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Sharp Conditions for the CLT of Linear Processes in a Hilbert Space

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In this paper we study the behavior of sums of a linear process $X_k = \sum_{j=-\infty}^{\infty} a_j(\xi_{k-j})$ associated to a strictly stationary sequence $\{\xi_k\}_{k\in\mathbb{Z}}$ with values in a real separable Hilbert space and $\{a_k\}_{k\in\mathbb{Z}}$ are linear operators from H to H. One of the results is that $\sum_{i=1}^{n} X_i/\sqrt{n}$ satisfies the CLT provided $\{\xi_k\}_{k\in\mathbb{Z}}$ are i.i.d. centered having finite second moments and $\sum_{j=-\infty}^{\infty} \|a_j\|_{L(H)} < \infty$. We shall provide an example which shows that the condition on the operators is essentially sharp. Extensions of this result are given for sequences of weak dependent random variables $\{\xi_k\}_{k\in\mathbb{Z}}$, under minimal conditions.

KEY WORDS: Central limit theorem; linear process in Hilbert space.

1. INTRODUCTION

Let H be a separable real Hilbert space with the norm $\|\cdot\|_H$ generated by an inner product, $\langle\cdot,\cdot\rangle_H$ and let $\{e_k\}_{k\geqslant 1}$ be an orthonormal basis in H. Let $L(H)$ be the class of bounded linear operators from H to H and denote by $\|\cdot\|_{L(H)}$ its usual norm. Let $\{\xi_k\}_{k\in\mathbb{Z}}$ be a strictly stationary sequence of H-valued random variables, and the $\{a_k\}_{k\in\mathbb{Z}}$ be a sequence of operators, $a_k\in L(H)$. We define the stationary Hilbert space process by:

$$X_k = \sum_{j=-\infty}^{\infty} a_j(\xi_{k-j}) \tag{1.1}$$

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provided the series is convergent in some sense (in the following, we suppose the brackets to soothe the notations). Notice that if $\sum_{j=-\infty}^{\infty} \|a_j\|_{L(H)}^2 < \infty$ and $\{\xi_k\}_{k \in \mathbb{Z}}$ are i.i.d. centered in $L_2(H)$, then it is well known that the series in (1.1) is convergent in $L_2(H)$ and almost surely (Araujo and Giné,[1] Chap. 3.2). The sequence $\{X_k\}_{k \geq 1}$ is a natural extension of the multivariate linear processes (Brockwell and Davis,[5] Chap. 11). These types of processes with values in functional spaces also facilitate the study of estimation and forecasting problems for several classes of continuous time processes. For details we mention Bosq,[4] Mourid,[13] and Merlevède.[12]

We define

$$S_n = \sum_{k=1}^{n} X_k \tag{1.2}$$

In this paper we investigate the CLT for $\{S_n\}_{n \geq 1}$. When $\{\xi_k\}_{k \in \mathbb{Z}}$ are real valued i.i.d. random variables and $\{a_k\}_{k \in \mathbb{Z}}$ is a numerical sequence, the following theorem is valid (Ibragimov and Linnik,[10] Th.18.6.5).

Theorem 1. Let $\{\xi_k\}_{k \in \mathbb{Z}}$ be a sequence of i.i.d. centered random variables having finite second moment and let $\{a_k\}_{k \in \mathbb{Z}}$ be a sequence of numbers such that

$$\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty \tag{1.3}$$

Define X_k by (1.1), S_n by (1.2), and assume $E\,|S_n|^2 \to \infty$, as $n \to \infty$. Then

$$\frac{S_n}{\sqrt{E\,|S_n|^2}} \xrightarrow{\mathscr{D}} N(0, 1), \qquad \text{as} \quad n \to \infty \tag{1.4}$$

In studying the infinite-dimensional space case, our question was to what extent Theorem 1 remains valid in the new context when we replace $\{\xi_k\}_{k \in \mathbb{Z}}$ by a infinite-dimensional space valued random variables, the constants by linear bounded operators and absolute values by the corresponding norms. To see new possible quality effects, we consider a simplest case of infinite dimensional Hilbert space H. It turns out that Theorem 1 does not remain valid to its full extent in this case. In fact we establish a CLT under the condition $\sum_{k=-\infty}^{\infty} \|a_k\|_{L(H)} < \infty$ replacing (1.3) and the normalization $\sqrt{n}$ in (1.4).

