Rosenthal-type inequalities for the maximum of partial sums of stationary processes and examples

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Abbreviated Title: Rosenthal-type inequalities for stationary processes

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Summary

The aim of this paper is to propose new Rosenthal-type inequalities for moments of order p larger than 2 of the maximum of partial sums of stationary sequences including martingales and their generalizations. As in the recent results by Peligrad *et al.* (2007) and Rio (2009), the estimates of the moments are expressed in terms of the norms of projections of partial sums. The proofs of the results are essentially based on a new maximal inequality generalizing the Doob's maximal inequality for martingales and dyadic induction. Various applications are also provided.

1 Introduction

For independent random variables, the Rosenthal inequalities relate moments of order higher than 2 of partial sums of random variables to the variance of partial sums. One variant of this inequality is the following (see Rosenthal (1970), p. 279): let $(X_k)_k$ be independent and centered real valued random variables with finite moments of order p, $p \geq 2$. Then for every positive integer n,

$$\mathbf{E}\big(\max_{1\leq j\leq n} |S_j|^p\big) \ll \sum_{k=1}^{n} \mathbf{E}(|X_k|^p) + \big(\sum_{k=1}^{n} \mathbf{E}(X_k^2)\big)^{p/2}, \quad (1)$$

where $S_j = \sum_{k=1}^{j} X_k$. Unless otherwise specified, throughout the paper the notation $a_n \ll b_n$ means that there exists a numerical constant C_p depending only on p (and not on the underlying random variables and neither on n) such that $a_n \leq C_p b_n$, for all positive integers n.

Besides of being useful to compare the norms $\mathbf{L}^p$ and $\mathbf{L}^2$ of partial sums, these inequalities are important tools for obtaining a variety of results, including tightness of the empirical process (see the proof of Theorem 22.1 in Billingsley (1968)), convergence rates with respect to the strong law of large numbers (see for instance Wittmann (1985)) or almost sure invariance principles (see Wu (2007) and Gouëzel (2010) for recent results). Since the 70's, there has been a great amount of works which extended the inequality (1) to dependent sequences. See, for instance among many others: Peligrad (1985) and Shao (1995) for the case of ρ-mixing sequences; Shao (1988), Peligrad (1989) and Utev (1991) for the case of ϕ-mixing sequences; Peligrad and Gut (1999) and Utev and Peligrad (2003) for interlaced mixing; Theorem 2.2 in Viennet (1997) for β-mixing processes; Theorem 6.3 in Rio (2000) for the strongly mixing case; Dedecker (2001) and Rio (2009) for projective criteria.

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The main goal of the paper is to generalize the Rosenthal inequality from sequences of independent variables to stationary dependent sequences including martingales, allowing then to consider examples that are not necessarily dependent in the sense of the dependence structures mentioned above.

In order to present our results, let us first introduce some notations and definitions used all along the paper.

Notation 1 *Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and let $T : \Omega \mapsto \Omega$ be a bijective bi-measurable transformation preserving the probability $\mathbf{P}$. Let $\mathcal{F}_0$ be a σ-algebra of $\mathcal{A}$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$. We then define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$ and the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$, where X_0 is a real-valued random variable. The sequence will be called adapted to the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ if X_0 is $\mathcal{F}_0$-measurable. The following notations will also be used: $\mathbf{E}_k(X) = \mathbf{E}(X|\mathcal{F}_k)$ and the norm in $\mathbf{L}^p$ of X is denoted by $||X||_p$. Let $S_n = \sum_{j=1}^n X_j$.*

In the rest of this section the sequence $(X_i)_{i\in\mathbb{Z}}$ is assumed to be stationary and adapted to $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ and the variables are in $\mathbf{L}^p$.

If $(X_k)_k$ are stationary martingale differences, the martingale form of the inequality (1) is

$$|| \max_{1\le j\le n} |S_j| \;||_p \ll n^{1/p}||X_1||_p + \Big\| \sum_{k=1}^n \mathbf{E}_{k-1}(X_k^2)\Big\|_{p/2}^{1/2} \text{ for any } p \ge 2, \tag{2}$$

(see Burkholder (1973)). One of our goals is to replace the last term in this inequality with a new one containing terms of the form $\|\mathbf{E}_0(S_n^2)\|_{p/2}$. The reason for introducing this term comes from the fact that for many stationary sequences $\|\mathbf{E}_0(S_n^2)\|_{p/2}$ is closer to the variance of partial sums. In addition, we are interested to point out a Rosenthal-type inequality for a larger class of stationary adapted sequences that includes the martingale differences as a special case.

Two recent results by Peligrad and Utev (2005) and Wu and Zhao (2008) show that

$$|| \max_{1\le j\le n} |S_j| \;||_p \ll n^{1/p}\left(||X_1||_p + \sum_{k=1}^n \frac{1}{k^{1+1/p}}||\mathbf{E}_0(S_k)||_p \right) \text{ for any } 1 \le p \le 2.$$

To find a suitable extension of this inequality for $p > 2$, the first step in our approach is to establish the following maximal inequality that has interest in itself:

$$\| \max_{1\le j\le n} |S_j| \;\|_p \ll n^{1/p}\left(\max_{1\le j\le n} \|S_j\|_p/n^{1/p} + \sum_{k=1}^n \frac{1}{k^{1+1/p}}||\mathbf{E}_0(S_k)||_p \right) \text{ for any } p > 1. \tag{3}$$

This inequality can be viewed as generalization of the well-known Doob's maximal inequality for martingales. For a more precise version than (3), with constants specified, see our inequality (7).

Then, we combine the inequality (3) with several inequalities for $\|S_n\|_p$ that will further be established in this paper.

As we shall see in Section 3.1, by a direct approach using dyadic induction combined with the maximal inequality (3), we shall prove that, for any $p > 2$,

$$\| \max_{1\le j\le n} |S_j| \;\|_p \ll n^{1/p}\left(||X_1||_p + \sum_{k=1}^n \frac{1}{k^{1+1/p}}||\mathbf{E}_0(S_k)||_p + \Big(\sum_{k=1}^n \frac{1}{k^{1+2\delta/p}}\|\mathbf{E}_0(S_k^2)\|_{p/2}^{\delta}\Big)^{1/(2\delta)} \right), \tag{4}$$

where $\delta = \min(1, 1/(p-2))$. For $2 < p \le 3$ our inequality provides a maximal form for Theorem 3.1 in Rio (2009). When $p \ge 4$, we shall see that the last term in the right hand side dominates the second term, so that the second term can be omitted in this case. Inequality (4) shows that in order to relate $\| \max_{1\le j\le n} |S_j|\|_p$ to the vector $(\|S_j\|_2)_{1\le j\le n}$ we have to control $\sum_{k=1}^n k^{-(p+1)/p}||\mathbf{E}_0(S_k)||_p$ and $\sum_{k=1}^n k^{-(p+2\delta)/p}\|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2}^{\delta}$.

In Section 3.3.1, we study the case of stationary martingale difference sequences showing that for all even powers $p \geq 4$, the inequality (4) holds with $\delta = 2/(p-2)$. This result is possible for stationary martingale differences with the help of a special symmetrization for martingales initiated by Kwapień and Woyczynski (1991). In addition, by using martingale approximation techniques, we obtain, for any even integer p, another form of Rosenthal-type inequality than (4) for stationary adapted processes (see the section 3.3.2), that gives, for instance, better results for functionals of linear processes with independent innovations.

We also investigate the situation when the conditional expectation with respect to both the past and the future of the process is used. For instance, when $p \geq 4$ is an even integer, and the process is reversible, then the inequality (4) holds (see Theorem 9 and Corollary 29) with $\delta = 1$.

In Section 3.2 we show that our inequalities imply the Burkholder-type inequality as stated in Theorem 1 of Peligrad, Utev and Wu (1997). For the sake of applications in Section 3.4 we express the terms that appear in our Rosenthal inequalities in terms of individual summands.

Our paper is organized as follows. In Section 2, we prove a new maximal inequality allowing to relate the moments of the maximum of partial sums of an adapted sequence, that is not necessarily stationary, to the moments of its partial sums. The maximal inequality (3) combined with moment estimates allows us to obtain the Rosenthal-type inequalities stated in Theorems 6 and 9 of Section 3.1. Section 3.3 is devoted to Rosenthal-type inequalities for even powers for the special case of stationary martingale differences and to an application to stationary processes via a martingale approximation technique. In Section 4, we give other applications of the maximal inequalities stated in Section 2 and provide examples for which we compute the quantities involved in the Rosenthal-type inequalities of Section 3. One of the applications presented in this section is a Bernstein inequality for the maximum of partial sums for strongly mixing sequences, that extends the inequality in Merlevède *et al.* (2009). The applications are given to Arch models, to functions of linear processes and reversible Markov chains. In Section 5, we apply the inequality (4) to estimate the random term of the $\mathbf{L}^p$-integrated risk of kernel estimators of the unknown marginal density of a stationary sequence that is assumed to be β-mixing in the weak sense (see the definition 32). Some technical results are postponed to the Appendix.

2 Maximal inequalities for adapted sequences

The next proposition is a generalization of the well-known Doob's maximal inequality for martingales to adapted sequences. It states that the moment of order p of the maximum of the partial sums of an adapted process can be compared to the corresponding moment of the partial sum plus a correction term which is zero for martingale differences sequences. The proof is based on convexity and chaining arguments.

Proposition 2 *Let $p > 1$ and $q = p/(p-1)$. Let Y_i, $1 \leq i \leq 2^r$ be real random variables in $\mathbf{L}^p$, where r is a positive integer. Assume that the random variables are adapted to an increasing filtration $(\mathcal{F}_i)_i$. Let $S_r = Y_1 + \cdots + Y_r$. Then the following inequality holds:*

$$\| \max_{1 \leq i \leq 2^r} |S_i| \, \|_p \leq q \|S_{2^r}\|_p + q \sum_{l=0}^{r-1} \Big(\sum_{k=1}^{2^{r-l}-1} ||\mathbf{E}(S_{(k+1)2^l} - S_{k2^l} | \mathcal{F}_{k2^l})||_p^p \Big)^{1/p} . \qquad (5)$$

Corollary 3 *In the stationary case, we get that for any integer $r \geq 1$,*

$$\| \max_{1 \leq i \leq 2^r} |S_i| \, \|_p \leq q \|S_{2^r}\|_p + q 2^{r/p} \sum_{l=0}^{r-1} 2^{-l/p} ||\mathbf{E}(S_{2^l} | \mathcal{F}_0)||_p \ .$$

Remark 4 *The inequality in Corollary 3 easily implies that*

$$\| \max_{1\le i\le n} |S_i| \, \|_p \le 2q \max_{1\le m\le n} \|S_m\|_p + (2^{1/p}q) n^{1/p} \sum_{l=0}^{r-1} 2^{-l/p} ||\mathbf{E}(S_{2^l}|\mathcal{F}_0)||_p \,, \tag{6}$$

for any integer $n \in [2^{r-1}, 2^r[$, where r is a positive integer. Moreover, due to the subadditivity of the sequence $\big(||\mathbf{E}(S_n|\mathcal{F}_0)||_p\big)_{n\ge 1}$, according to Lemma 37, we also have that for any positive integer n,

$$\| \max_{1\le i\le n} |S_i| \, \|_p \le 2q \max_{1\le m\le n} \|S_m\|_p + (q \frac{2^{2+2/p}}{2^{1+1/p}-1}) n^{1/p} \sum_{j=1}^{n} j^{-1-1/p} ||\mathbf{E}(S_j|\mathcal{F}_0)||_p \,. \tag{7}$$

The inequalities (6) and (7) are true even if the variables are not centered.

For several applications involving exponential bounds we point out the following proposition.

Proposition 5 *Let $p > 1$ and $q = p/(p-1)$. Let $(Y_i)_{i\ge 1}$, be real random variables in $\mathbf{L}^p$. Assume that the random variables are adapted to an increasing filtration $(\mathcal{F}_i)_i$. Let $S_n = Y_1 + \cdots + Y_n$. Let φ be a nondecreasing, non negative, convex and even function. Then for any positive real x, and any positive integer r, the following inequality holds*

$$\mathbf{P}(\max_{1\le i\le 2^r} |S_i| \ge 2x) \le \frac{1}{\varphi(x)} \mathbf{E}(\varphi(S_{2^r})) + q^p x^{-p} \Big(\sum_{l=0}^{r-1} \Big(\sum_{k=1}^{2^{r-l}-1} ||\mathbf{E}(S_{(k+1)2^l} - S_{k2^l}|\mathcal{F}_{k2^l})||_p^p \Big)^{1/p} \Big)^p . \tag{8}$$

Assume in addition that there exists a positive real M such that $\sup_i \|Y_i\|_\infty \le M$. Then for any positive real x, and any positive integer r, the following inequality holds

$$\mathbf{P}(\max_{1\le i\le 2^r} |S_i| \ge 4x) \le \frac{1}{\varphi(x)} \mathbf{E}(\varphi(S_{2^r})) + q^p x^{-p} \Big(\sum_{l=0}^{r-1} \Big(\sum_{k=1}^{2^{r-l}-1} ||\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l}|\mathcal{F}_{k2^l})||_p^p \Big)^{1/p} \Big)^p , \tag{9}$$

with $v = [x/M]$ (where $[\,.\,]$ denotes the integer part).

Proof of Propositions 2 and 5. Denote $S^*_{2^r} = \max_{1\le i\le 2^r} |S_i|$. For any $m \in [0, 2^r - 1]$, we have that

$$S_{2^r-m} = \mathbf{E}(S_{2^r}|\mathcal{F}_{2^r-m}) - \mathbf{E}(S_{2^r} - S_{2^r-m}|\mathcal{F}_{2^r-m}) \,.$$

So,

$$S^*_{2^r} \le \max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r}|\mathcal{F}_{2^r-m})| + \max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r} - S_{2^r-m}|\mathcal{F}_{2^r-m})| \,. \tag{10}$$

Since $(\mathbf{E}(S_{2^r}|\mathcal{F}_k))_{k\ge 1}$ is a martingale, we shall use Doob's maximal inequality to deal with the first term in the right-hand side of (10). Hence, since φ is a nondecreasing, non negative, convex and even function, we get that

$$\mathbf{P}\big(\max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r}|\mathcal{F}_{2^r-m})| \ge x\big) \le \frac{1}{\varphi(x)} \mathbf{E}(\varphi(S_{2^r})) \,, \tag{11}$$

and also that

$$\| \max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r}|\mathcal{F}_{2^r-m})| \|_p \le q \|S_{2^r}\|_p \,. \tag{12}$$

Write now m in basis 2 as follows:

$$m = \sum_{i=0}^{r-1} b_i(m) 2^i, \quad \text{where } b_i(m) = 0 \text{ or } b_i(m) = 1\,.$$

Set $m_l = \sum_{i=l}^{r-1} b_i(m) 2^i$ and notice that for any $p \geq 1$, we have

$$|\mathbf{E}(S_{2^r} - S_{2^r - m} | \mathcal{F}_{2^r - m})|^p \leq \Big(\sum_{l=0}^{r-1} |\mathbf{E}(S_{2^r - m_{l+1}} - S_{2^r - m_l} | \mathcal{F}_{2^r - m})| \Big)^p\,.$$

Hence setting

$$\alpha_l = \Big(\sum_{k=1}^{2^{r-l}-1} ||\mathbf{E}(S_{(k+1)2^l} - S_{k2^l} | \mathcal{F}_{k2^l})||_p^p \Big)^{1/p} \text{ and } \lambda_l = \frac{\alpha_l}{\sum_{l=0}^{r-1} \alpha_l}\,,$$

we get by convexity

$$\begin{aligned} |\mathbf{E}(S_{2^r} - S_{2^r - m} | \mathcal{F}_{2^r - m})|^p &\leq \sum_{l=0}^{r-1} \lambda_l^{1-p} |\mathbf{E}(S_{2^r - m_{l+1}} - S_{2^r - m_l} | \mathcal{F}_{2^r - m})|^p \\ &\leq \sum_{l=0}^{r-1} \lambda_l^{1-p} |\mathbf{E}(|\mathbf{E}(S_{2^r - m_{l+1}} - S_{2^r - m_l} | \mathcal{F}_{2^r - m_l})| \big| \mathcal{F}_{2^r - m})|^p\,. \end{aligned}$$

Now $m_l \neq m_{l+1}$ only if $b_l(m) = 1$, and in that case $m_l = k_m 2^l$ with k_m odd. It follows that

$$\begin{aligned} |\mathbf{E}(S_{2^r - m_{l+1}} - S_{2^r - m_l} | \mathcal{F}_{2^r - m_l})| &\leq \max_{1 \leq k \leq 2^{r-l}, k \text{ odd}} |\mathbf{E}(S_{2^r - (k-1)2^l} - S_{2^r - k2^l} | \mathcal{F}_{2^r - k2^l})| \\ &:= A_{r,l}\,. \end{aligned}$$

Hence, using the fact that if $|X| \leq |Y|$ then $\mathbf{E}(|X||\mathcal{F}) \leq \mathbf{E}(|Y||\mathcal{F})$, we get that

$$\| \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r} - S_{2^r - m} | \mathcal{F}_{2^r - m})| \|_p^p \leq \sum_{l=0}^{r-1} \lambda_l^{1-p} \mathbf{E}(\max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(A_{r,l} | \mathcal{F}_{2^r - m})|)^p\,.$$

Notice that $(\mathbf{E}(A_{r,l} | \mathcal{F}_k))_{k \geq 1}$ is a martingale and by Doob's maximal inequality, we obtain

$$\mathbf{E}(\max_{0 \leq m \leq 2^r - 1} \mathbf{E}(A_{r,l} | \mathcal{F}_{2^r - m}))^p \leq q^p \|A_{r,l}\|_p^p \leq q^p \alpha_l^p\,.$$

Using the definition of λ_l, it follows that

$$\begin{aligned} &\| \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r} - S_{2^r - m} | \mathcal{F}_{2^r - m})| \|_p^p \leq q^p \big(\sum_{l=0}^{r-1} \alpha_l \big)^p \\ &\leq q^p \Big(\sum_{l=0}^{r-1} \Big(\sum_{k=1}^{2^{r-l}-1} ||\mathbf{E}(S_{(k+1)2^l} - S_{k2^l} | \mathcal{F}_{k2^l})||_p^p \Big)^{1/p} \Big)^p\,. \end{aligned} \tag{13}$$

Starting from (10) and using (11) and (13) combined with Markov's inequality, Inequality (8) of Proposition 5 follows. To end the proof of Proposition 2, we start from (10) and consider the bounds (12) and (13).

