Condition number of a square matrix with i.i.d. columns drawn from a convex body

Radoslaw Adamczak[1] Olivier Guédon
Alexander E. Litvak Alain Pajor
Nicole Tomczak-Jaegermann[2]

Abstract

We study the smallest singular value of a square random matrix with i.i.d. columns drawn from an isotropic log-concave distribution. An important example is obtained by sampling vectors uniformly distributed in an isotropic convex body. We deduce that the condition number of such matrices is of the order of the size of the matrix and give an estimate on its tail behavior.

AMS 2010 Classification: primary 52A23, 46B06, 60B20, 60E15; secondary 52A20, 46B09
Keywords: condition number, convex bodies, log-concave distributions, isotropic distributions, random matrix, norm of a random matrix, smallest singular number.

1 Introduction

The behavior of the smallest singular value of random matrices with i.i.d. random entries attracted a lot of attention over the years. For Gaussian entries the problem has been investigated in [9] and [23], whereas for more general models of random matrices with i.i.d. entries, major results were

[1]A part of this work was done when this author held a postdoctoral position at the Department of Mathematical and Statistical Sciences, University of Alberta in Edmonton, Alberta. The position was sponsored by the Pacific Institute for the Mathematical Sciences.

[2]This author holds the Canada Research Chair in Geometric Analysis.

recently obtained in [16, 20, 21, 24]. In asymptotic geometry one is interested in sampling vectors uniformly distributed in a convex body. In particular the entries are not necessarily independent. In this paper, we study the more general case when the columns are i.i.d. random vectors with an isotropic log-concave distribution. Our main result is a deviation inequality for the smallest singular value.

The first results concerning the smallest singular value of large random matrices were obtained in [9] and [23]. In these papers, the authors considered matrices with independent standard Gaussian entries and proved the following theorem (below $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$).

Theorem 1.1 ([9, 23]) *Let $n \geq 1$ and let Γ be an $n \times n$ random matrix with i.i.d. $\mathcal{N}(0,1)$ entries. Then, for any $\varepsilon \geq 0$,*

$$\mathbb{P}(\inf_{x\in S^{n-1}} |\Gamma x| \leq \varepsilon n^{-1/2}) \leq C\varepsilon,$$

where C is an absolute constant.

Recently a lot of effort has been devoted to proving counterparts of the above theorem for matrices with general i.i.d. entries. In this paper we pursue a different line of investigation, motivated by applications in asymptotic convex geometry. We consider random matrices with i.i.d. columns distributed according to a log-concave isotropic probability measure. An important example is obtained by sampling vectors uniformly distributed in a isotropic convex body. For the formal definition of such random vectors see Section 2.1. Since Gaussian measures are log-concave, matrices with i.i.d. standard Gaussian entries may be considered a special case of this more general model.

Our main result is

Theorem 1.2 *Let $n \geq 1$ and let Γ be an $n \times n$ matrix with independent columns drawn from an isotropic log-concave probability μ. For every $\varepsilon \in (0,1)$,*

$$\mathbb{P}\Big(\inf_{x\in S^{n-1}} |\Gamma x| \leq c\varepsilon n^{-1/2}\Big) \leq C\varepsilon + C\exp(-cn^c), \quad (1)$$

where $c > 0$ and $C < \infty$ are absolute constants.

Moreover, there exists an absolute constant $\alpha > 0$, such that for all $\varepsilon \in (0,1)$,

$$\mathbb{P}\Big(\inf_{x\in S^{n-1}} |\Gamma x| \leq c\varepsilon n^{-1/2}\Big) \leq C\varepsilon^{n/(n+2)} \log^{\alpha}(2/\varepsilon). \quad (2)$$

Note that by absolute continuity of log-concave variables, for a fixed n the left hand side of (1) and (2) tends to 0 as $\varepsilon \to 0$. This is not the case for the right hand side of (1) but it is true for the right hand side of (2).

When combined with estimates for the operator norm of the matrix Γ, obtained recently in [2], the above theorem also yields a corollary about the tail behavior of the so called *condition number* of the matrix Γ (denoted by $\kappa(\Gamma)$). The question about its behavior for random matrices was raised by Smale [22] in connection with stability of numerical algorithms for solving large systems of linear equations. Theorem 1.2 implies that for random matrices Γ with independent log-concave isotropic columns, $\kappa(\Gamma) \leq Cn$, similarly as for matrices with independent Gaussian entries ([9, 23]).

The article is organized in the following way. In Section 2, after presenting some preliminary facts (Subsection 2.2) we prove Theorem 1.2 (Subsection 2.3). Similarly as in [16, 20, 21], the proof is based on the splitting of the sphere S^{n-1} into two regions, the set of vectors whose norm is concentrated in a small number of coordinates and the others vectors. The main difficulty lies in the fact that we have only independence of the column vectors of the matrix and not of all the entries. Therefore, we need new arguments to obtain estimates of small ball probabilities. This is where the log-concavity of the random vectors is essential.

We conclude Section 2 with Corollary 2.14 in flavor of Theorem 1.1 (however with a slightly worse dependence on ε) and a tail inequality for the condition number of Γ (Corollary 2.15).

In Section 3 we investigate the isotropic constant of isotropic log-concave measures (defined below in (3)), a quantity of major importance in convex geometry. Although our estimates do not appear in the present proof of Theorem 1.2, they can be used for related problems. In particular inequalities for the isotropic constant of the convolution of isotropic log-concave measures is of independent interest.

The main results of this paper were announced in [1] (in a weaker form).

2 Smallest singular value

2.1 Basic definitions and notation

Throughout the paper $|\cdot|$ denotes the Euclidean norm and $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{R}^n$.