We provide an example showing that the condition $\sum_{k=-\infty}^{\infty} \|a_k\|_{L(H)}$ is essentially sharp, i.e., if this condition fails, without any additional assumptions on behavoor or either the sequence of oprators $\{a_k\}_{k \in \mathbb{Z}}$

or eigenvalues of the covariance operator of ξ_0, the tightness of $\{S_n/\sqrt{E\,\|S_n\|_H^2}\}_{n\geq 1}$ may fail and no analogue of Theorem 1 is possible. Finally, we extend the CLT to strong mixing sequences of random variables under a certain condition which combines the tail distribution of $\|\xi_0\|_H$ with the size of the strong mixing coefficients. This result extends in two directions the CLT of Doukhan *et al.*,[7] which is optimal for real valued random variables.

2. RESULTS

We shall first establish Theorem 2.

Theorem 2. Assume $\{\xi_k\}_{k\in\mathbb{Z}}$ is a sequence of H-valued i.i.d. centered random variables such that $0<E\,\|\xi_0\|_H^2<\infty$, and let $\{a_k\}_{k\in\mathbb{Z}}$ be a sequence of linear bounded oprators on H. Define $\{X_k\}_{k\in\mathbb{Z}}$ by (1.1) and S_n by (1.2). Assume

$$\sum_{j=-\infty}^{\infty} \|a_j\|_{L(H)}<\infty \tag{2.1}$$

Then

$$\frac{S_n}{\sqrt{n}} \xrightarrow{\mathscr{D}} N(0, AC_{\xi_0}A^*), \qquad \text{as} \quad n\to\infty \tag{2.2}$$

where N is an H-valued Gaussian random variable, C_{ξ_0} denotes the covariance operator of ξ_0, $A=\sum_{j=-\infty}^{\infty} a_j$ and A^* is the adjoint oprator of A.

This result is essentially sharp as we can see from Theorem 3.

Theorem 3. Let $\{t_k\}_{k\in\mathbb{Z}}$ be a sequence of positive numbers satisfying

$$\sum_{k=-\infty}^{\infty} t_k=\infty \qquad \text{and} \qquad \sum_{k=-\infty}^{\infty} t_k^2<\infty \tag{2.3}$$

Then there is a sequence of i.i.d. H-valued random variables $\{\xi_j\}_{j\in\mathbb{Z}}$ which is centered, with $0<E\,\|\xi_0\|_H^2<\infty$ and there is a sequence of operators $\{a_k\}_{k\in\mathbb{Z}}$ satisfying $\|a_k\|_{L(H)}=t_k$, such that

$$\left\{\frac{S_n}{\sqrt{n}}\right\}_{n\geq 1} \text{ is not tight and } \frac{E\,\|S_n\|_H^2}{n}\to\infty \qquad \text{as} \quad n\to\infty \tag{2.4}$$

and

$$\left\{\frac{S_n}{\sqrt{E\,\|S_n\|_H^2}}\right\}_{n\geqslant 1} \quad \text{is not tight} \tag{2.5}$$

Remark 1. This theorem shows by (2.4) that the condition (2.1) in Theorem 2 is essentially sharp in the sense that if it is violated, the conclusion of Theorem 2 does not hold anymore. The conclusion (2.5) shows that even if we change the normalization the convergence to a random element does not hold, so a result similar to (1.4) cannot be obtained without additional assumptions on the operators.

Theorem 1 can be easily extended to the dependent sequences of random variables because its proof is based on the following decomposition result which is interesting in itself, and still valid in the Banach space context.