We turn now to the proof of Inequality (9). We start from (10) and we write that

$$S^*_{2^r} \le \max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r}|\mathcal{F}_{2^r-m})| + \max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r+v} - S_{2^r+v-m}|\mathcal{F}_{2^r-m})|$$
$$+ \max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r+v} - S_{2^r}|\mathcal{F}_{2^r-m})| + \max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r+v-m} - S_{2^r-m}|\mathcal{F}_{2^r-m})| \,.$$

By the fact that the variables are uniformly bounded by M, we then derive that

$$S^*_{2^r} \le \max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r}|\mathcal{F}_{2^r-m})| + \max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r+v} - S_{2^r+v-m}|\mathcal{F}_{2^r-m})| + 2vM \,.$$

Since $vM \le x$, it follows that

$$\mathbf{P}(S^*_{2^r} \ge 4x) \le \mathbf{P}(\max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r}|\mathcal{F}_{2^r-m})| \ge x) \tag{14}$$
$$+\mathbf{P}(\max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r+v} - S_{2^r+v-m}|\mathcal{F}_{2^r-m})| \ge x) \,.$$

Using chaining arguments, convexity and the Doob's maximal inequality, as above, we infer that for any $p > 1$,

$$\| \max_{0\le m\le 2^r-1} |\mathbf{E}(S_{2^r+v} - S_{2^r+v-m}|\mathcal{F}_{2^r-m})| \ \|_p^p \tag{15}$$
$$\le q^p \Big(\sum_{l=0}^{r-1} \Big(\sum_{k=1}^{2^{r-l}-1} ||\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l}|\mathcal{F}_{k2^l})||_p^p \Big)^{1/p} \Big)^p \,.$$

Starting from (14) and using (11) and (15) combined with Markov's inequality, Inequality (9) of Proposition 5 follows. ⋄

3 Moment inequalities for the maximum of partial sums under projective conditions

3.1 Rosenthal-type inequalities for stationary processes

Using a direct approach that combines the maximal inequality (7) and the Lemma 36, we obtain the following Rosenthal inequality for the maximum of the partial sums of a stationary process for all powers $p > 2$.

Theorem 6 *Let $p > 2$ be a real number and let $(X_i)_{i\in\mathbb{Z}}$ be an adapted stationary sequence in the sense of Notation 1. Then, for any positive integer n, the following inequality holds:*

$$\mathbf{E}\big(\max_{1\le j\le n} |S_j|^p\big) \ll n\mathbf{E}(|X_1|)^p + cn \left(\sum_{k=1}^n \frac{1}{k^{1+1/p}} ||\mathbf{E}_0(S_k)||_p \right)^p + n \left(\sum_{k=1}^n \frac{1}{k^{1+2\delta/p}} \|\mathbf{E}_0(S_k^2)\|_{p/2}^\delta \right)^{p/(2\delta)} ,$$

where $\delta = \min(1, 1/(p-2))$ and $c = 1$. When $p \ge 4$ we can take $c = 0$ by enlarging the constant involved.

Comment 7 *1. Notice that for $2 < p \le 3$ the inequality holds with $\delta = 1$ and therefore it provides a maximal form for Theorem 3.1 in Rio (2009).*
2. It is interesting to indicate the monotonicity of the right-hand side of the inequality in δ. To be more precise, for any $0 < \delta \le \gamma \le 1$, the following inequality holds:

$$\Big(\sum_{k=1}^n k^{-1-2\gamma/p} \|\mathbf{E}_0(S_k^2)\|_{p/2}^\gamma \Big)^{1/\gamma} \le 2^{(1+\gamma)(\gamma-\delta)/(\delta\gamma)} \Big(\sum_{k=1}^n k^{-1-2\delta/p} \|\mathbf{E}_0(S_k^2)\|_{p/2}^\delta \Big)^{1/\delta} \,.$$

To see this, we notice the subadditivity property $\|\mathbf{E}_0(S_{i+j}^2)\|_{p/2} \le 2\|\mathbf{E}_0(S_i^2)\|_{p/2} + 2\|\mathbf{E}_0(S_j^2)\|_{p/2}$ *and apply then the item 3 of Lemma 37 with* $C = 2$.

3. On the other hand, for any $0 < \delta < 1$ *and any* $\gamma > 1/\delta - 1$, *by Hölder's inequality, there exists a positive constant* C *depending on* p, γ *and* δ, *such that*

$$\Big(\sum_{k=1}^{n} k^{-1-2\delta/p}\|\mathbf{E}_0(S_k^2)\|_{p/2}^{\delta}\Big)^{p/(2\delta)} \le C\Big(\sum_{k=1}^{n} k^{-1-2/p}(\log k)^{\gamma}\ \|\mathbf{E}_0(S_k^2)\|_{p/2}\Big)^{p/2}. \qquad (16)$$

4. As a matter of fact we shall prove first the inequality from Theorem 6 in a slightly different form which is equivalent up to multiplicative constants: for any positive integer r *and any integer* n *such that* $2^{r-1} \le n < 2^r$, *(*δ *and* c *as above)*

$$\mathbf{E}\big(\max_{1\le j\le n}|S_j|^p\big) \ll n\mathbf{E}(|X_1|)^p + cn\Big(\sum_{k=0}^{r-1} 2^{-k/p}\|\mathbf{E}_0(S_{2^k})\|_p\Big)^p + n\Big(\sum_{k=0}^{r-1} 2^{-2k\delta/p}\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{\delta}\Big)^{p/(2\delta)}. \qquad (17)$$

5. In the traditional Rosenthal inequality for martingales, the lower bound and the upper one have the same order of magnitude (we refer to the paper by Hitczenko (1990) for results on the best constants). In general the upper bound obtained in Theorem 6 cannot be a lower bound for $\mathbf{E}(|S_n|^p)$ *(up to some constants) without any additional assumptions. Indeed, if* $X_k = Z_0 \circ T^{k-1} - Z_0 \circ T^k$ *where* Z_0 *is* $\mathcal{F}_0$*-measurable and in* $\mathbf{L}^p$, *then* $\mathbf{E}(|S_n|^p) \le 2^p\|Z_0\|_p^p$, *whereas the upper bound given in Theorem 6 is of order* n. *However, we notice that Theorem 6 is sharp in some sense. Indeed, a simple lower bound for* $\|S_n\|_p$ *can be obtained via the inequality:* $\|\mathbf{E}_0(S_n^2)\|_{p/2} \le \|S_n\|_p^2$. *Hence for* $p > 2$, *if* $\|\mathbf{E}_0(S_n^2)\|_{p/2}$ *is of order* n^{γ} *with* $\gamma > 2/p$, *then the lower bound,* $\|\mathbf{E}_0(S_n^2)\|_{p/2}^{p/2}$ *and the term in the right hand side of the inequality of Theorem 6 have the same order of magnitude (to see this, we also use the fact that* $\|\mathbf{E}_0(S_n)\|_p^2 \le \|\mathbf{E}_0(S_n^2)\|_{p/2}$*).*

With applications to Markov processes in mind, by conditioning with respect to both the future and the past of the process, our next result gives an alternative inequality than the one given in Theorem 6 when p is an even integer. For this case, the power δ appearing in Theorem 6 is always equal to one. Before stating the result, we first introduce the following notation to define the additional nonincreasing filtration that we consider.

Notation 8 *Let* $\bar{\mathcal{F}}_0$ *be a* σ*-algebra of* $\mathcal{A}$ *satisfying* $T^{-1}(\bar{\mathcal{F}}_0) \subseteq \bar{\mathcal{F}}_0$. *We then define the nonincreasing filtration* $(\bar{\mathcal{F}}_i)_{i\in\mathbb{Z}}$ *by* $\bar{\mathcal{F}}_i = T^{-i}(\bar{\mathcal{F}}_0)$. *In what follows, we use the notation* $\bar{\mathbf{E}}_k(Y) = \mathbf{E}(Y|\bar{\mathcal{F}}_k)$.

Theorem 9 *Let* $p \ge 4$ *be an even integer and let* X_0 *be a real valued random variable such that* $\|X_0\|_p < \infty$ *and measurable with respect to* $\mathcal{F}_0$ *and to* $\bar{\mathcal{F}}_0$. *We construct the stationary sequence* $(X_i)_{i\in\mathbb{Z}}$ *as in Notation 1. Then for any integer* n,

$$\mathbf{E}\big(\max_{1\le j\le n}|S_j|^p\big) \ll n\mathbf{E}(|X_1|^p) + n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+1/p}}\big(\|\mathbf{E}_0(S_k)\|_p + \|\bar{\mathbf{E}}_{k+1}(S_k)\|_p\big)\Big)^p$$
$$+ n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+2/p}}\big(\|\mathbf{E}_0(S_k^2)\|_{p/2} + \|\bar{\mathbf{E}}_{k+1}(S_k^2)\|_{p/2}\big)\Big)^{p/2}.$$

As a corollary to the proof of Theorem 9 we obtain:

Theorem 10 *Let $p \geq 4$ be a real number and let X_0 be a real valued random variables such that $\|X_0\|_p < \infty$ and measurable with respect to $\mathcal{F}_0$ and to $\bar{\mathcal{F}}_0$. We construct the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ as in Notation 1. Then for any integer n,*

$$\mathbf{E}\Big(\max_{1\leq j\leq n}|S_j|^p\Big) \ll n\mathbf{E}(|X_1|^p) + n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+1/p}}\big(\|\mathbf{E}_0(S_k^2)\|_{p/2}^{1/2} + \|\bar{\mathbf{E}}_{k+1}(S_k^2)\|_{p/2}^{1/2}\big)\Big)^p .$$

This theorem is also valid for $2 < p < 4$. In this range however, according to Item 2 of Comment 7, Theorem 6 gives better bounds.

Proof of Theorem 6. The proof of this theorem is based on dyadic induction and involves several steps. With the notation $a_n = ||S_n||_p$ we shall establish a recurrence formula: for any positive integer r,

$$a_{2n}^p \leq 2a_n^p + 2c_1 a_n^{p-1}\|\mathbf{E}_0(S_n)\|_p + 2c_2 a_n^{p-2\delta}\|\mathbf{E}_0(S_n^2)\|_{p/2}^{\delta} . \tag{18}$$

where c_1 and c_2 are positive constants depending only on p. Before proving it, let us show that (18) implies our result.

Lemma 11 *Assume that for some $0 < \delta \leq 1$ the recurrence formula (18) holds. Then the inequalities (17) and (4) hold with the same δ.*

Let us prove the lemma. From inequality (18), by recurrence on the first term, we obtain for any positive integer r,

$$a_{2^r}^p \leq 2^r\Big(a_{2^0}^p + c_1\sum_{k=0}^{r-1}2^{-k}a_{2^k}^{p-1}\|\mathbf{E}_0(S_{2^k})\|_p + c_2\sum_{k=0}^{r-1}2^{-k}a_{2^k}^{p-2\delta}\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{\delta}\Big) . \tag{19}$$

We shall establish first the inequality (17). Due to the maximal inequality (6), it suffices to prove that the inequality is satisfied for $\max_{1\leq j\leq n}\mathbf{E}(|S_j|^p)$ instead of $\mathbf{E}\big(\max_{1\leq j\leq n}|S_j|^p\big)$.

The proof is divided in several steps. The goal is to establish that for any positive integer r and any integer n such that $2^{r-1} \leq n < 2^r$,

$$\max_{1\leq j\leq n}\mathbf{E}\big(|S_j|^p\big) \ll n\mathbf{E}(|X_1|)^p + cn\Big(\sum_{k=0}^{r-1}2^{-k/p}\|\mathbf{E}_0(S_{2^k})\|_p\Big)^p + n\Big(\sum_{k=0}^{r-1}2^{-2k\delta/p}\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{\delta}\Big)^{p/2\delta} . \tag{20}$$

With the notation $B_r = \max_{0\leq k\leq r}(a_{2^k}^p/2^k)$, starting from (19), we get

$$B_r \leq a_{2^0}^p + c_1 B_r^{1-1/p}\sum_{k=0}^{r-1}2^{-k/p}\|\mathbf{E}_0(S_{2^k})\|_p + c_2 B_r^{1-2\delta/p}\sum_{k=0}^{r-1}2^{-2k\delta/p}\ \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{\delta} .$$

Therefore, taking into account that either $B_r \leq 3a_{2^0}^p$ or $B_r^{1/p} \leq 3c_1\sum_{k=0}^{r-1}2^{-k/p}\|\mathbf{E}_0(S_{2^k})\|_p$ or $B_r^{2\delta/p} \leq 3c_2\sum_{k=0}^{r-1}2^{-2k\delta/p}\ \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{\delta}$, we derive that

$$a_{2^r}^p \leq 2^r\Big(3a_{2^0}^p + \Big(3c_1\sum_{k=0}^{r-1}2^{-k/p}\|\mathbf{E}_0(S_{2^k})\|_p\Big)^p + \Big(3c_2\sum_{k=0}^{r-1}2^{-2k\delta/p}\ \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{\delta}\Big)^{p/2\delta}\Big) . \tag{21}$$

Let now $2^{r-1} \leq n < 2^r$ and write its binary expansion:

$$n = \sum_{k=0}^{r-1}2^k b_k \text{ where } b_{r-1} = 1 \text{ and } b_k \in \{0,1\} \text{ for } k = 0,\dots,r-2 . \tag{22}$$

Notice that

$$S_n = \sum_{k=0}^{r-1} b_k T_{2^k} \text{ where } T_{2^k} = \sum_{i=n_{k-1}+1}^{n_k} X_i,\ n_k = \sum_{j=0}^{k} b_j 2^j \text{ and } n_{-1} = 0\,.$$

Hence, by stationarity,

$$\|S_n\|_p \le \sum_{k=0}^{r-1} b_k \|T_{2^k}\|_p \le \sum_{k=0}^{r-1} b_k \|S_{2^k}\|_p\,.$$

Then, by using (21) and the fact that $\sum_{k=0}^{r-1} b_k 2^{k/p} \le 2^{r/p}/(1-2^{-1/p})$, we derive the inequality (20) for $\mathbf{E}(|S_n|^p)$ and also for $\max_{1\le j\le n}\mathbf{E}\big(|S_j|^p\big)$. The inequality (17) follows now by the maximal inequality (6).

We indicate now how to derive from (17) the inequality stated in Theorem 6. Notice that, by stationarity, for any integers i and j,

$$\|\mathbf{E}_0(S_{i+j})\|_p \le \|\mathbf{E}_0(S_i)\|_p + \|\mathbf{E}_0(S_j)\|_p\,,$$

and also that for any $0<\delta\le 1$,

$$\|\mathbf{E}_0(S_{i+j}^2)\|_{p/2}^{\delta} \le 2^{\delta}\|\mathbf{E}_0(S_i^2)\|_{p/2}^{\delta} + 2^{\delta}\|\mathbf{E}_0(S_j^2)\|_{p/2}^{\delta}\,.$$

Using Item 1 of Lemma 37, it follows that

$$\sum_{k=0}^{r-1} 2^{-k/p}\|\mathbf{E}_0(S_{2^k})\|_p \ll \sum_{k=1}^{n} k^{-1-1/p}\ \|\mathbf{E}_0(S_k)\|_p\,,$$

and

$$\sum_{k=0}^{r-1} 2^{-2k\delta/p}\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{\delta} \ll \sum_{k=1}^{n} k^{-1-2\delta/p}\|\mathbf{E}_0(S_k^2)\|_{p/2}^{\delta}\,. \tag{23}$$

The results follows by the above considerations via the inequality (17). $\diamond$

End of the proof of Theorem 6. It remains to establish the recurrence formula (18). We divide the proof in three cases according to the values of p. Denote $\bar S_n = X_{n+1}+\cdots+X_{2n}$.