For an $n \times n$ matrix Γ, let $s_1(\Gamma) \geq s_2(\Gamma) \geq \ldots \geq s_n(\Gamma)$ be the singular values of Γ, i.e. the eigenvalues of the matrix $\sqrt{\Gamma\Gamma^*}$. In particular

$$s_1(\Gamma) = \|\Gamma\| = \sup_{x \in S^{n-1}} |\Gamma x|$$

and if the matrix is invertible

$$s_n(\Gamma) = \frac{1}{\|\Gamma^{-1}\|} = \inf_{x \in S^{n-1}} |\Gamma x|.$$

The condition number of a square matrix Γ is defined as

$$\kappa(\Gamma) = \|\Gamma\| \|\Gamma^{-1}\| = \frac{s_1(\Gamma)}{s_n(\Gamma)}.$$

Let us now describe the model of a random matrix we are interested in. Recall that a non-negative function $f \colon \mathbb{R}^n \to \mathbb{R}$ is called log-concave if for all $x, y \in \mathbb{R}^n$ and all $\theta \in (0,1)$, $f((1-\theta)x + \theta y) \geq f(x)^{1-\theta} f(y)^{\theta}$. In this paper, a probability measure μ on $\mathbb{R}^n$ is said to be log-concave if it has density f, which is log-concave. It is called isotropic if it has mean zero and its covariance matrix is the identity, equivalently, for any $y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \langle x, y \rangle^2 f(x) dx = |y|^2.$$

A random vector X in $\mathbb{R}^n$ is called log-concave (resp. isotropic) if its distribution is log-concave (resp. isotropic).

For the rest of the paper, unless stated otherwise, X will denote a log-concave isotropic random vector in $\mathbb{R}^n$, $X_1, \ldots, X_n$ independent copies of X and Γ an $n \times n$ matrix with columns $X_1, \ldots, X_n$.

2.2 Preliminary facts

In this section we collect some basic facts concerning general log-concave probability measures and random matrices, which will be used in the proof of Theorem 1.2.

2.2.1 Some basic facts on log-concave probability

Recall first that the ${}_1$ Orlicz norm of a random variable X is defined as

$$\|X\|_{\ 1} = \inf \left\{ C > 0 \,;\ \mathbb{E} \exp\left(|X|/C\right) \leq 2 \right\}.$$

One of the most important properties of log-concave random vectors is the comparison of their second moment and $_1$ norm. The following well known Borell's lemma (see [5]) implies that for isotropic log-concave vectors the $_1$ norms of norm one linear functionals are bounded by a universal constant.

Lemma 2.1 *Let $X \in \mathbb{R}^n$ be a centered random vector with a log-concave distribution. Then for every $y \in S^{n-1}$*

$$\|\langle X, y\rangle\|_{\ _1} \leq C_1 \,(\mathbb{E}|\langle X, y\rangle|^2)^{1/2},$$

where C_1 is universal constant.

We will also need some recent results concerning the concentration and tail behavior of the Euclidean norm of an isotropic log-concave random vector.

Theorem 2.2 ([15]) *Let $1 \leq n \leq N$ be integers and let $X_1, \ldots, X_N \in \mathbb{R}^n$ be isotropic random vectors with log-concave densities. There exist numerical positive constants C_2 and $c_2 \in (0, \frac{1}{2})$ such that for all $\theta \in (0,1)$ and $N \leq \exp(c_2 \theta^{C_2} n^{c_2})$,*

$$\mathbb{P}\left(\max_{i \leq N} \left|\frac{|X_i|^2}{n} - 1\right| \geq \theta\right) \leq C_2 \exp(-c_2 \theta^{C_2} n^{c_2}).$$

The tail estimate for the Euclidean norm of isotropic log-concave random vectors is given by the following result.

Theorem 2.3 ([18]) *Let $N, n \geq 1$ be integers and let $X_1, \ldots, X_N \in \mathbb{R}^n$ be isotropic random vectors with log-concave densities. Then there exists an absolute positive constant C_3 such that whenever $N \leq \exp(\sqrt{n})$ then for every $K \geq 1$ one has*

$$\max_{i \leq N} |X_i| \leq C_3 K \sqrt{n}$$

with probability at least $1 - \exp(-K\sqrt{n})$.

Actually the above theorems were stated originally not for maxima but just for a single log-concave vector. However the versions presented above are following easily by the union bound.

To handle small values of ε in (2) we will need a small ball estimate for isotropic log-concave measures. It will involve the parameter

$$L_\mu = f(0)^{1/n}, \tag{3}$$

where f is the density of μ. For a log-concave isotropic random vector we let L_X denote the isotropic constant of its distribution.

Furthermore, if μ is an isotropic probability measure uniformly distributed on a symmetric convex body K then L_μ is the so-called isotropic constant of K. The question whether L_μ is bounded by a universal constant is one of the most important open problems of convex geometry. For our purposes we need a relatively easy polynomial (in n) bound on L_μ, given in the following lemma (which is a consequence of John's theorem). We refer to [6] and [14] for the best bounds (see also [8] and [19] for the non-symmetric case).

Lemma 2.4 *There exists a positive constant C_4 such that for any n and any log-concave isotropic probability measure μ on $\mathbb{R}^n$*

$$L_\mu \le C_4 n^{1/2}.$$

We will use a small ball inequality given by the following simple lemma.

Lemma 2.5 *Let X be an isotropic log-concave random vector in $\mathbb{R}^n$. Then for all $\rho \in (0,1)$*

$$\mathbb{P}(|X| \le \rho\sqrt{n}) \le C_5^n L_X^n \rho^n,$$

where C_5 is a universal constant.

Proof. It is well known that there exists an absolute constant $C > 0$ such that $|B_2^n| \le C^n n^{-n/2}$. Also, by Theorem 4 in [10], $\|f\|_\infty \le e^n f(0) = e^n L_X^n$ (note that if the vector X is symmetric, we obviously have $\|f\|_\infty = f(0)$). Thus

$$\mathbb{P}(|X| \le \rho\sqrt{n}) = \int_{\rho\sqrt{n}B_2^n} f(x)dx \le \|f\|_\infty \rho^n n^{n/2} |B_2^n| \le C_5^n L_X^n \rho^n.$$

□

2.2.2 The operator norm of a random matrix

Since the proof of Theorem 1.2 involves approximation of arbitrary vectors in B_2^n by vectors from ε-nets, we need to control the operator norm of the matrix Γ. To this end we will use the following result from [2] (it is an immediate consequence of Corollary 3.8 there).

