Metric entropy of convex hulls in Banach spaces

Bernd Carl   Ioanna Kyrezi *   Alain Pajor

September 15, 2001

Abstract

In this paper we present diverse methods for estimating the covering number of a precompact subset of a Banach space when we already know the entropy of the set of its extremal points. In the case of a Hilbert space we estimate also the Gelfand diameters of the subset.

0 Introduction

Let (M,d) be a metric space and $B(x_o;\varepsilon) := \{x \in M \,|\, d(x_o,x) \leq \varepsilon\}$ the closed ε-ball with centre x_o. For a bounded set $A \subset M$ let $N(A;\varepsilon)$ be the covering number of A by ε-balls of M which means:

$$N(A;\varepsilon) := \inf\left\{N : \exists x_1,\dots,x_N \in M \;\text{ such that }\; A \subset \bigcup_{k=1}^{N} B(x_k;\varepsilon)\right\}.$$

We denote the entropy numbers of A by

$$\varepsilon_n(A) := \inf\{\varepsilon \geq 0 : N(A;\varepsilon) \leq n\}$$

and the (dyadic) entropy numbers by

$$e_n(A) := \varepsilon_{2^{n-1}}(A)\ , \qquad n = 1,2,\dots\ .$$

We show in this paper how the rate of decay of entropy numbers $\varepsilon_n(A)$ of a precompact set A of a Banach space X reflects the rate of decay of dyadic entropy numbers $e_n(\mathrm{co}(A))$ of the (symmetric) absolute convex hull $\mathrm{co}(A)$ of A. We obtain optimal results for arbitrary Banach spaces and for Hilbert spaces as well as for Banach spaces of type $p, 1 < p \leq 2$. It will be convenient to couch the arguments in a more general context and in the

*Research supported by Hellenic S.S.F.

language of entropy numbers of operators. For a (bounded linear) operator $u : X \to Y$ between two Banach spaces X and Y the n-th dyadic entropy number $e_n(u)$ of u is defined by

$$e_n(u) := e_n(u(B_X)) \quad \text{where} \quad B_X \quad \text{is the unit ball of} \quad X.$$

If $\ell_1(A)$ denotes the Banach space of all summable families of real numbers $(\xi_t)_{t\in A}$ over the index set A with the norm given by

$$||(\xi_t)|| = \sum_{t\in A} |\xi_t|,$$

then the entropy numbers of the absolute convex hull $e_n(\text{co}(A))$ of a bounded set A of a Banach space X can be expressed in terms of entropy numbers of operators

$$e_n(\text{co}(A)) = e_n(u)$$

where $u : \ell_1(A) \to X$ is the operator defined on the canonical basis $(e_t)_{t\in A}$ of $\ell_1(A)$ by $u(e_t) = t$. This problem can be embedded into a more general frame. Namely, we investigate entropy numbers of operators $u : \ell_1(M) \to X$ mapping $\ell_1(M)$ over a precompact metric space (M, d) into a Banach space X. Furthermore, for some proofs we also consider the "dual" situation where we study entropy numbers $e_n(u)$ of operators $u : X \to \ell_\infty(M)$ from a Banach space X into the space $\ell_\infty(M)$ of all bounded number families over M with the norm

$$||(\xi_t)|| = \sup_{t\in M} |\xi_t|.$$

This situation is in so far universal that the entropy numbers $e_n(v)$ of a compact operator $v : X \to Y$ between Banach space X and Y are always shared by the entropy numbers $e_n(u)$ of a compact operator $u : X \to C[a, b]$ with values in the space $C[a, b]$ of continuous functions over a bounded and closed intervall $[a, b]$ in the sense that

$$\frac{1}{2} e_n(v) \leq e_n(u) \leq 2e_n(v).$$

This fact indicates why we also study the asymptotic behaviour of entropy numbers of operators

$$u : X \to C(M)$$

with values into the space of continuous functions $C(M)$ over a compact metric space M. We will see how the geometry of the Banach space X, the entropy numbers $\varepsilon_n(M)$ of the underlying compact metric space M and the smoothness of the operator u in terms of the modulus of continuity $\omega(u; \delta)$ of u,

$$\omega(u; \delta) := \sup_{||x||\leq 1} \sup\{|u(x)(s) - u(x)(t)| : s, t \in M, d(s, t) \leq \delta\},$$

enter the estimates of the entropy numbers $e_n(u)$ of the operator u. The problem of estimating entropy numbers of the convex hull co(A) is embedded into this more general situation. Indeed, for the operator $u : \ell_1(A) \to X$ with $u(e_t) = t$ we have that for the dual operator $u^* : X^* \to \ell_\infty(A)$,

$$\omega(u^*; \delta) \leq \delta.$$

Hence, the problem arises now to relate $\omega(u^*; \delta)$ with the dyadic entropy numbers $e_n(u)$ and $e_n(u^*)$ of u. This problem of independent interest will be studied in sections 2 and 3 and the applications to entropy numbers of convex hulls will be given in sections 4, 5 and 6 where we also present some alternative proofs and new approaches to entropy estimates as well. In this process we refine and extend a result of Dudley [D] on the entropy of convex hulls in Hilbert spaces. For our purpose we need some preliminaries. In the following we always consider real Banach spaces X and denote by X^* the dual Banach space. For a (bounded linear) operator $u : X \to Y$ between Banach spaces X and Y the n-th approximation number, the n-th Gelfand number and the n-th Tichomirov number (symmetrized approximation number) of u are defined by

$$\begin{aligned} a_n(u) &:= \inf\{||u - v|| \mid v : X \to Y,\ \operatorname{rank}(v) < n\} \\ c_n(u) &:= \inf\{||u|_E|| \mid E \subset X,\ \operatorname{codim}(E) < n\} \end{aligned}$$

and

$$t_n(u) := a_n(I_Y u Q_X),$$

respectively, where $I_Y : Y \to \ell_\infty(B_{Y^*})$ is the canonical embedding and $Q_X : \ell_1(B_X) \to X$ the canonical surjection. We always have

$$t_n(u) \leq c_n(u) \leq a_n(u) \quad \text{and} \quad t_n(u) = t_n(u^*), a_n(u) \leq a_n(u^*) \ ,$$

where u^* is the dual operator of u. Moreover, if $u : \ell_2^n \to Y$, we define

$$\ell(u) := \Big(\int_{\mathbb{R}^n} ||u(x)||^2 d\gamma_n(x)\Big)^{1/2},$$

where γ_n is the canonical Gaussian probability measure on $\mathbb{R}^n$, and for any $v : X \to \ell_2^n$ we define

$$\ell^*(v) := \sup\Big\{|\operatorname{trace}(vu)| \ \Big| \ u : \ell_2^n \to X,\ \ell(u) \leq 1\Big\}.$$

A Banach space X is called K-convex if there is a constant $C \geq 1$ such that for all integer $n \geq 1$ and all operators $u : X \to \ell_2^n$,

$$\ell(u^*) \leq C\ell^*(u) \ .$$

The smallest constant C is called the K-convexity constant. Finally, a Banach space X is said to have type $p, 1 \leq p \leq 2$, if there is a constant C such that for all finite sets $(x_i)_{i=1}^n \subset X$,

$$\int_0^1 \|\sum_{i=1}^n r_i(t)x_i\|dt \leq C(\sum_{i=1}^n \|x_i\|^p)^{1/p} ,$$

where $(r_i)_{i=1}^\infty$ are the Rademacher functions. The smallest constant

$$\tau_p(X) = \inf C$$

such that the above inequality is satisfied is called the type p constant of X.

1 Basic tools

In this section we give diverse inequalities for estimating entropy numbers and approximation quantities. They will be very useful for estimating the entropy of convex hulls in subsequent sections. We start with a general inequality of [CH] in the refined version of [DJ].

Theorem 1.1 *For every $\varepsilon > 0$, there is $b(\varepsilon) > 0$ such that for all Banach spaces X, Y, all operators $u : X \to Y$ and all integer $n \geq 1$,*

$$e_{[(1+\varepsilon)n]}(u) \leq b(\varepsilon) \sup_{1\leq k\leq n} 2^{-\frac{n}{k}} \left(\prod_{i=1}^{k} t_i(u) \right)^{1/k} .$$

As a consequence of this inequality we can reprove a well-known inequality in the following theorem.

