E}\left(Y_{j,n}^2|\mathcal{F}_{j-1,n}^Y\right)-\left(\mathbb{E}(Y_{j,n}|\mathcal{F}_{j-1,n}^Y)\right)^2. \tag{4.33}$$

Since $b_{n,m}^{-2}\sum_{j=1}^{k_{nt}}\mathbb{E}\left(\mathbb{E}(Y_{j,n}|\mathcal{F}_{j-1,n}^Y)\right)^2\leqslant\sigma_{m\,q_n}^{-2}\mathbb{E}\left(\mathbb{E}(S_{m\,q_n}|\mathcal{M}_{-q_n})\right)^2$, using (4.30) and stationarity, we infer that (4.26) will hold in $\mathbb{L}^1$ if we prove that

$$\lim_{n\to\infty}\mathbb{E}\left|\sum_{j=1}^{k_{nt}}\mathbb{E}\left(\frac{Y_{j,n}^2}{b_{n,m}^2}\middle|\mathcal{F}_{j-1,n}^Y\right)-t\,\eta\right|=0. \tag{4.34}$$

By using stationarity, we get

$$\mathbb{E}\left|\sum_{j=1}^{k_{nt}}\mathbb{E}\left(\frac{Y_{j,n}^2}{b_{n,m}^2}\middle|\mathcal{F}_{j-1,n}^Y\right)-t\,\eta\right|\leqslant k_{nt}\,\mathbb{E}\left|\mathbb{E}\left(\frac{S_{m\,q_n}^2}{b_{n,m}^2}\middle|\mathcal{M}_{-q_n}\right)-\frac{t}{k_{nt}}\,\eta\right|. \tag{4.35}$$

Now, from Decomposition (4.20), we get

$$S_{p_n}^2=\sum_{k=1}^{m}U_{k,n}^2+2\sum_{k=2}^{m}\sum_{\ell=1}^{k-1}U_{k,n}U_{\ell,n}. \tag{4.36}$$

It follows from this decomposition and Inequality (4.35), that

$$\begin{aligned}\mathbb{E}\left|\sum_{j=1}^{k_{nt}}\mathbb{E}\left(\frac{Y_{j,n}^2}{b_{n,m}^2}\middle|\mathcal{F}_{j-1,n}^Y\right)-t\,\eta\right|&\leqslant k_{nt}\,\mathbb{E}\left|\sum_{k=1}^{m}\mathbb{E}\left(\frac{U_{k,n}^2}{b_{n,m}^2}\middle|\mathcal{M}_{-q_n}\right)-\frac{t}{k_{nt}}\,\eta\right|\\&\quad+2k_{nt}\,\mathbb{E}\left|\sum_{k=2}^{m}\sum_{\ell=1}^{k-1}\mathbb{E}\left(\frac{U_{k,n}U_{\ell,n}}{b_{n,m}^2}\middle|\mathcal{M}_{-q_n}\right)\right|\end{aligned} \tag{4.37}$$

To treat the first term in the right-hand side, we notice that η is shift invariant ($\eta=\eta\circ T$) almost surely. Then by stationarity, we infer that

$$\begin{aligned}k_{nt}\,\mathbb{E}\left|\sum_{k=1}^{m}\mathbb{E}\left(\frac{U_{k,n}^2}{b_{n,m}^2}\middle|\mathcal{M}_{-q_n}\right)-\frac{t}{k_{nt}}\,\eta\right|&\leqslant m\,k_{nt}\,\mathbb{E}\left|\mathbb{E}\left(\frac{S_{q_n}^2}{b_{n,m}^2}\middle|\mathcal{M}_{-q_n}\right)-\frac{t}{m\,k_{nt}}\,\eta\right|\\&=\frac{m\,k_{nt}\,\sigma_{q_n}^2}{b_{n,m}^2}\,\mathbb{E}\left|\mathbb{E}\left(\frac{S_{q_n}^2}{\sigma_{q_n}^2}\middle|\mathcal{M}_{-q_n}\right)-\frac{t\,b_{n,m}^2}{m\,k_{nt}\,\sigma_{q_n}^2}\,\eta\right|\\&\leqslant\frac{m\,k_{nt}\,\sigma_{q_n}^2}{b_{n,m}^2}\,\mathbb{E}\left|\mathbb{E}\left(\frac{S_{q_n}^2}{\sigma_{q_n}^2}\middle|\mathcal{M}_{-q_n}\right)-\eta\right|+\mathbb{E}|\eta|\left|1-\frac{t\,b_{n,m}^2}{m\,k_{nt}\,\sigma_{q_n}^2}\right|.\end{aligned}$$

Using the definition of $b_{n,m}^2$ together with (4.12) and Item (c) of (4.1), we infer that both of the terms in the right-hand side converge to zero, as $n \to \infty$. Then we have shown that

$$\lim_{n\to\infty} k_{nt}\, \mathbb{E}\Big| \sum_{k=1}^{m} \mathbb{E}\left(\frac{U_{k,n}^2}{b_{n,m}^2} \Big| \mathcal{M}_{-q_n} \right) - \frac{t}{k_{nt}}\, \eta \Big| = 0. \tag{4.38}$$

On the other hand, we have

$$\frac{k_{nt}}{b_{n,m}^2} \mathbb{E} \left| \sum_{j=3}^{m} \sum_{\ell=1}^{j-2} \mathbb{E}(U_{j,n} U_{\ell,n} | \mathcal{M}_{-q_n}) \right| \leqslant \frac{k_{nt}}{b_{n,m}^2} \sum_{j=2}^{m} \mathbb{E} \big| S_{(j-2)q_n}\, \mathbb{E}\big(U_{j,n} | \mathcal{M}_{(j-2)q_n}\big) \big|.$$

Next, Cauchy–Schwarz's inequality combined with stationarity yields

$$\frac{k_{nt}}{b_{n,m}^2} \mathbb{E} \left| \sum_{j=3}^{m} \sum_{\ell=1}^{j-2} \mathbb{E}(U_{j,n} U_{\ell,n} | \mathcal{M}_{-q_n}) \right| \leqslant \frac{k_{nt}}{k_n} \frac{\sigma_{q_n}^2}{\sigma_{m\,q_n}^2} \frac{\|\mathbb{E}(S_{q_n} | \mathcal{M}_{-q_n})\|_2}{\sigma_{q_n}} \sum_{j=2}^{m} \frac{\sigma_{(j-2)q_n}}{\sigma_{q_n}}.$$

Hence by using (4.12), we derive , as $n \to \infty$, that

$$\frac{k_{nt}}{b_{n,m}^2} \mathbb{E} \left| \sum_{j=3}^{m} \sum_{\ell=1}^{j-2} \mathbb{E}(U_{j,n} U_{\ell,n} | \mathcal{M}_{-q_n}) \right| \ll m \frac{\|\mathbb{E}(S_{q_n} | \mathcal{M}_{-q_n})\|_2}{\sigma_{q_n}},$$

which converges to zero by (4.32). Then we have shown that

$$\lim_{n\to\infty} \frac{k_{nt}}{b_{n,m}^2} \mathbb{E} \Big| \sum_{j=3}^{m} \sum_{\ell=1}^{j-2} \mathbb{E}(U_{j,n} U_{\ell,n} | \mathcal{M}_{-q_n}) \Big| = 0. \tag{4.39}$$

Moreover, we clearly have

$$\begin{aligned} \mathbb{E} \left| \sum_{j=1}^{m-1} \mathbb{E}(U_{j,n} U_{j+1,n} | \mathcal{M}_{-q_n}) \right| &\leqslant m \mathbb{E} \big| \mathbb{E}\big(S_{q_n}(S_{2q_n} - S_{q_n}) | \mathcal{M}_{-q_n}\big) \big| \\ &\leqslant m \mathbb{E} \big| S_{q_n} \mathbb{E}(S_{2q_n} - S_{q_n} | \mathcal{M}_{q_n}) \big| \\ &\leqslant m \sigma_{q_n} \|\mathbb{E}(S_{q_n} | \mathcal{M}_0)\|_2. \end{aligned}$$

Taking into account (4.12) and Item (d) of (4.1), we get that

$$\lim_{n\to\infty} \frac{k_{nt}}{b_{n,m}^2} \mathbb{E} \Big| \sum_{j=1}^{m-1} \mathbb{E}(U_{j,n} U_{j+1,n} | \mathcal{M}_{-q_n}) \Big| = 0,$$

which combined with (4.39) entails that

$$\lim_{n\to\infty} k_{nt}\, \mathbb{E}\Big| \sum_{k=2}^{m} \sum_{\ell=1}^{k-1} \mathbb{E}\left(\frac{U_{k,n} U_{\ell,n}}{b_{n,m}^2} \Big| \mathcal{M}_{-q_n} \right) \Big| = 0. \tag{4.40}$$

Inequality (4.37) together with (4.38) and (4.40) prove (4.34).