Theorem 2.6 *There exist positive constants C_6 and c_6 such that for any $n \geq 1$ and $K \geq 1$,*

$$\mathbb{P}(\|\Gamma\| \geq C_6 K\sqrt{n}) \leq \exp\left(-c_6 K\sqrt{n}\right).$$

2.2.3 Compressible and incompressible vectors

The proof of Theorem 1.2 relies on splitting the sphere S^{n-1} into several regions (following [16, 20, 21], where an analogous construction was carried on in the case of matrices with independent entries). We use the following notation from [21].

$$\begin{aligned} Sparse = Sparse(\delta) &= \{x \in \mathbb{R}^n \colon\ |\mathrm{supp}(x)| \leq \delta n\}, \\ Comp = Comp(\delta,\rho) &= \{x \in S^{n-1} \colon\ \mathrm{dist}(x, Sparse(\delta)) \leq \rho\}, \\ Incomp = Incomp(\delta,\rho) &= S^{n-1} \backslash Comp(\delta,\rho). \end{aligned}$$

To control the behavior of $|\Gamma x|$ for $x \in Incomp(\delta,\rho)$ we will use the following lemma.

Lemma 2.7 ([21]) *Let Γ be any random matrix. Let $X_1, X_2, \ldots, X_n$ denote the column vectors of Γ and let H_k denote the span of all column vectors, except the k-th. Then for every $\rho, \delta \in (0,1)$ and every $\varepsilon > 0$ one has*

$$\mathbb{P}\big(\inf_{x \in Incomp(\delta,\rho)} |\Gamma x| < \varepsilon\rho n^{-1/2}\big) \leq \frac{1}{\delta n}\sum_{k=1}^{n} \mathbb{P}(\mathrm{dist}(X_k, H_k) < \varepsilon).$$

The case of compressible vectors requires different tools. To handle it we will need the following result from [3] (see Theorem 3.3 therein).

Theorem 2.8 *Let $n \geq 1$ and m, N be integers such that $1 \leq m \leq \min(N,n)$. Let $X_1, \ldots, X_N \in \mathbb{R}^n$ be independent $_1$ random vectors and let*

$$= \max_{i \leq N} \sup_{x \in B_2^n} \|\langle X_i, x\rangle\|_{\ 1}.$$

Let $\theta \in (0,1)$*,* $K, K' \geq 1$ *and set* $\xi = \quad K + K'$*. Then*

$$\sup_{\substack{x \in S^{N-1} \\ |\mathrm{supp}\ x| \leq m}} \left| \left| \sum_{i=1}^{N} x_i X_i \right|^2 - n \right| \leq C_7 \xi^2 \sqrt{mn} \log \quad \frac{eN}{m\sqrt{\frac{m}{n}}} \Bigg) + \theta n$$

holds with probability larger than

$$\begin{aligned} 1 \quad - \quad & C_7 \exp \quad -c_7 K \sqrt{m} \log \quad \frac{eN}{m\sqrt{\frac{m}{n}}} \Bigg)\Bigg) \\ - \quad & \mathbb{P}\left(\max_{i \leq N} |X_i| \geq K'\sqrt{n} \right) - \mathbb{P}\left(\max_{i \leq N} \left| \frac{|X_i|^2}{n} - 1 \right| \geq \theta \right), \end{aligned}$$

where $C_7, c_7 > 0$ *are universal constants.*

Corollary 2.9 *There exist universal positive constants* c_8, C_8 *such that for any* $\delta \in (0, c_8)$*,*

$$\mathbb{P}(\inf_{\substack{x \in Sparse(\delta) \\ |x|=1}} |\Gamma x| < c_8 \sqrt{n}) \leq C_8 \exp(-c_8 \delta^{C_8} n^{c_8}).$$

Proof Since by Lemma 2.1, $\max_{i \leq N} \sup_{x \in B_2^n} \|\langle X_i, x\rangle\|_{\ \ 1}$ is bounded by a universal constant, Theorem 2.8 applied with $N = n$, $m = \delta n$, $K = 1$, $K' = C_3$ (where C_3 is the constant from Theorem 2.3) and $\theta = \sqrt{\delta}\log(e/\delta^{3/2})$ gives

$$\inf_{\substack{x \in Sparse(\delta) \\ |x|=1}} |\Gamma x|^2 \geq n(1 - C\sqrt{\delta}\log(e/\delta^{3/2}))$$

with probability at least

$$\begin{aligned} 1 \quad - \quad & C_7 \exp\left(-c_7 \sqrt{\delta n} \log(e/\delta^{3/2})\right) \\ - \quad & \mathbb{P}\left(\max_{i \leq N} |X_i| \geq C_3 \sqrt{n} \right) - \mathbb{P}\left(\max_{i \leq N} \left| \frac{|X_i|^2}{n} - 1 \right| \geq \sqrt{\delta}\log(e/\delta^{3/2}) \right), \end{aligned}$$

which by Theorems 2.3 and 2.2 is greater than

$$1 - C_8 \exp(-c_8 \delta^{C_8} n^{c_8})$$

for appropriately chosen constants $c_8, C_8 \in (0, \infty)$. □

2.3 Proof of Theorem 1.2

Proposition 2.10 *For all* $\rho, \delta, \varepsilon \in (0,1)$ *we have*

$$\mathbb{P}(\inf_{x \in Incomp(\delta,\rho)} |\Gamma x| \leq \rho \varepsilon n^{-1/2}) \leq \frac{C_9}{\delta}\varepsilon, \tag{4}$$

where C_9 *is an absolute constant.*

Proof The column vectors of Γ are the vectors $X_1, \ldots, X_n$. For fixed k, let X_k^* be a (unit) normal to H_k - the hyperplane spanned by $\{X_i \colon\ i \neq k\}$. Then X_k^* and X_k are independent. Using Fubini's theorem we obtain

$$\mathbb{P}(\mathrm{dist}(X_k, H_k) < \varepsilon) = \mathbb{P}(|\langle X_k^*, X_k \rangle| < \varepsilon) = \mathbb{E}_{X_k^*}\mathbb{P}_{X_k}(|\langle X_k^*, X_k \rangle| < \varepsilon).$$

By Lemma 2.7, the proof of (4) is reduced to the estimate $\mathbb{P}_{X_k}(|\langle X_k^*, X_k \rangle| < \varepsilon)$, for a fixed $1 \leq k \leq n$, where X_k^* is a random vector of norm 1, independent of X_k. For each fixed value of X_k^*, $\langle X_k^*, X_k \rangle$ is a one-dimensional isotropic log-concave random variable, therefore its density g is bounded by a universal constant (this follows by a simple straightforward argument, alternatively it may be seen e.g. as a special case of Lemma 2.4 with $n = 1$). This implies

$$\mathbb{P}_{X_k}(|\langle X_k^*, X_k \rangle| < \varepsilon) = \int_{-\varepsilon}^{\varepsilon} g(t)dt \leq C\varepsilon$$