Theorem 1.2 ([C1]) *Let $\alpha > 0$, then there is a constant $\rho_\alpha \geq 1$ such that for all operators $u : X \to Y$ between Banach spaces X, Y and all integer $n \geq 1$,*

$$\sup_{1\leq k\leq n} k^\alpha \max\{e_k(u); e_k(u^*)\} \leq \rho_\alpha \sup_{1\leq k\leq n} k^\alpha t_k(u).$$

Proof. From the inequality in Theorem 1.1 we, in particular, get

$$e_{2n}(u) \leq b(1) \sup_{1\leq k\leq n} 2^{-\frac{n}{k}} \left(\prod_{i=1}^{k} t_i(u) \right)^{1/k} .$$

Because

$$\prod_{i=1}^{k} t_i(u) \leq \left(\prod_{i=1}^{k} i^{-\alpha} \right) \left(\sup_{1\leq i\leq k} i^\alpha t_i(u) \right)^k = (k!)^{-\alpha} \left(\sup_{1\leq i\leq k} i^\alpha t_i(u) \right)^k ,$$

we infer

$$\begin{aligned} e_{2n}(u) &\leq b(1) \left(\sup_{1\leq k\leq n} 2^{-\frac{n}{k}} (k!)^{-\frac{\alpha}{k}} \right) \sup_{1\leq i\leq n} i^\alpha t_i(u) \\ &\leq c_\alpha n^{-\alpha} \sup_{1\leq i\leq n} i^\alpha t_i(u) \quad \text{for} \quad n \in \mathbb{N}. \end{aligned}$$

The monotonicity of the entropy numbers and $t_i(u) = t_i(u^*)$ yield with a new constant $\rho_\alpha \geq 1$ the estimate

$$\max\{e_n(u); e_n(u^*)\} \leq \rho_\alpha n^{-\alpha} \sup_{1\leq i\leq n} i^\alpha t_i(u), \quad n \in \mathbb{N},$$

and therefore the desired estimate

$$\sup_{1\leq k\leq n} k^\alpha \max\{e_k(u); e_k(u^*)\} \leq \rho_\alpha \sup_{1\leq k\leq n} k^\alpha t_k(u)$$

for all integer $n \geq 1$. □

We also need a further variant of the preceding theorem.

Theorem 1.3 *Let (b_n) be a positive and increasing sequence with the property that there exists a constant $\gamma \geq 1$ and for all $n \in \mathbb{N}, b_{2n} \leq \gamma b_n$. Then there exists a constant $c(\gamma) \geq 1$ such that for all operators $u : X \to Y$ between Banach spaces X, Y and all integer $n \geq 1$,*

$$\sup_{1\leq k\leq n} b_k e_k(u) \leq c(\gamma) \sup_{1\leq k\leq n} b_k t_k(u) \quad .$$

Proof. The inequality in Theorem 1.2 is equivalent to

$$\sup_{0\leq j\leq N} 2^{j\alpha} e_{2^j}(u) \leq \rho_0(\alpha) \sup_{0\leq j\leq N} 2^{j\alpha} t_{2^j}(u), \quad N = 1, 2, \dots .$$

Put $\alpha = \log_2 \gamma$. Then $b_{2n} \leq \gamma b_n = 2^\alpha b_n$ implies

$$\frac{2^{j\alpha}}{b_{2^j}} \leq \gamma \frac{2^{j\alpha}}{b_{2^{j+1}}} = \frac{2^{(j+1)\alpha}}{b_{2^{j+1}}},$$

i.e. the sequence $\frac{2^{j\alpha}}{b_{2^j}}$ is monotonously increasing. Hence,

$$\begin{aligned} 2^{N\alpha} e_{2^N}(u) &\leq \rho_0(\alpha) \sup_{0\leq j\leq N} \frac{2^{j\alpha}}{b_{2^j}} \sup_{0\leq j\leq N} b_{2^j} t_{2^j}(u) \\ &\leq \rho_0(\alpha) \frac{2^{N\alpha}}{b_{2^N}} \sup_{0\leq j\leq N} b_{2^j} t_{2^j}(u) \end{aligned}$$

or

$$b_{2^N} e_{2^N}(u) \leq \rho_0(\alpha) \sup_{1\leq j\leq N} b_{2^j} t_{2^j}(u), \quad N = 1, 2, \dots \quad .$$

Now for arbitrary integer n we choose N such that $2^{N-1} \leq n \leq 2^N$ and get

$$\begin{aligned} b_n e_n(u) &\leq b_{2^N} e_{2^{N-1}}(u) \leq \gamma b_{2^{N-1}} e_{2^{N-1}}(u) \leq \\ &\leq \gamma \rho_0(\alpha) \sup_{1\leq j\leq N-1} b_{2^j} t_{2^j}(u) \leq \\ &\leq \gamma \rho_0(\alpha) \sup_{1\leq j\leq n} b_j t_j(u) \quad \text{for} \quad n = 1, 2, \dots, \end{aligned}$$

yielding the desired inequality

$$\sup_{1\leq j\leq n} b_j e_j(u) \leq \rho(\gamma) \sup_{1\leq j\leq n} b_j t_j(u) \quad \text{for} \quad n = 1, 2, \dots,$$

where $\rho(\gamma) = \gamma \rho_0(\log_2 \gamma)$. □

The next theorem gives duality inequalities for entropy numbers.

Theorem 1.4 ([BPST]) *Let X or Y be a K-convex Banach space, then for every $\alpha > 0$ there is a constant $c \geq 1$ which may depend on α and the K-convexity constant such that for all compact operators $u : X \to Y$ the inequalities*

$$\frac{1}{c} \sup_{1\leq k\leq n} k^\alpha e_k(u) \leq \sup_{1\leq k\leq n} k^\alpha e_k(u^*) \leq c \sup_{1\leq k\leq n} k^\alpha e_k(u)$$

for $n = 1, 2, \dots,$ are satisfied.

A refined version of a Sudakov-type inequality is given in the following theorem.

Theorem 1.5 ([PT]) *There is a constant $c \geq 1$ such that for all Banach spaces X, all integer $n \geq 1$ and all operators $u : X \to \ell_2^n$ the inequality*

$$k^{\frac{1}{2}} c_k(u) \leq c \int_{\mathbb{R}^n} \|u^*(x)\| d\gamma_n(x) \leq c\,\ell(u^*)$$

holds for $k = 1, 2, \ldots$.

Moreover, we need the following inequality.

Theorem 1.6 ([P1]) *If $(Z_1, \ldots, Z_n)$ is a finite Gaussian process, then*

$$\mathbb{E} \sup_{1 \leq i \leq n} Z_i \leq C \sup_{1 \leq i \neq j \leq n} \|Z_i - Z_j\| (\log\, n)^{1/2},$$

where $C \geq 1$ is a universal constant.

Finally, the following theorem is a nonpublished result by B. Maurey the proof of which can be found in [P2]. We use the formulation given in [C2].

Theorem 1.7 *Let X be a Banach space of type p, $1 < p \leq 2$. Then for all integers such that $1 \leq k \leq n$ and all operators $S : \ell_1^n \to X$ the estimate*

$$e_k(S) \leq c_p \tau_p(X) \left(\frac{\log(\frac{n}{k} + 1)}{k} \right)^{1 - \frac{1}{p}} \|S\|$$

is satisfied, where $c_p \geq 1$ is a constant depending only on p.