The case $2<p\le 3$ was discussed in Rio (2009). We give here a shorter alternative proof. We apply inequality (85) of Lemma 36 with $x=S_n$ and $y=\bar S_n$. Then, by taking the expectation and using stationarity and properties of conditional expectation, we obtain

$$\mathbf{E}(|S_{2n}|)^p \le 2\mathbf{E}(|S_n|)^p + p\mathbf{E}(|S_n|^{p-1}\mathrm{sign}(S_n)\mathbf{E}_n(\bar S_n)) + C_p^2\mathbf{E}(|S_n|^{p-2}\mathbf{E}_n(\bar S_n^2))\,,$$

where $C_p^2 = p(p-1)/2$. This inequality combined with Hölder's inequality gives

$$a_{2n}^p \le 2a_n^p + pa_n^{p-1}\|\mathbf{E}_0(S_n)\|_p + C_p^2 a_n^{p(1-2/p)}\|\mathbf{E}_0(S_n^2)\|_{p/2}$$

and therefore (18) holds with $\delta=1$, $c_1=2^{-1}p$, and $c_2=2^{-1}C_p^2$.

Assume now that $p\in]3,4[$. Using the inequality (86) of Lemma 36 with $x=S_n$ and $y=\bar S_n$, taking the expectation, and using Lemma 34, we get by stationarity that for any positive integer r,

$$a_{2n}^p \le 2a_n^p + pa_n^{p-1}\|\mathbf{E}_0(S_n)\|_p + C_p^2 a_n^{p(1-2/p)}\|\mathbf{E}_0(S_n^2)\|_{p/2} + 2p(p-2)^{-1}a_n^{p-2/(p-2)}||\mathbf{E}_0(S_n^2)||_{p/2}^{1/(p-2)}\,.$$

Since $\|\mathbf{E}_0(S_n^2)\|_{p/2} \le a_n^{2(p-3)/(p-2)}\|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/(p-2)}$, it follows that

$$a_{2n}^p \le 2a_n^p + pa_n^{p-1}\|\mathbf{E}_0(S_n)\|_p + 4pa_n^{p-2/(p-2)}||\mathbf{E}_0(S_n^2)||_{p/2}^{1/(p-2)}\,.$$

It follows that (18) holds with $\delta = 1/(p-2)$, $c_1 = 2^{-1}p$, and $c_2 = 2p$.

It remains to prove the inequality (4) for $p \geq 4$. Using the inequality (87) of Lemma 36 with $x = S_n$ and $y = \bar{S}_n$, and taking the expectation, we get by stationarity that

$$a_{2n}^p \leq 2a_n^p + 4^p \mathbf{E}\big(|S_n|^{p-1}|\bar{S}_n| + |\bar{S}_n|^{p-1}|S_n|\big) .$$

Using Lemma 34 together with stationarity, it follows that

$$\mathbf{E}(|S_n||\bar{S}_n|^{p-1}) \leq a_n^{p-2/(p-2)} ||\mathbf{E}_0(S_n^2)||_{p/2}^{1/(p-2)} ,$$

and that

$$\mathbf{E}(\ |S_n|^{p-1}|\bar{S}_n|) \leq a_n^{p-1} ||\mathbf{E}_0(S_n^2)||_{p/2}^{1/2} \leq a_n^{p-2/(p-2)} ||\mathbf{E}_0(S_n^2)||_{p/2}^{1/(p-2)} .$$

From these estimates we deduce

$$a_{2n}^p \leq 2a_n^p + 2(4^p) a_n^{p-2/(p-2)} \|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/(p-2)} ,$$

and then (18) holds with $\delta = 1/(p-2)$, $c_1 = 0$, and $c_2 = 4^p$. Therefore, in this case (20) holds with $c = 0$. Then by the maximal inequality (6), the inequality (17) holds with $c = 1$. We show now that, in this case, the second term in the inequality can be bounded up to a multiplicative constant by the third term. By Jensen's inequality and since in this case $\delta < 1/2$, we have

$$\sum_{k=0}^{r-1} 2^{-k/p} ||\mathbf{E}_0(S_{2^k})||_p \leq \Big(\sum_{k=0}^{r-1} 2^{-(2k/p)(1/2)} ||\mathbf{E}_0(S_{2^k}^2)||_{p/2}^{1/2}\Big)^2 \leq \Big(\sum_{k=0}^{r-1} 2^{-(2k/p)\delta} \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{\delta}\Big)^{1/\delta} . \quad (24)$$

We then finish the proof by using (23). $\diamond$

Proof of Theorem 9. Denote $\bar{S}_n = X_{n+1} + \cdots + X_{2n}$. Starting from the inequality (88) of Lemma 36 applied with $x = S_n$ and $y = \bar{S}_n$ and using the notation $a_n = ||S_n||_p$, by stationarity, we get that

$$a_{2n}^p \leq 2a_n^p + p\big(\mathbf{E}(S_n^{p-1}\bar{S}_n) + \mathbf{E}(S_n\bar{S}_n^{p-1})\big) + 2^p\big(\mathbf{E}(S_n^{p-2}\bar{S}_n^2) + \mathbf{E}(\bar{S}_n^{p-2}S_n^2)\big) . \quad (25)$$

By using Hölder's inequality and recurrence, we then derive that for any positive integer r,

$$a_{2^r}^p \leq 2^r a_{2^0}^p + 2^{-1}p \sum_{k=0}^{r-1} 2^{r-k} a_{2^k}^{p-1} \big(\|\mathbf{E}_0(S_{2^k})\|_p + \|\bar{\mathbf{E}}_{2^k+1}(S_{2^k})\|_p\big)$$
$$+ 2^{p-1} \sum_{k=0}^{r-1} 2^{r-k} a_{2^k}^{p-2} \big(\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2} + \|\bar{\mathbf{E}}_{2^k+1}(S_{2^k}^2)\|_{p/2}\big) .$$

By using the arguments of the proof of Lemma 11, we get for $2^{r-1} \leq n < 2^r$,

$$\mathbf{E}\big(\max_{1\leq j\leq n} |S_j|^p\big) \ll n\mathbf{E}(|X_1|^p) + n\Big(\sum_{k=1}^{r-1} 2^{-k/p}\big(\|\mathbf{E}_0(S_{2^k})\|_p + \|\bar{\mathbf{E}}_{k+1}(S_{2^k})\|_p\big)\Big)^p$$
$$+ n\Big(\sum_{k=1}^{n} 2^{-2k/p}\big(\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2} + \|\bar{\mathbf{E}}_{k+1}(S_{2^k}^2)\|_{p/2}\big)\Big)^{p/2} .$$

Noticing in addition that, by stationarity, for any integer i and j

$$\|\mathbf{E}_0(S_{i+j})\|_p \leq \|\mathbf{E}_0(S_i)\|_p + \|\mathbf{E}_0(S_j)\|_p \ , \ \|\bar{\mathbf{E}}_{i+j+1}(S_{i+j})\|_p \leq \|\bar{\mathbf{E}}_{i+1}(S_i)\|_p + \|\bar{\mathbf{E}}_{j+1}(S_j)\|_p ,$$

$$\|\bar{\mathbf{E}}_{i+j+1}(S_{i+j}^2)\|_{p/2} \le 2\|\bar{\mathbf{E}}_{i+1}(S_i^2)\|_{p/2} + 2\|\bar{\mathbf{E}}_{j+1}(S_j^2)\|_{p/2}\,, \tag{26}$$

and that

$$\|\mathbf{E}_0(S_{i+j}^2)\|_{p/2} \le 2\|\mathbf{E}_0(S_i^2)\|_{p/2} + 2\|\mathbf{E}_0(S_j^2)\|_{p/2}\,. \tag{27}$$

We obtain the desired result by using Lemma 37. ⋄

Proof of Theorem 10. To prove this theorem we apply the inequality (87) of Lemma 36 with $x = S_n$ and $y = \bar{S}_n$, where $\bar{S}_n = X_{n+1} + \cdots + X_{2n}$. With the notation $a_n = ||S_n||_p$, we then have by stationarity that

$$a_{2n}^p \le 2a_n^p + 4^p\big(\mathbf{E}(|S_n|^{p-1}|\bar{S}_n|) + \mathbf{E}(|S_n||\bar{S}_n|^{p-1})\big)\,.$$

By conditioning and then applying Jensen's inequality followed by Hölder's inequality we obtain

$$\begin{aligned} a_{2n}^p &\le 2a_n^p + 4^p\big(\mathbf{E}(|S_n|^{p-1}\mathbf{E}_n^{1/2}(\bar{S}_n^2)) + \mathbf{E}(|\bar{S}_n|^{p-1}\bar{\mathbf{E}}_{n+1}^{1/2}(S_n^2))\big) \\ &\le 2a_n^p + 4^p\ a_n^{p-1}\big(\|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/2} + \|\bar{\mathbf{E}}_{n+1}(S_n^2)\|_{p/2}^{1/2}\big)\,. \end{aligned}$$

By recurrence, we then derive that for any positive integer r,

$$a_{2^r}^p \le 2^r\Big(a_0^p + 2^{2p-1}\sum_{k=0}^{r-1} 2^{-k} a_{2^k}^{p-1}\big(\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{1/2} + \|\bar{\mathbf{E}}_{2^k+1}(S_{2^k}^2)\|_{p/2}^{1/2}\big)\Big)\,.$$

The proof is completed by the arguments developed in the proof of Lemma 11 and by using Lemma 37 via the inequalities (26) and (27) . ⋄

3.2 Relation with the Burkholder-type Inequality.

Next lemma shows how to compare $\|\mathbf{E}_0(S_n^2)\|_{p/2}$ with quantities involving only $\|\mathbf{E}_0(S_n)\|_p$.

Lemma 12 *Let $p \ge 2$ be a real number and let (X_n) be an adapted stationary sequence in the sense of Notation 1. Then, for any positive integer n,*

$$\|\mathbf{E}_0(S_n^2)\|_{p/2} \ll n\|\mathbf{E}_0(X_1^2)\|_{p/2} + n\left(\sum_{j=1}^{n}\frac{\|\mathbf{E}_0(S_j)\|_p}{j^{3/2}}\right)^2\,. \tag{28}$$

As a consequence of the above lemma, we get that for any $0 < \delta \le 1$ and any real $p > 2$,

$$n\left(\sum_{j=1}^{n}\frac{\|\mathbf{E}_0(S_j^2)\|_{p/2}^{\delta}}{j^{1+2\delta/p}}\right)^{p/(2\delta)} \ll n^{p/2}\|\mathbf{E}_0(X_1^2)\|_{p/2}^{p/2} + n^{p/2}\left(\sum_{j=1}^{n}\frac{\|\mathbf{E}_0(S_j)\|_p}{j^{3/2}}\right)^p\,.$$

Theorem 6 then implies the following Burkholder-type inequality that was established by Peligrad, Utev and Wu (2007, Theorem 1):

Corollary 13 *Let $p > 2$ be a real number and let (X_n) be an adapted stationary sequence in the sense of Notation 1. Then, for any integer n,*

$$\mathbf{E}\big(\max_{1\le j\le n}|S_j|^p\big) \ll n^{p/2}\mathbf{E}(|X_1|^p) + n^{p/2}\Big(\sum_{j=1}^{n}\frac{\|\mathbf{E}_0(S_j)\|_p}{j^{3/2}}\Big)^p\,.$$

Proof of Lemma 12. We shall first prove that for any positive integer k,

$$\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2} \le 2^{k+1}\|\mathbf{E}_0(X_1^2)\|_{p/2} + 2^{k+2}\left(\sum_{j=0}^{k-1}\frac{\|\mathbf{E}_0(S_{2^j})\|_p}{2^{j/2}}\right)^2 . \tag{29}$$

By using the notation $\bar{S}_n = X_{n+1} + \cdots + X_n$ and the fact that $S_{2n}^2 = S_n^2 + \bar{S}_n^2 + 2S_n\bar{S}_n$, we get, by stationarity, that

$$\|\mathbf{E}_0(S_{2n}^2)\|_{p/2} \le 2\|\mathbf{E}_0(S_n^2)\|_{p/2} + 2\|\mathbf{E}_0\big(S_n\mathbf{E}_n(\bar{S}_n)\big)\|_{p/2} .$$

Now, Cauchy-Schwarz inequality applied twice, gives

$$\|\mathbf{E}_0\big(S_n\mathbf{E}_n(\bar{S}_n)\big)\|_{p/2} \le \|\mathbf{E}_0^{1/2}(S_n^2)\mathbf{E}_0^{1/2}(\mathbf{E}_n^2(\bar{S}_n))\|_{p/2} \le \|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/2}\|\mathbf{E}_0(S_n)\|_p .$$

Hence, setting $b_n = \|\mathbf{E}_0(S_n^2)\|_{p/2}$, it follows that

$$b_{2n} \le 2b_n + 2b_n^{1/2}\|\mathbf{E}_0(S_n)\|_p .$$

By recurrence, this gives that

$$b_{2^k} \le 2^k b_0 + \sum_{j=0}^{k-1} 2^{k-j} b_{2^j}^{1/2}\|\mathbf{E}_0(S_{2^j})\|_p .$$

With the notation $B_k = \max_{0\le j\le k} 2^{-j} b_{2^j}$, we derive that

$$B_k \le 2\max\Big(b_0, B_k^{1/2}\sum_{j=0}^{k-1} 2^{-j/2}\|\mathbf{E}_0(S_{2^j})\|_p\Big) ,$$

implying that

$$2^{-k}b_{2^k} \le B_k \le 2b_0 + 2^2\Big(\sum_{j=0}^{k-1} 2^{-j/2}\|\mathbf{E}_0(S_{2^j})\|_p\Big)^2 .$$

This ends the proof of the inequality (29).

We turn now to the proof of (28). Let r be the positive integer such that $2^{r-1} \le n < 2^r$. Starting with the binary expansion (22), and using Minkowski's inequality twice, first with respect to the conditional expectation, and second with respect to the norm in $\mathbf{L}^p$, we get by stationarity that

$$\|\mathbf{E}_0(S_n^2)\|_{p/2} \le \Big(\sum_{k=0}^{r-1} b_k\|(\mathbf{E}_0(S_{2^k}^2))^{1/2}\|_p\Big)^2 \le \Big(\sum_{k=0}^{r-1}\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{1/2}\Big)^2 .$$

Using then the inequality (29), we derive that

$$\|\mathbf{E}_0(S_n^2)\|_{p/2} \ll n\|\mathbf{E}_0(X_1^2)\|_{p/2} + n\left(\sum_{j=0}^{r-1}\frac{\|\mathbf{E}_0(S_{2^j})\|_p}{2^{j/2}}\right)^2 . \tag{30}$$

Since $(\|\mathbf{E}_0(S_n)\|_p)_{n\ge1}$ is subadditive, using Item 1 of Lemma 37, Inequality (28) follows from (30). ⋄

3.3 Rosenthal inequalities for martingales and the case of even powers.

3.3.1 The martingale case

For any real $p > 2$, Theorem 6 applied to stationary martingale differences gives the following inequality:

$$\mathbf{E}\big(\max_{1\le j\le n}|S_j|^p\big) \ll n\mathbf{E}(|X_1|^p) + n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+2\delta/p}}\|\mathbf{E}_0(S_k^2)\|_{p/2}^{\delta}\Big)^{p/(2\delta)},$$

where $\delta = \min(1, 1/(p-2))$.

Since for stationary martingale differences we have $\mathbf{E}(S_n^2) = n\mathbf{E}(X_1^2)$, we can express the inequality in the following form useful for applications:

$$\mathbf{E}\big(\max_{1\le j\le n}|S_j|^p\big) \ll n^{p/2}(\mathbf{E}(X_1^2))^{p/2} + n\mathbf{E}(|X_1|^p) + n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+2\delta/p}}\|\mathbf{E}_0(S_k^2)-\mathbf{E}(S_k^2)\|_{p/2}^{\delta}\Big)^{p/(2\delta)}. \quad (31)$$

As we shall see in the next result, for a stationary sequence $(d_i)_{i\in\mathbf{Z}}$ of martingale differences in $\mathbf{L}^p$ with $p \ge 4$ an even integer, this inequality can be sharpened since it holds with $\delta = 2/(p-2)$ (see Comment 7). As a consequence, we recover, in case $p = 4$, the inequality (1.6) stated in Rio (2009) that was obtained for variables in $\mathbf{L}^q$ with $q = p/2$, by using the classical Burkholder's inequality combined with Theorem 3 in Wu and Zhao (2008). Notice that the inequality (1.6) stated in Rio (2009) cannot be generalized for $p > 4$ since Theorem 3 in Wu and Zhao (2008) is only valid for variables in $\mathbf{L}^q$ with $1 < q \le 2$.

Theorem 14 *Let $p \ge 4$ be an even integer and let d_0 be a real random variable in $\mathbf{L}^p$, measurable with respect to $\mathcal{F}_0$ and such that $\mathbf{E}(d_0|\mathcal{F}_{-1}) = 0$. Let $d_i = d_0 \circ T^i$ and $S_n = \sum_{i=1}^n d_i$. Then for any integer n,*

$$\mathbf{E}\big(\max_{1\le j\le n}|S_j|^p\big) \ll n\ \mathbf{E}(|d_1|^p) + n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+4/p(p-2)}}\|\mathbf{E}_0(S_k^2)\|_{p/2}^{2/(p-2)}\Big)^{p(p-2)/4}.$$

The technique that makes this result possible is a special symmetrization for martingales initiated by Kwapień and Woyczynski (1991).