Thus $\mathbb{P}(\mathrm{dist}(X_k, H_k) < \varepsilon) \leq C\varepsilon$ and the proof of (4) is completed. □

Lemma 2.11 *Let* Γ *be any* $n \times n$ *matrix and* $\rho, \delta \in (0,1)$*. For any* $M \geq 1$*, if*

$$\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \leq \rho\sqrt{n}$$

and $\|\Gamma\| \leq M\sqrt{n}$*, then* $\inf_{y \in Sparse(\delta), |y|=1} |\Gamma y| \leq 4\rho\sqrt{n}$*.*

Proof Assume that there exists $x \in Comp(\delta, \rho/(2M))$, such that $|\Gamma x| \leq \rho\sqrt{n}$ and $\|\Gamma\| \leq M\sqrt{n}$. Then by the definition of "compressible vectors", there exists $y \in Sparse(\delta)$, such that $|x - y| \leq \rho/(2M)$. We have

$$|\Gamma y| \leq |\Gamma x| + |\Gamma(y - x)| \leq \rho\sqrt{n} + \sqrt{n}\rho/2 \leq 2\rho\sqrt{n}.$$

Moreover $|y| \geq 1 - |y - x| \geq 1/2$, which means that

$$\inf_{y \in Sparse(\delta), |y|=1} |\Gamma y| \leq 4\rho\sqrt{n}.$$

□

The above lemma together with Corollary 2.9 immediately yield the following statement, describing the case of compressible vectors for the estimate (1).

Proposition 2.12 *There exist absolute constants $C_{10} < \infty$, $c_{10} \in (0,1)$ such that for any $M > 1$ and $\delta \in (0, c_{10}]$*

$$\mathbb{P}(\inf_{x \in Comp(\delta, c_{10}/(2M))} |\Gamma x| \le c_{10}\sqrt{n} \;\&\; \|\Gamma\| \le M\sqrt{n}) \le C_{10}\exp(-c_{10}\delta^{C_{10}} n^{c_{10}}).$$

To prove the second part of Theorem 1.2, estimate (2), we will use another lower bound on $\inf_{x \in Comp(\delta, c_1/(2M))} |\Gamma x|$.

Proposition 2.13 *For any $M > 1$ and $\delta, \rho \in (0,1)$ we have*

$$\mathbb{P}(\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \le \rho\sqrt{n} \;\&\; \|\Gamma\| \le M\sqrt{n}) \le C_{11}^n n^{n/2} M^{\delta n} \rho^{(1-\delta)n},$$

where C_{11} is an absolute constant.

Proof. By Lemma 2.11,

$$\begin{aligned} &\mathbb{P}(\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \le \rho\sqrt{n} \;\&\; \|\Gamma\| \le M\sqrt{n}) \\ &\quad \le \mathbb{P}(\inf_{x \in Sparse(\delta), |x|=1} |\Gamma x| \le 4\rho\sqrt{n} \;\&\; \|\Gamma\| \le M\sqrt{n}). \end{aligned}$$

Let $k = \lfloor \delta n \rfloor$ (where $\lfloor \cdot \rfloor$ denotes the integer part of a real number), let Γ_δ be the $n \times k$ matrix with independent columns $X_1, \dots, X_k$. Then by the union bound, the right-hand side of the above inequality is less than or equal to

$$\binom{n}{k} \mathbb{P}(\inf_{z \in S^{k-1}} |\Gamma_\delta z| \le 4\rho\sqrt{n} \;\&\; \|\Gamma_\delta\| \le M\sqrt{n}).$$

Since the columns of Γ_δ are independent, for any vector $z \in S^{k-1}$, $\Gamma_\delta z$ is again a log-concave isotropic random vector in $\mathbb{R}^n$ (the log-concavity follows by the Prekopa-Leindler inequality, whereas isotropicity is elementary). Therefore, by Lemma 2.5 and Lemma 2.4, for every $z \in S^{k-1}$ we have

$$\mathbb{P}(|\Gamma_\delta z| \le 5\rho\sqrt{n}) \le 5^n C_4^n C_5^n n^{n/2} \rho^n.$$

Let now $\varepsilon = \rho/M$ and let $\mathcal{N}$ be an ε-net for the Euclidean norm in the set S^{k-1}. We can choose $\mathcal{N}$ of cardinality not greater than $(3/\varepsilon)^{\delta n}$. By approximation

$$\mathbb{P}\Big(\inf_{z\in S^{k-1}} |\Gamma_\delta z| \le 4\rho\sqrt{n} \;\&\; \|\Gamma_\delta\| \le M\sqrt{n}\Big) \le \mathbb{P}(\exists_{z\in\mathcal{N}}\, |\Gamma_\delta z| \le 5\rho\sqrt{n}).$$

By the union bound, the latter probability is less then or equal to

$$\begin{aligned}|\mathcal{N}| \max_{z\in S^{k-1}} \mathbb{P}(|\Gamma_\delta z| \le 5\rho\sqrt{n}) &\le \Big(\frac{3}{\varepsilon}\Big)^{\delta n} (5C_4C_5)^n n^{n/2}\rho^n \\ &= \Big(\frac{3M}{\rho}\Big)^{\delta n} (5C_4C_5)^n n^{n/2}\rho^n.\end{aligned}$$

Since $\binom{n}{k} \le (en/k)^k \le (e/\delta)^{\delta n} \le C^n$ for some universal constant C, we conclude that

$$\mathbb{P}(\inf_{x\in Comp(\delta,\rho/(2M))} |\Gamma x| \le \rho\sqrt{n} \;\&\; \|\Gamma\| \le M\sqrt{n}) \le C_{11}^n M^{\delta n} n^{n/2}\rho^{(1-\delta)n}.$$

□

Proof of Theorem 1.2. We apply Proposition 2.12 with $\delta = c_{10}$ and $M = C_6$, where C_6 is the constant from Theorem 2.6, to obtain

$$\mathbb{P}(\inf_{x\in Comp(c_{10},c_{10}/(2C_6))} |\Gamma x| \le c_{10}\sqrt{n} \;\&\; \|\Gamma\| \le C_6\sqrt{n}) \le C_{10}\exp(-c_{10}^{1+C_{10}} n^{c_{10}}).$$