Estimates of this type will be referred to as local estimates of entropy numbers. Furthermore, we say that a Banach space X is of *entropy type* $\beta \geq 0$ if there is a constant $c \geq 0$ such that for all integer $n \geq 1$ and all $x_1, \ldots, x_n \in X$ the (dyadic) entropy numbers of the absolute convex hull $\mathrm{co}(x_1, \ldots, x_n)$ can be estimated by

$$e_n(\mathrm{co}(x_1, \ldots, x_n)) \leq c n^{-\beta} \max_{1 \leq i \leq n} \|x_i\|.$$

In terms of operators this inequality states that for all integer $n \geq 1$ and all operators $S : \ell_1^n \to X$,

$$e_n(S) \leq c n^{-\beta} \|S\|.$$

From Dvoretzky's theorem we infer that the parameter β is bounded by $\frac{1}{2}$. Moreover, by Pisier's (cf. [P1]) characterization of K-convex Banach spaces we can easily show that a Banach space of entropy type $\beta > 0$ is always K-convex.

A characterization of weak type p spaces, $1 \leq p \leq 2$, (for a definition see [P1] or [DJ]) in terms of entropy numbers has been given in [Pa] for $p = 2$ and by Defant and Junge [DJ] for $1 < p < 2$. They proved that a Banach space is of weak type p, $1 < p \leq 2$, if and only if it is of entropy type β with $\beta = 1 - \frac{1}{p}$. Much more, Defant and Junge showed in [DJ] that in the case $0 < \beta < \frac{1}{2}$ a Banach space X is of entropy type β if there is a constant $c \geq 1$ such that for all integer $n \geq 1$ and all operators $S : \ell_1^n \to X$ an estimate as in Theorem 1.7

$$e_k(S) \leq c \left(\frac{\log(\frac{n}{k} + 1)}{k} \right)^{\beta} \|S\|, \quad 1 \leq k \leq n,$$

is satisfied. In the case $\beta = \frac{1}{2}$ such an estimate remains true at least for Banach spaces of type 2 (see Theorem 1.7). Finally, we give local estimates for entropy numbers of operators from X into ℓ_∞^n.

Theorem 1.8 *Let $0 < \beta \leq \frac{1}{2}$. If X is a Banach space such that there is a constant $c \geq 1$ and for all integer $n \geq 1$ and all operators $S : \ell_1^n \to X^*$ with values in the dual space X^* of X the estimate*

$$e_k(S) \leq c \left(\frac{\log(\frac{n}{k} + 1)}{k} \right)^{\beta} \|S\|, \quad 1 \leq k \leq n,$$

is satisfied, then for all integer $n \geq 1$ and all operators $T : X \to \ell_\infty^n$ the estimate

$$e_k(T) \leq \tilde{c} \left(\frac{\log(\frac{n}{k} + 1)}{k} \right)^{\beta} \|T\|, \quad 1 \leq k \leq n,$$

is true, where $\tilde{c} \geq 1$ is a constant which may depend on β and the K-convexity constant of X. In particular, if X is a Banach space such that X^* is of type p, then this estimate is valid for $\beta = 1 - \frac{1}{p}$.

Proof. Using the inequality of Theorem 1.4 with the parameter $\alpha = 2\beta$ we immediately check the first assertion of the theorem. The remaining part follows from Theorem 1.7. $\square$

Remark. The assertion of Theorem 1.8 remains valid if X is a Banach space such that X^* is of weak type p with $1 < p < 2$, (cf. [DJ]).

2 Entropy of $C(M)$-valued operators

As already announced in the introduction the entropy behaviour of a compact operator is also reflected by a $C(M)$-valued operator on a compact metric space M. So $C(M)$-valued operators are universal for our purpose. In the sequel we relate the entropy of such operators with the modulus of continuity of an operator. Recall that the modulus of continuity $\omega(f;\delta)$ of a bounded function f on a metric space (M,d) is defined by

$$\omega(f;\delta) := \sup\{|f(t)-f(s)| \mid s,t \in M, d(s,t) \le \delta\}$$

for $0 \le \delta < \infty$. Given an operator $u : X \to C(M)$ from a Banach space X into the space $C(M)$ of all continuous functions on a compact metric space (M,d) we put

$$\omega(u;\delta) := \sup_{||x||\le 1} \omega(u(x);\delta), \tag{1}$$

calling $\omega(u;\delta)$ the *modulus of continuity of the operator* u. This definition in fact makes sense for all $\delta \ge 0$ since

$$\omega(u;\delta) \le 2\|u\|.$$

By the Arzela-Ascoli theorem we have that $u : X \to C(M)$ is compact if and only if

$$\lim_{\delta\to 0} \omega(u;\delta) = 0. \tag{2}$$

Inserting the entropy numbers $\varepsilon_n(M)$ of the underlying compact metric space M, the rate of decrease of the sequence $\omega(u;\varepsilon_n(M))$ indicates the degree of compactness of the operator u related to the degree of compactness of the underlying metric space M. Moreover, the modulus of continuity $\omega(u;\varepsilon_n(M))$ dominates the approximation numbers of u by an inequality

$$a_{n+1}(u) \le \omega(u;\varepsilon_n(M)),\ \ n = 1,2,\ldots, \tag{3}$$

for compact operators $u : X \to C(M)$ (cf. [CS]). If one omits the compactness of the operator u one can replace the approximation numbers by the Gelfand numbers (cf. [RS]),

$$c_{n+1}(u) \le \omega(u;\varepsilon_n(M)), \quad n = 1,2,\ldots\ . \tag{4}$$

Combining this estimate with Theorem 1.1 we obtain a general inequality dominating the entropy numbers by terms of the modulus of continuity.

Theorem 2.1 *For every $\varepsilon > 0$, there is $b(\varepsilon) > 0$ such that for all Banach spaces X, all compact metric spaces M, all operators $u : X \to C(M)$ and all integer $n \ge 1$ the inequality*

$$\max\{e_{[(1+\varepsilon)n]}(u), e_{[(1+\varepsilon)n]}(u^*)\} \le b(\varepsilon) \sup_{1\le k\le n} 2^{-\frac{n}{k}} \left(\prod_{i=0}^{k-1} \omega(u;\varepsilon_i(M))\right)^{1/k} \tag{5}$$

is valid, where we put $\omega(u;\varepsilon_0(M)) := \|u\|$.

Remark. Theorem 2.1 remains true if we consider operators $u : X \to \ell_\infty(M)$ from a Banach space X into the space $\ell_\infty(M)$ over a precompact metric space. Moreover, if we take into account the geometry of the underlying Banach space X, the smoothness of the operator u and the rate of decrease of the sequence $(\varepsilon_n(M))$ we are able to give much better estimates for the entropy numbers $e_n(u : X \to C(M))$. Before we state a result for power decrease of $\varepsilon_n(M)$ we introduce the useful notion of a Lipschitz-continuous operator (1-Hölder continuous in [CS]). An operator $u : X \to C(M)$ is called *Lipschitz-continuous* if

$$|u| := \sup_{\delta > 0} \frac{\omega(u;\delta)}{\delta} < \infty.$$

The set

$$\mathrm{Lip}(X, C(M))$$

of all Lipschitz-continuous operators becomes a Banach space under the norm

$$\mathrm{Lip}(u) := \max\{\|u\|; |u|\}.$$

Now we can state the following theorem.

Theorem 2.2 *Let M be a compact metric space and $\rho, \alpha > 0$ such that for all integer $n \geq 1$,*

$$\varepsilon_n(M) \leq \rho n^{-\alpha} \tag{6}$$

is satisfied. Moreover, let X be a Banach space with the property that there is a constant $\beta > 0$ such that for each $\varepsilon > 0$ there is a constant $c(\varepsilon) \geq 1$ and for all integer $n \geq 1$ and all operators $S : X \to \ell_\infty^n$ the local estimate

$$e_k(S) \leq c(\varepsilon)\|S\|k^{-\beta}\left(\frac{n}{k}\right)^{\varepsilon} \quad \text{for} \quad 1 \leq k \leq n \tag{7}$$

is valid. Then for $u \in \mathrm{Lip}(X, C(M))$ we have the estimate

$$e_n(u) \leq cn^{-\beta-\alpha}\mathrm{Lip}(u) \quad \text{for} \quad n = 1, 2, \ldots, \tag{8}$$

where $c = c(\rho, \alpha, \beta)$ is a constant depending on ρ, α and β.

As an immediate consequence of Theorem 2.2 we obtain the following result.