Proposition 1. Let $\{\xi_k\}_{k\in\mathbb{Z}}$ be a sequence of H-valued random variables. Assume there is a constant $K>0$ such that for every sequence of linear bounded operators $\{d_k\}_{k\in\mathbb{Z}}$ on H, and for every $-\infty<p<q<\infty$,

$$E\left\|\sum_{j=p}^{q} d_j\xi_j\right\|_H^2 \leqslant K\sum_{j=p}^{q}\|d_j\|_{L(H)}^2 \tag{2.6}$$

Let $\{a_k\}_{k\in\mathbb{Z}}$ be a sequence of linear bounded operators satisfying (2.1). Then, the series in (1.1) is convergent in $L_2(H)$. In addition we have

$$\frac{1}{n}E\left\|\sum_{k=1}^{n} X_k - A\sum_{k=1}^{n}\xi_k\right\|_H^2 \to 0, \qquad \text{as} \quad n\to\infty \tag{2.7}$$

where A denotes $\sum_{j=-\infty}^{\infty} a_j$.

Remark 2. It is very easy to see that Theorem 1 will be a consequence of Proposition 1 because under the conditions imposed on $\{\xi_k\}_{k\in\mathbb{Z}}$ in Theorem 1, (2.6) is obviously satisfied. Theorem 1 results by the known CLT for $1/\sqrt{n}\sum_{k=1}^{n}\xi_k$ where $\{\xi_k\}_{k\in\mathbb{Z}}$ are H-valued i.i.d. r.v.'s centered and having finite second moment (see Ledoux and Talagrand,[11] Chap. 10), by the fact that A is a continuous operator under (2.1), and by Billingsley,[3] Thm. 4.1. Again, by the same arguments Theorem 1 remains valid when we replace an i.i.d. sequence of H-valued r.v.'s by an i.i.d. sequence of B-valued r.v.'s where B is a separable Banach space of type 2.

Remark 3. The condition (2.6) is also satisfied for a variety of dependent sequences $\{\xi_k\}_{k\in\mathbb{Z}}$ such are strictly stationary H-valued martingale differences with $0 < E\,\|\xi_0\|_H^2 < +\infty$ and various kinds of mixing sequences of H-valued random variables under certain conditions. In these cases the conclusion (2.7) of Proposition 1 holds and by Billingsley,[3] Thm. 4.1, the behavior of $(\sum_{k=1}^n X_k/\sqrt{n})$ is reduced under (2.1) to the study of the behavior of $(\sum_{k=1}^n \xi_k/\sqrt{n})$. In this paper, we shall treat only the case of strongly mixing sequences. Other weak dependent classes of random variables, including ρ-mixing and interlaced mixing, will be discussed elsewhere.

Definition 1. Given two σ algebras $\mathscr{A}$ and $\mathscr{B}$, we define the α-mixing coefficient by

$$\alpha(\mathscr{A}, \mathscr{B}) = \sup_{A\in\mathscr{A},\, B\in\mathscr{B}} |P(AB) - P(A)\,P(B)|$$

Definition 2. Let $\{\xi_k\}_{k\in\mathbb{Z}}$ be a strictly stationary sequence of H-valued random variables. We call the sequence strongly mixing if $\alpha_n \to 0$, as $n\to\infty$, α_n being defined by $\alpha_n = \alpha(\mathscr{F}_0, \mathscr{G}_n)$, where $\mathscr{F}_0 = \sigma\{\xi_k,\ k\leqslant 0\}$ and $\mathscr{G}_n = \sigma\{\xi_k, k\geqslant n\}$.

Denote by $\alpha^{-1}(u)$ the inverse function of $\alpha(x) = \alpha_{[x]}$, where $[x]$ is the integer part of x.

Theorem 4. Let $\{\xi_k\}_{k\in\mathbb{Z}}$ be a strongly mixing sequence of strictly stationary H-valued centered random variables. Denote by $Q_{\|\xi_0\|_H}(x)$ the quantile function of $\|\xi_0\|_H$, i.e., the inverse function of $G(x) = P(\|\xi_0\|_H > x)$ and assume

$$\int_0^1 \alpha^{-1}(u)\, Q^2_{\|\xi_0\|_H}(u)\, du < \infty \tag{2.8}$$

Then for every $i, j\geqslant 1$

$$\lim_{n\to\infty} \frac{1}{n} E\left(\sum_{k=1}^n \langle \xi_k, e_i\rangle_H \cdot \sum_{k=1}^n \langle \xi_k, e_j\rangle_H\right) = \sigma_{ij} \text{ exists}$$

and

$$\frac{\sum_{i=1}^n \xi_i}{\sqrt{n}} \xrightarrow{\mathscr{D}} N(0, T) \tag{2.9}$$

where N is a centered Gaussian random variable with values in H and with covariance operator $T=(\sigma_{i,j})$; $i,j\geqslant 1$. If (2.1) is satisfied, then