Thus, by Theorem 2.6 with $K = 1$, we get for some positive universal constants C and c,

$$\begin{aligned}&\mathbb{P}(\inf_{x\in Comp(c_{10},c_{10}/(2C_6))} |\Gamma x| \le c_{10}\sqrt{n}) \\ &\quad\le C_{10}\exp(-c_{10}^{1+C_{10}} n^{c_{10}}) + \exp(-c_6\sqrt{n}) \le Ce^{-cn^c}.\end{aligned}$$

On the other hand, Proposition 2.10 with $\delta = c_{10}$ and $\rho = c_{10}/(2C_6)$ gives for any $\varepsilon \in (0,1)$,

$$\mathbb{P}(\inf_{x\in Incomp(c_{10},c_{10}/(2C_6))} |\Gamma x| \le c_{10}(2C_6)^{-1}\varepsilon n^{-1/2}) \le \frac{C_9}{c_{10}}\varepsilon.$$

Since clearly $\varepsilon n^{-1/2} \le n^{1/2}$ and $Incomp(c_{10}, c_{10}/(2C_6)) \cup Comp(c_{10}, c_{10}/(2C_6)) = S^{n-1}$ the above inequalities imply (1).

Let us now turn to the proof of (2). Let c be the constant in (1). For $\varepsilon \ge \exp(-cn^c)$, the statement follows from (1), we can therefore assume that $\varepsilon < \exp(-cn^c)$. We shall apply Proposition 2.13 with parameters

- δ, where $\delta < 1/2$ is a small absolute constant to be determined later on,
- $M = C_6 \max(c_6^{-1}, 1) \log(1/\varepsilon)$, where C_6, c_6 are constants from Theorem 2.6,
- $\rho = \varepsilon^{1/(1-\delta)n} / \log^{\alpha}(1/\varepsilon)$, where α is another constant whose value will be fixed later.

We obtain

$$\begin{aligned}\mathbb{P}(&\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \leq \rho\sqrt{n} \;\&\; \|\Gamma\| \leq M\sqrt{n}) \\ &\leq C_{11}^n n^{n/2} M^{\delta n} \rho^{(1-\delta)n} \\ &\leq C^n n^{n/2} \log^{\delta n}(1/\varepsilon) \log^{-\alpha(1-\delta)n}(1/\varepsilon)\varepsilon,\end{aligned}$$

where $C = C_{11} C_6 \max\{c_6^{-1}, 1\}$.

Obviously we may also assume that $\varepsilon < \delta$ which together with the previous assumption on ε implies that $\log(1/\varepsilon) > \max\{cn^c, \log(1/\delta)\}$. Therefore for δ small enough and α large enough (depending on δ) we get

$$\mathbb{P}(\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \leq \rho\sqrt{n} \;\&\; \|\Gamma\| \leq M\sqrt{n}) \leq \varepsilon. \tag{5}$$

Notice now that for $n > 1$ and δ small enough we have $1/(1-\delta)n < 2/3$. Thus for some constant c_α, we also have $c_\alpha \varepsilon \leq \rho$. Therefore, we have

$$\mathbb{P}(\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \leq c_\alpha \varepsilon n^{-1/2}) \leq \mathbb{P}(\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \leq \rho\sqrt{n}) \leq 2\varepsilon,$$

where in the second inequality we used (5) combined with Theorem 2.6 and the definition of M.

Now, Proposition 2.10 gives

$$\mathbb{P}\Big(\inf_{x \in Incomp(\delta, \rho/(2M))} |\Gamma x| \leq c\Big(\varepsilon^{1+1/(1-\delta)n} / \log^{\alpha+2}(1/\varepsilon)\Big) n^{-1/2}\Big) \leq C\varepsilon/\delta.$$

Since $\delta < 1/2$, the last two inequalities imply (2).

□

2.4 Consequences

In this section we consider consequences of Theorem 1.2. First we show that as a corollary to (2) we can obtain a bound on the smallest singular value of Γ with a slightly worse dependence on ε than in Theorem 1.1, valid for every n. Next we consider the condition number of Γ and show that with overwhelming probability it is of order n.

Corollary 2.14 *For any $\delta \in (0,1)$ there exists C_δ, depending only on δ, such that for $n \geq 1$ and $\varepsilon \in (0,1)$*

$$\mathbb{P}(\inf_{x \in S^{n-1}} |\Gamma x| \leq \varepsilon n^{-1/2}) \leq C_\delta \varepsilon^{1-\delta}. \tag{6}$$

Proof Let us fix $\delta \in (0,1)$. For large n, the inequality (6) follows from (2). It is therefore enough to show that for every n there exists C, such that

$$\mathbb{P}(\inf_{x \in S^{n-1}} |\Gamma x| \leq \varepsilon n^{-1/2}) \leq C \varepsilon^{1-\delta}. \tag{7}$$

In what follows we will work with fixed n and use the letter C to denote a positive number depending only on n and δ. Its value may change from line to line.

The measure $\mathbb{P}$ is a Borel measure on the space of $n \times n$ matrices, which we will identify with $\mu^{\otimes n}$ on $\mathbb{R}^{n^2}$. By Theorem 2.6,

$$\mathbb{P}(\|\Gamma\| \geq C \log(1/\varepsilon)) \leq \varepsilon. \tag{8}$$

Moreover, by Lemma 2.4, the density of the measure $\mathbb{P}$ is bounded (by a number depending on n). Therefore

$$\begin{aligned} &\mathbb{P}(\{\Gamma\colon \inf_{x \in S^{n-1}} |\Gamma x| \leq \varepsilon n^{-1/2} \ \&\ \|\Gamma\| < C \log(1/\varepsilon)\}) \\ &\quad \leq C |\{\Gamma\colon \inf_{x \in S^{n-1}} |\Gamma x| \leq \varepsilon n^{-1/2} \ \&\ \|\Gamma\| < C \log(1/\varepsilon)\}|, \end{aligned} \tag{9}$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^{n^2}$. Now

$$\begin{aligned} &|\{\Gamma\colon \inf_{x \in S^{n-1}} |\Gamma x| \leq \varepsilon n^{-1/2} \ \&\ \|\Gamma\| < C \log(1/\varepsilon)\}| \\ &\quad = C^{n^2} \log^{n^2}(1/\varepsilon) |\{\Gamma\colon \inf_{x \in S^{n-1}} |\Gamma x| \leq \varepsilon C^{-1} \log^{-1}(1/\varepsilon) n^{-1/2} \ \&\ \|\Gamma\| < 1\}|. \end{aligned} \tag{10}$$