Proposition 2.3 *Let M be a compact metric space with the property that there are $\rho, \alpha > 0$ and for all integer $n \geq 1$,*

$$\varepsilon_n(M) \leq \rho n^{-\alpha}.$$

If X is a Banach space such that the dual Banach space X^* is of type p, $1 < p \leq 2$, then for $u \in \mathrm{Lip}(X, C(M))$ we have the entropy estimate

$$e_n(u) \leq cn^{-(1-\frac{1}{p})-\alpha}\,\mathrm{Lip}(u) \quad \text{for} \quad n = 1, 2, \ldots, \tag{9}$$

where c is a constant which may depend on ρ, α, p, and the K-convexity constant of X. In particular, for $X = L_q$, $1 < q < \infty$, we have

$$e_n(u) \leq cn^{-\min\{1-\frac{1}{q};\frac{1}{2}\}-\alpha}\,\mathrm{Lip}(u) \quad \textit{for} \quad n = 1, 2, \ldots. \tag{10}$$

Proof. The assertion follows from the local estimates of entropy numbers in Theorem 1.8 and Theorem 2.2. □

Remarks.

1) The assertion of Proposition 2.3 remains valid if we replace type p by weak type p for $1 < p < 2$ (cf. Remark after Theorem 1.8).

2) Theorem 2.2 and Proposition 2.3 remain true if we consider operators $u \in \mathrm{Lip}(X, \ell_\infty(M))$ where M is a precompact metric space satisfying the property (6) of Theorem 2.2.

3 Entropy of operators from $\ell_1(M)$

In this section we give corresponding results to section 2 for operators $u : \ell_1(M) \to X$ from $\ell_1(M)$ over a precompact metric space M into a Banach space X. The corresponding statement to Theorem 2.1 reads as follows.

Theorem 3.1 *Let $\varepsilon > 0$, then there is a constant such that for all Banach spaces X, all precompact metric spaces M, all operators $u : \ell_1(M) \to X$ and all integer $n \geq 1$,*

$$e_{[(1+\varepsilon)n]}(u) \leq b(\varepsilon) \sup_{1\leq k\leq n} 2^{-\frac{n}{k}} \left(\prod_{i=0}^{k-1} \omega(u^*; \varepsilon_i(M))\right)^{1/k} \tag{11}$$

where we put $\omega(u^*, \varepsilon_0(M)) := \|u^*\| = \|u\|$.

Proof. The assertion follows from

$$t_{k+1}(u) = t_{k+1}(u^*) \leq \omega(u^*; \varepsilon_k(M))$$

and Theorem 2.1. □

Before we state a corresponding result to Theorem 2.2 we need the following notion. Let (M, d) be a precompact metric space. An operator u belongs to $\mathrm{Lip}(\ell_1(M), X)$ and is said to be Lipschitz-continuous if

$$u^* \in \mathrm{Lip}(X^*, \ell_\infty(M)).$$

This is equivalent to say that

$$\sup\left\{\frac{\|u(e_t) - u(e_s)\|}{d(s,t)} \;\Big|\; s, t \in M,\ s \neq t\right\} < \infty,$$

where (e_t) is the canonical basis of $\ell_1(M)$. By

$$\mathrm{Lip}(u) := \mathrm{Lip}(u^*) = \max\{\|u\|; |u^*|\}$$

the space $\mathrm{Lip}(\ell_1(M), X)$ becomes a Banach space. Furthermore, we need also the following duality relations.

Lemma 3.2 *Let X be a Banach space and $\beta > 0$. Then the following estimates are equivalent:*

(i) *For each $\varepsilon > 0$ there is $c(\varepsilon) \geq 1$ such that for all integer $n \geq 1$ and all $T : \ell_1^n \to X$,*

$$e_k(T) \leq c(\varepsilon)\|T\|k^{-\beta}(\frac{n}{k})^\varepsilon \quad \textit{for} \quad 1 \leq k \leq n.$$

(ii) *For each* $\varepsilon > 0$ *there is* $c(\varepsilon) \geq 1$ *such that for all integer* $n \geq 1$ *and all* $T : \ell_1^n \to X^{**}$,

$$e_k(T) \leq c(\varepsilon)\|T\|k^{-\beta}(\frac{n}{k})^\varepsilon \quad \textit{for} \quad 1 \leq k \leq n.$$

(iii) *For each* $\varepsilon > 0$ *there is* $c(\varepsilon) \geq 1$ *such that for all integer* $n \geq 1$ *and all* $T : X^* \to \ell_\infty^n$,

$$e_k(T) \leq c(\varepsilon)\|T\|k^{-\beta}(\frac{n}{k})^\varepsilon \quad \textit{for} \quad 1 \leq k \leq n.$$

Proof. First we show that (i) is equivalent to (ii). Indeed, because of

$$e_k(T : \ell_1^n \to X) \leq 2e_k(T^{**} : \ell_1^n \to X^{**}),$$

property (ii) implies (i). The converse implication follows by Lindenstrauss and Rosenthal's principle of local reflexivity (cf. [LT]) as follows: given $T : \ell_1^n \to X^{**}$, then there exists an isomorphism

$$S : T(\ell_1^n) \to X_0 \subset X \quad \text{with} \quad \|S\|\|S^{-1}\| \leq 2.$$

Let $T_0 : \ell_1^n \to T(\ell_1^n)$ the restriction of T with $T = I_{T(\ell_1^n)}^{X^{**}} T_0$, then by (i) we infer

$$\begin{aligned} e_k(T) &\leq e_k(T_0) = e_k(S^{-1}ST_0) \leq \|S^{-1}\|e_k(ST_0) \leq \\ &\leq 2\|S^{-1}\|e_k(I_{X_0}^X ST_0) \leq 2\|S^{-1}\|c(\varepsilon)\|I_{X_0}^X ST_0\|k^{-\beta}(\frac{n}{k})^\varepsilon \\ &\leq 2c(\varepsilon)\|S^{-1}\|\|S\|\|T_0\|k^{-\beta}(\frac{n}{k})^\varepsilon \\ &\leq 4c(\varepsilon)\|T\|k^{-\beta}(\frac{n}{k})^\varepsilon \quad \text{for} \quad 1 \leq k \leq n, \end{aligned}$$

where $I_{X_0}^X : X_0 \hookrightarrow X$ stands for the canonical embedding. The equivalence of (i) and (iii) follows from Theorem 1.4 if we choose for $\varepsilon > 0$ the parameter $\alpha = \beta + \varepsilon$. □

Now we are able to formulate the corresponding theorem to Theorem 2.2.

Theorem 3.3 *Let* M *be a precompact metric space such that there are* $\rho, \alpha > 0$ *and for all* $n \in I\!N$,

$$\varepsilon_n(M) \leq \rho n^{-\alpha}.$$

Moreover, let X *be a Banach space with the property that there is a constant* $\beta > 0$ *such that for each* $\varepsilon > 0$ *there is a constant* $c(\varepsilon) \geq 1$ *and for all* $n \in I\!N$ *and all operators* $S : \ell_1^n \to X$,

$$e_k(S) \leq c(\varepsilon)\|S\|\, k^{-\beta}(\frac{n}{k})^\varepsilon \quad \textit{for} \quad 1 \leq k \leq n \;. \tag{12}$$

Then for $u \in \mathrm{Lip}(\ell_1(M), X)$ *we have the estimate*

$$e_n(u) \leq cn^{-\beta-\alpha}\,\mathrm{Lip}(u) \quad \textit{for} \quad n = 1, 2, ..., \tag{13}$$

where c *is a constant depending on* ρ, α *and* β.