Step 5. Now we prove (4.10). Notice first that according to (4.12), it is easy to see that $\lim_{m\to\infty}\limsup_{n\to\infty}\frac{\sqrt{(n/q_n)}\sigma_{q_n}}{b_{n,m}}=1$ which implies (4.10) with $B_n=\sqrt{(n/q_n)}\sigma_{q_n}$. Now we turn to the proof of (4.10) with $B_n=\sqrt{\frac{\pi}{2}}\,\mathbb{E}|S_n|$. First we set

$$a_{n,m}=\left|\frac{b_{n,m}}{\sqrt{\frac{\pi}{2}}\,\mathbb{E}|S_n|}-1\right|,$$

and we write for all $\varepsilon>0$,

$$\mathbb{P}\left(\sup_{t\in[0,1]}\left|\frac{S_{[nt]}}{\sqrt{\frac{\pi}{2}}\,\mathbb{E}|S_n|}-\frac{S_{[nt]}}{b_{n,m}}\right|\geqslant\varepsilon\right)\leqslant\mathbb{P}\left(\sup_{t\in[0,1]}a_{n,m}\left|\frac{S_{[nt]}-M_n'(t)}{b_{n,m}}\right|\geqslant\varepsilon/2\right)$$
$$+\mathbb{P}\left(\sup_{t\in[0,1]}a_{n,m}\left|\frac{M_n'(t)}{b_{n,m}}\right|\geqslant\varepsilon/2\right). \tag{4.41}$$

Assume that we can prove that for all $m\geqslant 2$,

$$\lim_{n\to\infty}a_{n,m}\leqslant\frac{1}{m-1}, \tag{4.42}$$

then, for n large enough,

$$\mathbb{P}\left(\sup_{t\in[0,1]}a_{n,m}\left|\frac{S_{[nt]}-M_n'(t)}{b_{n,m}}\right|\geqslant\varepsilon/2\right)\leqslant\mathbb{P}\left(\sup_{t\in[0,1]}\left|\frac{S_{[nt]}-M_n'(t)}{b_{n,m}}\right|\geqslant\varepsilon/2\right),$$

and using (4.9), it will follow that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{t\in[0,1]}a_{n,m}\left|\frac{S_{[nt]}-M_n'(t)}{b_{n,m}}\right|\geqslant\varepsilon/2\Big)=0. \tag{4.43}$$

On the other hand, Doob's inequality yields

$$\mathbb{P}\Big(\sup_{t\in[0,1]}a_{n,m}\left|\frac{M_n'(t)}{b_{n,m}}\right|\geqslant\varepsilon/2\Big)\leqslant\frac{4\,a_{n,m}^2}{\varepsilon^2}\,\frac{\mathbb{E}(M_n'(1))^2}{b_{n,m}^2}=\frac{4\,a_{n,m}^2}{\varepsilon^2}.$$

Then if (4.42) holds, we will obviously get

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{t\in[0,1]}a_{n,m}\left|\frac{M_n'(t)}{b_{n,m}}\right|\geqslant\varepsilon/2\Big)=0. \tag{4.44}$$

Next starting from (4.41) and considering (4.43) and (4.44), (4.10) follows with $B_n = \sqrt{\frac{\pi}{2}}\,\mathbb{E}|S_n|$. It remains to show (4.42). With this aim, we infer that (4.42) will hold if we can prove that

$$\lim_{n\to\infty}\left|\frac{\mathbb{E}|S_n|}{\sigma_{m\,q_n}\sqrt{k_n}} - \sqrt{\frac{2}{\pi}}\right| \leqslant \frac{1}{m}. \tag{4.45}$$

To prove (4.45), we first notice that $\left\{\frac{M_n'(1)}{b_{n,m}}\right\}_{n\geqslant 1}$ is obviously an uniformly integrable family. This remark combined with (4.8) entails that

$$\frac{\mathbb{E}|M_n'(1)|}{b_{n,m}} \to \mathbb{E}|\eta|\mathbb{E}|N| = \sqrt{\frac{2}{\pi}}, \text{ as } n\to\infty, \tag{4.46}$$

(see Theorem 3.5 in Billingsley[(3)]).

Now (4.46) yields

$$\lim_{n\to\infty}\left|\frac{\mathbb{E}|S_n|}{b_{n,m}} - \sqrt{\frac{2}{\pi}}\right| = \lim_{n\to\infty}\left|\frac{\mathbb{E}|S_n|}{b_{n,m}} - \frac{\mathbb{E}|M_n'(1)|}{b_{n,m}}\right| \leqslant \lim_{n\to\infty}\mathbb{E}\left|\frac{S_n}{b_{n,m}} - \frac{M_n'(1)}{b_{n,m}}\right|.$$

Now, by analyzing the proof of (4.9), we infer that (4.45) will holds if we can prove that

$$\lim_{n\to\infty}\mathbb{E}\left(\frac{R_n(1)}{b_{n,m}}\right)^2 = 0. \tag{4.47}$$

If we denote by ℓ_n the number of terms in $R_n(1)$, by using of (2.1), the standard properties of the slowly varying function and the fact that $\ell_n \leqslant (m+1)q_n$, we derive that for every positive ϵ,

$$\frac{\sigma_{\ell_n}^2}{k_n\sigma_{m\,q_n}^2} \ll \left(\frac{\ell_n}{mq_n}\right)^{1-\epsilon}\frac{1}{k_n} = o(1), \text{ as } n\to\infty,$$

which ends the proof of (4.47) and hence of (4.45).

4.2. Proofs of Theorems 1 and 2

We first start by the following technical lemma which is interesting in itself.

Lemma 1. Assume that (2.1) holds and

$$\|\mathbb{E}(S_n|\mathcal{M}_{-n})\|_2 = o(\sigma_n) \text{ as } n\to\infty, \tag{4.48}$$

then (2.5) holds.

Proof of Lemma 1. We start with the binary expansion

$$n=\sum_{k=0}^{r-1} a_k 2^k \text{ where } a_{r-1}=1 \text{ and } a_k\in\{0,1\}.$$

Then, we have the following representation

$$S_n=\sum_{j=0}^{r-1} a_j T_{2^j} \qquad \text{where } T_{2^j}=S_{n_j}-S_{n_{j-1}},\ n_j=\sum_{k=0}^{j} a_k 2^k,\ \ n_{-1}=0.$$

Applying this representation, by the properties of the norm, we easily derive that

$$\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2\leqslant \sum_{j=0}^{r-1}\|\mathbb{E}(S_{2^j}|\mathcal{M}_0)\|_2. \tag{4.49}$$

Now by stationnarity,

$$\|\mathbb{E}(S_{2^j}|\mathcal{M}_0)\|_2\leqslant \|\mathbb{E}(S_{2^{j-1}}|\mathcal{M}_0)\|_2+\|\mathbb{E}(S_{2^{j-1}}|\mathcal{M}_{-2^{j-1}})\|_2.$$

It follows by recurrence that

$$\|\mathbb{E}(S_{2^j}|\mathcal{M}_0)\|_2\leqslant \sum_{k=1}^{j-1}\|\mathbb{E}(S_{2^k}|\mathcal{M}_{-2^k})\|_2.$$

Starting from this inequality and using (4.48), we derive that

$$\|\mathbb{E}(S_{2^j}|\mathcal{M}_0)\|_2\leqslant \sum_{k=1}^{N}\sigma_{2^k}+\epsilon_N\sigma_{2^j}\sum_{k=N+1}^{j-1}\frac{\sigma_{2^k}}{\sigma_{2^j}},$$

where ϵ_N is such that $\lim_{N\to\infty}\epsilon_N=0$.

Now from (2.1) and the properties of the slowly varying functions, we infer that $\sum_{k=N+1}^{j-1}\frac{\sigma_{2^k}}{\sigma_{2^j}}<C$, where C is a constant not depending on N and j. Then, by first letting j to tend to infinity and after N, it follows easily that

$$\|\mathbb{E}(S_{2^j}|\mathcal{M}_0)\|_2=o(\sigma_{2^j}). \tag{4.50}$$

Now starting from (4.49) and using (4.50) together with preceding arguments it follows that $\|\mathbb{E}(S_n|\mathcal{M}_0)\|_2=o(\sigma_{2^{r-1}})$. Now since $2^{r-1}\leqslant n<2^r$, it follows from (2.1) and the properties of the slowly varying functions that $\sigma_{2^{r-1}}=O(\sigma_n)$. This completes the proof of the lemma. □

The proof of Theorem 1 (resp. Theorem 2) is an application of Proposition 3 (Proposition 4) by taking the sequence q_n very "close" to n such that $q_n \to \infty$, $\lim_{n\to\infty} \frac{n\sigma_{q_n}^2}{q_n\sigma_n^2} = 1$ and Item (a) of (4.1) holds. We also use Lemma 1 and the fact that (2.2) combined with (2.3) imply (4.48).

4.3. Proof of Corollary 1

The proof of this corollary is divided in several steps.