Thus the problem reduces to bounding from above the Lebesgue measure of the set of linear contractions with the smallest singular value not exceeding $\varepsilon C^{-1} \log^{-1}(1/\varepsilon) n^{-1/2}$. This can be done directly, however for convenience we will reduce the problem to the case of the Gaussian measure, which will allow us to use Theorem 1.1. Let γ_{n^2} denote the standard Gaussian measure on $\mathbb{R}^{n^2}$ and note that on the unit ball $\{\Gamma\colon \|\Gamma\| \le 1\}$ the density of γ_n^2 is bounded away from 0 (by a number depending on n). Thus

$$
\begin{aligned}
&\{\Gamma\colon \inf_{x \in S^{n-1}} |\Gamma x| \le \varepsilon C^{-1} \log^{-1}(1/\varepsilon) n^{-1/2} \ \&\ \|\Gamma\| < 1\}| \\
&\le C \gamma_{n^2}(\{\Gamma\colon \inf_{x \in S^{n-1}} |\Gamma x| \le \varepsilon C^{-1} \log^{-1}(1/\varepsilon) n^{-1/2}\}) \\
&\le C \varepsilon \log^{-1}(1/\varepsilon),
\end{aligned}
$$

where in the last inequality we used Theorem 1.1. Combining the above inequality with (8), (9), and (10) proves (7). □

Let us now state the result concerning the tail decay of the condition number of Γ.

Corollary 2.15 *For every $\delta > 0$ there exists C_δ such that for all $t > 0$,*

$$
\mathbb{P}(\kappa(\Gamma) \ge nt) \le \frac{C_\delta}{t^{1-\delta}}.
$$

Proof By Corollary 2.14 and Theorem 2.6 for some large absolute constant C,

$$
\begin{aligned}
\mathbb{P}(\kappa(\Gamma) \ge nt) &\le \mathbb{P}(\inf_{x \in S^{n-1}} |\Gamma x| < C^{-1} t^{-1+\delta/2} n^{-1/2}) + \mathbb{P}(\|\Gamma\| \ge C t^{\delta/2} \sqrt{n}) \\
&\le \frac{\tilde{C}_\delta}{t^{1-\delta}} + \exp(-c_6 t^{\delta/2}) \le \frac{C_\delta}{t^{1-\delta}}.
\end{aligned}
$$

□

3 Isotropic constant of a sum of i.i.d. random vectors in $\mathbb{R}^n$

The first version of Theorem 1.2 (as announced in [1]) involved the isotropic constant of column vectors of the matrix Γ. The argument was based on the small-ball estimate given in Lemma 2.5 and a modified version of Proposition 2.13 (see Proposition 3.2 below). Its proof required the control of the isotropic constant of the convolution of isotropic log-concave measures. Since a theorem providing such control is of independent interest, we present it now together with the proof.

Theorem 3.1 *Let $X_1, \ldots, X_n$ be i.i.d. random vectors in $\mathbb{R}^n$ distributed according to a symmetric isotropic log-concave probability μ, let $x \in S^{n-1}$ and $Z = x_1 X_1 + \ldots + x_n X_n$. Then $L_Z \leq C L_\mu$, where C is a universal constant.*

As a corollary we may obtain the following strengthening of Proposition 2.13. The only change with respect to the proof of Proposition 2.13 is using Theorem 3.1 instead of Lemma 2.4 to bound the isotropic constant of the convolution.

Proposition 3.2 *Let Γ be an $n \times n$ random matrix with independent columns $X_1, \ldots, X_n$ distributed according to a symmetric isotropic log-concave probability μ. Then for any $M > 1$ and $\delta, \rho \in (0,1)$ we have*

$$\mathbb{P}\Big(\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \leq \rho\sqrt{n} \ \&\ \|\Gamma\| \leq M\sqrt{n}\Big) \leq C^n L_\mu^n M^{\delta n} \rho^{(1-\delta)n},$$

where C is an absolute constant. In particular, there exist constants $c_1, c_2 > 0$ such that for every $M > 1$ and $\delta, \rho \in (0,1)$, satisfying

$$\rho \leq \Big(\frac{c_1}{M^\delta L_\mu}\Big)^{\frac{1}{1-\delta}}$$

we have

$$\mathbb{P}\Big(\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \leq \rho\sqrt{n} \ \&\ \|\Gamma\| \leq M\sqrt{n}\Big) \leq e^{-c_2 n}.$$

The proof of Theorem 3.1 is based on Lemma 3.3 below, which is a slightly modified version of a result of Gluskin and Milman [11], giving an ℓ_2 lower bound for the norm defined on $\mathbb{R}^n$ by

$$\|(\lambda_1, \ldots, \lambda_n)\| = \quad \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \Big\| \sum_{i=1}^m \lambda_i x_i \Big\|_K^2 \prod_{i=1}^m f_i(x_i) dx_i \Big)^{1/2},$$

where $f_1, \dots, f_n$ are densities of probability on $\mathbb{R}^n$, and on a result of Junge [13], which relates the isotropy constant of convolved log-concave probability measures with the norm $\|\cdot\|$.

Let $X_1, \dots, X_n$ be independent isotropic log-concave symmetric random vectors in $\mathbb{R}^n$. Let $x \in S^{n-1}$ and set

$$Z = x_1 X_1 + \ldots + x_n X_n.$$

Then it is well-known that Z is also an isotropic log-concave symmetric random vector in $\mathbb{R}^n$.

Recall that K is called a star body whenever $tK \subset K$ for all $0 \leq t \leq 1$, and in such a case $\|\cdot\|_K$ denotes its Minkowski functional, i.e. $\|x\|_K = \inf\{t > 0 \colon xt^{-1} \in K\}$.