Proof. By Lemma 3.2 we get that the estimate (12) is equivalent to the estimate (7) of Theorem 2.2. Therefore, by Theorem 2.2 and the Remark at the end of section 2 we obtain for the dual operator $u^* \in \mathrm{Lip}(X^*, \ell_\infty(M))$ of u the estimate

$$e_n(u^*) \leq cn^{-\beta-\alpha}\,\mathrm{Lip}(u^*) = c\, n^{-\beta-\alpha}\,\mathrm{Lip}(u)$$

for $n = 1, 2, \ldots$. Finally, using the duality inequality in Theorem 1.4 with the exponent $\beta + \alpha$ we check the desired estimate for the entropy numbers of u,

$$e_n(u) \leq cn^{-\beta-\alpha}\,\mathrm{Lip}(u), \quad n = 1, 2, \ldots\ ,$$

where c is a constant which may depend on ρ, α, β and the K-convexity constant of X. However, an alternative but direct proof of this inequality can be given along a similar line as in the proof of Theorem 5.10.1 in [CS] by taking Lemma 5.10.2 of [CS] in a dual version. The advantage of this proof is that it yields a constant c not depending on the K-convexity constant of the Banach space X. □

Remark. If X is a Banach space of entropy type β with $0 < \beta < \frac{1}{2}$, then by section 1, X guarantees the entropy estimate (12) of Theorem 3.3.

As an application of Theorem 3.3 we get a corresponding statement to Proposition 2.3.

Proposition 3.4 *Let M be a a precompact metric space such that there are $\rho, \alpha > 0$ and for all integer $n \geq 1$,*

$$\varepsilon_n(M) \leq \rho n^{-\alpha}.$$

If X is a Banach space of type p, $1 < p \leq 2$, then for $u \in \mathrm{Lip}(\ell_1(M), X)$ we have the entropy estimate,

$$(14) \qquad e_n(u) \leq cn^{-(1-\frac{1}{p})-\alpha}\,\mathrm{Lip}(u), \quad n = 1, 2, \ldots,$$

where c is a constant which may depend on ρ, α, p and $\tau_p(X)$. In particular, for $X = L_q$, $1 < q < \infty$, we have

$$(15) \qquad e_n(u) \leq cn^{-\min\{1-\frac{1}{q};\frac{1}{2}\}-\alpha}\,\mathrm{Lip}(u), \quad n = 1, 2, \ldots.$$

Proof. The assertion follows from Theorem 1.7 and Theorem 3.3. □

Remark. Proposition 3.4 remains true if we replace type p by weak type p in the case $1 < p < 2$ (cf. section 1).

4 Entropy of convex hulls in Banach spaces

This section is devoted to entropy estimates of dyadic entropy numbers $e_n(\mathrm{co}(A))$ of the absolute convex hull of a precompact subset $A \subset X$ of a Banach space X by the entropy numbers $\varepsilon_n(A)$ of A. We can give the following general inequality.

Proposition 4.1 *For every $\varepsilon > 0$ there exists a constant $b(\varepsilon) \geq 1$ such that for all Banach spaces X, all precompact subsets $A \subset X$ and all integer $n \geq 1$ the following inequality is true,*

$$e_{[(1+\varepsilon)n]}(\mathrm{co}(A)) \leq b(\varepsilon) \sup_{1\leq k\leq n} 2^{-\frac{n}{k}} \left(\prod_{i=0}^{k-1} \varepsilon_i(A) \right)^{1/k},$$

where $\varepsilon_0(A) := \sup_{t\in A} \|t\|$.

Proof. Define an operator $u : \ell_1(A) \to X$ by

$$u(e_t) = t \quad \text{for} \quad t \in A.$$

Then

$$u^*(a)(t) = \langle t, a\rangle \quad \text{for} \ \ t \in A \quad \text{and} \quad a \in X^*.$$

Because of

$$|u^*(a)(t) - u^*(a)(s)| = |\langle t - s, a\rangle| \leq \|t - s\| \ \|a\|$$

we get

$$\omega(u^*;\delta) = \sup_{\|a\|\leq 1} \sup\{|u^*(a)(t) - u^*(a)(s)| : s,t \in A, \|s - t\| \leq \delta\} \leq \delta$$

Consequently,

$$\omega(u^*;\varepsilon_n(A)) \leq \varepsilon_n(A), \ n = 1, 2, \ldots,$$

and the desired inequality follows from Theorem 3.1 with

$$\omega(u^*;\varepsilon_o(A)) = \|u\| = \sup_{t\in A} \|t\|.$$

□

Proposition 4.2 . *Let (b_n) be a positive increasing sequence with the property that there exists a constant $\gamma \geq 1$ and for all $n \in I\!N$, $b_{2n} \leq \gamma b_n$. Then there exists a constant $c(\gamma)$ such that for all Banach spaces X, all precompact subsets $A \subset X$ and all integer $n \geq 1$ the inequality*

$$\sup_{1\leq k\leq n} b_k e_k(\mathrm{co}(A)) \leq c(\gamma)(b_1 \sup_{t\in A} \|t\| + \gamma \sup_{1\leq k\leq n} b_k \varepsilon_k(A))$$

holds.

Proof. For the operator $u : \ell_1(A) \to X$ defined as in the proof of the previous proposition, $u(e_t) = t$, we have already

$$t_{k+1}(u) \leq \omega(u^*; \varepsilon_k(A)) \leq \varepsilon_k(A).$$

combining this estimate with the inequality of Theorem 1.3 and using $b_{n+1} \leq b_{2n} \leq \gamma b_n$ we get

$$\begin{aligned}
\sup_{1\leq k\leq n} b_k e_k(u) &\leq c(\gamma) \sup_{1\leq k\leq n} b_k t_k(u) \leq c(\gamma) \sup_{1\leq k\leq n+1} b_k t_k(u) \leq \\
&\leq c(\gamma)(b_1\|u\| + \sup_{1\leq k\leq n} b_{k+1} t_{k+1}(u)) \leq \\
&\leq c(\gamma)(b_1 \sup_{t\in A}\|t\| + \sup_{1\leq k\leq n} b_{k+1}\varepsilon_k(A)) \leq \\
&\leq c(\gamma)(b_1 \sup_{t\in A}\|t\| + \gamma \sup_{1\leq k\leq n} b_k\varepsilon_k(A)).
\end{aligned}$$

□

As a consequence of the previous proposition we get the following result.

Proposition 4.3 *Let (s_n) be a positive decreasing sequence with the property that there exists a constant $\gamma \geq 1$ and for all $n \in I\!N$, $s_n \leq \gamma s_{2n}$. Then there exists a constant $\rho(\gamma)$ such that for all Banach spaces X and all precompact subsets $A \subset X$ satisfying the estimate*

$$\varepsilon_n(A) \leq s_n \quad \textit{for all} \quad n \in I\!N,$$

the inequality

$$e_n(\mathrm{co}(A)) \leq \rho(\gamma)\left(\frac{1}{s_1}\sup_{t\in A}\|t\| + \frac{1}{\gamma}\right) s_n(A)\ , \qquad n = 1, 2, \ldots,$$

is true.

Proof. The sequence $b_n := \frac{1}{s_n}$ guarantees the property $b_{2n} \leq \frac{1}{\gamma} b_n$ of Proposition 4.2. Hence, we infer by this proposition that

$$\frac{1}{s_n} e_n(\mathrm{co}(A)) \leq c(\frac{1}{\gamma})(\frac{1}{s_1}\sup_{t\in A}\|t\| + \frac{1}{\gamma})$$

yielding the desired inequality. □

Remark. The estimate in Proposition 4.3 is asymptotically optimal. Indeed, for the set

$$A = \{s_n e_n : n \in I\!N, n \geq 1\} \subset \ell_1,$$

where (s_n) is the sequence of Proposition 4.3 and (e_n) the canonical basis of ℓ_1, we have $\varepsilon_n(A) \leq s_n$. By a result of Gordon-König-Schütt [GKS] we immediately check

$$\frac{1}{2} s_n \leq e_n(\mathrm{co}(A)) \leq c s_n, \quad n = 1, 2, \ldots\ ,$$

for some universal constant c.

As an application of the previous proposition we get, for example, the following result.