Proposition 15 *Assume that $(e_k)_k$ are stationary martingale differences adapted to an increasing filtration $(\mathcal{F}_k)_k$ that are conditionally symmetric (the distribution of e_k given $\mathcal{F}_{k-1}$ is equal to the distribution of $-e_k$ given $\mathcal{F}_{k-1}$). Assume, in addition, that the e_k's are conditionally independent given a sigma algebra $\mathcal{G}$ and such that the law of e_k given $\mathcal{G}$ is the same as the law of e_k given $\mathcal{F}_{k-1}$. Let $S_n = \sum_{i=1}^n e_i$. Then for any even integer $p \ge 4$ and any integer $n \ge 1$,*

$$\mathbf{E}\big(\max_{1\le j\le n}|S_j|^p\big) \ll n\ \mathbf{E}(|e_1|^p) + n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+4/p(p-2)}}\|\mathbf{E}_0(S_k^2)\|_{p/2}^{2/(p-2)}\Big)^{p(p-2)/4}. \quad (32)$$

Proof of Proposition 15. Due to the Doob's maximal inequality, we have that $\|\max_{1\le j\le n}|S_j|\|_p \le q\|S_n\|_p$ where $q = p(p-1)^{-1}$. Then, it suffices to show that the inequality (32) holds for $\mathbf{E}(|S_n|^p)$. We shall base this proof again on dyadic induction. Denote $\bar{S}_n = e_{n+1} + ... + e_{2n}$ and $a_n = \|S_n\|_p$.

We start from the inequality (25). Since the sequence of martingale differences (e_k) is conditionally symmetric and conditionally independent given a master sigma algebra $\mathcal{G}$, we have $\mathbf{E}(S_n^{p-1}\bar{S}_n) + \mathbf{E}(S_n\bar{S}_n^{p-1}) = 0$ and therefore

$$a_{2n}^p \le 2a_n^p + 2^p\big(\mathbf{E}(S_n^{p-2}\bar{S}_n^2) + \mathbf{E}(\bar{S}_n^{p-2}S_n^2)\big).$$

Using Lemma 34, we have that

$$\mathbf{E}(S_n^{p-2}\bar{S}_n^2) \le a_n^{p-2}\|\mathbf{E}_0(S_n^2)\|_{p/2} \text{ and } \mathbf{E}(S_n^2\bar{S}_n^{p-2}) \le a_n^{p-4/(p-2)}\|\mathbf{E}_0(S_n^2)\|_{p/2}^{2/(p-2)} .$$

Therefore, by combining all these bounds, we obtain for every even integer $p \ge 4$,

$$\begin{aligned} a_{2n}^p &\le 2a_n^p + 2^p\big(a_n^{p-2}\|\mathbf{E}_0(S_n^2)\|_{p/2} + a_n^{p-4/(p-2)}\|\mathbf{E}_0(S_n^2)\|_{p/2}^{2/(p-2)}\big) \\ &\le 2a_n^p + 2^{p+1}a_n^{p-4/(p-2)}\|\mathbf{E}_0(S_n^2)\|_{p/2}^{2/(p-2)} . \end{aligned}$$

We end the proof by using Lemma 11. ⋄

Proof of Theorem 14.

We consider our martingale differences sequence $(d_k)_k$ and we construct two decoupled tangent versions $(e_k)_k$ and $(\tilde{e}_k)_k$ to $(d_k)_k$ that are $\mathcal{G}-$conditionally independent between them (here $\mathcal{G} = \sigma(\{d_i\})$). This means that these two sequences are martingale differences with the additional property that the conditional distribution of d_k given $\mathcal{F}_{k-1}$ is equal to the distribution of e_k given $\mathcal{F}_{k-1}$ and also to the distribution of $\tilde{e}_k$ given $\mathcal{F}_{k-1}$ (see Definition 6.1.4 in de la Peña and Giné (1999) for the definition of a decoupled tangent sequence, and their Proposition 6.1.5. for the crucial fact that decoupled sequences always exist. We refer also to their Remark 6.1.6, for the construction of $(e_k)_k$ and $(\tilde{e}_k)_k$). Therefore, for any even integer p,

$$\mathbf{E}\Big(\sum_{i=1}^n d_i\Big)^p = \mathbf{E}\Big(\sum_{i=1}^n (d_i - \mathbf{E}(e_i|\mathcal{G})\Big)^p \le \mathbf{E}\Big(\sum_{i=1}^n (d_i - e_i)\Big)^p .$$

Now we use Corollary 6.6.8. in de la Peña and Giné (1999) (see also Zinn (1985)). Since $(e_i - \tilde{e}_i)_i$ is a decoupled tangent sequence of $(d_i - e_i)_i$, it follows that

$$\mathbf{E}\Big(\sum_{i=1}^n d_i\Big)^p \ll \mathbf{E}\Big(\sum_{i=1}^n (e_i - \tilde{e}_i)\Big)^p .$$

Notice that the distribution of $e_i - \tilde{e}_i$ is conditionally symmetric given $\mathcal{G}$. Therefore, using the Doob's maximal inequality and applying Proposition 15, we obtain that for every even integer $p \ge 4$ and any integer n,

$$\mathbf{E}\Big(\max_{1\le k\le n}\big(\sum_{i=1}^k d_i\big)^p\Big) \ll \mathbf{E}\Big(\sum_{i=1}^n (e_i - \tilde{e}_i)\Big)^p \tag{33}$$

$$\ll n\ \mathbf{E}(|e_1 - \tilde{e}_1|^p) + n\Big(\sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}}\Big\|\mathbf{E}_0\Big(\Big(\sum_{i=1}^k (e_i - \tilde{e}_i)\Big)^2\Big)\Big\|_{p/2}^{2/(p-2)}\Big)^{p(p-2)/4} .$$

Notice now that $\sum_{i=1}^k \mathbf{E}(e_i^2|\mathcal{F}_{i-1}) = \sum_{i=1}^k \mathbf{E}(d_i^2|\mathcal{F}_{i-1})$ since both quantities are obtained using only the conditional distributions of the d_i's and e_i's respectively, and these two sequences are tangent. Tangency also implies that d_i and e_i have the same distributions. Hence $\|d_1\|_p = \|e_1\|_p$. For the same reasons, we also have $\sum_{i=1}^k \mathbf{E}(\tilde{e}_i^2|\mathcal{F}_{i-1}) = \sum_{i=1}^k \mathbf{E}(d_i^2|\mathcal{F}_{i-1})$ and $\|d_1\|_p = \|\tilde{e}_1\|_p$. Therefore, $\|e_1 - \tilde{e}_1\|_p \le 2\|d_1\|_p$ and

$$\begin{aligned} \Big\|\mathbf{E}_0\Big(\Big(\sum_{i=1}^k (e_i - \tilde{e}_i)\Big)^2\Big)\Big\|_{p/2} &\le 2\Big\|\mathbf{E}_0\Big(\sum_{i=1}^k \mathbf{E}(e_i^2|\mathcal{F}_{i-1})\Big)\Big\|_{p/2} + 2\Big\|\mathbf{E}_0\Big(\sum_{i=1}^k \mathbf{E}(\tilde{e}_i^2|\mathcal{F}_{i-1})\Big)\Big\|_{p/2} \\ &= 4\Big\|\mathbf{E}_0\Big(\sum_{i=1}^k \mathbf{E}(d_i^2|\mathcal{F}_{i-1})\Big)\Big\|_{p/2} = 4\Big\|\sum_{i=1}^k \mathbf{E}_0(d_i^2)\Big\|_{p/2} . \end{aligned}$$

Theorem 14 follows by introducing these bounds in the inequality (33). ⋄

3.3.2 Application to stationary processes via martingale approximation

Theorem 14 together with the martingale approximation provide an alternative Rosenthal-type inequality involving the projection operator, very useful for analyzing linear processes.

Next lemma is a slight reformulation of the martingale approximation result that can be found in the paper by Wu and Woodroofe (Theorem 1, 2004). See also Zhao and Woodroofe (2008) and Gordin and Peligrad (2011).

Lemma 16 *Let $p \geq 1$ and let (X_n) be an adapted stationary sequence in the sense of Notation 1. Then there is a triangular array of row-wise stationary martingale differences satisfying*

$$D_0^n = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{E}_1(S_i) - \mathbf{E}_0(S_i)) \quad ; D_k^n = D_0^n \circ T^k \tag{34}$$

such that for any $1 \leq k \leq n$ we have

$$S_k = M_k^n + R_k^n \quad \text{where } M_k^n = \sum_{i=1}^{k} D_i^n \tag{35}$$

and

$$\max_{1\leq k\leq n} \|R_k^n\|_p \leq 2\|X_0\|_p + \frac{3}{n}\sum_{i=1}^{n}\|\mathbf{E}_0(S_i)\|_p \,.$$

We state now the Rosenthal-type inequality that we shall establish with the help of the approximation result above.

Theorem 17 *Let $p \geq 4$ be an even integer and let $(X_i)_{i\in\mathbb{Z}}$ be as in Theorem 6. Then the following inequality is valid: for any integer n,*

$$\mathbf{E}\big(\max_{1\leq j\leq n}|S_j|^p\big) \ll n\|X_0\|_p^p + n^{1-p}\Big(\sum_{i=1}^{n}\|\mathbf{E}_0(S_i)\|_p\Big)^p + n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+4/p(p-2)}}\|\mathbf{E}_0(S_k^2)\|_{p/2}^{2/(p-2)}\Big)^{p(p-2)/4} .$$

Remark 18 *Theorems 6 and 17 are in general not comparable. Indeed, for $p \geq 4$, Theorem 6 applies with $\delta = 1/(p-2)$ so the last term of the inequality stated in Theorem 17 can be bounded by the last term in the inequality from Theorem 6 (see Item 2 of Comment 7). However the second term in Theorem 17 gives additional contribution.*

Proof of Theorem 17. We bound first $\max_{1\leq k\leq n}\mathbf{E}(|S_k|^p)$. By the martingale approximation of Lemma 16 combined with Theorem 14, we get that

$$\max_{1\leq k\leq n}\mathbf{E}(|S_k|^p) \ll \max_{1\leq k\leq n}\mathbf{E}(|M_k^n|^p) + \max_{1\leq k\leq n}\mathbf{E}(|R_k^n|^p) \ll n\,\mathbf{E}(|D_0^n|^p) + \|X_0\|_p^p + \Big(\frac{1}{n}\sum_{i=1}^{n}\|\mathbf{E}_0(S_i)\|_p\Big)^p + n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+4/p(p-2)}}\|\mathbf{E}_0((M_k^n)^2)\|_{p/2}^{2/(p-2)}\Big)^{p(p-2)/4} .$$

It remains to analyze the first and the last term. By (35) applied with $k = 1$, we notice that $D_1^n = X_1 - R_1^n$ and by the triangle inequality,

$$||D_0^n||_p^p \leq (||X_0||_p + ||R_1^n||_p)^p \ll ||X_0||_p^p + \frac{1}{n^p}\Big(\sum_{i=1}^{n}\|\mathbf{E}_0(S_i)\|_p\Big)^p .$$

For analyzing the last term, we use the fact that

$$\mathbf{E}_0((M_k^n)^2) \leq 2\mathbf{E}_0(S_k^2) + 2\mathbf{E}_0((R_k^n)^2)\,.$$

By using Lemma 16 it follows that

$$\|\mathbf{E}_0((M_k^n)^2)\|_{p/2} \ll \|\mathbf{E}_0(S_k^2)\|_{p/2} + \|X_0\|_p^2 + n^{-2}\Big(\sum_{i=1}^{n}\|\mathbf{E}_0(S_i)\|_p\Big)^2\,,$$

and overall

$$\max_{1\leq k\leq n}\mathbf{E}(|S_k|^p) \leq n\|X_0\|_p^p + n^{1-p}\Big(\sum_{i=1}^{n}\|\mathbf{E}_0(S_i)\|_p\Big)^p +$$
$$n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+4/p(p-2)}}\|\mathbf{E}_0(S_k^2)\|_{p/2}^{2/(p-2)}\Big)^{p(p-2)/4}\,.$$

We apply now the inequality (7) that has as effect the addition of a forth term, namely: $n\Big(\sum_{k=1}^{n}k^{-1-1/p}\|\mathbf{E}_0(S_k)\|_p\Big)^p$. However we can express our inequality without including this term because it can be bounded, up to a multiplicative constant, by $n^{1-p}\Big(\sum_{i=1}^{n}\|\mathbf{E}_0(S_i)\|_p\Big)^p$. Indeed, notice that it is enough to show that for a certain universal constant C,

$$\max_{1\leq i\leq n}\|\mathbf{E}_0(S_i)\|_p \leq \frac{C}{n}\sum_{i=1}^{n}\|\mathbf{E}_0(S_i)\|_p\,.$$

To prove it, we first notice that

$$\max_{1\leq i\leq n}\|\mathbf{E}_0(S_i)\|_p \leq \max_{1\leq i\leq [n/2]}\|\mathbf{E}_0(S_i)\|_p + \max_{[n/2]<i\leq n}\|\mathbf{E}_0(S_i)\|_p$$

and that for any $i\in\{1,\cdots,[n/2]\}$,

$$\|\mathbf{E}_0(S_i)\|_p \leq \|\mathbf{E}_0(S_{i+[n/2]})\|_p + \|\mathbf{E}_0(S_{i+[n/2]}-S_i)\|_p\,.$$

Therefore, by the properties of conditional expectation and stationarity, it follows that

$$\max_{1\leq i\leq n}\|\mathbf{E}_0(S_i)\|_p \leq \|\mathbf{E}_0(S_{[n/2]})\|_p + 2\max_{[n/2]<i\leq n}\|\mathbf{E}_0(S_i)\|_p\,.$$

To complete the proof, it remains to apply the inequality (92) to the subadditive sequence $(\|\mathbf{E}_0(S_i)\|_p)_{i\geq 1}$.
$\diamond$

3.4 Rosenthal inequality in terms of individual summands.

For the sake of applications in this section we indicate how to estimate the terms that appear in our Rosenthal inequalities in terms of individual summands and formulate some specific inequalities. By substracting $\mathbf{E}(S_k^2)$ and applying the triangle inequality we can reformulate all the inequalities in terms of the quantities $\mathbf{E}(S_k^2)$, $||\mathbf{E}_0(S_k)||_p$ and $\|\mathbf{E}_0(S_k^2)-\mathbf{E}(S_k^2)\|_{p/2}$. Next lemma proposes a simple way to estimate these quantities in terms of coefficients in the spirit of Gordin (1969).

Lemma 19 *Under the stationary setting assumptions in Notation 1, we have the following estimates:*

$$\mathbf{E}(S_k^2) \leq 2k\sum_{j=0}^{k-1}|\mathbf{E}(X_0X_j)|\,, \tag{36}$$

$$||\mathbf{E}_0(S_k)||_p \le \sum_{\ell=1}^{n} ||\mathbf{E}_0(X_\ell)||_p\,, \tag{37}$$

and

$$\|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2} \le 2\sum_{i=1}^{k}\sum_{j=0}^{k-i} \|\mathbf{E}_0(X_iX_{i+j}) - \mathbf{E}(X_iX_{i+j})\|_{p/2} \tag{38}$$

$$\le 2\sum_{i=1}^{k}\sum_{j=0}^{k-i} \sup_{\ell\ge 0} \|\mathbf{E}_0(X_iX_{i+\ell}) - \mathbf{E}(X_iX_{i+\ell})\|_{p/2} \wedge (2\|X_0\mathbf{E}_0(X_j)\|_{p/2})$$

$$\le 4\sum_{j=1}^{k} j\|X_0\mathbf{E}_0(X_j)\|_{p/2} + 2\sum_{i=1}^{k} i \sup_{j\ge i} \|\mathbf{E}_0(X_iX_j) - \mathbf{E}(X_iX_j)\|_{p/2}\,.$$

Mixing coefficients are useful to continue the estimates from Lemma 19. We refer to the books by Bradley (2007, Theorem 4.13 via Remark 4.7, VI), Rio (2000, Theorem 2.5 and Appendix, Section C) and Dedecker *et al.* (2007, Remark 2.5 and Ch 3) for various estimates of the coefficients involved in Lemma 19 and examples. We shall also provide applications and explicit computations of the quantities involved.

We formulate the following proposition:

Proposition 20 *Let $p > 2$ be a real number and let $(X_i)_{i\in\mathbf{Z}}$ be a stationary sequence of real-valued random variables in $\mathbf{L}_p$ adapted to an increasing filtration $(\mathcal{F}_i)$. For any $j \ge 1$, let*

$$\lambda(j) = \max\big(\|X_0\mathbf{E}_0(X_j)\|_{p/2}, \sup_{i\ge j} \|\mathbf{E}_0(X_iX_j) - \mathbf{E}(X_iX_j)\|_{p/2}\big)\,. \tag{39}$$

Then for every positive integer n,

$$||\max_{1\le j\le n} |S_j|\,||_p \ll n^{1/2}\Big(\sum_{k=0}^{n-1} |\mathbf{E}(X_0X_k)|\Big)^{1/2} + n^{1/p}||X_1||_p$$

$$+cn^{1/p}\sum_{k=1}^{n} \frac{1}{k^{1/p}}||\mathbf{E}_0(X_k)||_p + n^{1/p}\Big(\sum_{k=1}^{n} \frac{1}{k^{(2/p)-1}}(\log k)^\gamma \lambda(k)\Big)^{1/2}.$$

where γ can be taken $\gamma = 0$ for $2 < p \le 3$ and $\gamma > p-3$ for $p > 3$; $c = 1$ for $2 < p < 4$ and $c = 0$ for $p \ge 4$. The constant that is implicitly involved in the notation $\ll$ depends on p and γ but it does not depend on n and on the X_i's.

