ON A GEOMETRIC INEQUALITY

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Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a normed space and $K = K(X) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ its unit ball. We denote $|K| = \mathrm{Vol}\, K$, where Vol means the standard Lebesgue volume on $\mathbb{R}^n$ equipped with the standard Euclidean structure. Let $\lambda_i \in \mathbb{R}$ and $\overline{\lambda} = (\lambda_i)_{i=1}^m$. Consider the following symmetric norm $|||\cdot|||$ on $\mathbb{R}^m$,

$$|||\overline{\lambda}|||_K = \int\limits_{x_1 \in K} \cdots \int\limits_{x_m \in K} \Big\|\sum_1^m \lambda_i x_i\Big\| \frac{dx_1 \cdots dx_m}{|K|^m}. \tag{1.1}$$

At the Denmark Conference on Probability in Banach spaces (June, 1986), the following question was asked (by V.M.):

Is it true that, for any X (or, in other terms, for any centrally symmetric compact body $K \subset \mathbb{R}^n$ and the norm generated by this body), the symmetric norm $|||\cdot|||_K$ is, up to $\log n$, close to the standard Euclidean norm?

Moreover, is it true that under cotype condition on X this norm is equivalent to the Euclidean norm?

Precisely:

Problem. Does there exist a constant $C = C\big(q, C_q(X)\big)$ depending only on $2 \leq q < \infty$ and on the cotype q constant $C_q(X)$, such that

$$\frac{1}{C}\Big(\sum_1^m \lambda_i^2\Big)^{1/2} \leq |||\overline{\lambda}|||_K \leq C\Big(\sum_1^m \lambda_i^2\Big)^{1/2} \;?$$

Note that the question appears first as some kind of a generalization of the Khinchine inequality $\operatorname*{Ave}_{\varepsilon_i=\pm1} \left|\sum_1^m \varepsilon_i \lambda_i\right| \approx \left(\sum_1^m \lambda_i^2\right)^{1/2}$, complementary to the Kahane-type generalization: in the Kahane inequality the numbers λ_i are replaced by vectors $x_i \in X$ and, as in the original Khinchine inequality, an equivalence between different L_p-norms is established:

$$\Big\|\sum_1^m \varepsilon_i x_i\Big\|_{L_p(X)} \overset{\text{def}}{\equiv} \Big(\operatorname*{Ave}_{\varepsilon_i=\pm1} \Big\|\sum_1^m \varepsilon_i x_i\Big\|^p\Big)^{1/p} \leq C(p) \Big\|\sum_1^m \varepsilon_i x_i\Big\|_{L_1(X)}.$$