Step 1. We proof here that (2.7) is satisfied. With this aim, it suffices to show that

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} \sigma_n^{-2}\mathbb{E}((\bar{S}_n - \lambda\sigma_n)_+^2) = 0, \tag{4.51}$$

where $\bar{S}_n = \max_{1\leqslant i\leqslant n} |S_i|$. From Proposition 1(a) in Dedecker and Rio[8] applied to the sequences $(X_i)_{i\in\mathbb{Z}}$ and $(-X_i)_{i\in\mathbb{Z}}$ and stationarity we derive that

$$\mathbb{E}((\bar{S}_n - \lambda\sigma_n)_+^2) \leqslant 8\,n\mathbb{E}(X_0^2\mathbb{I}_{\bar{S}_n>\lambda\sigma_n}) + 16\sum_{k=1}^{n-1}\mathbb{E}\big|X_0\mathbb{E}(S_k|\mathcal{M}_0)\mathbb{I}_{\bar{S}_n>\lambda\sigma_n}\big|. \tag{4.52}$$

Now the condition (2.8) entails that for each $k\geqslant 1$, $\sigma_k^{-2}k\mathbb{E}\big|X_0\mathbb{E}(S_k|\mathcal{M}_0)\big| \leqslant K$ where K is a positive constant. Hence, from (4.52) with $\lambda = 0$ we get that

$$\sigma_n^{-2}\mathbb{E}(\bar{S}_n)^2 \leqslant 8\mathbb{E}(X_0^2)n\sigma_n^{-2} + 16K\sigma_n^{-2}\sum_{k=1}^{n-1}\frac{\sigma_k^2}{k}.$$

Now since $\sigma_n^2 = nh(n)$ where $h(n)$ is slowly varying in the strong sense, we have: $\lim_{x\to\infty} \frac{\int_0^x h(t)dt}{xh(x)} = 1$ and $\lim_{t\to\infty} \frac{\inf_{t\leqslant x\leqslant 2t} h(x)}{\sup_{t\leqslant x\leqslant 2t} h(x)} = 1$ (see Relation (b) page 14 in Bradley[6]). Then $\lim_{n\to\infty} \frac{1}{nh(n)}\sum_{i=1}^n h(i) = 1$, which implies that

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2}\sum_{i=1}^{n-1}\frac{\sigma_i^2}{i} = 1. \tag{4.53}$$

Consequently, for each $n\geqslant 2$, $\frac{1}{\sigma_n^2}\sum_{i=1}^{n-1}\frac{\sigma_i^2}{i} \leqslant M$ where M is a positive constant. This last consideration together with the fact that $\lim_{n\to\infty}\frac{\sigma_n^2}{n} > 0$ entails that there exists a positive constant C such that

$$\sigma_n^{-2}\mathbb{E}(\bar{S}_n)^2 \leqslant C.$$

It follows that

$$\mathbb{P}(\bar{S}_n > \lambda\sigma_n) \leqslant C\lambda^{-2}, \tag{4.54}$$

which converges to zero as λ tends to infinity.
Hence from (4.54), (4.52) and (2.8), we easily get that

$$\sigma_n^{-2}\mathbb{E}((\bar{S}_n - \lambda\sigma_n)_+^2) \leqslant \delta(\lambda)$$

for some nonincreasing function δ satisfying $\lim_{\lambda\to\infty}\delta(\lambda) = 0$. This proves (4.51).

Step 2. We prove here that

$$\|\mathbb{E}(S_n|\mathcal{M}_0)\|_1 = o(\sigma_n) \text{ as } n\to\infty, \tag{4.55}$$

which obviously entails (2.3).

By following the proof of Proposition 3 in Dedecker and Merlevède[9] (proof of s2(b)), replacing $\sqrt{n}$ by σ_n and taking into account the fact that $\liminf_{n\to\infty}\frac{\sigma_n^2}{n} > 0$, we infer that

$$\|\mathbb{E}(S_n|\mathcal{M}_0)\|_1 = o(\sigma_n) \text{ as } n\to\infty, \tag{4.56}$$

as soon as $\sigma_n^{-1}\|X_0\mathbb{E}(S_n|\mathcal{M}_0)\|_1$ converges to 0, as $n\to\infty$. This last convergence obviously holds by (2.8) and the fact that $\lim_{n\to\infty}\frac{\sigma_n^2}{n^2} = 0$.

Step 3. We finish the proof by proving that (2.4) holds. We then need to show that

$$\lim_{n\to\infty}\|\sigma_n^{-2}\mathbb{E}(S_n^2|\mathcal{M}_{-n}) - \eta\|_1 = 0, \tag{4.57}$$

where $\eta = c^{-1}\mathbb{E}(X_0^2|\mathcal{I}) + 2\mathbb{E}(\mu|\mathcal{I})$.

Notice first that

$$\sigma_n^{-2}\mathbb{E}(S_n^2|\mathcal{M}_{-n}) = \sigma_n^{-2}\sum_{i=1}^{n}\mathbb{E}(X_i^2|\mathcal{M}_{-n}) + 2\sigma_n^{-2}\sum_{i=1}^{n-1}\mathbb{E}(X_i(S_n - S_i)|\mathcal{M}_{-n}).$$

By the ergodic theorem, the reverse martingale theorem in $\mathbb{L}^1$ and the fact that $\lim_{n\to\infty}\frac{\sigma_n^2}{n} = c$, we get that

$$\sigma_n^{-2}\sum_{i=1}^{n}\mathbb{E}(X_i^2|\mathcal{M}_{-n}) \to c^{-1}\mathbb{E}(X_0^2|\mathcal{I}) \text{ in } \mathbb{L}^1, \text{ as } n\to\infty. \tag{4.58}$$

Then to finish the proof of (4.57), it remains to show that

$$\left\|\sigma_n^{-2}\sum_{i=1}^{n-1}\mathbb{E}(X_i(S_n - S_i)|\mathcal{M}_{-n}) - \mathbb{E}(\mu|\mathcal{I})\right\|_1 \to 0, \text{ as } n\to\infty. \tag{4.59}$$

Observe that, by stationarity and the properties of the conditional expectation,

$$\left\| \sigma_n^{-2} \sum_{i=1}^{n-1} \mathbb{E}(X_i(S_n - S_i)|\mathcal{M}_{-n}) - \mathbb{E}(\mu|\mathcal{I}) \right\|_1$$
$$\leqslant \left| \frac{1}{\sigma_n^2} \sum_{i=1}^{n-1} \frac{\sigma_i^2}{i} - 1 \right| \mathbb{E}|\mu| + \sigma_n^{-2} \sum_{i=1}^{n-1} \frac{\sigma_i^2}{i} \mathbb{E}\left| \frac{i}{\sigma_i^2} \mathbb{E}(X_0 S_i|\mathcal{M}_{-i}) - \mathbb{E}(\mu|\mathcal{M}_{-i}) \right|$$
$$+ \sigma_n^{-2} \mathbb{E}\left| \sum_{i=1}^{n-1} \mathbb{E}(\mu \circ T^i|\mathcal{M}_{-n}) - \mathbb{E}(\mu|\mathcal{I}) \right|. \tag{4.60}$$

According to (2.8), we get that

$$\sigma_n^{-2} \sum_{i=1}^{n-1} \frac{\sigma_i^2}{i} \mathbb{E}\left| \frac{i}{\sigma_i^2} \mathbb{E}(X_0 S_i|\mathcal{M}_{-i}) - \mathbb{E}(\mu|\mathcal{M}_{-i}) \right| \leqslant \frac{K}{\sigma_n^2} \sum_{i=1}^{N} \frac{\sigma_i^2}{i} + \varepsilon_N \left(\frac{1}{\sigma_n^2} \sum_{i=N}^{n-1} \frac{\sigma_i^2}{i} \right), \tag{4.61}$$

where ε_N is such that $\lim_{N\to\infty} \varepsilon_N = 0$. The last term in (4.60) is convergent to 0 when n tends to infinity by the ergodic theorem, the reverse martingale theorem in $\mathbb{L}^1$ and the fact that $\lim_{n\to\infty} \frac{\sigma_n^2}{n} = c$.

By taking into account (4.60) and (4.61), (4.59) holds by first letting n to tend to infinity and after N and by using (4.53). This completes the proof of the corollary.

4.4. Proof of Theorem 3

In order to prove Theorem 3, we shall verify the conditions of Proposition 4. For the sake of clarity, we have divided the proof of Theorem 3 in several steps. First we give and prove some preliminary lemmas. Then, we make a selection of a sequence of integers q_n which converges to infinity and finally, we check that this selection satisfies the assumptions of Proposition 4.