Proof of Proposition 20. The proof of this proposition is basically a combination of Theorem 6 and Lemma 19. By the triangle inequality

$$||\mathbf{E}_0(S_k^2)||_{p/2} \le \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2} + \mathbf{E}_0(S_k^2)\,.$$

By (36), for any $p > 2$ and any $\delta > 0$, we easily obtain

$$\left(\sum_{k=1}^{n} \frac{1}{k^{1+2\delta/p}}(\mathbf{E}(S_k^2))^\delta\right)^{1/(2\delta)} \ll n^{1/2-1/p}\Big(\sum_{j=0}^{n-1} |\mathbf{E}(X_0X_j)|\Big)^{1/2}.$$

Then, we use inequality (37) and changing the order of summation

$$\sum_{k=1}^{n} \frac{1}{k^{1+1/p}}||\mathbf{E}_0(S_k)||_p \ll \sum_{k=1}^{n} \frac{1}{k^{1/p}}||\mathbf{E}_0(X_k)||_p\,.$$

Now for the situation $0<\delta<1$, by Hölder's inequality,

$$\Big(\sum_{k=1}^n \frac{1}{k^{1+2\delta/p}}\|\mathbf{E}_0(S_k^2)-\mathbf{E}(S_k^2)\|_{p/2}^{\delta}\Big)^{p/(2\delta)} \ll \Big(\sum_{k=1}^n \frac{(\log k)^{\gamma}}{k^{1+2/p}}\|\mathbf{E}_0(S_k^2)-\mathbf{E}(S_k^2)\|_{p/2}\Big)^{p/2} .$$

where $\gamma>1/\delta-1$. We continue the proof by using (38) and get

$$\sum_{k=1}^n \frac{(\log k)^{\gamma}}{k^{1+2/p}}\|\mathbf{E}_0(S_k^2)-\mathbf{E}(S_k^2)\|_{p/2} \ll \sum_{k=1}^n \frac{(\log k)^{\gamma}}{k^{(2/p)-1}}\lambda(k)\,.$$

Proposition 20 follows by using Theorem 6 combined with all the above upper bounds. $\diamond$

We give now a consequence of Theorem 6 that will be used in one of our applications. The proof is omitted since it is in the spirit of the proof of Proposition 20; namely, we use Lemma 19 combined with Hölder's inequality, and the fact that $||\mathbf{E}_0(S_k)||_p \le ||\mathbf{E}_0(S_k^2)-\mathbf{E}(S_k^2)||_{p/2}^{1/2}+\big(\mathbf{E}(S_k^2)\big)^{1/2}$.

Proposition 21 *Let $p>2$ be a real number and let $(X_i)_{i\in\mathbf{Z}}$ be a stationary sequence of real-valued random variables in $\mathbf{L}_p$ adapted to an increasing filtration $(\mathcal{F}_i)$. Let $(\lambda(j))_{j\ge1}$ be defined by (39). For every positive integer n, the following inequality holds: for any $\varepsilon>0$,*

$$\mathbf{E}\big(\max_{1\le j\le n}|S_j|^p\big) \ll n^{p/2}\Big(\sum_{k=0}^{n-1}|\mathbf{E}(X_0X_k)|\Big)^{p/2} + n\mathbf{E}(|X_1|^p) + n\sum_{k=1}^n k^{p-2+\varepsilon}\lambda^{p/2}(k)\,.$$

The constant that is implicitly involved in the notation $\ll$ depends on p and ε but it does not depend on n.

4 Applications and examples

As we have seen, Propositions 2 and 5 give a direct approach to compare the moments of order p of the maximum of the partial sums to the moments of order p of the partial sum. We start this section by presenting two additional applications of these propositions to the convergence of maximum of partial sums and to the maximal Bernstein inequality for dependent structures. In the last three examples, we apply our results on the Rosenthal-type inequalities to different classes of processes. In all the examples given in Sections 4.3, 4.4 and 4.5, we show that, for real numbers p larger than 2, the order of magnitude of $\|\max_{1\le k\le n}|S_k|\|_p$ is essentially given by the order of magnitude of $\|S_n\|_2$ (or by a bound of it).

4.1 Convergence of the maximum of partial sums in $\mathbf{L}^p$.

Corollary 22 *Let $p>1$ and let $(X_i)_{i\in\mathbf{Z}}$ be a strictly stationary sequence of centered real-valued random variables in $\mathbf{L}^p$ adapted to an increasing and stationary filtration $(\mathcal{F}_i)_{i\in\mathbf{Z}}$. Assume that*

$$\lim_{n\to\infty} n^{-1/p}\|S_n\|_p = 0\,. \tag{40}$$

Assume in addition that

$$\sum_{n\ge1}\frac{\|\mathbf{E}(S_n|\mathcal{F}_0)\|_p}{n^{1+1/p}} < \infty\,. \tag{41}$$

Then

$$\lim_{n\to\infty} n^{-1/p}\|\max_{1\le k\le n}|S_k|\,\|_p = 0\,. \tag{42}$$

Remark 23 *This corollary is particularly useful for studying the asymptotic behavior of a partial sum via a martingale approximation. Assume there exists a strictly stationary sequence* $(d_i)_{i\in\mathbf{Z}}$ *of martingale differences with respect to* $(\mathcal{F}_i)_{i\in\mathbf{Z}}$ *that are in* $\mathbf{L}^p$*, such that* $\lim_{n\to\infty} n^{-1/p}\|S_n-\sum_{i=1}^n d_i\|_p = 0$ *. Then, if the condition (41) holds for the sequence* $(X_i)_{i\in\mathbf{Z}}$*, by a construction in Woodroofe and Zhao (2008) and by the uniqueness of the martingale approximation, the sequence* $(X_i-d_i)_{i\in\mathbf{Z}}$ *is still a strictly stationary sequence and by our theorem* $\lim_{n\to\infty} n^{-1/p}\|\max_{1\le k\le n}|S_k-\sum_{i=1}^k d_i|\|_p=0$*. As a matter of fact, for* $p=2$*, our corollary leads to the functional form of the central limit theorem for* $\{n^{-1/2}S_{[nt]}, t\in[0,1]\}$ *(see also Theorem 1.1 in Peligrad and Utev, 2005).*

Proof of Corollary 22.

Let m be an integer and $k=k_{n,m}=[n/m]$ (where $[x]$ denotes the integer part of x).

The initial step of the proof is to divide the variables in blocks of size m and to make the sums in each block. Let

$$X_{i,m}=\sum_{j=(i-1)m+1}^{im} X_j\ ,\ i\ge 1.$$

Notice first that

$$\|\sup_{t\in[0,1]}\big|\sum_{j=1}^{[nt]}X_j-\sum_{i=1}^{[kt]}X_{i,m}\big|\|_p\le\|\sup_{t\in[0,1]}\big|\sum_{i=[kt]m+1}^{[nt]}X_i\big|\|_p\le m\|\max_{1\le i\le n}X_i\|_p\,.$$

Since for every $\varepsilon>0$,

$$\mathbf{E}(\max_{1\le i\le n}|X_i|^p)\le\varepsilon^p+\sum_{i=1}^n\mathbf{E}\big(|X_i|^p\mathbf{1}_{\{|X_i|>\varepsilon\}}\big)\,,$$

and since $\|X_i\|_p<\infty$ for all i, we derive that $\lim_{n\to\infty}\|\max_{1\le i\le n}X_i\|_p/n^{1/p}=0$. Hence, in order to prove (42) it remains to show that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\ n^{-1/p}\|\sup_{t\in[0,1]}\big|\sum_{i=1}^{[kt]}X_{i,m}\big|\|_p=0\,. \tag{43}$$

Applying Proposition 2 to the variables $(X_{i,m})_{1\le i\le k}$ which are adapted with respect to $\mathcal{F}_{im}$, and taking into account Remark 4, we get that

$$\|\sup_{t\in[0,1]}\big|\sum_{i=1}^{[kt]}X_{i,m}\big|\|_p\ll\max_{1\le j\le k}\|\sum_{\ell=1}^{jm}X_\ell\|_p+k^{1/p}\sum_{j=1}^k\frac{||\mathbf{E}(S_{jm}|\mathcal{F}_0)||_p}{j^{1+1/p}}\,,$$

where for the last term we used the fact that for any positive integer u, $\|\mathbf{E}\big(\sum_{j=um2^\ell+1}^{(u+1)m2^\ell}X_j|\mathcal{F}_{um2^\ell}\big)\|_p=\|\mathbf{E}\big(\sum_{j=1}^{m2^\ell}X_j|\mathcal{F}_0\big)\|_p$. Condition (40) implies that

$$\max_{1\le j\le k}\|\sum_{\ell=1}^{jm}X_\ell\|_p=o((km)^{1/p})=o(n^{1/p})\,.$$

Now, by the subadditivity of the sequence $\big(||\mathbf{E}(S_n|\mathcal{F}_0)||_p\big)_{n\ge1}$ and applying Lemma 37 we have

$$n^{-1/p}k^{1/p}\sum_{j=1}^k\frac{||\mathbf{E}(S_{jm}|\mathcal{F}_0)||_p}{j^{1+1/p}}\ll\sum_{\ell=1}^m\frac{||\mathbf{E}(S_\ell|\mathcal{F}_0)||_p}{(\ell+m)^{1+1/p}}+\sum_{j\ge m}\frac{||\mathbf{E}(S_j|\mathcal{F}_0)||_p}{j^{1+1/p}}\,. \tag{44}$$

Hence, by (41) and by using the dominated convergence theorem for discrete measures applied to the first term in the right-hand side of (44), we get that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\ n^{-1/p}k^{1/p}\sum_{j=1}^{k}\frac{||\mathbf{E}(S_{jm}|\mathcal{F}_0)||_p}{j^{1+1/p}}=0\,,$$

which ends the proof of the corollary. ⋄

4.2 Maximal exponential inequalities for strong mixing sequences.

Let us first recall the definition of strongly mixing sequences, introduced by Rosenblatt (1956): For any two σ algebras $\mathcal{A}$ and $\mathcal{B}$, we define the α-mixing coefficient by

$$\alpha(\mathcal{A},\mathcal{B})=\sup_{A\in\mathcal{A},B\in\mathcal{B}}|\mathbf{P}(A\cap B)-\mathbf{P}(A)\mathbf{P}(B)|\ .$$

Let $(X_k, k\geq 1)$ be a sequence of real-valued random variables defined on $(\Omega,\mathcal{A},\mathbf{P})$. This sequence will be called strongly mixing if

$$\alpha(n):=\sup_{k\geq 1}\alpha\left(\mathcal{F}_k,\mathcal{G}_{k+n}\right)\to 0\text{ as }n\to\infty\,, \tag{45}$$

where $\mathcal{F}_j:=\sigma(X_i, i\leq j)$ and $\mathcal{G}_j:=\sigma(X_i, i\geq j)$ for $j\geq 1$.

In 2009, Merlevède, Peligrad and Rio have proved (see their Theorem 2) that for a strongly mixing sequence of centered random variables satisfying $\sup_{i\geq 1}\|X_i\|_\infty\leq M$ and for a certain $c>0$

$$\alpha(n)\leq\exp(-cn)\,, \tag{46}$$

the following Bernstein-type inequality is valid: there is a constant C depending only on c such that for all $n\geq 2$,

$$\mathbf{P}(|S_n|\geq x)\leq\exp\big(-\frac{Cx^2}{v^2n+M^2+xM(\log n)^2}\big)\,, \tag{47}$$

where

$$v^2=\sup_{i>0}\Big(\mathrm{Var}(X_i)+2\sum_{j>i}|\mathrm{Cov}(X_i,X_j)|\Big)\,. \tag{48}$$

Proving the maximal version of the inequality (47) cannot be handled directly neither using Theorem 2.2 in Móricz, Serfling and Stout (1982) nor using Theorem 1 in Kevei and Mason (2011) since the right-hand side of (47) does not satisfy the assumptions of these papers. However, an application of Proposition 5 leads to the maximal version of Theorem 2 in Merlevède, Peligrad and Rio (2009).

Corollary 24 *Let $(X_j)_{j\geq 1}$ be a sequence of centered real-valued random variables. Suppose that there exists a positive M such that $\sup_{i\geq 1}\|X_i\|_\infty\leq M$ and that the strongly mixing coefficients $(\alpha(n))_{n\geq 1}$ of the sequence satisfy (46). Then there exist constants $C=C(c)$ and $K=K(M,c)$ such that for all integer $n\geq 2$ and all real $x>K\log n$,*

$$\mathbf{P}(\max_{1\leq k\leq n}|S_k|\geq x)\leq\exp\big(-\frac{Cx^2}{v^2n+M^2+xM(\log n)^2}\big)\,. \tag{49}$$

Proof of Corollary 24. We first apply Inequality (9) of Proposition 5 with $p=2$ and $\varphi(x)=e^{t|x|}$ where t is a positive integer. According to Theorem 2 in Merlevède, Peligrad and Rio (2009), there exist positive constants C_1 and C_2 depending only on c such that for all $n \geq 2$ and any positive t such that $t < \frac{1}{C_1 M (\log n)^2}$, the following inequality holds:

$$\log \mathbf{E}\big(\exp(tS_n)\big) \leq \frac{C_2 t^2 (nv^2 + M^2)}{1 - C_1 t M (\log n)^2}\,.$$

Then an optimization on t gives that there is a constant C_3 depending only on c such that for all $n \geq 2$ and any positive real x,

$$\mathbf{P}\big(\max_{1\leq k\leq n} |S_k| \geq 4x\big) \leq \exp\Big(-\frac{C_3 x^2}{v^2 n + M^2 + xM(\log n)^2}\Big)$$
$$+4x^{-2}\Big(\sum_{l=0}^{r-1}\Big(\sum_{k=1}^{2^{r-l}-1} ||\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l}|\mathcal{F}_{k2^l})||_2^2\Big)^{1/2}\Big)^2\,, \tag{50}$$

where r is the positive integer satisfying $2^{r-1} < n \leq 2^r$ and $v = [x/M]$.

It remains to bound up the second term in the right-hand side of the above inequality. Notice first that for any centered variable Z such that $\|Z\|_\infty \leq B$, Ibragimov's covariance inequality (see Theorem 1.11 in Bradley, 2007) gives

$$\|\mathbf{E}(Z|\mathcal{F})\|_2^2 = \mathrm{Cov}(\mathbf{E}(Z|\mathcal{F}), Z) \leq 4B^2 \alpha(\mathcal{F}, \sigma(Z))\,.$$

Therefore, applying this last estimate with $Z = S_{v+(k+1)2^l} - S_{v+k2^l}$ and $\mathcal{F} = \mathcal{F}_{k2^l}$, we get that

$$||\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l}|\mathcal{F}_{k2^l})||_2^2 \leq 4M^2 2^{2l}\alpha(v)\,.$$

implying that

$$\Big(\sum_{l=0}^{r-1}\Big(\sum_{k=1}^{2^{r-l}-1} ||\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l}|\mathcal{F}_{k2^l})||_2^2\Big)^{1/2}\Big)^2 \leq 4M^2 2^{2r}(\sqrt{2}+1)^2\alpha(v)\,.$$

Since $2^{2r} \leq 4n^2$ and $[x/M] \geq x/(2M)$, for $x \geq 2M$, by using (46), we get that, for any $x \geq 2M$,

$$\Big(\sum_{l=0}^{r-1}\Big(\sum_{k=1}^{2^{r-l}-1} ||\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l}|\mathcal{F}_{k2^l})||_2^2\Big)^{1/2}\Big)^2 \leq 3(2^5)M^2 n^2 \exp(-cx/(2M))\,. \tag{51}$$

Starting from (50) and using (51), we then derive that for any $x \geq 2M \max(1, 4c^{-1}\log n)$,

$$\mathbf{P}(S^*_{2^r} \geq 4x) \leq \exp\Big(-\frac{C_3 x^2}{v^2 n + M^2 + xM(\log n)^2}\Big) + 96\exp\Big(-\frac{xc}{4M}\Big)\,,$$

proving the inequality (49). $\diamond$

4.3 Application to Arch models.

Theorem 6 applies to the case where $(X_i)_{i\in\mathbf{Z}}$ has an ARCH(∞) structure as described by Giraitis *et al.* (2000), that is

$$X_n = \sigma_n \eta_n\,, \text{ with } \sigma_n^2 = c + \sum_{j=1}^{\infty} c_j X_{n-j}^2\,, \tag{52}$$

where $(\eta_n)_{n\in\mathbf{Z}}$ is a sequence of i.i.d. centered random variables such that $\mathbf{E}(\eta_0^2)=1$, and where $c\geq 0$, $c_j\geq 0$, and $\sum_{j\geq 1} c_j<1$. Notice that $(X_i)_{i\in\mathbf{Z}}$ is a stationary sequence of martingale differences adapted to the filtration $(\mathcal{F}_i)$ where $\mathcal{F}_i=\sigma(\eta_k, k\leq i)$.