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{n}} \xrightarrow{\mathscr{D}} N(0, ATA^*) \tag{2.10}$$

where X_k is defined by (1.1), $A=\sum_{j=-\infty}^{\infty} a_j$ and A^* denotes the adjoint operator of A.

Remark 4. Condition (2.8) is the same that the one introduced by Rio.[15] Thus, (2.9) is an extension to the Hilbert space of a result due to Doukhan *et al.*[7] which is optimal in R in the case of arithmetic rates of mixing.

Furthermore, noticing that condition (2.8) is equivalent to the usual condition $E\|\xi_0\|_H^2<\infty$ for m-dependent and in particular independent sequences, the conclusion (2.10) of Theorem 4 is an extension of Theorem 1 to weak dependent sequences.

As a consequence of the application No. 3 in Doukhan *et al.*[7] and to Theorem 4 we can formulate Corollary 1 which is an extension of Theorem 2.2, (c), in Peligrad and Utev:[14]

Corollary 1. Let $\{\xi_k\}_{k\in\mathbb{Z}}$ be a strictly stationary centered sequence of H-valued random variables such that for some $0<\delta\leqslant\infty$, $E\|\xi_0\|_H^{2+\delta}<\infty$ and $\sum_{i=1}^{\infty} i^{2/\delta}\alpha_i<\infty$. Then the conclusions of Theorem 4 hold.

3. PROOFS

In order to prove Proposition 1 which is the basic step in the proof of Theorems 2 and 4, we shall establish the following lemma.

Lemma 1. Let $\{b_k\}_{k\in\mathbb{Z}}$ be a sequence of elements in a Banach space $(B, \|\cdot\|_B)$ such that

$$\sum_{k=-\infty}^{\infty} \|b_k\|_B<\infty \tag{3.1}$$

and

$$\sum_{k=-\infty}^{\infty} b_k=0 \tag{3.2}$$

Then we have

$$\frac{1}{n}\sum_{j=-\infty}^{\infty}\left\|\sum_{i=1-j}^{n-j} b_i\right\|_B^2 \to 0 \qquad \text{as} \quad n\to\infty \tag{3.3}$$

Proof. Denote by $D_n := \sum_{|j|\geq n} \|b_j\|_B$. By taking into account (3.1) we observe that

$$\frac{1}{n}\sum_{|j|\geq 2n}\left\|\sum_{i=1-j}^{n-j} b_i\right\|_B^2 \leq \left(\sum_{|j|\geq n}\|b_j\|_B\right)\frac{1}{n}\sum_{j=-\infty}^{\infty}\left(\sum_{i=1-j}^{n-j}\|b_i\|_B\right)$$

$$:= D_n \sum_{j=-\infty}^{\infty}\|b_j\|_B \to 0 \qquad \text{as} \quad n\to\infty \tag{3.4}$$

Now for a fixed x in the interval $[-2, 2]$, we define

$$h_n(x) = \left\|\sum_{i=1-[nx]}^{n-[nx]} b_i\right\|_B^2$$

One can easily see that, under the conditions (3.1) and (3.2), for every $x\neq 1$ we have $h_n(x)\to 0$, as $n\to\infty$ and $0\leq h_n(x)\leq (\sum_{i=-\infty}^{\infty}\|b_i\|_B)^2$. Hence by Lebesgue's dominated convergence theorem, we obtain

$$\frac{1}{n}\sum_{j=-2n}^{2n-1}\left\|\sum_{i=1-j}^{n-j} b_i\right\|_B^2 = \int_{-2}^{2} h_n(x)\,dx \to 0 \qquad \text{as} \quad n\to\infty \tag{3.5}$$

Therefore the conclusion (3.3) is a consequence of (3.4) and (3.5).