4.4.1. Preparatory Material

Lemma 2. Assume that $\sigma_n^2 \to \infty$ and (2.5) holds. Then

$$\max_{1\leqslant \ell \leqslant n} \mathbb{E}\big(\mathbb{E}(S_\ell|\mathcal{M}_0)\big)^2 = o(\sigma_n^2), \quad \text{as } n \to \infty. \tag{4.62}$$

Proof of Lemma 2: Let $\varepsilon > 0$. Since $\mathbb{E}\big(\mathbb{E}(S_n\,|\mathcal{M}_0)\big)^2 = o(\sigma_n^2)$, as $n \to \infty$, there exists n_0 such that for all $\ell \geqslant n_0$,

$$\frac{\|\mathbb{E}(S_\ell\,|\mathcal{M}_0)\|_2}{\sigma_n} \leqslant \varepsilon. \tag{4.63}$$

Notice now that the stationarity obviously implies that for all $1 \leqslant \ell \leqslant n$,

$$\sigma_n^2 = \sigma_\ell^2 + \sigma_{n-\ell}^2 + 2\mathbb{E}\left(S_{n-\ell}\sum_{k=n-\ell+1}^{n} X_k\right). \tag{4.64}$$

Using Cauchy–Schwarz's inequality together with (4.63), we get for $n_0 \leqslant \ell \leqslant n$,

$$\left|\mathbb{E}\left(S_{n-\ell}\sum_{k=n-\ell+1}^{n} X_k\right)\right| = \left|\mathbb{E}\left(S_{n-\ell}\,\mathbb{E}\left(\sum_{k=n-\ell+1}^{n} X_k|\mathcal{M}_{n-\ell}\right)\right)\right|$$
$$\leqslant \sigma_{n-\ell}\,\|\mathbb{E}(S_\ell\,|\mathcal{M}_0)\|_2 \leqslant \varepsilon\,\sigma_{n-\ell}\,\sigma_\ell.$$

Starting from (4.64) and using the last inequality, we derive that

$$\sigma_n^2 \geqslant \sigma_\ell^2 + \sigma_{n-\ell}^2 - 2\varepsilon\,\sigma_{n-\ell}\,\sigma_\ell \geqslant (\sigma_\ell^2 + \sigma_{n-\ell}^2)(1-\varepsilon) \geqslant \sigma_\ell^2(1-\varepsilon).$$

Therefore,

$$\begin{aligned}
&\frac{\max_{1\leqslant \ell \leqslant n}\mathbb{E}\big(\mathbb{E}(S_\ell|\mathcal{M}_0)\big)^2}{\sigma_n^2} \\
&= \frac{\max_{1\leqslant \ell < n_0}\mathbb{E}\big(\mathbb{E}(S_\ell|\mathcal{M}_0)\big)^2}{\sigma_n^2} + \frac{\max_{n_0\leqslant \ell \leqslant n}\mathbb{E}\big(\mathbb{E}(S_\ell|\mathcal{M}_0)\big)^2}{\sigma_n^2} \\
&\leqslant \frac{\max_{1\leqslant \ell < n_0}\mathbb{E}\big(\mathbb{E}(S_\ell|\mathcal{M}_0)\big)^2}{\sigma_n^2} + \frac{\max_{n_0\leqslant \ell \leqslant n}\varepsilon^2\sigma_\ell^2}{\sigma_n^2} \\
&\leqslant \frac{\max_{1\leqslant \ell < n_0}\mathbb{E}\big(\mathbb{E}(S_\ell\,|\mathcal{M}_0)\big)^2}{\sigma_n^2} + \frac{\varepsilon^2}{1-\varepsilon},
\end{aligned}$$

which converges to zero by letting first n to tend to infinity and after by letting ε to tend to zero. □

The next lemma deals with a moment inequality for the maximum of partial sums of real random variables and is a direct consequence of Theorem 2.5 in Rio[32] and of a result of Móricz.[25]

Lemma 3. Assume that $\{X_k, k \in \mathbb{Z}\}$ is a strictly stationary sequence of real zero mean random variables, such that for each $k \geqslant 1$, $\mathbb{P}(|X_k| \leqslant T) = 1$. Then

$$\mathbb{E}\left(\sum_{i=1}^{m} X_i\right)^4 \leqslant 32T^2m^2\Big\{\mathbb{E}(X_0)^2 + \sup_{\ell \in [1,m]} \mathbb{E}\big(\mathbb{E}(S_\ell \,|\mathcal{M}_0)\big)^2\Big\}.$$

In addition, there exists an universal constant C such that

$$\mathbb{E}\left(\max_{1\leqslant j\leqslant m}\left|\sum_{i=1}^{j} X_i\right|^4\right) \leqslant CT^2m^2\Big\{\mathbb{E}(X_0)^2 + \sup_{\ell \in [1,m]} \mathbb{E}\big(\mathbb{E}(S_\ell \,|\mathcal{M}_0)\big)^2\Big\}.$$

Remark 9. C can be chosen equal to 50×10^3.

Proof of Lemma 3: By using Theorem 2.5 of Rio[(32)] combined with stationarity and the fact that the sequence is bounded, we successively get for all $a \geqslant 0$ and all $m \geqslant 1$

$$\begin{aligned}\mathbb{E}\left(\sum_{i=a+1}^{a+m} X_i\right)^4 &\leqslant 16m^2 \sup_{\ell \in [1,m]} \mathbb{E}\Big(\big|X_0\big(X_0 + \mathbb{E}(S_{\ell-1}|\mathcal{M}_0)\big)\big|^2\Big)\\ &\leqslant 16m^2T^2 \sup_{\ell \in [2,m]} \mathbb{E}\Big(\big|X_0 + \mathbb{E}(S_{\ell-1}|\mathcal{M}_0)\big|^2\Big)\\ &\leqslant 32T^2m^2\Big\{\mathbb{E}(X_0)^2 + \sup_{\ell \in [1,m]} \mathbb{E}\big(\mathbb{E}(S_\ell \,|\mathcal{M}_0)\big)^2\Big\}.\end{aligned}$$

Now set $g(m) := 4\sqrt{2}Tm\Big(\mathbb{E}(X_0)^2 + \sup_{\ell \in [1,m]} \mathbb{E}\big(\mathbb{E}(S_\ell \,|\mathcal{M}_0)\big)^2\Big)^{1/2}$. Observe that for all $1 \leqslant m < m+n$, we obviously have $g(m) + g(n) \leqslant g(m+n)$. Then applying Theorem 1 in Móricz (1976), we get that for all $a \geqslant 0$ and all $m \geqslant 1$

$$\begin{aligned}\mathbb{E}\left(\max_{a+1\leqslant j\leqslant a+m}\left|\sum_{i=1}^{j} X_i\right|^4\right) &\leqslant 32(1-2^{-1/4})^{-4}T^2m^2\\ &\quad\times\Big\{\mathbb{E}(X_0)^2 + \sup_{\ell \in [1,m]} \mathbb{E}\big(\mathbb{E}(S_\ell \,|\mathcal{M}_0)\big)^2\Big\}\end{aligned}$$

and the desired result holds. □

4.4.2. The Selection of the Sequence q_n.

Let us start with some useful remarks: Since (under (2.1)) $\sigma_n^2 \to \infty$, one has that

$$\lim_{u\to 0^+} Q(u) > 0 \text{ (the limit may be } \infty). \tag{4.65}$$

Indeed, since $\sigma_n^2 \to \infty$, one has that $\mathbb{P}(|X_0| > 0) > 0$. (Otherwise, one would have $X_0 = 0$ a.s., and $S_n = 0$ a.s. for all $n \geqslant 1$, contradicting the fact that $\sigma_n^2 \to \infty$). For $u_1 = (1/2)\mathbb{P}(|X_0| > 0)$, one has that $u_1 < \mathbb{P}(|X_0| > 0)$ and hence $0 < Q(u_1)$. Now (4.65) follows from the fact that Q is a nonincreasing function.

On the other hand, for all $A \in (0, 1)$,

$$\mathbb{E}(X_0^2) = \int_0^1 Q^2(u)du \leqslant A^{-1}\int_0^A Q^2(u)du.$$

Then, it follows that

$$\mathbb{E}(X_0^2)\frac{n^2}{\sigma_n^2}G\Big(\frac{\|\mathbb{E}(S_n|\mathcal{M}_{-n})\|_1}{n}\Big) \leqslant \frac{n^2}{\sigma_n^2}\int_0^{G\left(\frac{\|\mathbb{E}(S_n|\mathcal{M}_{-n})\|_1}{n}\right)} Q^2(u)du. \tag{4.66}$$

According to (2.1) and to (2.10), Inequality (4.66) yields that

$$\lim_{n\to\infty} G\Big(\frac{\|\mathbb{E}(S_n|\mathcal{M}_{-n})\|_1}{n}\Big) = 0. \tag{4.67}$$

Set now

$$T_n = Q \circ G\Big(\frac{\|\mathbb{E}(S_{n-1}|\mathcal{M}_{-(n-1)})\|_1}{n-1}\Big). \tag{4.68}$$

By combining (4.65) with (4.67), it is clear that

$$\lim_{n\to\infty} T_n > 0 \text{ (the limit may be } \infty). \tag{4.69}$$

We are now in the position to give the selection of the sequence of integers q_n: let q_n be the greatest nonnegative integer such that

$$\frac{q_n^3 T_{q_n}^2}{\sigma_{q_n}^2} \leqslant n. \tag{4.70}$$

Using (4.69) and (2.1), it is easy to see that $q_n \to \infty$, as $n \to \infty$ (and hence $q_n = 0$ for at most finitely many n), and also that $q_n = o(n)$. In addition the definition of q_n obviously implies

$$\frac{(q_n+1)^3\, T^2_{q_n+1}}{\sigma^2_{q_n+1}} > n. \tag{4.71}$$

4.4.3. End of the Proof of Theorem 3.

Clearly Condition (2.4) implies Item (c) of (4.1). Then, in order to apply Proposition 3, it remains to show that the selection of q_n described in Section 4.4.2 entails also that the other conditions are satisfied.