Let $p>2$ and assume that $\|\eta_0\|_p<\infty$. Notice first that

$$\|\mathbf{E}(X_j^2|\mathcal{F}_0)-\mathbf{E}(X_0^2)\|_{p/2}=\|\mathbf{E}(\sigma_j^2|\mathcal{F}_0)-\mathbf{E}(\sigma_j^2)\|_{p/2}\,. \tag{53}$$

In addition, since $\mathbb{E}(\eta_0^2)=1$ and $\sum_{j\geq 1} c_j<1$, the unique stationary solution to (52) is given by Giraitis *et al.* (2000):

$$\sigma_n^2=c+c\sum_{\ell=1}^{\infty}\sum_{j_1,\dots,j_\ell=1}^{\infty} c_{j_1}\dots c_{j_\ell}\eta_{n-j_1}^2\dots\eta_{n-(j_1+\cdots+j_\ell)}^2\,. \tag{54}$$

Starting from (53) and using (54), one can prove that

$$\|\mathbf{E}(X_j^2|\mathcal{F}_0)-\mathbf{E}(X_0^2)\|_{p/2}\leq 2c\|\eta_0\|_p^2\sum_{\ell=1}^{\infty}\ell\kappa^{\ell-1}\sum_{i=[j/\ell]}^{\infty}c_i\,,$$

where $\kappa=\|\eta_0\|_p^2\sum_{j\geq 1}c_j$ (see Section 6.6 in Dedecker and Merlevède (2011) for more detailed computations). Therefore, if

$$\|\eta_0\|_p^2\sum_{j\geq 1}c_j<1\,, \tag{55}$$

for any $p>2$ and $\delta\in]0,1]$ we obtain

$$\sum_{k=1}^{n}\frac{1}{k^{1+2\delta/p}}||\mathbf{E}_0(S_k^2)-\mathbf{E}(S_k^2)||_{p/2}^{\delta}\leq c_{p,\delta,X}\sum_{k=1}^{n}\frac{1}{k^{1+2\delta/p}}\Big(\sum_{j=1}^{n}\sum_{i=j}^{\infty}c_i\Big)^{\delta}\,,$$

where $c_{p,\delta,X}$ is a positive constant depending depending only on p, δ, c, κ and $\|\eta_0\|_p$. Applying Theorem 6 for the martingale case, we then get the following corollary:

Corollary 25 *Let $X=(X_i)_{i\in\mathbf{Z}}$ be defined by (52) and $S_n=\sum_{i=1}^n X_i$. Let $p>2$ and assume that (55) is satisfied.*
(1) If we assume that $\sum_{j\geq n}c_j=O(n^{-b})$ for $b>1-2/p$ then, for any integer n,

$$\mathbf{E}\big(\max_{1\leq j\leq n}|S_j|^p\big)\ll\big(n\mathbf{E}(X_0^2)\big)^{p/2}+n(\mathbf{E}(|X_0|^p)+b_{p,X})\,,$$

where $b_{p,X}$ is a positive constant depending on p and on the underlying sequence X, but not depending on n.
(2) If we assume $\sum_{j\geq n}c_j=O(n^{-b})$ for $b>0$ then

$$\limsup_{n\to\infty}\; n^{-1/2}\|\max_{1\leq k\leq n}\;|S_k|\;\|_p\leq a_p\left(\mathbf{E}(X_0^2)\right)^{1/2}<\infty\,,$$

where a_p is a constant depending only on p.

4.4 Application to functions of linear processes.

Let $(a_i)_{i\in\mathbf{Z}}$ be a sequence of real numbers in ℓ^2 and $(\varepsilon_i)_{i\in\mathbf{Z}}$ be a sequence of i.i.d. random variables in $\mathbf{L}^2$. Define

$$X_k=h\Big(\sum_{i\in\mathbf{Z}}a_i\varepsilon_{k-i}\Big)-\mathbf{E}\Big(h\Big(\sum_{i\in\mathbf{Z}}a_i\varepsilon_{k-i}\Big)\Big)\,. \tag{56}$$

Denote by $w_h(.,M)$ the modulus of continuity of the function h on the interval $[-M,M]$, that is

$$w_h(t,M)=\sup\{|h(x)-h(y)|,|x-y|\le t,|x|\le M,|y|\le M\}\,.$$

We shall establish the following result:

Corollary 26 *Let $X=(X_k)_{k\in\mathbf{Z}}$ be defined by (56). Assume that h is γ-Hölder on any compact set, with $w_h(t,M)\le Ct^\gamma M^\alpha$, for some $C>0$, $\gamma\in]0,1]$ and $\alpha\ge 0$. Let $p>2$ and assume that $\mathbf{E}(|\varepsilon_0|^{2\vee(\alpha+\gamma)p})<\infty$.*
1) If p is an even integer and for $\lambda>p/2-2$ if $p>4$ and $\lambda=0$ if $p=4$,

$$\sum_{i\ge 1} i^{1-2/p}(\log i)^\lambda\Big(\sum_{j\ge i}a_j^2\Big)^{\gamma/2}<\infty\,, \tag{57}$$

then, for any integer n,

$$\mathbf{E}\big(\max_{1\le j\le n}|S_j|^p\big)\ll\big(\mathbf{E}(S_n^2)\big)^{p/2}+n(\mathbf{E}(|X_0|^p)+b_{p,X})\,,$$

where $b_{p,X}$ is a positive constant depending on p and on the underlying sequence X, but not depending on n.
2) If for some $\eta>1$

$$\big(\sum_{j\ge i}a_j^2\big)^{\gamma/2}\ll i^{-\eta}\,, \tag{58}$$

then

$$\limsup_{n\to\infty}\; n^{-1/2}\|\max_{1\le k\le n}\;|S_k|\;\|_p\le a_p\big(\sum_{k\in\mathbf{Z}}\mathbf{E}(X_0X_k)\big)^{1/2}<\infty\,,$$

where a_p is a constant depending only on p.

Remark 27 *The proof of the first item of the above result is based on Theorem 17. Our proof reveals that an application of Theorem 6 would involve a more restrictive condition on λ, namely $\lambda>p-3$.*

As a preliminary step in the proof of Corollary 26 we state the following proposition, which is a direct consequence of the proof of Proposition 4.2 and of Theorem 4.2 in Dedecker, Merlevède and Rio (2009), page 988.

Proposition 28 *Let $(X_i)_{i\in\mathbf{Z}}$ be as in Corollary 26 and $p\ge 2$. Let $(\varepsilon'_i)_{i\in\mathbf{Z}}$ be an independent copy of $(\varepsilon_i)_{i\in\mathbf{Z}}$ and denote $V_0=\sum_{i\ge 0}a_i\varepsilon_{-i}$ and*

$$M_{1,i}=|V_0|\vee\Big|\sum_{0\le j<i}a_j\varepsilon_{-j}+\sum_{j\ge i\ge 0}a_j\varepsilon'_{-j}\Big|\quad\text{and}\quad \tilde{w}_h(i)=\big\|w_h\big(\big|\sum_{k\ge i}a_k\varepsilon_{-k}\big|,M_{1,i}\big)\big\|_p\,.$$

Then for any $i,j\ge 0$,

$$||\mathbf{E}_0(X_i)||_p\le 2\tilde{w}_h(i)\,,$$

and

$$\|\mathbf{E}_0(X_iX_{j+i})-\mathbf{E}(X_iX_{j+i})\|_{p/2}\le 2||X_0||_p\min\Big(\tilde{w}_h(i)+\tilde{w}_h(i+j),\tilde{w}_h(j-[j/2])\Big)\,.$$

Moreover, if $w_h(t,M)$ and ε_0 are as in Corollary 26, then for all $i\ge 0$

$$\tilde{w}_h(i)\le K(\sum_{j\ge i}a_i^2)^{\gamma/2}\,,$$

where K is a constant depending on p, C, α, γ, $\|\varepsilon_0\|_{2\vee(\alpha+\gamma)p}$ and on $\sum_{j\ge 0}a_j^2$.

Proof of Corollary 26. Notice first that by Proposition 28 and the relation (58) we get

$$\sum_{k\geq 1} ||\mathbf{E}_0(X_k)||_p < \infty\,. \tag{59}$$

For any even integer $p \geq 4$, applying Theorem 17 we obtain, via the relation (59) and the triangular inequality, that

$$\mathbf{E}\Big(\max_{1\leq j\leq n}|S_j|^p\Big) \ll n\,(1+\mathbf{E}(|X_0|^p)) + n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+4/p(p-2)}}(\mathbf{E}(S_k^2))^{2/(p-2)}\Big)^{p(p-2)/4}$$
$$+\, n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+4/p(p-2)}}||\mathbf{E}_0(S_k^2)-\mathbf{E}(S_k^2)\;||_{p/2}^{2/(p-2)}\Big)^{p(p-2)/4}\,.$$

Notice now that the relation (59) implies the so called coboundary decomposition (see Theorem 5.4 in Hall and Heyde, 1980); namely, there is a constant $K_{p,X}$ such that for all $n \geq 1$

$$S_n = M_n + R_n \text{ with } ||R_n||_p \leq 2\sum_{k\geq 1}||\mathbf{E}_0(X_k)||_p = K_{p,X}\,,$$

where M_n is a martingale in $\mathbf{L}^p$ with stationary differences. It easily follows that for all $1 \leq k \leq n$,

$$\frac{\mathbf{E}(S_k^2)}{k} \leq 4\frac{\mathbf{E}(S_n^2)}{n} + 2\frac{\mathbf{E}(R_k^2)}{k} + 4\frac{\mathbf{E}(R_n^2)}{n} \leq 4\frac{\mathbf{E}(S_n^2)}{n} + 6\frac{K_{p,X}}{k}\,.$$

So,

$$n\Big(\sum_{k=1}^{n}\frac{1}{k^{1+4/p(p-2)}}(\mathbf{E}(S_k^2))^{2/(p-2)}\Big)^{p(p-2)/4} \ll (\mathbf{E}(S_n^2))^{p/2} + K_{p,X}\;n\,.$$

Therefore, the first part of the corollary follows if we prove that (57) implies that

$$\sum_{k\geq 1}\frac{1}{k^{1+4/p(p-2)}}||\mathbf{E}_0(S_k^2)-\mathbf{E}(S_k^2)\;||_{p/2}^{2/(p-2)} < \infty\,. \tag{60}$$

Notice now that by Hölder's inequality (when $p > 4$) and (38), in order to prove (60), it suffices to prove that for $\lambda > p/2-2$ if $p > 4$ and $\lambda = 0$ if $p = 4$,

$$\sum_{k\geq 1}\frac{(\log k)^{\lambda}}{k^{1+2/p}}\sum_{i=1}^{k}\sum_{j=0}^{k-i}||\mathbf{E}_0(X_iX_{j+i})-\mathbf{E}(X_iX_{j+i})||_{p/2} < \infty\,.$$

The first part of Corollary 26 is then a consequence of Proposition 28. The second part follows in the same way by using Theorem 6, simple computations and the fact that under (59), $\lim_{n\to\infty} n^{-1}\mathbf{E}(S_n)^2 = \sum_{k\in\mathbf{Z}}\mathbf{E}(X_0X_k)$. $\diamond$

4.5 Application to a stationary reversible Markov chain.

First we want to mention that all our results can be formulated in the Markov chain setting. We assume that $(\zeta_n)_{n\in\mathbb{Z}}$ denotes a stationary Markov chain defined on a probability space $(\Omega,\mathcal{A},\mathbf{P})$ with values in a measurable space $(E,\mathcal{E})$. The marginal distribution and the transition kernel are denoted by $\pi(A) = \mathbf{P}(\zeta_0 \in A)$ and $Q(\zeta_0, A) = \mathbf{P}(\zeta_1 \in A|\,\zeta_0)$. In addition Q denotes the operator acting via $(Qf)(\zeta) = \int_E f(s)Q(\zeta, ds)$. Next, let f be a function on E such that $\int_E |f|^p d\pi < \infty$ and $\int_E f d\pi = 0$.

Denote by $\mathcal{F}_k$ the σ–field generated by ζ_i with $i \le k$, $X_i = f(\zeta_i)$, and $S_n(f) = \sum_{i=1}^{n} X_i$. Notice that any stationary sequence $(Y_k)_{k\in\mathbb{Z}}$ can be viewed as a function of a Markov process $\zeta_k = (Y_i; i \le k)$, for the function $g(\zeta_k) = Y_k$.

The Markov chain is called reversible if $Q = Q^*$, where Q^* is the adjoint operator of Q. In this setting, an application of Theorem 9 gives the following estimate:

Corollary 29 *Let (ζ_n) be a reversible Markov chain. For any even integer $p \ge 4$ and any positive integer n,*

$$\mathbf{E}\big(\max_{1\le k\le n} |S_k(f)|^p\big) \ll n\mathbf{E}(|f(\zeta_1)|^p) + n\Big(\sum_{k=1}^{n} \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k(f))\|_p\Big)^p$$
$$+n\Big(\sum_{k=1}^{n} \frac{1}{k^{1+2/p}} \|\mathbf{E}_0(S_k^2(f)) - \mathbf{E}(S_k^2(f))\|_{p/2}\Big)^{p/2} + n\Big(\sum_{k=1}^{n} \frac{1}{k^{1+2/p}} \mathbf{E}(S_k^2(f))\Big)^{p/2}.$$

Moreover using Theorem 10 we obtain:

Corollary 30 *Let (ζ_n) be a reversible Markov chain. For any real number $p > 4$ and any positive integer n,*

$$\mathbf{E}\big(\max_{1\le k\le n} |S_k(f)|^p\big) \ll n\mathbf{E}(|f(\zeta_1)|^p)$$
$$+n\Big(\sum_{k=1}^{n} \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k^2(f)) - \mathbf{E}(S_k^2(f))\|_{p/2}^{1/2}\Big)^p + n\Big(\sum_{k=1}^{n} \frac{1}{k^{1+1/p}} ||S_k(f)||_2\Big)^p.$$

This corollary is also valid for any real $2 < p \le 4$. For this range however, according to the comment 7, Theorem 6 (respectively Corollary 29) gives a better bound for $p \in]2, 4[$ (respectively for $p = 4$).

For a particular example let $E = [-1, 1]$ and let υ be a symmetric atomless law on E. The transition probabilities are defined by

$$Q(x, A) = (1 - |x|)\delta_x(A) + |x|\upsilon(A),$$

where δ_x denotes the Dirac measure. Assume that $\theta = \int_E |x|^{-1}\upsilon(dx) < \infty$. Then there is a unique invariant measure

$$\pi(dx) = \theta^{-1}|x|^{-1}\upsilon\,(dx)$$

and the stationary Markov chain $(\zeta_i)_i$ is reversible and positively recurrent.

Assume the following assumption on the measure υ: there exists a positive constant c such that for any $x \in [0, 1]$,

$$\frac{d\upsilon}{dx}(x) \le cx^{p/2-1}(\log(1 + 1/x))^{-\lambda} \text{ for some } \lambda > 0. \tag{61}$$

As an application of Corollary 30 we shall establish:

Corollary 31 *Let $p > 2$ be a real number and let $f(-x) = -f(x)$ for any $x \in E$. Assume that $|f(x)| \le C|x|^{1/2}$ for any x in E and a positive constant C.*
(1) Assume in addition that (61) is satisfied for $\lambda > p$. Then for any integer n,

$$\mathbf{E}\big(\max_{1\le k\le n} |S_k(f)|^p\big) \ll n^{p/2}\Big(\int_0^1 f^2(x)x^{-2}\upsilon(dx)\Big)^{p/2} + n(\mathbf{E}(|f(\zeta_0)|^p) + b_{p,\lambda,c,C}), \tag{62}$$

where $b_{p,\lambda,c,C}$ is a positive constant depending on p, λ, c and C.
(2) Assume now (61) is relaxed to

$$\frac{d\upsilon}{dx}(x) \le cx^a \text{ for some } a > 0 \; .$$

Then

$$\limsup_{n\to\infty} n^{-1/2} \| \max_{1\le k\le n} |S_k(f)| \, \|_p \le a_p \Big(\int_0^1 f^2(x) x^{-2} \upsilon(dx) \Big)^{1/2} \, .$$

where a_p is a constant depending only on p.

Notice that this example of reversible Markov chain has been considered by Rio (2009, Section 4) under a slightly more stringent condition on the measure than (61). Corollary 31 then extends Proposition 4.1 (b) in Rio (2009) to all real numbers $p > 2$.

Proof of Corollary 31.