3.1. Proof of Proposition 1

Because of (2.6) it is easy to see that the series $\sum_{j=-\infty}^{\infty} a_j\xi_{k-j}$ is convergent in $L_2(H)$ under (2.1). Denote by X_k a representative of this limit in $L_2(H)$ and note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n}\sum_{m=-\infty}^{\infty} a_m\xi_{k-m} = \sum_{j=-\infty}^{\infty}\left(\sum_{k=1}^{n} a_{k-j}\right)\xi_j \tag{3.6}$$

By partitioning the last sum in (3.6) into two sums, one with j between 1 and n, and another containing all the other terms, we get the representation

$$\sum_{k=1}^{n} X_k - A\sum_{j=1}^{n}\xi_j = \sum_{j=-\infty}^{\infty}\left(\sum_{k=1}^{n} b_{k-j}\right)\xi_j \tag{3.7}$$

where

$$b_0 = a_0 - A \quad \text{and} \quad b_i = a_i \quad \text{for } |i| \geqslant 1 \tag{3.8}$$

Now by condition (2.6) and Fatou Lemma, we deduce from (3.7)

$$\frac{1}{n} E \left\| \sum_{k=1}^{n} X_k - A \sum_{j=1}^{n} \xi_j \right\|_H^2 \leqslant K \frac{1}{n} \sum_{j=-\infty}^{\infty} \left\| \sum_{i=1-j}^{n-j} b_i \right\|_{L(H)}^2$$

Notice that the operators $\{b_i\}_{i \in \mathbb{Z}}$ being defined by (3.8) satisfy the conditions of Lemma 1 by (2.1). Therefore the conclusion of this proposition follows by applying Lemma 1.

3.2. Proof of Theorem 3

Let us construct the sequences $\{\xi_k\}_{k \in \mathbb{Z}}$ and $\{a_k\}_{k \in \mathbf{Z}}$.

Set $T_n := \sum_{|k| \leqslant n} t_k$.

Let $\{N_{k,j},\ k \in \mathbb{Z},\ j \in \mathbb{Z}\}$ be a triangular array of i.i.d. real random variables with a standard normal distribution, and let $\{e_k\}_{k \geqslant 1}$ be an orthonormal basis of the Hilbert space H. We construct the operators $\{a_k\}_{k \in \mathbb{Z}}$, continuous linear and symmetric by:

$$a_k(e_p) = t_k \exp(-|k|/p) \cdot e_p := a_k(p) \cdot e_p, \qquad p \geqslant 1$$

At this step, notice that $\|a_k\|_{L(H)} = t_k$ and then by the properties of $\{t_k\}_{k \in \mathbb{Z}}$, $\sum_{k=-\infty}^{\infty} \|a_k\|_{L(H)} = \infty$.

Set $z_n = 1/(1 + T_{[n/3]})$ and $g_n = \sqrt{z_n - z_{n+1}}$. Now define the H-valued i.i.d. random variables $\{\xi_k\}_{k \in \mathbb{Z}}$ by $\xi_k = \sum_{p=1}^{\infty} g_p N_{p,q} \cdot e_p$.

The $\{\xi_k\}$'s are obviously centered H-valued random variables with $0 < E\, \|\xi_0\|_H^2 < \infty$. Moreover, we have

$$S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{j=-\infty}^{\infty} a_j \xi_{k-j} = \sum_{p=1}^{\infty} S_n(p) \cdot e_p$$

where $S_n(p) = g_p \sum_{k=1}^{n} \sum_{j=-\infty}^{\infty} a_j(p)\, N_{p,k-j}$.