Step 1. We first verify Item (a) of (4.1). With this aim, we use (4.71) and stationarity to write that

$$\sqrt{\frac{n}{q_n}}\,\frac{\mathbb{E}|\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})|}{\sigma_{q_n}} \ll \frac{(q_n+1)\, T_{q_n+1}}{\sigma_{q_n+1}\sigma_{q_n}}\mathbb{E}|\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})|. \tag{4.72}$$

Now by (2.1) it follows that

$$\sigma^2_{q_n} \sim \sigma^2_{q_n+1}, \text{ as } n \to \infty. \tag{4.73}$$

Starting from Inequality (4.72) and using (4.73), we derive that

$$\sqrt{\frac{n}{q_n}}\,\frac{\mathbb{E}|\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})|}{\sigma_{q_n}} \ll \frac{q_n^2}{\sigma^2_{q_n}}\,\frac{\|\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})\|_1}{q_n}\, Q\circ G\Big(\frac{\|\mathbb{E}(S_{q_n}|\mathcal{M}_{-q_n})\|_1}{q_n}\Big),$$

which converges to zero according to (2.10) by using the fact that $Q\circ G(\cdot)$ is a nonincreasing function. Hence Item (a) of (4.1) holds.

Step 2. We shall verify now that the conditions of Theorem 3 entails that Condition (2.5) holds, and then Item (d) of (4.1) also. With this aim notice first that (2.10) together with (2.1) entail that

$$\|\mathbb{E}(S_n|\mathcal{M}_{-n})\|_1 = o(\sigma_n) \text{ as } n \to \infty. \tag{4.74}$$

Now (2.4) clearly implies that $\{\sigma_n^2 \mathbb{E}^2(S_n|\mathcal{M}_{-n})\}$ is uniformly integrable which combined with (4.74) entails that (4.48) holds. Hence according to Lemma 1 Condition (2.5) is satisfied.

Step 3. We end the proof by verifying Condition (4.3). With this aim, we need to use a truncation argument. We truncate the variables X_i in the following way:

$$X_i' = \begin{cases} X_i \mathbb{I}\big(|X_i| \leqslant T_{q_n}\big) - \mathbb{E}X_i\mathbb{I}\big(|X_i| \leqslant T_{q_n}\big) & \text{if } X_i \text{ is an unbounded sequence} \\ X_i & \text{if ess.sup}|X_i| = T \text{ a.s.} \end{cases}$$

$$\text{and} \tag{4.75}$$

$$X_i'' = \begin{cases} X_i \mathbb{I}\big(|X_i| > T_{q_n}\big) - \mathbb{E}X_i\mathbb{I}\big(|X_i| > T_{q_n}\big) & \text{if } X_i \text{ is an unbounded sequence} \\ 0 & \text{if ess.sup}|X_i| = T \text{ a.s.} \end{cases}$$

Define $S'_n=\sum_{i=1}^n X'_i$ and $S''_n=\sum_{i=1}^n X''_i$. Now observe that for any $\varepsilon>0$

$$\begin{aligned}
&\sigma_{q_n}^{-2}\mathbb{E}\left(\max_{1\leqslant i\leqslant q_n} S_i^2\mathbb{I}\big(\sigma_{q_n}^{-1}\max_{1\leqslant i\leqslant q_n}|S_i|\geqslant\varepsilon(n/q_n)^{1/2}\big)\right)\\
&\quad=\sigma_{q_n}^{-2}\mathbb{E}\left(\max_{1\leqslant i\leqslant q_n}(S'_i+S''_i)^2\mathbb{I}\big(\sigma_{q_n}^{-1}\max_{1\leqslant i\leqslant q_n}|S'_i+S''_i|\geqslant\varepsilon(n/q_n)^{1/2}\big)\right)\\
&\quad\leqslant 2\sigma_{q_n}^{-2}\mathbb{E}\left(\max_{1\leqslant i\leqslant q_n}(S'_i)^2\mathbb{I}\big(\sigma_{q_n}^{-1}\max_{1\leqslant i\leqslant q_n}|S'_i|\geqslant\frac{\varepsilon}{2}(n/q_n)^{1/2}\big)\right)\\
&\qquad+2\sigma_{q_n}^{-2}\mathbb{E}\big(\max_{1\leqslant i\leqslant q_n}(S''_i)^2\big).
\end{aligned}\tag{4.76}$$

We first treat the last term of the inequality. Using Proposition 1 in Dedecker and Rio[8], we get:

$$\sigma_{q_n}^{-2}\mathbb{E}\big(\max_{1\leqslant i\leqslant q_n}(S''_i)^2\big)\leqslant 8\sigma_{q_n}^{-2}\sum_{i=1}^{q_n}\mathbb{E}(X''_i)^2+16\sigma_{q_n}^{-2}\sum_{i=1}^{q_n}\mathbb{E}\Big|X''_i\mathbb{E}(S''_{q_n}-S''_i\Big|\mathcal{M}_i)\Big|.$$

Hence by stationarity

$$\sigma_{q_n}^{-2}\mathbb{E}\left(\max_{1\leqslant i\leqslant q_n}(S''_i)^2\right)\leqslant 24\,\sigma_{q_n}^{-2}q_n^2\mathbb{E}(X''_0)^2\leqslant 48\,\sigma_{q_n}^{-2}q_n^2\mathbb{E}\big(X_0^2\mathbb{I}(|X_0|>T_{q_n})\big).\tag{4.77}$$

Observe that

$$Q_{|X_0|\mathbb{I}(|X_0|>T_{q_n})}(u)=Q(u)\mathbb{I}(u<\ P(|X_0|>T_{q_n})).\tag{4.78}$$

It follows that

$$\mathbb{E}\big(X_0^2\mathbb{I}(|X_0|>T_{q_n})\big)=\int_0^{\frac{\|\mathbb{E}(S_{q_n-1}|\mathcal{M}_{-(q_n-1)})\|_1}{q_n-1}}Q\circ G(u)du.\tag{4.79}$$

Using (4.73) together with (4.77) and (4.79), we derive from (2.10) that

$$\lim_{n\to\infty}\sigma_{q_n}^{-2}\mathbb{E}\big(\max_{1\leqslant i\leqslant q_n}(S''_i)^2\big)=0.\tag{4.80}$$

Now we compute the first term in (4.76). We obviously have for all $\varepsilon>0$,

$$\sigma_{q_n}^{-2}\mathbb{E}\left(\max_{1\leqslant i\leqslant q_n}S'^2_i\mathbb{I}\big(\max_{1\leqslant i\leqslant q_n}|S'_i|\geqslant\varepsilon\sigma_{q_n}(n/q_n)^{1/2}\big)\right)\leqslant\frac{1}{\varepsilon^2}\frac{q_n\mathbb{E}\left(\max_{1\leqslant i\leqslant q_n}S'^4_i\right)}{n\sigma_{q_n}^4}.$$

Lemma 3 together with (4.70) insure that

$$\frac{q_n}{n\sigma_{q_n}^4}\mathbb{E}\big(\max_{1\leqslant i\leqslant q_n} S_i'^4\big) \ll \frac{q_n^3 T_{q_n}^2}{n\sigma_{q_n}^4}\Big\{\mathbb{E}(X_0^2)+\sup_{\ell\in[1,q_n]}\mathbb{E}\big(\mathbb{E}(S_\ell'|\mathcal{M}_0)\big)^2\Big\}$$
$$\ll \frac{1}{\sigma_{q_n}^2}\Big\{\mathbb{E}(X_0^2)+\sup_{\ell\in[1,q_n]}\mathbb{E}\big(\mathbb{E}(S_\ell'|\mathcal{M}_0)\big)^2\Big\}. \quad (4.81)$$

Now notice that

$$\sigma_{q_n}^{-2}\sup_{\ell\in[1,q_n]}\mathbb{E}\big(\mathbb{E}(S_\ell'|\mathcal{M}_0)\big)^2\leqslant 2\sigma_{q_n}^{-2}\sup_{\ell\in[1,q_n]}\mathbb{E}\big(\mathbb{E}(S_\ell|\mathcal{M}_0)\big)^2+2\sigma_{q_n}^{-2}\sup_{\ell\in[1,q_n]}\mathbb{E}(S_\ell'')^2. \quad (4.82)$$

The last term of the above inequality has been shown to tend to zero (see (4.80)), whereas to show that the first one also converges to zero, we use the step 2 of the proof together with Lemma 2. Then we have shown that for all $\varepsilon>0$,

$$\lim_{n\to\infty}\sigma_{q_n}^{-2}\mathbb{E}\left(\max_{1\leqslant i\leqslant q_n} S_i'^2\mathbb{I}\big(\max_{1\leqslant i\leqslant q_n}|S_i'|\geqslant\varepsilon\sigma_{q_n}(n/q_n)^{1/2}\big)\right)=0, \quad (4.83)$$

which combined with (4.80) and (4.76) entails that (4.3) holds.