To get this result we shall apply Corollary 30. We start by noticing that f being an odd function we have

$$\mathbf{E}(f(\zeta_k)|\zeta_0) = (1-|\zeta_0|)^k f(\zeta_0) \text{ a.s.} \tag{63}$$

Therefore, for any $j \ge 0$,

$$\mathbf{E}(X_0 X_j) = \mathbf{E}(f(\zeta_0)\mathbf{E}(f(\zeta_j)|\zeta_0)) = \theta^{-1} \int_E f^2(x)(1-|x|)^j |x|^{-1} \upsilon(dx) \, .$$

Then, by the inequality (36),

$$\begin{aligned} \mathbf{E}(S_k^2(f)) &\le 2k\theta^{-1} \Big(\int_0^1 f^2(x) x^{-1} \upsilon(dx) + 2 \sum_{j=1}^{k-1} \int_0^1 f^2(x)(1-x)^j x^{-1} \upsilon(dx) \Big) \qquad (64) \\ &\le 2k\theta^{-1} \Big(\int_0^1 f^2(x) x^{-1} \upsilon(dx) + 2 \int_0^1 f^2(x) x^{-2} \upsilon(dx) \Big) \, . \end{aligned}$$

Next, we give an upper bound of the quantity $\|\mathbf{E}_0(S_n^2(f)) - \mathbf{E}(S_n^2(f))\|_{p/2}$. By using the fact that for any positive k, $\pi Q^k = \pi$ we have

$$\|\mathbf{E}_0(S_n^2(f)) - \mathbf{E}(S_n^2(f))\|_{p/2} \le \sum_{k=1}^n \Big(\int_E \Big| (\delta_x Q^k - \pi)\big(f^2 + 2f \sum_{k=1}^{n-k} Q^k f\big) \Big|^{p/2} \pi(dx) \Big)^{2/p}$$

(see also the inequality (4.12) in Rio (2009)). Now, by using the relation (63), one can prove that for any $x \in E$,

$$\Big(f^2 + 2f \sum_{k=1}^n Q^k f \Big)(x) = f^2(x) \big(1 + 2(1-(1-|x|)^n)(|x|^{-1} - 1)\big) \, ,$$

(see the computations in Rio (2009) leading to his relation (4.13)). Then, since $|f(x)| \le C|x|^{1/2}$, it follows that

$$\sup_{x\in E} \big| f^2(x) + 2f(x) \sum_{k=1}^n Q^k f(x) \big| \le 2C^2 \, .$$

Therefore, for any $p > 2$,

$$\|\mathbf{E}_0(S_n^2(f)) - \mathbf{E}(S_n^2(f))\|_{p/2} \le 4C^2 \sum_{k=1}^n \Big(\int_E \|Q^k(x,\cdot) - \pi(\cdot)\| \pi(dx) \Big)^{2/p} \, , \tag{65}$$

where $\|\mu(\cdot)\|$ denotes the total variation of the signed measure μ (see also the inequality (4.15) in Rio (2009)). We estimate next the coefficients of absolute regularity β_n as defined in (66). Let $a = p/2-1$. We shall prove now that under (61), there exists a positive constant K depending on a, c, and λ, such that

$$2\beta_n := \int_E \|Q^n(x,\cdot) - \pi(\cdot)\|\pi(dx) \le K n^{-a} (\log n)^{-\lambda} \,. \tag{66}$$

Notice first that by Lemma 2, page 75, in Doukhan, Massart and Rio (1994), we have that

$$\beta_n \le 3 \int_E (1-|x|)^{[n/2]} \pi(dx) \,. \tag{67}$$

Let $k \ge 2$ be an integer. Clearly, for any $\alpha \in]0,1[$,

$$\int_0^1 (1-x)^k \pi(dx) \le c \int_0^{k^{-\alpha}} (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx + c \int_{k^{-\alpha}}^1 (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx \,. \tag{68}$$

Notice now that

$$\int_0^{k^{-\alpha}} (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx \le (\alpha \log k)^{-\lambda} \int_0^1 (1-x)^k x^{a-1} dx \,.$$

Hence, by the properties of the Beta and Gamma functions,

$$\lim_{k\to\infty} k^a (\log k)^{\lambda} \int_0^{k^{-\alpha}} (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx \le \alpha^{-\lambda} a \Gamma(a) \,. \tag{69}$$

On the other hand, we have that

$$\int_{k^{-\alpha}}^1 (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx \le (\log 2)^{-\lambda} (1-k^{-\alpha})^k \int_0^1 x^{a-1} dx \,,$$

and then, since $\alpha < 1$, we easily obtain

$$\lim_{k\to\infty} k^a (\log k)^{\lambda} \int_{k^{-\alpha}}^1 (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx = 0 \,. \tag{70}$$

Starting from (67) and taking into account (68), (69) and (70), (66) follows. Then, by using the inequality (65) combined with (66), we derive that

$$\|\mathbf{E}_0(S_n^2(f)) - \mathbf{E}(S_n^2(f))\|_{p/2} \le C^2 K_{p,c,\lambda} n^{2/p} (\log n)^{-2\lambda/p} \,,$$

where $K_{p,c,\lambda}$ is a positive constant depending on p, c and λ. Therefore, for any $\delta \in]0,1]$, there exists a positive constant $b_{p,\lambda,c,C}$ depending on p, λ, c and C, such that

$$\sum_{k=1}^n \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k^2(f)) - \mathbf{E}(S_k^2(f))\|_{p/2}^{1/2} \le b_{p,\lambda,c,C} \text{ for } \lambda > p \,. \tag{71}$$

Considering the estimates (64) and (71), the first part of Corollary 31 follows from an application of Corollary 30 (taking also into account the comment after its statement for $2 < p \le 4$). To get the second part, we also apply Corollary 30 but with the upper bound $\beta_n = O(n^{-a})$ instead of (66). $\diamond$

5 Application to density estimation

In this section, we estimate the $\mathbf{L}^p$-integrated risk for $p \geq 4$, for the kernel estimator of the unknown marginal density f of a stationary sequence $(Y_i)_{i\geq 0}$.

Applying our theorem 6, we shall show that if the coefficients of dependence $((\beta_{2,Y}(k))_{k\geq 1}$ (see the definition 32 below) of the sequence $(Y_i)_{i\in\mathbf{Z}}$ satisfy $\beta_{2,Y}(k) = O(n^{-a})$ for $a > p-1$, then the bound of the $\mathbf{L}^p$-norm of the random term of the risk is of the same order of magnitude as the optimal one obtained in Bretagnolle and Huber (1979) in the independence setting (see their corollary 2), provided that the density is bounded and the kernel K satisfies the assumption $\mathbf{A}_p$ below.

Assumption $\boldsymbol{A}_p$. *K is a BV (bounded variation) function such that*

$$\int_{\mathbf{R}} |K(u)|du < \infty \text{ and } \int_{\mathbf{R}} |K(u)|^p du < \infty\,.$$

Definition 32 *Let $(Y_i)_{i\in\mathbf{Z}}$ be a stationary sequence of real valued random variables, and let $\mathcal{F}_0 = \sigma(Y_i, i \leq 0)$. For any positive i and j, define the random variables*

$$b(\mathcal{F}_0, i, j) = \sup_{(s,t)\in\mathbf{R}^2} |\mathbf{P}(Y_i \leq t, Y_j \leq s|\mathcal{F}_0) - \mathbf{P}(Y_i \leq t, Y_j \leq s)|\,.$$

Define now the coefficient

$$\beta_{2,Y}(k) = \sup_{i\geq j\geq k} \mathbf{E}(b(\mathcal{F}_0, i, j))\,.$$

Proposition 33 *Let $p \geq 4$ and K be any real function satisfying assumption $\boldsymbol{A}_p$. Let $(Y_i)_{i\geq 0}$ be a stationary sequence with unknown marginal density f such that $\|f\|_\infty < \infty$. Define*

$$X_{k,n}(x) = K(h_n^{-1}(x - Y_k)) \text{ and } f_n(x) = \frac{1}{nh_n}\sum_{k=1}^n X_{k,n}(x)\,,$$

where $(h_n)_{n\geq 1}$ is a sequence of positive real numbers.
(1) Assume that there exists a positive constant c such that for some $\eta > 0$ and all $n \geq 1$,

$$\beta_{2,Y}(n) \leq cn^{-(p-1+\eta)}\,. \tag{72}$$

Then there exist positive constants C_1 and C_2 depending on p, η and c such that for any positive integer n,

$$\begin{aligned}\mathbf{E}\int_{\mathbf{R}} |f_n(x) - \mathbf{E}(f_n(x))|^p dx \leq C_1(nh_n)^{-p/2}||dK||^{p/2}||f||_\infty^{p/2-1}\Big(\int_{\mathbf{R}}|K(u)|du\Big)^{p/2} \\ +C_2(nh_n)^{1-p}\Big(\int_{\mathbf{R}}|K(u)|^p du + ||dK||\int_{\mathbf{R}}|K(u)|^{p-1}du + ||dK||^2\int_{\mathbf{R}}|K(u)|^{p-2}du\Big)\,,\end{aligned} \tag{73}$$

where $||dK||$ is the total variation norm of the measure dK.
2) Assume now in addition that $nh_n \to \infty$. Then we also have the following asymptotic result:

$$\limsup_{n\to\infty}\ (nh_n)^{-p/2}\mathbf{E}\int_{\mathbf{R}} |f_n(x) - \mathbf{E}(f_n(x))|^p dx \leq a_{p,c,\eta}||dK||^{p/2}||f||_\infty^{p/2-1}\Big(\int_{\mathbf{R}}|K(u)|du\Big)^{p/2}\,,$$

where where $a_{p,c,\eta}$ is a constant depending only on p, c and η.

The bound obtained in Proposition 33 can be also compared to the one obtained in Theorem 3.3 in Viennet (1997) under the assumption that the strong β-mixing coefficients in the sense of Rozanov and Volkonskii (1959) of the sequence $(Y_i)_{i\in\mathbf{Z}}$, denoted by $\beta_\infty(k)$, satisfy: $\sum_{k\geq 1} k^{p-2}\beta_\infty(k) < \infty$. Our condition is then comparable to the one imposed by Viennet (1997) but less restrictive in the sense that many processes are such that the sequence $\beta_{2,Y}(n)$ tends to zero as $n \to \infty$ which is not the case for $\beta_\infty(n)$ (see the examples given in Dedecker and Prieur (2007)). Notice also that, for $p = 2$, the inequality (73) is proven in Dedecker and Prieur (2005, Proposition 3) under the summability of the weak β-dependence coefficients. In addition, when $p = 3$, under the condition $\sum_{k\geq 1} k\beta_{2,Y}(k) < \infty$, it can be proven that the inequality (73) still holds by applying the Rosenthal-type inequality in Dedecker (2001) (see also Proposition 3 in Dedecker and Prieur (2007) for a version of the Dedecker's inequality in case of stationary sequences), however his inequality does not lead to (73) when $p > 3$.

When the random variables Y_i are a function of an i.i.d. sequence, namely, $Y_i = g(..., \varepsilon_{i-1}, \varepsilon_i)$ where g is a measurable function and $(\varepsilon_n, n \in \mathbf{Z})$ are i.i.d. random variables, Wu, Huang and Huang (Theorem 1, 2010) obtained an upper bound of similar order as in (73) (for $p > 1$) under conditions imposed to the so called "predictive dependent measures". When in addition the variable ε_0 has a density with bounded derivatives up to order 2, the method is especially effective for short memory linear processes with independent innovations (see Section 4.1 in Wu, Huang and Huang (2010)). Our Proposition 33 complements the above cited results since the coefficients $\beta_{2,Y}(n)$ can be estimated without assuming that ε_0 has a density. For instance, we obtain the upper bound (73) for $Y_i = \sum_{k\geq 0} 2^{-k-1}\varepsilon_{i-k}$ and the ε_k's are i.i.d. Bernoulli random variables with parameter 1/2 (see Section 6.1 of Dedecker and Prieur (2007) for computations of $\beta_{2,Y}(n)$). In addition, our Proposition 33 applies even for situations when the variables Y_i are not assumed to be a function of an i.i.d. sequence. We refer for instance to Dedecker and Prieur (2009, Theorem 3.1) who gave an upper bound of the coefficients $\beta_{2,Y}(n)$ of the Markov chain associated to an intermittent map.

If we assume now that f has a derivative of order s, where $s \geq 1$ is an integer and that the following bound holds for the bias term:

$$\int_{\mathbf{R}} |f(x) - \mathbf{E}(f_n(x))|^p dx \leq M h_n^{sp} \|f^{(s)}\|_p^p \,, \tag{74}$$

where M is a constant depending on the kernel K, then the choice of $(nh_n)^{p/2} h_n^{sp} = O(1)$ leads to the following estimate:

$$\mathbf{E}\int_{\mathbf{R}} |f_n(x) - f(x)|^p dx = O(n^{-sp/(2s+1)})\,. \tag{75}$$

We mention that (74) holds for any Parzen Kernel of order s (see Section 4 in Bretagnolle and Huber (1979)). We also mention that if we only assume that $\sum_{k\geq 1} k^{p-2}\beta_{2,Y}(k) < \infty$ instead of (72) in Proposition 33 then the inequality (73) is valid with $(nh_n)^{1-p}n^\varepsilon$ (for any $\varepsilon > 0$) replacing $(nh_n)^{1-p}$ in the second term of the right-hand side. In this situation, the bound (74) combined with a choice of h_n of order $n^{-1/(1+2s)}$ still leads to the estimate (75).

Proof of Proposition 33.

Setting $X_{i,n}(x) = K((x - Y_i)/h_n) - \mathbf{E}(K((x - Y_i)/h_n))$, we have that

$$\mathbf{E}\int_{\mathbf{R}} |f_n(x) - \mathbf{E}(f_n(x))|^p dx \leq (nh_n)^{-p} \int_{\mathbf{R}} \mathbf{E}\Big|\sum_{i=1}^n X_{i,n}(x)\Big|^p dx\,. \tag{76}$$

Starting from (76) and applying Proposition 21 to the stationary sequence $(X_{i,n}(x))_{i\in\mathbf{Z}}$, Proposition 33 follows provided we establish the following bounds (in what follows C is a positive constant which may vary from line to line and that may depend on p, c and η but not on n):

$$\int_{\mathbf{R}} \mathbf{E}|X_{1,n}(x)|^p dx \leq 2^{p+1} h_n \int_{\mathbf{R}} |K(u)|^p du\,, \tag{77}$$

$$\int_{\mathbf{R}} \Big(\sum_{j=0}^{n-1} |\mathbf{E}(X_{0,n}(x)X_{j,n}(x))| \Big)^{p/2} dx \le C h_n^{p/2} ||dK||^{p/2} ||f||_\infty^{(p/2)-1} \Big(\int_{\mathbf{R}} |K(u)| du \Big)^{p/2} , \tag{78}$$

and that for $\varepsilon > 0$ small enough,

$$\sum_{j=1}^{n} j^{p-2+\varepsilon} \int_{\mathbf{R}} \|X_{0,n}(x)\mathbf{E}_0(X_{j,n}(x))\|_{p/2}^{p/2} dx \le C h_n ||dK|| \int_{\mathbf{R}} |K(u)|^{p-1} du \,, \tag{79}$$

and

$$\sum_{j=1}^{n} j^{p-2+\varepsilon} \sup_{i \ge j} \int_{\mathbf{R}} \|\mathbf{E}_0(X_{i,n}(x)X_{j,n}(x)) - \mathbf{E}(X_{i,n}(x)X_{j,n}(x))\|_{p/2}^{p/2} dx \tag{80}$$

$$\le C h_n ||dK||^2 \int_{\mathbf{R}} |K(u)|^{p-2} du \,.$$

In what follows, we shall prove these bounds. Notice first that

$$\int_{\mathbf{R}} \mathbf{E}|X_{1,n}(x)|^p dx \le 2^{p+1} \int_{\mathbf{R}} \int_{\mathbf{R}} |K((x-y)h_n^{-1})|^p f(y) dx dy \,,$$

proving (77) by the change of variables $u = (x-y)h_n^{-1}$. To prove (78), we first apply Item 1 of Lemma 35 implying that

$$\sum_{j=0}^{n-1} |\mathbf{E}(X_{0,n}(x)X_{j,n}(x))| \le ||dK|| \, \mathbf{E}\big(\widetilde{b}(\mathcal{F}_0, n) \, |K((x-Y_0)/h_n)|\big) \,,$$

where $\widetilde{b}(\mathcal{F}_0, n) = \sum_{j=0}^{n-1} b(\mathcal{F}_0, j, j)$. An application of Hölder's inequality as done in Viennet (1997) at the bottom of page 474, then gives

$$\int_{\mathbf{R}} \Big(\sum_{j=0}^{n-1} |\mathbf{E}(X_{0,n}(x)X_{j,n}(x))| \Big)^{p/2} dx \le h_n^{p/2} \|f\|_\infty^{p/2-1} \mathbf{E}\big(\widetilde{b}(\mathcal{F}_0, n)\big)^{p/2} \Big(\int_{\mathbf{R}} |K(u)| du \Big)^{p/2} .$$

This proves (78) since $\mathbf{E}\big(\widetilde{b}(\mathcal{F}_0, n)\big)^{p/2} \le C \sum_{k=1}^{n} k^{p-2} \beta_{2,Y}(k)$ and $\sum_{k=1}^{n} k^{p-2} \beta_{2,Y}(k) = O(1)$ by condition (72).
We turn now to the proof of (79). With this aim we notice that

$$\|X_{0,n}(x)\mathbf{E}_0(X_{j,n}(x))\|_{p/2}^{p/2} = \mathbf{E}(Z_0(x)\mathbf{E}_0(X_{j,n}(x))) = \mathbf{E}(Z_0(x)X_{j,n}(x)) \,,$$

where $Z_0(x) = |X_{0,n}|^{p/2} |\mathbf{E}_0(X_{j,n}(x))|^{p/2-1} \mathrm{sign}\big(\mathbf{E}_0(X_{j,n}(x))\big)$. Then, by using Item 1 of Lemma 35, we derive that

$$\|X_{0,n}(x)\mathbf{E}_0(X_{j,n}(x))\|_{p/2}^{p/2} = \mathrm{Cov}\Big(Z_0(x), K((x-Y_j)/h_n)\Big) \le ||dK|| \, \mathbf{E}\big(b(\mathcal{F}_0, j, j) \, |Z_0(x)|\big) \,. \tag{81}$$

Notice now that by using the elementary inequality: $x^\alpha y^{1-\alpha} \le x+y$ valid for $\alpha \in [0,1]$ and nonnegative x and y, we get that $|Z_0(x)| \le \big(|X_{0,n}(x)| + |\mathbf{E}_0(X_{j,n}(x))|\big)^{p-1}$. Therefore, some computations involving Jensen's inequality lead to

$$\int_{\mathbf{R}} |Z_0(x)| dx \le 4^p h_n \int_{\mathbf{R}} |K(u)|^{p-1} du \,, \tag{82}$$

where $||dK||$ is the total variation norm of the measure dK. Starting from (81), we end the proof of (79) by taking into account (82) and the fact that

$$\sum_{j=1}^{n} j^{p-2+\varepsilon}\mathbf{E}(b(\mathcal{F}_0, j, j)) \le \sum_{j=1}^{n} j^{p-2+\varepsilon}\beta_{2,Y}(j)$$

is convergent by condition (72) for any $\varepsilon < \eta$.