Corollary 4.4 *Let $A \subset X$ be a precompact subset of a Banach space X with the property that for some $\alpha \geq 0,\ \beta \geq 0$,*

$$\varepsilon_n(A) \leq n^{-\alpha}(\log(n+1))^{-\beta} \quad \textit{for} \quad n = 1, 2, \dots \ .$$

Then there exists a positive constant $c_{\alpha,\beta}$ depending on α and β such that

$$e_n(\mathrm{co}(A)) \leq c_{\alpha,\beta} n^{-\alpha}(\log(n+1))^{-\beta} \textit{for} \quad n = 1, 2, \dots \ .$$

In the case of rapidely decreasing sequences $\varepsilon_n(A)$ we can check the following

Corollary 4.5 *Let $A \subset X$ be a precompact subset of a Banach space X with the property that for some $\gamma, \sigma > 0$,*

$$\varepsilon_n(A) \leq 2^{-\gamma(n+1)^\sigma} \quad \textit{for} \quad n = 1, 2, \dots \ .$$

Then for each $\varepsilon > 0$ there exists a constant $b(\varepsilon)$ such that for all integer $n \geq 1$,

$$e_{[(1+\varepsilon)n]}(\mathrm{co}(A)) \leq b(\varepsilon) 2^\gamma (1 + \sup_{t \in A} \|t\|) 2^{-[(1+\frac{1}{\sigma})(\frac{\gamma\sigma}{\sigma+1})^{\frac{1}{\sigma+1}} n^{\frac{\sigma}{\sigma+1}}]} .$$

Proof. By Proposition 4.1 we obtain

$$\begin{aligned}
e_{[(1+\varepsilon)n]}(\mathrm{co}(A)) &\leq b(\varepsilon) \sup_{1 \leq k \leq n} 2^{-\frac{n}{k}} \left(\prod_{i=0}^{k-1} \varepsilon_i(A) \right)^{1/k} \leq \\
&\leq b(\varepsilon)(1 + \sup_{t \in A} \|t\|) \sup_{1 \leq k \leq n} 2^{-\frac{n}{k}} \left(\prod_{i=1}^{k-1} \varepsilon_i(A) \right)^{1/k} \leq \\
&\leq b(\varepsilon)(1 + \sup_{t \in A} \|t\|) 2^\gamma \sup_{1 \leq k \leq n} 2^{-\frac{n}{k} - \frac{\gamma}{k} \sum_{i=1}^{k} i^\sigma} \leq \\
&\leq b(\varepsilon)(1 + \sup_{t \in A} \|t\|) 2^\gamma 2^{-\inf_{1 \leq k \leq n} \left[\frac{n}{k} + \frac{\gamma}{k} \sum_{i=1}^{k} i^\sigma \right]} .
\end{aligned}$$

Because of

$$\begin{aligned}
\inf_{1 \leq k \leq n} \left[\frac{n}{k} + \frac{\gamma}{k} \sum_{i=1}^{k} i^\sigma \right] &\geq \inf_{1 \leq k \leq n} \left[\frac{n}{k} + \frac{\gamma}{\sigma+1} k^\sigma \right] \geq \\
&\geq (1 + \frac{1}{\sigma}) \left(\frac{\gamma\sigma}{\sigma+1} \right)^{\frac{1}{\sigma+1}} n^{\frac{\sigma}{\sigma+1}}
\end{aligned}$$

we get the desired estimate. □

Remark. The estimate in Corollary 4.5 is asymptotically optimal. Indeed, again by Gordon-König-Schütt [GKS] we get for the precompact set

$$A = \{2^{-\gamma(n+1)^\sigma} e_n : n \in I\!N\} \subset \ell_1$$

of ℓ_1 that

$$\varepsilon_n(A) \leq 2^{-\gamma(n+1)^\sigma}$$

and

$$c_o 2^{-(1+\frac{1}{\sigma})(\frac{\gamma\sigma}{\sigma+1})^{\frac{1}{\sigma+1}} n^{\frac{\sigma}{\sigma+1}}} \leq e_n(\mathrm{co}(A)) \leq c_1 2^{-(1+\frac{1}{\sigma})(\frac{\gamma\sigma}{\sigma+1})^{\frac{1}{\sigma+1}} n^{\frac{\sigma}{\sigma+1}}},$$

where $c_o, c_1 > 0$ are positive constants not depending on n.

5 Entropy of convex hulls in Hilbert spaces

In this section we first consider the entropy behaviour of the convex hull $\mathrm{co}(A)$, as well as its Gelfand numbers, in the case where the covering number of A is of power type. If A is a precompact subset of a Hilbert space H, we define the operator $u : \ell_1(A) \to H$ by $u(e_t) = t$ for $t \in A$ and we set $c_n(\mathrm{co}(A)) = c_n(u)$. The following proposition is an improvement over results from [D], [BP] and [T].

Proposition 5.1 *Let $A \subset H$ be a precompact subset of the unit ball of a Hilbert space with the property that there are constants $\rho, \alpha > 0$ such that*

$$\varepsilon_n(A) \leq \rho n^{-\alpha} \quad \textit{for} \quad n = 1, 2, \ldots \quad .$$

Then there exists a positive constant $c_{\rho,\alpha}$ such that we have the estimate

$$\max\{c_n(\mathrm{co}(A)); e_n(\mathrm{co}(A))\} \leq c_{\rho,\alpha} n^{-\frac{1}{2}-\alpha} \quad \textit{for} \quad n = 1, 2, \ldots \quad .$$

Consequently, if the covering number is of power type

$$N(A; \varepsilon) \leq c_0 \varepsilon^{-\frac{1}{\alpha}} \quad \textit{for} \quad \varepsilon \downarrow 0,$$

then

$$\ln N(\mathrm{co}(A); \varepsilon) \leq c_1 \varepsilon^{-\frac{2}{1+2\alpha}} .$$

Proof. For the operator $u : \ell_1(A) \to H$ defined by $u(e_t) = t$ for $t \in A$ we have as in the proof of Proposition 4.1 that

$$\omega(u^*; \delta) \leq \delta$$

and therefore

$$\mathrm{Lip}(u) = \max\{\|u\|; \sup_{\delta > 0} \frac{\omega(u^*; \delta)}{\delta}\} \leq \max\{\sup_{t \in A} \|t\|; 1\}.$$

Since a Hilbert space has type 2 we infer from Proposition 3.4 the desired estimate

$$e_n(\mathrm{co}(A)) = e_n(u) \leq c n^{-\frac{1}{2}-\alpha} \, \mathrm{Lip}(u) \leq c \max\{1; \sup_{t \in A} \|t\|\} \, n^{-\frac{1}{2}-\alpha}.$$

Now from a result in [PTJ], for $\alpha > 0$,

$$\sup_{n \geq 1} n^{\alpha+\frac{1}{2}} c_n(u) \leq c_\alpha \sup_{n \geq 1} n^{\alpha+\frac{1}{2}} e_n(u)$$

where c_α depends only on α. This gives the estimate for Gelfand numbers.

The above result is asymptotically optimal. Indeed, for the precompact subset

$$A = \{n^{-\alpha} e_n : n \in I\!N\} \subset \ell_2,$$

where (e_n) is the canonical basis of ℓ_2, we have

$$\varepsilon_n(A) \le n^{-\alpha}$$

and

$$c_o n^{-\frac{1}{2}-\alpha} \le e_n(\text{co}(A)) \le c_1 n^{-\frac{1}{2}-\alpha}, \quad n = 1, 2, \ldots,$$

with positive constants $c_0, c_1 > 0$ (cf. [BP] or [C3]). □

Now we turn to the case where the decay of $\varepsilon_n(A)$ is of logarithmic type; $(\ln(n+1))^{-\alpha}$, or the covering number is of exponential type. For this purpose we give two general results.

Proposition 5.2 *Let A be a precompact subset of the unit ball of a Hilbert space H, then there exists an absolute constant $C > 0$ such that for all $k \ge 1$, we have*

$$\sqrt{k}\, c_k(\text{co}(A)) \le C \inf_{\varepsilon > 0} \Big(\int_{\frac{\varepsilon}{4}}^{1} \sqrt{\ln N(A;\delta)}\, d\delta + \sqrt{k}\, \varepsilon \Big).$$

Remark. By the same method we can prove the next inequality concerning the covering numbers:

$$\varepsilon \sqrt{\ln N(\text{co(A)};\varepsilon)} \le C \int_{\frac{\varepsilon}{4}}^{1} \sqrt{\ln N(A;\delta)}\, d\delta.$$

Proof of Proposition 5.2.