4.5. Proof of Corollary 3.

We first need a preliminary lemma.

Lemma 4. Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space and $\mathcal{M}$ be a σ-algebra of $\mathcal{A}$. Let X and Y be two integrable random variables such that $|XY|$ is integrable. The following inequality holds:

$$\mathbb{E}|X\mathbb{E}(Y|\mathcal{M})|\leqslant\frac{3}{2}\left\{\int_0^{\|\mathbb{E}(Y|\mathcal{M})\|_1} Q_X\circ G_X(u)du+\int_0^{\|\mathbb{E}(Y|\mathcal{M})\|_1} Q_Y\circ G_Y(u)du\right\}.$$

Hence if $Q_X=Q_Y=Q$,

$$\mathbb{E}|X\mathbb{E}(Y|\mathcal{M})|\leqslant 3\int_0^{\|\mathbb{E}(Y|\mathcal{M})\|_1} Q\circ G(u)du. \quad (4.84)$$

Remark 10. Note that an inequality quite similar to (4.84) is stated in Dedecker and Doukhan[10] (see their Proposition 1 and its proof). However, compared to their result, we do not need to assume that X is a $\mathcal{M}$-measurable random variable.

Proof of Lemma 4. First we set $T = Q_X \circ G_X(\|\mathbb{E}(Y|\mathcal{M})\|_1)$. Then we write

$$\mathbb{E}|X\mathbb{E}(Y|\mathcal{M})| = \mathbb{E}\big|X\mathbb{I}\big(|X| \leqslant T\big)\mathbb{E}(Y|\mathcal{M})\big| + \mathbb{E}\big|X\mathbb{I}\big(|X| > T\big)\mathbb{E}(Y|\mathcal{M})\big|.$$

Next using Cauchy–Schwarz's inequality, we derive that

$$\begin{aligned}\mathbb{E}|X\mathbb{E}(Y|\mathcal{M})| &\leqslant \|\mathbb{E}(Y|\mathcal{M})\|_1 Q_X \circ G_X(\|\mathbb{E}(Y|\mathcal{M})\|_1) \\ &\quad + \Big(\mathbb{E}\big(X^2\mathbb{I}(|X| > T)\big)\Big)^{1/2} \|\mathbb{E}(Y|\mathcal{M})\|_2 \\ &\leqslant \|\mathbb{E}(Y|\mathcal{M})\|_1 Q_X \circ G_X(\|\mathbb{E}(Y|\mathcal{M})\|_1) \\ &\quad + \frac{1}{2}\mathbb{E}\big(X^2\mathbb{I}(|X| > T)\big) + \frac{1}{2}\|\mathbb{E}(Y|\mathcal{M})\|_2^2. \qquad (4.85)\end{aligned}$$

To treat the first term in the right-hand side, we use the fact that $Q_X \circ G_X(\cdot)$ is a nonincreasing function. Then

$$\|\mathbb{E}(Y|\mathcal{M})\|_1 Q_X \circ G_X(\|\mathbb{E}(Y|\mathcal{M})\|_1) \leqslant \int_0^{\|\mathbb{E}(Y|\mathcal{M})\|_1} Q_X \circ G_X(u)\,du. \qquad (4.86)$$

Now, from the observation (4.78), we infer that

$$\mathbb{E}\big(X^2\mathbb{I}(|X| > T)\big) \leqslant \int_0^{G_X(\|\mathbb{E}(Y|\mathcal{M})\|_1)} Q_X^2(u)\,du,$$

and using the change-of-variable: $v = H_X(u)$, we get

$$\mathbb{E}\big(X^2\mathbb{I}(|X| > T)\big) \leqslant \int_0^{\|\mathbb{E}(Y|\mathcal{M})\|_1} Q_X \circ G_X(v)\,dv. \qquad (4.87)$$

Starting from Inequality (4.85) and using (4.86)and (4.87), we derive that

$$\mathbb{E}|X\mathbb{E}(Y|\mathcal{M})| \leqslant \frac{3}{2}\int_0^{\|\mathbb{E}(Y|\mathcal{M})\|_1} Q_X \circ G_X(u)\,du + \frac{1}{2}\|\mathbb{E}(Y|\mathcal{M})\|_2^2. \qquad (4.88)$$

Now, by using (4.88) with $X = Y$, we get that

$$\|\mathbb{E}(Y|\mathcal{M})\|_2^2 \leqslant \mathbb{E}|Y\mathbb{E}(Y|\mathcal{M})| \leqslant \frac{3}{2}\int_0^{\|\mathbb{E}(Y|\mathcal{M})\|_1} Q_Y \circ G_Y(u)\,du + \frac{1}{2}\|\mathbb{E}(Y|\mathcal{M})\|_2^2,$$

and hence

$$\frac{1}{2}\,\|\mathbb{E}(Y|\mathcal{M})\|_2^2 \leqslant \frac{3}{2}\int_0^{\|\mathbb{E}(Y|\mathcal{M})\|_1} Q_Y \circ G_Y(u)du. \tag{4.89}$$

Inequality (4.88) together with (4.89) give the desired result. □

We turn now to the proof of Theorem 3. First, due to the fact that $\{\|\mathbb{E}(X_n|\mathcal{M}_0)\|_1, n \geqslant 1\}$ is a nonincreasing sequence, we notice that

$$n^{-1}\,\|\mathbb{E}(S_n|\mathcal{M}_{-n})\|_1 \leqslant \|\mathbb{E}(X_n|\mathcal{M}_0)\|_1.$$

Starting from this inequality, we infer that (2.10) holds provided (2.11) does.

It remains to prove now that Condition (2.11) implies (2.1). According to Remark 2, this will hold if we prove that (2.5) is satisfied.
Notice that

$$\frac{\mathbb{E}\big(\mathbb{E}(S_n|\mathcal{M}_0)\big)^2}{\sigma_n^2} \leqslant \frac{2}{\sigma_n^2}\sum_{i=1}^{n}\sum_{j=1}^{i}\mathbb{E}|X_j\,\mathbb{E}(X_i|\mathcal{M}_0)|.$$

Next, using Lemma 4 together with the fact that, by stationarity, $Q_{X_i} = Q_{X_j} = Q$, we derive that

$$\frac{\mathbb{E}\big(\mathbb{E}(S_n|\mathcal{M}_0)\big)^2}{\sigma_n^2} \leqslant \frac{6}{\sigma_n^2}\sum_{i=1}^{n} i \int_0^{\|\mathbb{E}(X_i|\mathcal{M}_0)\|_1} Q \circ G(u)du, \tag{4.90}$$

which converges to zero provided that Condition (2.11) holds.

4.6. Proof of Theorem 4

By using the fact that $Q \circ G(\cdot)$ is nonincreasing and Lemma 1 in Dedecker and Doukhan (2003), Condition (3.2) implies (2.11). Then Theorem 4 will follow from Corollary 3 if we can prove that

$$\lim_{n\to\infty} \sigma_n^{-2}\|\mathbb{E}(S_n^2|\mathcal{M}_{-n}) - \mathbb{E}(S_n^2)\|_1 = 0. \tag{4.91}$$

But by using stationarity, we get that

$$\begin{aligned}\sigma_n^{-2}\|\mathbb{E}(S_n^2|\mathcal{M}_{-n})-\mathbb{E}(S_n^2)\|_1 &\leqslant \sigma_n^{-2}\sum_{i=n+1}^{2n}\sum_{j=n+1}^{2n}\|\mathbb{E}(X_iX_j|\mathcal{M}_0)-\mathbb{E}(X_iX_j)\|_1\\ &\leqslant 2\sigma_n^{-2}\sum_{i=n+1}^{2n}\sum_{j=i}^{2n}\|\mathbb{E}(X_iX_j|\mathcal{M}_0)-\mathbb{E}(X_iX_j)\|_1.\end{aligned} \tag{4.92}$$

By using Rio's covariance inequality (1993) and the fact that $Q_{|X_iX_j-\mathbb{E}(X_iX_j)|}(u)\leqslant Q_{|X_iX_j|}(u)+|\mathbb{E}(X_iX_j)|$, and that for every $A\in(0,1)$: $\int_0^A Q_{|X_iX_j|}(u)du\leqslant\int_0^A Q^2(u)du$ (see Lemma 2.1 in Rio (2000)), it is easy to infer that for $j\geqslant i$,

$$\|\mathbb{E}(X_iX_j|\mathcal{M}_0)-\mathbb{E}(X_iX_j)\|_1\leqslant 8\int_0^{\alpha_{2,\infty}(i)} Q^2(u)du.$$

Then starting from (4.92) and since $(\alpha_{2,\infty}(i))_{i\geqslant 1}$ is nonincreasing, we derive that

$$\sigma_n^{-2}\|\mathbb{E}(S_n^2|\mathcal{M}_{-n})-\mathbb{E}(S_n^2)\|_1\leqslant 16n^2\sigma_n^{-2}\int_0^{\alpha_{2,\infty}(n)} Q^2(u)du,$$

which converges to zero as soon as (3.2) holds by using again the fact that $(\alpha_{2,\infty}(i))_{i\geqslant 1}$ is nonincreasing. This ends the proof of (4.91) and of the theorem.