Lemma 3.3 *Let $f_1, \dots, f_m$ be densities of probability measures on $\mathbb{R}^n$ and let $K \subset \mathbb{R}^n$ be a star body containing the origin in its interior. Then for all $\lambda_1, \dots, \lambda_m$ we have*

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \Big\| \sum_{i=1}^m \lambda_i x_i \Big\|_K^2 \prod_{i=1}^m f_i(x_i) dx_i \geq \frac{n}{n+2} \; |K|^{-2/n} \; \sum_{i=1}^m \lambda_i^2 r_i^2, \qquad (11)$$

where $r_i^2 = \int_0^\infty |\{x \colon f_i(x) \geq t\}|^{1+2/n} \, dt \geq \|f_i\|_\infty^{-2/n}$.

Proof. Let us recall that the symmetric decreasing rearrangement of a function $f \colon \mathbb{R}^n \to \mathbb{R}_+$ is a function $f^* \colon \mathbb{R}^n \to \mathbb{R}$, which is equidistributed with f (i.e. for all $t \in \mathbb{R}_+$, $|\{x \colon f^*(x) \geq t\}| = |(\{x \colon f(x) \geq t\}|)$ and for $x, y \in \mathbb{R}^n$, if $|x| \leq |y|$ then $f^*(x) \geq f^*(y)$.

If f is just a characteristic function of a set then f^* is the characteristic function of a ball of the same volume centered at the origin. In general one has the following "layer cake representation"

$$f^*(x) = \int_0^\infty [\mathbf{1}_{\{y \in \mathbb{R}^n \colon f(y) \geq t\}}]^*(x) dt. \qquad (12)$$

Let now D be the Euclidean ball of the same volume as K, centered at the origin. We will first prove the following inequality valid for all $t \in \mathbb{R}_+$

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mathbf{1}_{\{(x_i)_{i=1}^m \colon \|\sum_{i=1}^m \lambda_i x_i\|_K^2 \leq t\}} \prod_{i=1}^m f_i(x_i) dx_1 \ldots dx_m$$
$$\geq \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mathbf{1}_{\{(x_i)_{i=1}^m \colon \|\sum_{i=1}^m \lambda_i x_i\|_D^2 \leq t\}} \prod_{i=1}^m f_i^*(x_i) dx_1 \ldots dx_m. \qquad (13)$$

It is a corollary from the Brascamb-Lieb-Luttinger inequality [7], which asserts that for any functions $g_0, \dots, g_m \colon \mathbb{R}^n \to \mathbb{R}_+$ and any $(m+1) \times k$ matrix (a_{ij}), we have

$$\underbrace{\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n}}_{k} \prod_{i=0}^m g_i(\sum_{j=1}^k a_{ij}x_j) dx_1 \dots dx_k \le \underbrace{\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n}}_{k} \prod_{i=0}^m g_i^*(\sum_{j=1}^k a_{ij}x_j) dx_1 \dots dx_k.$$

Inequality (13) will follow if we substitute $g_0 = \mathbf{1}_{\sqrt{t}K}$ and $g_i = f_i$ for $i \ge 1$ (notice that $g_0^* = \mathbf{1}_{\sqrt{t}D}$) with the appropriate choice of the matrix (a_{ij}).

Since

$$\|x\|_K^2 = \int_0^\infty 2t(1 - \mathbf{1}_{\{\|x\|_K \le t\}}) dt,$$

we conclude from (13) and from the symmetry of the f_i^* that

$$\begin{aligned} &\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \Big\| \sum_{i=1}^m \lambda_i x_i \Big\|_K^2 \prod_{i=1}^m f_i(x_i) dx_i \\ &\ge \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \Big\| \sum_{i=1}^m \lambda_i x_i \Big\|_D^2 \prod_{i=1}^m f_i^*(x_i) dx_i \\ &= \sum_{i=1}^m \lambda_i^2 \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \|x_i\|_D^2 \prod_{i=1}^m f_i^*(x_i) dx_i. \end{aligned}$$

Since $|D|^{1/n} = |K|^{1/n}$, we have

$$\|x\|_D = \left(\frac{|B_2^n|}{|K|} \right)^{1/n} |x|.$$

Now we can use (12), to get

$$\begin{aligned} \int_{R^n} \|x\|_D^2 f_i^*(x) dx &= \left(\frac{|B_2^n|}{|K|} \right)^{2/n} \int_{\mathbb{R}^n} |x|^2 f_i^*(x) dx \\ &= \left(\frac{|B_2^n|}{|K|} \right)^{2/n} \int_0^\infty \int_{\mathbb{R}^n} |x|^2 (\mathbf{1}_{\{y \colon f_i(y) \ge t\}})^*(x) dx dt. \end{aligned}$$

The function $(\mathbf{1}_{\{y \colon f_i(y) \ge t\}})^*$ is the indicator of the ball of volume equal to $|\{y \colon f_i(y) \ge t\}|$, centered at the origin. Therefore, integrating in polar coordinates,

$$\int_{\mathbb{R}^n} |x|^2 (\mathbf{1}_{\{y:\, f_i(y) \geq t\}})^*(x)dx = \left(\frac{|\{y:\, f(y) \geq t\}|}{|B_2^n|} \right)^{1+2/n} \int_{B_2^n} |x|_2^2 dx$$
$$= \frac{n}{n+2} \frac{|\{y:\, f(y) \geq t\}|^{1+2/n}}{|B_2^n|^{2/n}}.$$

Thus

$$\int_{R^n} \|x\|_D^2 f_i^*(x)dx \geq \frac{n}{n+2} |K|^{-2/n} r_i^2,$$

where $r_i^2 = \int_0^\infty |\{x:\, f_i(x) \geq t\}|^{1+2/n} \, dt$. This concludes the proof of (11).