It remains to prove (80). We first write that

$$\|\mathbf{E}_0(X_{i,n}(x)X_{j,n}(x)) - \mathbf{E}(X_{i,n}(x)X_{j,n}(x))\|_{p/2}^{p/2} = \mathbf{E}((Z_0')^{(0)}(x)X_{i,n}(x)X_{j,n}(x)),$$

where the notation $X^{(0)}$ stands for $X^{(0)} = X - \mathbf{E}(X)$ and

$$Z_0'(x) = |\mathbf{E}_0(B_{i,j}(x))|^{p/2-1}\operatorname{sign}\big(\mathbf{E}_0(B_{i,j}(x))\big),$$

with $B_{i,j}(x) = X_{i,n}(x)X_{j,n}(x) - \mathbf{E}(X_{i,n}(x)X_{j,n}(x))$. Since the variables $X_{i,n}(x)$ and $X_{j,n}(x)$ are centered, an application of Item 2 of Lemma 35 then gives

$$\begin{aligned}&\|\mathbf{E}_0(X_{i,n}(x)X_{j,n}(x)) - \mathbf{E}(X_{i,n}(x)X_{j,n}(x))\|_{p/2}^{p/2} \\ &\le ||dK||^2\, \mathbf{E}\big(|Z_0(x)|(b(\mathcal{F}_0, i, i) + b(\mathcal{F}_0, j, j) + b(\mathcal{F}_0, i, j))\big).\end{aligned}$$

Notice now that since $p/2 - 1 \ge 1$, we can easily get

$$\int_{\mathbf{R}} |Z_0'(x)|dx \le c_p h_n \int_{\mathbf{R}} |K(u)|^{p-2}du,$$

where c_p is a positive constant depending on p. In addition

$$\sum_{j=1}^{n} j^{p-2+\varepsilon} \sup_{i\ge j} \mathbf{E}(b(\mathcal{F}_0, i, i) + b(\mathcal{F}_0, j, j) + b(\mathcal{F}_0, i, j)) \le 3\sum_{j=1}^{n} j^{p-2+\varepsilon}\beta_{2,Y}(j),$$

which is convergent by condition (72) for any $\varepsilon < \eta$. Then (80) holds and so does the proposition. $\diamond$

6 Appendix

This section is devoted to some technical lemmas. Next lemma gives estimates for terms of the type $\mathbf{E}(X_0^u X_1^{p-u})$.

Lemma 34 *Let p and u be real numbers such that $0 \le u \le p-2$. Let X_0 and X_1 be two positive identically distributed random variables. With the notation $a^p = \mathbf{E}(X_0^p)$, $\mathbf{E}_0(X_1) = \mathbf{E}(X_1|X_0)$ the following estimates hold*

$$\mathbf{E}(X_0^u X_1^{p-u}) \le a^{p-2u/(p-2)}||\mathbf{E}_0(X_1^2)||_{p/2}^{u/(p-2)}, \tag{83}$$

and

$$\mathbf{E}(X_0^{p-1}X_1) \le a^{p-1}||\mathbf{E}_0(X_1^2)||_{p/2}^{1/2}. \tag{84}$$

Proof of Lemma 34. The inequality (83) is trivial for $u = 0$. To prove it for $u = p-2$, it suffices to write that $\mathbf{E}(X_0^{p-2}X_1^2) = \mathbf{E}(X_0^{p-2}\mathbf{E}_0(X_1^2))$, and then to use Hölder's inequality.

We prove now the inequality (83) for $0 < u < p-2$. Select $x = (p/2-1)/u = (p-2)/2u$. Notice that $2x > 1$ and $p - u - 1/x > 0$ since $u < p-2$. Then, since the variables are identically distributed,

$$\mathbf{E}(X_0^u X_1^{p-u}) = \mathbf{E}(X_0^u X_1^{1/x} X_1^{p-u-1/x}) \leq ||X_0^u X_1^{1/x}||_{2x} ||X_1^{p-u-1/x}||_{2x/(2x-1)} \\ \leq \big(\mathbf{E}(X_0^{p-2} X_1^2)\big)^{u/(p-2)} (a^p)^{1-u/(p-2)} .$$

Now, again by Hölder's inequality applied with $x = p/(p-2)$ and $1 - 1/x = 2/p$,

$$\mathbf{E}(X_0^{p-2} X_1^2) = \mathbf{E}(X_0^{p-2} \mathbf{E}_0(X_1^2)) \leq \big(\mathbf{E}(X_0^p)\big)^{(p-2)/p} \big(\mathbf{E}(\mathbf{E}_0(X_1^2))^{p/2}\big)^{2/p} = a^{(p-2)} ||\mathbf{E}_0(X_1^2)||_{p/2} .$$

Overall

$$\mathbf{E}(X_0^u X_1^{p-u}) \leq a^u ||\mathbf{E}_0(X_1^2)||_{p/2}^{u/(p-2)} (a^p)^{1-u/(p-2)} \\ = a^{p-2u/(p-2)} ||\mathbf{E}_0(X_1^2)||_{p/2}^{u/(p-2)} ,$$

ending the proof of the inequality (83).

To prove the inequality (84), we use Hölder's inequality which entails that

$$\mathbf{E}(\ X_0^{p-1} X_1) \leq \mathbf{E}(\ X_0^{p-1} \mathbf{E}_0^{1/2}(X_1^2)) \leq a^{p-1} ||\mathbf{E}_0(X_1^2)||_{p/2}^{1/2} .$$

⋄

Next lemma gives covariance-type inequalities in terms of beta coefficients as defined in Definition 32.

Lemma 35 *Let Z be a $\mathcal{F}_0$-measurable real valued random variable and let h and g be two BV functions (denote by $||dh||$ (resp. $||dg||$) the total variation norm of the measure dh (resp. dg)). Denote $Z^{(0)} = Z - \mathbf{E}(Z)$, $h^{(0)}(Y_i) = h(Y_i) - \mathbf{E}(h(Y_i))$ and $g^{(0)}(Y_j) = g(Y_j) - \mathbf{E}(g(Y_j))$. Define the random variables $b(\mathcal{F}_0, i, j)$ as in Definition 32. Then*

1. $\big|\mathbf{E}\big(Z^{(0)} h^{(0)}(Y_i)\big)\big| = \big|\mathrm{Cov}(Z, h(Y_i))\big| \leq ||dh||\, \mathbf{E}\big(|Z| b(\mathcal{F}_0, i, i)\big)$.
2. $\big|\mathbf{E}\big(Z^{(0)} h^{(0)}(Y_i) g^{(0)}(Y_j)\big)\big| \leq ||dh||\ ||dg||\, \mathbf{E}\big(|Z|(b(\mathcal{F}_0, i, i) + b(\mathcal{F}_0, j, j) + b(\mathcal{F}_0, i, j)\big)$.

Proof of Lemma 35. Item 1 has been proven by Dedecker and Prieur (2005) (see Item 2 of their Proposition 1). Item 2 needs a proof. We first notice that

$$h^{(0)}(X) g^{(0)}(Y) = \iint (\mathbf{1}_{X \leq t} - F_X(t))(\mathbf{1}_{Y \leq s} - F_Y(s)) dh(t) dg(s) .$$

Therefore

$$\mathbf{E}\big(Z^{(0)} h^{(0)}(Y_i) g^{(0)}(Y_j)\big) \\ = \mathbf{E}\Big(Z \iint \Big(\mathbf{1}^{(0)}_{Y_i \leq t} \mathbf{1}^{(0)}_{Y_j \leq s} - \mathbf{E}\big(\mathbf{1}^{(0)}_{Y_i \leq t} \mathbf{1}^{(0)}_{Y_j \leq s}\big)\Big) dh(t) dg(s)\Big) \\ = \mathbf{E}\Big(Z \iint \mathbf{E}\Big(\mathbf{1}^{(0)}_{Y_i \leq t} \mathbf{1}^{(0)}_{Y_j \leq s} - \mathbf{E}\big(\mathbf{1}^{(0)}_{Y_i \leq t} \mathbf{1}^{(0)}_{Y_j \leq s}\big) | \mathcal{F}_0\Big) dh(t) dg(s)\Big) ,$$

which proves Item 2 by noticing that

$$|\mathbf{E}\Big(\mathbf{1}^{(0)}_{Y_i \leq t} \mathbf{1}^{(0)}_{Y_j \leq s} - \mathbf{E}\big(\mathbf{1}^{(0)}_{Y_i \leq t} \mathbf{1}^{(0)}_{Y_j \leq s}\big) | \mathcal{F}_0\Big)| \leq b(\mathcal{F}_0, i, i) + b(\mathcal{F}_0, j, j) + b(\mathcal{F}_0, i, j) .$$

⋄

Next lemma gives inequalities for $|x+y|^p$ for different ranges of p, where $p \geq 2$ is a real number.

Lemma 36 *1. Let x and y be two real numbers and $2 \le p \le 3$. Then*

$$|x+y|^p \le |x|^p + |y|^p + p|x|^{p-1}\mathrm{sign}(x)y + \frac{p(p-1)}{2}|x|^{p-2}y^2 \,. \tag{85}$$

2. Let x and y be two real numbers and $3 < p \le 4$. Then

$$|x+y|^p \le |x|^p + |y|^p + p|x|^{p-1}\mathrm{sign}(x)y + \frac{p(p-1)}{2}|x|^{p-2}y^2 + \frac{2p}{(p-2)}|x||y|^{p-1} \,. \tag{86}$$

3. Let x and y be two positive real numbers and $p \ge 1$ any real number. Then

$$(x+y)^p \le x^p + y^p + 4^p(x^{p-1}y + xy^{p-1}) \,. \tag{87}$$

4. Let x and y be two real numbers and p an even positive integer. Then

$$(x+y)^p \le x^p + y^p + p(x^{p-1}y + xy^{p-1}) + 2^p(x^2y^{p-2} + x^{p-2}y^2) \,. \tag{88}$$

Proof of Lemma 36. The inequality (85) was established in Rio (2007, Relation (3.3)) by using Taylor expansion with integral rest for evaluating the difference $|x+y|^p - |x|^p$. To prove the inequality (86), we also use the Taylor integral formula of order 2 that gives

$$|x+y|^p - |x|^p = p|x|^{p-1}\mathrm{sign}(x)y + C_p^2|x|^{p-2}y^2 + 2C_p^2y^2\int_0^1(1-t)(|x+ty|^{p-2} - |x|^{p-2})dt \,, \tag{89}$$

where $C_p^2 = p(p-1)/2$. Notice now that, for $3 < p \le 4$,

$$|x+ty|^{p-2} \le \frac{x^2 + 2|x||ty| + y^2}{(|x| + |ty|)^{4-p}} \le |x|^{p-2} + 2|x||ty|^{p-3} + |ty|^{p-2} \,.$$

Hence

$$\begin{aligned}
&2C_p^2y^2\int_0^1(1-t)(|x+ty|^{p-2} - |x|^{p-2})dt \\
&\le 2C_p^2|y|^p\int_0^1(1-t)t^{p-2}dt + 4C_p^2|x||y|^{p-1}\int_0^1(1-t)t^{p-3}dt \\
&= 2|y|^pC_p^2\frac{\Gamma(p-1)}{\Gamma(p+1)} + 4|x||y|^{p-1}C_p^2\frac{\Gamma(p-2)}{\Gamma(p)} = |y|^p + \frac{2p}{(p-2)}|x||y|^{p-1} \,.
\end{aligned}$$

Starting from (89) and using (**??**), the inequality (86) follows.

The inequality (87) was observed by Shao (1995, page 957). We shall establish now (88). We start by noticing that for any a, b two positive real numbers and $2 \le k \le p-2$ we have

$$a^{p-k}b^k \le \max(a^p(b/a)^{p-2}, a^p(b/a)^2) \le a^2b^{p-2} + a^{p-2}b^2 \,. \tag{90}$$

Now for p an even positive integer and x and y two real numbers, the Newton binomial formula gives

$$\begin{aligned}
(x+y)^p &= x^p + y^p + p(x^{p-1}y + xy^{p-1}) + \sum_{k=2}^{p-2} C_p^k x^{p-k}y^k \\
&\le x^p + y^p + p(x^{p-1}y + xy^{p-1}) + \sum_{k=2}^{p-2} C_p^k |x|^{p-k}|y|^k \,.
\end{aligned}$$

Whence, by (90) and the fact that $\sum_{k=0}^p C_p^k = 2^p$, the inequality (88) follows. ⋄

Lemma 37 *Let $(V_i)_{i\geq 0}$ be a sequence of non negative numbers such that $V_0 = 0$ and for all $i, j \geq 0$,*

$$V_{i+j} \leq C(V_i + V_j)\,, \tag{91}$$

where $C \geq 1$ is a constant not depending on i and j. Then

1. *For any integer $r \geq 1$, any integer n satisfying $2^{r-1} \leq n < 2^r$ and any real $q \geq 0$*

$$\sum_{i=0}^{r-1} \frac{1}{2^{iq}} V_{2^i} \leq C 2^{q+2}(2^{q+1}-1)^{-1} \sum_{k=1}^{n} \frac{V_k}{k^{1+q}}\,.$$

2. *For any positive integers k and m and any real $q > 0$,*

$$\sum_{j=1}^{k} \frac{1}{j^q} V_{jm} \leq 2^{q+1} C q^{-1} m^{q-1} \sum_{\ell=1}^{m} \frac{1}{(\ell+m)^q} V_\ell + 2Cq^{-1} m^{q-1} \sum_{\ell=m+1}^{km} \frac{1}{\ell^q} V_\ell\,.$$

3. *Let $0 < \delta \leq \gamma \leq 1$. Then for any real $q \geq 0$,*

$$\left(\sum_{k=1}^{n} \frac{1}{k^{1+q\gamma}} V_k^{\gamma}\right)^{1/\gamma} \leq 2^{1/\delta - 1/\gamma} C^{(\gamma-\delta)/\delta} \left(\sum_{k=1}^{n} \frac{1}{k^{1+q\delta}} V_k^{\delta}\right)^{1/\delta}\,.$$

Remark 38 *If $(V_i)_{i\geq 0}$ satisfies (91) with $C = 1$, then the sequence is said to be subadditive.*

Proof of Lemma 37.

The condition (91) implies that for any integer k and any integer $0 \leq j \leq k$,

$$V_k \leq C(V_j + V_{k-j}) \text{ and then that } (k+1)V_k \leq 2C \sum_{j=1}^{k} V_j\,. \tag{92}$$

Therefore for $2^{r-1} \leq n < 2^r$,

$$\sum_{i=0}^{r-1} \frac{1}{2^{iq}} V_{2^i} \leq 2C \sum_{j=1}^{2^{r-1}} V_j \sum_{i:2^i \geq j} \frac{1}{2^{i(q+1)}}\,,$$

proving Item 1. To prove Item 2, using again (92), it suffices to notice that

$$\begin{aligned}\sum_{j=1}^{k} \frac{1}{j^q} V_{jm} &\leq 2C \sum_{j=1}^{k} \frac{1}{(1+jm)j^q} \sum_{\ell=1}^{jm} V_\ell \\ &\leq 2Cm^{-1}(2m)^q \sum_{j=1}^{k} j^{-q-1} \sum_{\ell=1}^{m} \frac{1}{(\ell+m)^q} V_\ell + 2Cq^{-1} m^{q-1} \sum_{\ell=m+1}^{km} \frac{1}{\ell^q} V_\ell\,.\end{aligned}$$

To prove Item 3, we first notice that (91) entails that

$$V_{i+j}^{\gamma} \leq C^{\gamma}(V_i^{\gamma} + V_j^{\gamma}) \text{ and then that } (k+1)V_k^{\gamma} \leq 2C^{\gamma} \sum_{j=1}^{k} V_j^{\gamma}\,.$$

Then for any real $q \geq 0$,

$$k^{-q(\gamma-\delta)} V_k^{\gamma-\delta} \leq 2^{1-\delta/\gamma} C^{\gamma-\delta} \Big(\sum_{j=1}^{k} j^{-(1+q\gamma)} V_j^{\gamma} \Big)^{1-\delta/\gamma} . \tag{93}$$

Writing that $k^{-(1+q\gamma)} V_k^{\gamma} = (k^{-(1+q\delta)} V_k^{\delta})(k^{-q(\gamma-\delta)} V_k^{\gamma-\delta})$ and using (93), the following inequality holds

$$\sum_{k=1}^{n} \frac{1}{k^{1+q\gamma}} V_k^{\gamma} \leq 2^{1-\delta/\gamma} C^{\gamma-\delta} \Big(\sum_{k=1}^{n} k^{-(1+q\delta)} V_k^{\delta} \Big) \Big(\sum_{j=1}^{n} j^{-(1+q\gamma)} V_j^{\gamma} \Big)^{1-\delta/\gamma} ,$$

proving Item 3. $\diamond$

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