Notice that since the sequences $\{a_j(p)\}_{j \in \mathbb{Z}}$ satisfy (1.1) for any fixed p, we obtain by Theorem 2 that there exist nonnegative numbers c_p such that

$$\frac{1}{n} E S_n^2(p) \to c_p \qquad \text{as} \quad n \to \infty, \quad p \geqslant 1$$

Now, set $r_n := (1/n)\, E\, \|S_n\|^2$ and notice by construction that

$$\begin{aligned}
r_n &\geqslant \frac{1}{n} \sum_{p=n}^{\infty} ES_n^2(p) \\
&= \frac{1}{n} \sum_{p=n}^{\infty} g_p^2 \left[\sum_{j=-\infty}^{\infty} \left[\sum_{k=1+j}^{n+j} t_k \exp\left(\frac{-|k|}{p}\right)\right]^2 \right] \\
&\geqslant \frac{1}{n} \sum_{p=n}^{\infty} g_p^2 \left(\sum_{j:\, |j| \leqslant [n/3]+1} \left[\sum_{k:\, |k| \leqslant [n/3]} t_k \exp\left(\frac{-|k|}{p}\right)\right]^2 \right) \\
&\geqslant \frac{1}{n} \left(\exp\left(- \left[\frac{n}{3}\right] \cdot \frac{1}{n}\right)\right)^2 \left(\sum_{p=n}^{\infty} g_p^2 \right) \left(2\left[\frac{n}{3}\right] + 3\right) \cdot \left(\sum_{k:\, |k| \leqslant [n/3]} t_k \right)^2 \\
&\geqslant \frac{2}{3} \cdot e^{-2/3} z_n T_{[n/3]}^2
\end{aligned}$$

Then, $r_n \to \infty$, as $n \to \infty$ by the assumption $\sum_{k=-\infty}^{\infty} t_k = \infty$. Thus by this construction we notice that the cylindrical distributions of $\{S_n/\sqrt{n}\}_{n \geqslant 1}$ all have Gaussian limits. Moreover because $E\, \|S_n\|_H^2/n \to \infty$, as $n \to \infty$, it follows that all the cylindrical distributions of $\{S_n/\sqrt{E\, \|S_n\|_H^2}\}_{n \geqslant 1}$ are convergent to the cylindrical distributions of 0 regarded as an element of H. All these considerations show that none of the sequences $\{S_n/\sqrt{E\, \|S_n\|_H^2}\}_{n \geqslant 1}$ or $\{S_n/\sqrt{n}\}_{n \geqslant 1}$ can be tight.

3.3. Proof of Theorem 4

We base our proof on Lemma 2, which is an extension to Hilbert space valued random variables of Rio's covariance inequality.[15]

Lemma 2. Let X and Y be two H-valued variables with the quantile functions respectively $Q_{\|X\|_H}(x)$ and $Q_{\|Y\|_H}(y)$. Denote by $\bar{\alpha} = \alpha(\sigma(X), \sigma(Y))$. Then,

$$|E\langle X, Y\rangle_H - \langle EX, EY\rangle_H| \leqslant 18 \left\{ \int_0^{\bar{\alpha}} Q_{\|X\|_H}(u)\, Q_{\|Y\|_H}(u)\, du \right\}$$

Proof. We denote by $I = \int_0^{\bar{\alpha}} Q_{\|X\|_H}(u)\, Q_{\|Y\|_H}(u)\, du$. Let M and N be positive numbers and denote by

$$X^M = XI(\|X\|_H \leqslant M), \qquad X_M = XI(\|X\|_H > M)$$

$$Y^N = YI(\|Y\|_H \leqslant N), \qquad Y_N = YI(\|Y\|_H > N)$$

It is easy to see that

$$\begin{aligned}|E\langle X, Y\rangle_H - \langle EX, EY\rangle_H| \leqslant {} & |E\langle X^M, Y^N\rangle_H - \langle EX^M, EY^N\rangle_H| \\ & + |E\langle X^M, Y_N\rangle_H - \langle EX^M, EY_N\rangle_H| \\ & + |E\langle X_M, Y^N\rangle_H - \langle EX_M, EY^N\rangle_H| \\ & + |E\langle X_M, Y_N\rangle_H - \langle EX_M, EY_N\rangle_H| \\ \equiv {} & I_1 + I_2 + I_3 + I_4 \end{aligned} \tag{3.9}$$