4.7. Proof of Proposition 1

The proof is an application of Theorem 1 and is based on the fact that

$$\mathbb{E}(S_n|\mathcal{M}_0)=\sum_{j=0}^{\infty}(b_{n+j}-b_j)\xi_{-j}$$

and that

$$S_n=\sum_{j=1}^{n}b_{n-j}\xi_j+\sum_{j=0}^{\infty}(b_{n+j}-b_j)\xi_{-j}:=T_n+V_n. \tag{4.93}$$

Then $\mathbb{E}(\mathbb{E}(S_n|\mathcal{M}_0))^2 = \sum_{j=0}^{\infty}(b_{n+j}-b_j)^2\sigma^2$ and

$$\sigma_n^2 = \sigma^2\left(\sum_{k=0}^{n-1} b_k^2 + \sum_{j=0}^{\infty}(b_{n+j}-b_j)^2\right). \tag{4.94}$$

It follows that (3.4) and (3.5) imply (2.5) and then, according to Remark 2, (2.1) and (2.3) both hold. Now, in order to verify (2.4), notice that, by using the fact that $\mathbb{E}(\xi_i\xi_j|\mathcal{M}_{-n}) = 0$ if $i \neq j$ and if either one of the indexes i or j is strictly larger than $-n$, we obtain

$$\begin{aligned}\mathbb{E}(S_n^2|\mathcal{M}_{-n}) &= \sum_{j=0}^{\infty}(b_{n+j}-b_j)^2\mathbb{E}(\xi_{-j}^2|\mathcal{M}_{-n}) + \sum_{j=1}^{n} b_{n-j}^2\,\mathbb{E}(\xi_j^2|\mathcal{M}_{-n})\\ &\quad + \sum_{i=n}^{\infty}\ \sum_{j=n, j\neq i}^{\infty} (b_{n+j}-b_j)(b_{n+i}-b_i)\mathbb{E}(\xi_{-i}\xi_{-j}|\mathcal{M}_{-n})\\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Notice first that Condition (3.5) implies that $\lim_{n\to\infty}\dfrac{\mathbb{E}|I_1|}{\sigma_n^2} = 0$. Also,

$$\begin{aligned}\mathbb{E}|I_3| &\leqslant \mathbb{E}\Big(\sum_{i=n}^{\infty}(b_{n+i}-b_i)\xi_{-i}\Big)^2\\ &= \sigma^2\sum_{i=n}^{\infty}(b_{n+i}-b_i)^2.\end{aligned}$$

Then, by using once again Condition (3.5), we derive that $\lim_{n\to\infty}\dfrac{\mathbb{E}|I_3|}{\sigma_n^2} = 0$.

In order to deal with I_2 it is easy to see, by taking into account (4.94) and (3.5), that it is enough to prove that

$$\frac{\sum_{j=1}^{n} b_{n-j}^2\xi_j^2}{\sigma^2\sum_{j=0}^{n-1} b_j^2} \text{ is convergent to } \sigma^{-2}\mathbb{E}(\xi_0^2|\mathcal{I}) \text{ in } \mathbb{L}^1.$$

Since

$$\frac{\sum_{j=1}^{n} b_{n-j}^2\xi_j^2}{\sum_{j=0}^{n-1} b_j^2} = \frac{\sum_{k=0}^{n-1} b_k^2\xi_{n-k}^2}{\sum_{j=0}^{n-1} b_j^2} = \frac{\sum_{k=0}^{n-1} b_k^2\xi_{-k}^2}{\sum_{j=0}^{n-1} b_j^2}\circ T^n,$$

the conclusion will follow if we prove that

$$\frac{\sum_{k=0}^{n-1} b_k^2\xi_{-k}^2}{\sum_{j=0}^{n-1} b_j^2} \text{ is convergent to } \mathbb{E}(\xi_0^2|\mathcal{I}) \text{ in } \mathbb{L}^1.$$

This convergence will be a consequence of theorem 4.2.10 (ii) in Stout[35] if we establish that b_n^2 is a slowly varying function. Notice first that since $\sigma_n^2 = nh(n)$ with $h(n)$ a slowly varying function in the strong sense, then by using (4.94) and (3.5), $\sigma^2 \sum_{j=0}^{n-1} b_j^2$ is equivalent to $nh(n)$. Then, by using Theorem 1.9.8. in Bingham et al.[4] it is easy to show that b_n^2 is equal to a certain slowly varying function equivalent to $h(n)$.

By all the previous considerations, we derive that Condition (2.4) is verified as soon as Condition (3.5) holds.
We prove now that (3.5) entails (2.2). Since under (3.5), $\lim_{n\to\infty} \sigma_n^{-2} \mathbb{E}(V_n^2) = 0$, we only have to prove that

$$\frac{T_n'^2}{\sigma_n^2} \text{ is uniformly integrable.} \tag{4.95}$$

With this aim, for a given positive number T define the four variables

$$\xi_i' = \xi_i \mathbb{I}(|\xi_i| \leqslant T) \text{ and } \xi_i'' = \xi_i \mathbb{I}(|\xi_i| > T)$$
$$\zeta_i' = \xi_i' - \mathbb{E}(\xi_i' | \mathcal{M}_{i-1}) \text{ and } \zeta_i'' = \xi_i'' - \mathbb{E}(\xi_i'' | \mathcal{M}_{i-1}).$$

Since (ξ_i) is a sequence of martingale differences, it follows that $\xi_i = \zeta_i' + \zeta_i''$. Then $T_n = \sum_{i=1}^n b_{n-i} \zeta_i' + \sum_{i=1}^n b_{n-i} \zeta_i'' := T_n' + T_n''$. By taking into account this decomposition, we get for any positive A,

$$\mathbb{E}\left(\frac{T_n^2}{\sigma_n^2} \mathbb{I}(|T_n| \geqslant 2A\sigma_n)\right) \leqslant 2\mathbb{E}\Big(\frac{T_{n'}^2}{\sigma_n^2} \mathbb{I}(|T_n'| \geqslant A\sigma_n)\Big) + 2\mathbb{E}\left(\frac{T_n''^2}{\sigma_n^2}\right). \tag{4.96}$$

Notice first that

$$\mathbb{E}\left(\frac{T_n''^2}{\sigma_n^2}\right) = \frac{\sum_{i=0}^{n-1} b_i^2}{\sigma_n^2} \mathbb{E}(\zeta_0'')^2 \leqslant \frac{\sum_{i=0}^{n-1} b_i^2}{\sigma_n^2} \mathbb{E}(\xi_0^2 \mathbb{I}(|\xi_0| > T)). \tag{4.97}$$

Since under (3.5), $\sigma_n^2 \sim \sigma^2 \sum_{i=0}^{n-1} b_i^2$, as $n \to \infty$, it follows from (4.98), that

$$\lim_{T\to\infty} \sup_n \mathbb{E}\left(\frac{T_n''^2}{\sigma_n^2}\right) = 0. \tag{4.98}$$

We study now the first term in (4.96). With this aim, we compute $\mathbb{E}(T_n'^4)$. Using Burkholder inequality followed by Minkowsky inequality, we successively derive that there exists an universal constant C such that

$$\mathbb{E}(T_n'^4) \leqslant C\mathbb{E}\left(\sum_{i=0}^{n-1} b_{n-i}^2 \zeta_i'^2\right)^2 \leqslant C\left(\sum_{i=0}^{n-1} b_i^2\right)^2 \|\zeta_0'^2\|_2^2.$$

Since $\|\zeta_0'^2\|_2 = \|\zeta_0'\|_4^2 \leqslant (2\|\xi_0'\|_4)^2 \leqslant \left(2T^{1/2}\mathbb{E}(\xi_0^2)^{1/4}\right)^2$, it follows that

$$\mathbb{E}(T_n'^4) \leqslant 16CT^2\|\xi_0\|_2^2 \left(\sum_{i=0}^{n-1} b_i^2\right)^2.$$

This upper bound together with (4.94) yield that

$$\mathbb{E}\left(\frac{T_n'^2}{\sigma_n^2}\mathbb{I}\left(|T_n'| \geqslant A\sigma_n\right)\right) \leqslant \frac{1}{A^2}\frac{\mathbb{E}(T_n'^4)}{\sigma_n^4} \leqslant \frac{16C}{\sigma^4 A^2}T^2\|\xi_0\|_2^2.$$

It follows that

$$\lim_{A\to\infty}\sup_n \mathbb{E}\left(\frac{T_n'^2}{\sigma_n^2}\mathbb{I}(|T_n'| \geqslant A\sigma_n)\right) = 0. \tag{4.99}$$

(4.96) together with (4.98) and (4.99) entail (4.95). The proof of the proposition is now complete.