Let u be the operator defined in the beginning of this section. For any integer $\nu \ge 0$, let A_ν be a $2^{-\nu}$-net of A with $\text{card}(A_\nu) = N(A; 2^{-\nu})$. For every $t \in A$, let $\zeta_\nu(t)$ be an element of A_ν such that $\|\zeta_\nu(t) - t\| \le 2^{-\nu}$.

Let $N \in I\!N$, we define an operator $u_N : \ell_1(A) \to H$ by $u_N(e_t) = \zeta_N(t)$, $t \in A$. Then $u = u_N + u - u_N$ and

$$\sqrt{k}\, c_k(u) \le \sqrt{k}\, c_k(u_N) + \sqrt{k}\, c_1(u - u_N) \le \sqrt{k}\, c_k(u_N) + \sqrt{k}\, 2^{-N}.$$

Theorem 1.5 gives

$$\sqrt{k}\, c_k(u_N) \le C\ell(u_N^*) = C I\!E \sup_{t \in A_N} |Z_t|,$$

where $(Z_t)_{t \in A}$ is a centered Gaussian process with $\|Z_s - Z_t\|_{L_2} = \|s - t\|$.

Then $Z_t = Z_{\zeta_0(t)} + \sum_{\nu=1}^{N} Z_{\zeta_\nu(t)} - Z_{\zeta_{\nu-1}(t)}$. Since $Z_{\zeta_0(t)}$ does not depend on t (the diameter of A is ≤ 1) we have

$$\mathbb{E} \sup_{t\in A_N} |Z_t| \leq \sum_{\nu=1}^{N} \mathbb{E} \sup_{t\in A_N} |Z_{\zeta_\nu(t)} - Z_{\zeta_{\nu-1}(t)}|.$$

From the symmetry of the random variables and Theorem 1.6, we get

$$\begin{aligned}\mathbb{E} \sup_{t\in A} |Z_{\zeta_\nu(t)} - Z_{\zeta_{\nu-1}(t)}| &\leq C'[\ln(\mathrm{card}(A_\nu)\mathrm{card}(A_{\nu-1}))]^{\frac{1}{2}} (2^{-\nu} + 2^{-(\nu-1)}) \\ &\leq 3C'[2\ln N(A; 2^{-\nu})]^{\frac{1}{2}} 2^{-\nu}\end{aligned}$$

where $C' > 0$ is an absolute constant. This implies that

$$\mathbb{E} \sup_{t\in A_N} |Z_t| \leq 3\sqrt{2}C' \sum_{\nu=1}^{N} \sqrt{\ln N(A; 2^{-\nu})}\; 2^{-\nu}.$$

Let now $0 \leq \varepsilon \leq 1$ and define $N \in \mathbb{N}$ by the relation $2^{-N} \leq \varepsilon < 2^{-N+1}$, then we have

$$\mathbb{E} \sup_{t\in A_N} |Z_t| \leq 6\sqrt{2}C' \int_{\frac{\varepsilon}{4}}^{1} \sqrt{\ln N(A;\delta)}\, d\delta\,.$$

The last inequality gives that there exists $C > 0$ such that for all $\varepsilon > 0$ and all integer $k \geq 1$ we have

$$\sqrt{k}\, c_k(\mathrm{co}(A)) \leq C \int_{\frac{\varepsilon}{4}}^{1} \sqrt{\ln N(A;\delta)}\, d\delta + \sqrt{k}\varepsilon.$$

□

Proposition 5.3 *Let A be a precompact subset of a Hilbert space H, then there exists an absolute constant $C > 0$ such that for all integers $k, \ell \geq 1$, we have*

$$\sqrt{\ell}\, c_{k+\ell}(\mathrm{co}(A)) \leq C \int_{0}^{\varepsilon_k(A)} \sqrt{\ln N(A;\varepsilon)}\, d\varepsilon\,.$$

Proof. Let us write for simplicity ε_n instead of $\varepsilon_n(A)$. We shall prove the proposition for a finite set A. We fix $k \in \mathbb{N}$ and consider $A_k \subset A$ an ε_k-net of A such that $\mathrm{card}(A_k) \leq k$. Let $t \in A$ and denote by $\zeta(t)$ an element of A_k approximating t in the sense that $\|\zeta(t) - t\| \leq \varepsilon_k$. Define the operator $v : \ell_1(A) \to H$ by $v(e_t) = \zeta(t)$, $t \in A$. Since $\mathrm{rank}(v) \leq \mathrm{card}(A_k) \leq k$ we have

$$c_{k+\ell}(u) = c_{k+\ell}(u - v + v) \leq c_{k+1}(v) + c_\ell(u - v) = c_\ell(u - v).$$

Theorem 1.5 gives

$$\sqrt{\ell}\, c_{k+\ell}(u) \le C\ell((u-v)^*) = \mathbb{E} \sup_{t\in A} |Z_t - Z_{\zeta(t)}|,$$

where $(Z_t)_{t\in A}$ is a centered Gaussian process with $\|Z_s - Z_t\|_{L_2} = \|s-t\|$. Now we follow the chaining method from Dudley's inequality. For any $\nu \ge 1$, consider an $\varepsilon_k 2^{-\nu}$-net of A with $\mathrm{card}(A_\nu) = N(A; \varepsilon_k 2^{-\nu})$. For every $t \in A$, let $\zeta_\nu(t)$ be an element of A_ν such that $\|\zeta_\nu(t) - t\| \le \varepsilon_k 2^{-\nu}$. Put $\zeta_0 = \zeta$, then we have

$$|Z_t - Z_{\zeta(t)}| \le \sum_{\nu=1}^{\infty} |Z_{\zeta_\nu(t)} - Z_{\zeta_{\nu-1}(t)}|.$$

Therefore

$$\mathbb{E} \sup_{t\in A} |Z_t - Z_{\zeta(t)}| \le \sum_{\nu=1}^{\infty} \mathbb{E} \sup_{t\in A} |Z_{\zeta_\nu(t)} - Z_{\zeta_{\nu-1}(t)}|.$$

Using the symmetry of the random variables and Theorem 1.6, we obtain

$$\begin{aligned}\mathbb{E} \sup_{t\in A} |Z_{\zeta_\nu(t)} - Z_{\zeta_{\nu-1}(t)}| &\le C[\ln(\mathrm{card}(A_\nu)\mathrm{card}(A_{\nu-1}))]^{\frac{1}{2}} (\varepsilon_k 2^{-\nu} + \varepsilon_k 2^{-(\nu-1)}) \\ &\le 3C[2\ln N(A; \varepsilon_k 2^{-\nu})]^{\frac{1}{2}}\, \varepsilon_k 2^{-\nu}.\end{aligned}$$

This implies that

$$\mathbb{E} \sup_{t\in A} |Z_t - Z_{\zeta(t)}| \le \sum_{\nu=1}^{\infty} 3C\sqrt{2\ln N(A; \varepsilon_k 2^{-\nu})}\, \varepsilon_k 2^{-\nu} \le 6C \int_0^{\varepsilon_k} \sqrt{2\ln N(A;\varepsilon)}\, d\varepsilon,$$

which finally gives that for all integer $\ell \ge 1$,

$$\sqrt{\ell}\, c_{k+\ell}(u) \le 6\sqrt{2}C \int_0^{\varepsilon_k} \sqrt{(\ln N(A;\varepsilon))}\, d\varepsilon.$$

This accomplishes the proof. □

Let us give an application of the previous propositions.

Proposition 5.4 *Let $A \subset H$ be a precompact subset of the unit ball of a Hilbert space H such that for some constants $\rho, \alpha > 0$, we have*

$$\varepsilon_n(A) \le \rho(\ln(n+1))^{-\alpha} \quad \textit{for} \quad n = 1, 2, \ldots \ .$$

Then there exists a positive constant $c_{\rho,\alpha}$, such that we have the estimates

$$\max\{c_n(\mathrm{co}(A)); e_n(\mathrm{co}(A))\} \le c_{\rho,\alpha} n^{-\frac{1}{2}} (\ln(n+1))^{\frac{1}{2}-\alpha}$$

for $\alpha > \frac{1}{2}$ and $n = 1, 2, \ldots$, and

$$\max\{c_n(\mathrm{co}(A)); e_n(\mathrm{co}(A))\} \le c_{\rho,\alpha} n^{-\alpha}$$

for $0 < \alpha < \frac{1}{2}$ and $n = 1, 2, \ldots$.
These estimates are asymptotically optimal.