By Lemma 2 in Dehling and Philipp,[6] we get

$$I_1 \leqslant 10\bar{\alpha}MN \tag{3.10}$$

Set $M = Q_{\|X\|_H}(\bar{\alpha})$ and $N = Q_{\|Y\|_N}(\bar{\alpha})$. Now, by using the fact that $Q_{\|X\|_H}$ is a nonincreasing function, and $Q_{\|X\|_H}(U)$ is distributed as $\|X\|_H$, where U is a variable uniformly distributed on $[0, 1]$, we get

$$\begin{aligned} I_2 &\leqslant 2ME\,\|Y_N\|_H = 2Q_{\|X\|_H}(\bar{\alpha}) \int_0^{\bar{\alpha}} Q_{\|Y\|_H}(u)\,du \\ &\leqslant 2\int_0^{\bar{\alpha}} Q_{\|X\|_H}(u)\, Q_{\|Y\|_H}(u)\,du = 2I \end{aligned} \tag{3.11}$$

and the same bound is true for the term I_3. Let F and G be the quantile functions of r.v.'s $\|X_M\|_H$ and $\|Y_N\|_H$ respectively. One can easily check that $F(u) = Q_{\|X\|_H}(u)$ for $0 < u < \bar{\alpha}$ and $F(u) = M$ for $u > \bar{\alpha}$. Analogous observation is valid for the quantile function G. Further, we recall that by Fréchet's result[8,9] the maximum of expectation of a product of two real positive r.v.'s with given marginal distributions (with the quantile functions F and G) is equal to $E[F(U)\,G(U)]$, where U is uniformly distributed over $[0, 1]$ (see Bártfai[2] for a detailed proof.) Thus:

$$\begin{aligned} & |E\langle X_M, Y_N\rangle_H| + |\langle EX_M, EY_N\rangle_H| \\ &\quad\leqslant E(\|X_M\|_H\,\|Y_N\|_H) + E\,\|X_M\|_H E\,\|Y_N\|_H \\ &\quad\leqslant 2\int_0^1 F(u)\,G(u)\,du \\ &\quad= 2\int_0^{\bar{\alpha}} F(u)\,G(u)\,du + 2\int_{\bar{\alpha}}^1 F(u)\,G(u)\,du \\ &\quad= 2I + 2Q_{\|X\|_H}(\bar{\alpha})\,Q_{\|Y\|_H}(\bar{\alpha}) \leqslant 4I \end{aligned} \tag{3.12}$$

Then the result follows by inserting (2.10)–(2.12) in (2.9).

Now we shall establish the following extension of Rio's[15] Theorem 3 (the proof is standard and therefore omitted).

Lemma 3. Let $\{\zeta_i\}_{i\in\mathbb{Z}}$ be a sequence of centered H-valued random variables not necessarily stationary. Denote by

$$\bar{\alpha}(x)=\bar{\alpha}_{[x]}=\sup_{(i,j);\,|i-j|\geqslant[x]}\alpha(\sigma(\zeta_i),\sigma(\zeta_j))$$

and assume that for each $1\leqslant i\leqslant n$

$$\int_0^1 \bar{\alpha}^{-1}(u)\,Q^2_{\|\zeta_i\|_H}(u)\,du<\infty$$

Then for every $n\geqslant 1$

$$E\left\|\sum_{i=1}^n \zeta_i\right\|_H^2\leqslant 36\sum_{i=1}^n\int_0^1 \bar{\alpha}^{-1}(u)\,Q^2_{\|\zeta_i\|_H}(u)\,du$$

Set $T_n=\sum_{k=1}^n \xi_k$.

We shall establish first the tightness of the H-valued random variable $T_n/\sqrt{n}$. For that, it is enough to show that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{E\,\|T_n-P^mT_n\|_H^2}{n}=0 \tag{3.13}$$

where $P^m(x)$ is the projection of the vector x in H on the m first vectors of the othonormal basis $\{e_k\}_{k\geqslant1}$ of H.