It remains to show that $r_i^2 \geq \|f_i\|_\infty^{-2/n}$. We know that

$$\int_0^\infty |\{y:\, f_i(y) \geq t\}| \, dt = \int_0^{\|f_i\|_\infty} |\{y:\, f_i(y) \geq t\}| \, dt = 1.$$

Hence by Hölder inequality,

$$1 \leq \quad \int_0^{\|f_i\|_\infty} |\{y:\, f_i(y) \geq t\}|^{1+2/n} \, dt \Bigg)^{n/(n+2)} \|f_i\|_\infty^{2/(n+2)}.$$

Since $r_i^2 = \int_0^\infty |\{x:\, f_i(x) \geq t\}|^{1+2/n} \, dt = \int_0^{\|f_i\|_\infty} |\{x:\, f_i(x) \geq t\}|^{1+2/n} \, dt$, we get the desired inequality. □

Proof of Theorem 3.1. Let f be the density of μ and let g be the density of Z. By Lemma 2 in [13] there exists a star-shaped body $K \subset \mathbb{R}^n$, with 0 in its interior such that

$$g(0)^{1/n} |K|^{1/n} \left(\int_{\mathbb{R}^n} \|x\|_K^2 g(x)dx \right)^{1/2} \leq C,$$

for a certain universal constant C. On the other hand, by Lemma 3.3 we have

$$\left(\int_{\mathbb{R}^n} \|x\|_K^2 g(x)dx \right)^{1/2} = \left(\mathbb{E}\|Z\|_K^2 \right)^{1/2} = \left(\mathbb{E}\|x_1 X_1 + \ldots + x_n X_n\|_K^2 \right)^{1/2}$$
$$\geq \frac{c}{|K|^{1/n} f(0)^{1/n}} \left(\sum_{i=1}^n x_i^2 \right)^{1/2} = \frac{c}{|K|^{1/n} f(0)^{1/n}}.$$

Putting these two inequalities together concludes the proof. □

References

[1] R. Adamczak, O. Guédon, A.E. Litvak, A. Pajor, N. Tomczak-Jaegermann, *Smallest singular value of random matrices with independent columns*, C. R. Math. Acad. Sci. Paris 346 (2008), no. 15-16, 853–856.

[2] R. Adamczak, A.E. Litvak, A. Pajor, N. Tomczak-Jaegermann, *Quantitative estimates of the convergence of the empirical covariance matrix in Log-concave Ensembles.* Journal of AMS, to appear. Available at `http://arxiv.org/abs/0903.2323`

[3] R. Adamczak, A. Litvak, A. Pajor, N. Tomczak-Jaegermann, *Restricted isometry property of matrices with independent columns and neighborly polytopes by random sampling.* Available at `http://arxiv.org/abs/0904.4723`

[4] G. Aubrun, *Sampling convex bodies: a random matrix approach*, Proc. AMS 135, No. 5 (2007), 1293–1303.

[5] C. Borell, *Convex measures on locally convex spaces,* Ark. Mat. 12 (1974), 239–252.

[6] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, Lect. Notes Math. 1469 (1991), 127-137.

[7] H.J. Brascamp, E.H. Lieb, *Best constants in Youngs inequality, its converse, and its generalization to more than three functions.* Adv. Math. 20, no. 2, (1976), 151–173

[8] S. Dar, *On the isotropic constant of non-symmetric convex bodies*, Isr. J. Math. 97 (1997), 151-156.

[9] A. Edelman, *Eigenvalues and condition numbers of random matrices.* SIAM J. of Matrix Anal. and Applic., 9 (1988), pp. 543-560.

[10] M. Fradelizi, *Sections of convex bodies through their centroid.* Arch. Math. (Basel) 69 (1997), no. 6, 515–522.

[11] E. Gluskin, V. Milman, *Geometric probability and random cotype 2*, GAFA, 123–138, Lecture Notes in Math., 1850, Springer, Berlin, 2004.

[12] O. Guédon, M. Rudelson, *L_p-moments of random vectors via majorizing measures*, Adv. Math. 208 (2007), 798–823.

[13] M. Junge, *Volume estimates for log-concave densities with application to iterated convolutions*, Pacific J. Math. 169 (1995), 107–133.

[14] B. Klartag, *On convex perturbations with a bounded isotropic constant,* Geom. Func. Anal., 16, no. 6 (2006), 1274–1290.

[15] B. Klartag, *Power-law estimates for the central limit theorem for convex sets,* J. Funct. Anal., Vol. 245, (2007), 284–310.

[16] A. E. Litvak, A. Pajor, M. Rudelson, N. Tomczak-Jaegermann, *Smallest singular value of random matrices and geometry of random polytopes.* Adv. Math. 195 (2005), 491–523.

[17] S. Mendelson, A. Pajor, *On singular values of matrices with independent rows*, Bernoulli 12 (2006), 761–773.

[18] G. Paouris, *Concentration of mass on convex bodies*, Geom. Funct. Anal. 16, no. 5 (2006), 1021–1049.

[19] G. Paouris, *On the isotropic constant of non-symmetric convex bodies*, Lect. Notes Math. 1745 (2000), 239-243.

[20] M. Rudelson, *Invertibility of random matrices: norm of the inverse*, Annals of Math, 168, No. 2 (2008), 575-600.

[21] M. Rudelson, R. Vershynin, *The Littlewood-Offord Problem and invertibility of random matrices*, Adv. Math., 218, No. 2 (2008), 600-633.

[22] S. Smale, *On the efficiency of algorithms of analysis.* Bull. Amer. Math. Soc., 13 (1985), 87–121.

[23] S. Szarek, *Condition numbers of random matrices.* J. Complexity 7 (1991), no. 2, 131–149.

[24] T.Tao, V. Vu, *Inverse Littlewood-Offord theorems and the condition number of random discrete matrices*, Annals of Math, 169, No. 2 (2009), 595-632.

Radoslaw Adamczak,
Institute of Mathematics,
University of Warsaw,
Banacha 2, 02-097 Warszawa, Poland
e-mail: radamcz@mimuw.edu.pl

Olivier Guédon,
Université Pierre et Marie Curie, Paris 6,
Institut de Mathématiques de Jussieu,
4 place Jussieu, 75005 Paris, France
e-mail: guedon@math.jussieu.fr

Alexander E. Litvak,
Dept. of Math. and Stat. Sciences,
University of Alberta,
Edmonton, Alberta, Canada, T6G 2G1.
e-mail: alexandr@math.ualberta.ca

Alain Pajor,
Université Paris-Est
Équipe d'Analyse et Mathématiques Appliquées,
5, boulevard Descartes, Champs sur Marne,
77454 Marne-la-Vallée, Cedex 2, France
e-mail: Alain.Pajor@univ-mlv.fr

Nicole Tomczak-Jaegermann,
Dept. of Math. and Stat. Sciences,

University of Alberta,
Edmonton, Alberta, Canada, T6G 2G1.
e-mail: nicole.tomczak@ualberta.ca