4.8. Construction of Example 1

Our example is inspired by the construction of examples as in Herrndorf,[(16)] and also by the paper of Wu and Woodroofe.[(36)] Let us define two sequences $\{a_n, n \geqslant 0\}$ and $\{a_n', n \geqslant 0\}$ as follows:

$$a_0 = 0, a_1 = \frac{1}{\log 2} \text{ and } a_n = \frac{1}{\log(n+1)} - \frac{1}{\log n}, \text{ for } n \geqslant 2$$

and

$$a_0' = 1, a_1' = -1, a_2' = 0 \text{ and } a_n' = a_{n-2}, \text{ for } n \geqslant 3.$$

Let now $(\xi_i, i \in \mathbb{Z})$ be a sequence of independent, identically distributed and symmetric random variables such that

$$\mathbb{P}(\xi_0 > x) \sim \frac{1}{x^2 \log^{3/2} x}. \tag{4.100}$$

Define now two linear processes

$$X_k = \sum_{j\geqslant 0} a_j\, \xi_{k-j} \text{ and } X_k' = \sum_{j\geqslant 0} a_j'\, \xi_{k-j}.$$

Denote $\sigma_n^2 = \mathbb{E}(\sum_{k=1}^n X_k)^2$ and $\sigma_n'^2 = \mathbb{E}(\sum_{k=1}^n X_k')^2$. Since

$$\sum_{j=0}^{\infty}(b_{n+j} - b_j)^2 = O\left(\frac{n}{\log^3 n}\right) \text{ and } \sum_{k=0}^{n-1} b_k^2 \sim \frac{n}{(\log n)^2},$$

it follows that both of the linear processes $\{X_k, k\in\mathbb{Z}\}$ and $\{X'_k, k\in\mathbb{Z}\}$ satisfy the conditions of Proposition 1.

Now observe that

$$X'_{k+2} = X_k + \xi_{k+2} - \xi_{k+1},$$

then: $\sum_{k=1}^{n} X'_{k+2} = \sum_{k=1}^{n} X_k + \xi_{n+2} - \xi_2$. It follows that $\sigma_n'^2 \sim \sigma_n^2$. In addition, according to (4.100), classical computations yield that for every $\varepsilon > 0$,

$$\limsup_{n\to\infty} \mathbb{P}\left(\max_{1\leqslant k\leqslant n} |\xi_{k+2} - \xi_{k+1}| \geqslant \varepsilon\sqrt{n}/\log n\right) = 1. \tag{4.101}$$

As a consequence, it follows that the sequences $\dfrac{\sum_{i=1}^{[nt]} X_i}{\sigma_n}$ and $\dfrac{\sum_{i=1}^{[nt]} X'_i}{\sigma'_n}$ cannot satisfy the weak invariance principle at the same time. Indeed, if for instance the sequence $\dfrac{\sum_{i=1}^{[nt]} X_i}{\sigma_n}$ satisfies the weak invariance principle, then necessarily for every $\varepsilon \geqslant 0$, $\mathbb{P}(\max_{1\leqslant i\leqslant n} |X_i| \geqslant \varepsilon\sigma_n) \to 0$, as $n \to \infty$, and consequently from (4.101)

$$\limsup_{n\to\infty} \mathbb{P}\left(\max_{1\leqslant i\leqslant n} |X'_i| \geqslant \varepsilon\sigma'_n\right) \geqslant \limsup_{n\to\infty} \mathbb{P}\left(\max_{1\leqslant k\leqslant n} |\xi_{k+2} - \xi_{k+1}| \geqslant \varepsilon\sqrt{n}/\log n\right) = 1.$$

Then, for linear processes, the weak invariance principle cannot hold without additional assumptions to the conditions of Proposition 1.

4.9. Proof of Proposition 2

According to Remark 3, we have only to show (2.7), namely

$$\lim_{A\to\infty} \sup_n \mathbb{E}\left(\frac{\max_{1\leqslant i\leqslant n} S_i^2}{\sigma_n^2} \mathbb{I}(|\max_{1\leqslant i\leqslant n} S_i| \geqslant A\sigma_n)\right) = 0. \tag{4.102}$$

For this task, we shall show that there exists a finite constant K such that

$$\sup_n \mathbb{E}\left(\frac{|\max_{1\leqslant i\leqslant n} S_i|^{2+\delta}}{\sigma_n^{2+\delta}}\right) \leqslant K, \tag{4.103}$$

which obviously will imply (4.102). Let suppose that we can show that for every $k \geqslant 1$ and $m \geqslant 1$, there exists a positive constant K_δ depending only on δ, such that

$$\mathbb{E}|\sum_{i=k+1}^{k+m} X_i|^{2+\delta} \leqslant K_\delta \sigma_m^{2+\delta}, \tag{4.104}$$

According to the fact that under (3.4) and (3.5), $\sigma_n^2 = nh(n)$, where $h(n)$ is a slowly varying function, we can apply Lemma 3.7 in Peligrad[26] (her lemma is true with $2+\delta$ replacing 4) with $b(m) = K_\delta^{1/(2+\delta)} m^{\frac{\delta}{2(2+\delta)}} \sqrt{h(m)}$. It follows that for every $k \geqslant 1$ and $m \geqslant 1$, there exists a constant $\tilde{K}_\delta$ depending only on δ such that

$$\mathbb{E}\left(\max_{1\leqslant j\leqslant m}\left|\sum_{i=k+1}^{k+j} X_i\right|^{2+\delta}\right) \leqslant \tilde{K}_\delta m \left\{\sum_{j=1}^{\log m} \left(\left[\frac{m}{2^j}\right]\right)^{\frac{\delta}{2(2+\delta)}} \sqrt{h\left(\left[\frac{m}{2^j}\right]\right)}\right\}^{2+\delta}.$$

Then,

$$\mathbb{E}\left(\max_{1\leqslant j\leqslant m}\left|\sum_{i=k+1}^{k+j} X_i\right|^{2+\delta}\right) \leqslant \tilde{K}_\delta m^{(2+\delta)/2} (h(m))^{(2+\delta)/2} \left\{\sum_{j=1}^{\log m} \left(\left[\frac{1}{2^j}\right]\right)^{\frac{\delta}{2(2+\delta)}} \sqrt{\frac{h\left(\left[\frac{m}{2^j}\right]\right)}{h(m)}}\right\}^{2+\delta}. \tag{4.105}$$

Notice now that by the properties of the slowly varying function there exits $0 \leqslant \varepsilon < 2^{\delta/(2+\delta)} - 1$ such that for m big enough

$$\frac{h\left(\left[\frac{m}{2^j}\right]\right)}{h(m)} \leqslant (1+\varepsilon)^j. \tag{4.106}$$

Then if (4.104) holds, (4.103) follows from (4.105), (4.106) and the representation of σ_n^2. It remains to prove (4.104). By using stationarity and Decomposition (4.93), we get that for every $k \geqslant 1$ and $m \geqslant 1$,

$$\mathbb{E}\left|\sum_{i=k+1}^{k+m} X_i\right|^{2+\delta} = \mathbb{E}|S_m|^{2+\delta} \leqslant 2^{2+\delta}\left(\mathbb{E}|T_m|^{2+\delta} + \mathbb{E}|V_m|^{2+\delta}\right).$$

Applying now Burkholder inequality followed by Minkowski inequality, we derive that there exists a positive constant C_δ depending only on δ, such that

$$\mathbb{E}|T_m|^{2+\delta} \leqslant C_\delta \left(\sum_{i=0}^{n-1} b_i^2\right)^{(2+\delta)/2} \mathbb{E}|\xi_0|^{2+\delta}$$

and

$$\mathbb{E}|V_m|^{2+\delta} \leqslant C_\delta \left(\sum_{i=0}^{\infty} (b_{n+i} - b_i)^2\right)^{(2+\delta)/2} \mathbb{E}|\xi_0|^{2+\delta}.$$

Then by using (4.94), we get that for every $k \geqslant 1$ and $m \geqslant 1$,

$$\mathbb{E}|\sum_{i=k+1}^{k+m} X_i|^{2+\delta} \leqslant \frac{2^{3+\delta} C_\alpha \mathbb{E}|\xi_0|^{2+\delta}}{\sigma^{2+\delta}} \sigma_m^{2+\delta},$$

which establishes (4.104).

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