By Lemma 3, using the fact that $\{\xi_k\}_{k\in\mathbb{Z}}$ is a strictly stationary sequence of H-valued centered random variables, we obtain

$$\begin{aligned}\frac{E\,\|T_n-P^mT_n\|_H^2}{n}&=\frac{E\,\|\sum_{k=1}^n(\xi_k-P^m\xi_k)\|_H^2}{n}\\&\leqslant 36E\alpha^{-1}(U)\,Q^2_{\|\xi_0-P^m\xi_0\|_H}(U)\end{aligned}$$

where U is a random variable uniformly distributed on $[0,1]$.

Now, noticing that $\|\xi_0-P^m\xi_0\|_H\leqslant\|\xi_0\|_H$, we get

$$\alpha^{-1}(U)\,Q^2_{\|\xi_0-P^m\xi_0\|_H}(U)\leqslant\alpha^{-1}(U)\,Q^2_{\|\xi_0\|_H}(U)$$

and $\alpha^{-1}(U)\,Q^2_{\|\xi_0\|_H}(U)$ is integrable by assumption. Thus, by Lebesgue's dominated convergence theorem, we get (3.13).

Now, because of the tightness of $\{T_n/\sqrt{n}\}_{n\geq 1}$, in order to show (2.9) it is enough to prove that the cylindrical distributions have a Gaussian limiting distribution. Notice that (2.8) implies by Lemma 2 and stationarity that $\sum_{k=1}^{\infty} |E\langle \xi_0, e_i\rangle_H \langle \xi_k, e_j\rangle_H| < \infty$ for every $i, j \geq 1$. One again by stationarity this convergence implies that

$$\lim_{n\to\infty} \frac{1}{n} E\left(\sum_{k=1}^{n} \langle \xi_k, e_i\rangle_H \sum_{k=1}^{n} \langle \xi_k, e_j\rangle_H\right) = \sum_{k=-\infty}^{\infty} E\langle \xi_0, e_i\rangle_H \langle \xi_{|k|}, e_j\rangle_H = \sigma_{ij}$$

Let m be a fixed integer and let $\{v_j\}_{1\leq j\leq m}$ be real numbers with $\sup_{1\leq j\leq m} |v_j| = v$. Set $Z_k^m = \sum_{j=1}^{m} v_j \langle \xi_k, e_{i_j}\rangle_H$ and notice that $\{Z_k^m\}_{k\in\mathbb{Z}}$ is a strictly stationary sequence of real-valued centered random variables. Moreover, since $|Z_0^m| \leq mv \, \|\xi_0\|_H$, observe that

$$Q_{|Z_0^m|}(u) \leq mv Q_{\|\xi_0\|_H}(u)$$

It follows that $\int_0^1 \alpha^{-1}(u)\, Q^2_{|Z_0^m|}(u) < \infty$ by assumption (2.8). Thus, by using Theorem 1 in Doukhan *et al.*,[7] we obtain that $1/\sqrt{n} \sum_{k=1}^{n} Z_k^m$ has a normal limiting distribution with mean 0 and variance $\sum_{j=1}^{m} \sum_{i=1}^{m} v_i v_j \sigma_{ij}$. It follows that $P^m(T_n)/\sqrt{n}$ converges in distribution to a centered H-valued Gaussian random variable with the covariance matrix $T_m = (\sigma_{ij})_{m\times m}$ which completes the proof of (2.9).

In order to establish (2.10) we shall verify now the condition (2.6) of Proposition 1. By Lemma 3, for every $p < q$

$$E\left\|\sum_{j=p}^{q} d_j \xi_j\right\|_H^2 \leq 36 \sum_{j=p}^{q} \int_0^1 \alpha^{-1}(u)\, Q^2_{\|d_j\xi_j\|_H}(u)\, du$$

By the fact that $\|d_i \xi_i\|_H \leq \|d_i\|_{L(H)} \|\xi_i\|_H$, by stationarity and the properties of the quantile function, it follows that:

$$E\left\|\sum_{j=p}^{q} d_j \xi_j\right\|_H^2 \leq 36 \left(\sum_{j=p}^{q} \|d_j\|^2_{L(H)}\right) \int_0^1 \alpha^{-1}(u)\, Q^2_{\|\xi_0\|_H}(u)\, du$$

Therefore (2.6) is verified under (2.8) and the proof is complete. □

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