For the proof of Proposition 5.4 we will use the following lemma which is actually a consequence of Theorem 1.2 or 1.3 (cf. [CP]).

Lemma 5.5 *Let $\beta, \gamma > 0$, there exists $d_{\beta,\gamma}$ such that for any compact operator u between two Banach spaces, we have*

(i) $\frac{n^\beta}{(\ln(n+1))^\gamma} e_n(u) \le d_{\beta,\gamma} \sup\limits_{1\le k\le n} \frac{k^\beta}{(\ln(k+1))^\gamma} c_k(u), \quad n = 1, 2, \ldots.$

(ii) $n^\beta (\ln(n+1))^\gamma e_n(u) \le d_{\beta,\gamma} \sup\limits_{1\le k\le n} k^\beta (\ln(k+1))^\gamma c_k(u), \quad n = 1, 2, \ldots .$

Proof of Proposition 5.4.
First we treat the case $\alpha > \frac{1}{2}$. The relation $\varepsilon_n(A) \le \rho(\ln(n+1))^{-\alpha}$ implies that $N(A;\varepsilon) \le \exp(\frac{\rho}{\varepsilon})^{\frac{1}{\alpha}}$. Using Proposition 5.3 with $\ell = k$, we obtain

$$\begin{aligned} k^{\frac{1}{2}} c_{2k}(\mathrm{co}(A)) &\le C \int_0^{\varepsilon_k(A)} (\ln N(A;\varepsilon))^{1/2}\, d\varepsilon \le C\rho \int_0^{(\ln(k+1))^{-\alpha}} \varepsilon^{-\frac{1}{2\alpha}}\, d\varepsilon \\ &\le C\rho \frac{2\alpha}{2\alpha - 1} (\ln(k+1))^{\frac{1}{2}-\alpha}. \end{aligned}$$

Therefore

$$c_{2k}(\mathrm{co}(A)) \le C\rho \frac{2\alpha}{2\alpha-1} k^{-\frac{1}{2}} (\ln(k+1))^{\frac{1}{2}-\alpha}.$$

To get the estimate for entropy numbers, observe that A is contained in the unit ball of H. Then Lemma 5.5 (ii) applied to the operator u associated to A, as above, implies that

$$e_n(\mathrm{co}(A)) = e_n(u) \le C\rho \frac{2\alpha}{2\alpha-1} d_{\frac{1}{2},\alpha-\frac{1}{2}}\, n^{-\frac{1}{2}} (\ln(n+1))^{\frac{1}{2}-\alpha}$$

for $n = 1, 2, \ldots$.
The results are asymptotically optimal. Indeed, let

$$A = \{(\ln(n+1))^{-\alpha} e_n \,:\, n \in \mathbb{N}\} \subset \ell_2,$$

where (e_n) denotes the canonical basis of ℓ_2. Then

$$\varepsilon_n(A) \le (\ln(n+1))^{-\alpha}$$

and for $\Delta_n = \mathrm{co}(\{(\ln(k+1))^{-\alpha} e_k : n \le k \le n^2\})$ we have

$$\begin{aligned} c_n(\mathrm{co}(A)) \ge c_n(\Delta_n) &\ge (\ln(n^2+1))^{-\alpha} c_n(id : \ell_1^{n^2-n} \to \ell_2^{n^2-n}) \\ &\ge 2^{-\alpha} (\ln(n+1))^{-\alpha} c\, n^{-\frac{1}{2}} (\ln \frac{n^2-n}{n})^{1/2} \\ &\ge c\, 2^{-\alpha-\frac{1}{2}} n^{-\frac{1}{2}} (\ln(n+1))^{\frac{1}{2}-\alpha} \text{ for } n \ge 3, \end{aligned}$$

since by a result of Garnaev and Gluskin [GG], we have

$$c_n(id : \ell_1^m \to \ell_2^m) \geq c\, n^{-\frac{1}{2}}(\ln(\frac{m}{n}+1))^{\frac{1}{2}}$$

for $\ln(m+1) \leq n \leq m$ and where $c > 0$ is a numerical constant. The same reasonning shows that the estimate is also optimal for the entropy numbers. In this case, one use a result of Schütt [S] for the entropy numbers $e_n(id : \ell_1^{n^2-n} \to \ell_2^{n^2-n})$.

Now we turn to the case $0 < \alpha < \frac{1}{2}$. We use this time Proposition 5.2 and we get for all $0 < \varepsilon < 1$ and $k \geq 1$,

$$\sqrt{k} c_k(\mathrm{co}(A)) \leq C \int_{\frac{\varepsilon}{4}}^{1} (\ln \exp{(\frac{1}{\delta})^{\frac{1}{\alpha}}})^{\frac{1}{2}}\, d\delta + \sqrt{k}\varepsilon \leq C_\alpha \varepsilon^{1-\frac{1}{2\alpha}} + \sqrt{k}\varepsilon.$$

By choosing $\varepsilon = \frac{1}{k^\alpha}$, we get the desired estimate for $k \geq 1$

$$c_k(\mathrm{co}(A)) \leq c_\alpha k^{-\alpha}$$

where c_α is a positive constant depending only on α.

The estimate for entropy numbers now follows using Theorem 1.2.

The results are asymptotically optimal. Indeed, take again

$$A = \{(\ln(n+1))^{-\alpha} e_n : n \in I\!N\} \subset \ell_2.$$

Then if $\Delta_n = \mathrm{co}(\{(\ln(k+1))^{-\alpha} e_k : n \leq k \leq 2^n\})$ we get as before :

$$\begin{aligned} c_n(\mathrm{co}(A)) \geq c_n(\Delta_n) &\geq (\ln(2^n+1))^{-\alpha} c_n(id : \ell_1^{2^n-n} \to \ell_2^{2^n-n}) \\ &\geq 2^{-\alpha} n^{-\alpha} c\, n^{-\frac{1}{2}} (\ln(\frac{2^n-n}{n}))^{1/2} \\ &\geq 2^{-\alpha-\frac{1}{2}} c\, n^{-\alpha} \quad \text{for} \quad n \geq 3 \end{aligned}$$

where c is the constant above. Again, as before, the result is optimal for the entropy numbers, using similar estimates for entropy. □

Remarks.

1) What happens in the remaining case $\alpha = \frac{1}{2}$? Is the estimate of Proposition 5.4 still valid for $\alpha = \frac{1}{2}$?

2) We say that A satisfies the entropy condition of Dudley if

$$\int_0^{+\infty} (\ln N(A;\varepsilon))^{\frac{1}{2}}\, d\varepsilon < \infty.$$

There exists a subset A of a Hilbert space such that A satisfies this condition but $\mathrm{co}(A)$ does not. Indeed, take

$$A = \{(\ln(n+1))^{-\alpha} e_n : n \in I\!N\} \subset \ell_2 \quad \text{for} \quad \frac{1}{2} < \alpha < \frac{3}{2}.$$

The first part of Proposition 5.4 implies that A satisfies the entropy condition of Dudley but its convex hull does not.

6 Entropy of convex hulls in Banach spaces of type p

The first result is again devoted to the entropy behaviour of the convex hull $\mathrm{co}(A)$ in case the covering number of A is of power type. The corresponding result to Proposition 5.1 for type p spaces is the following proposition.

Proposition 6.1 *Let $A \subset X$ be a precompact subset of the unit ball of a Banach space of type p, $p > 1$, with the property that there are constants $\rho, \alpha > 0$ such that*

$$\varepsilon_n(A) \leq \rho n^{-\alpha} \quad for \quad n = 1, 2, \dots \;.$$

Then there exists a positive constant $c_{\rho,\alpha,p}$ such that for the dyadic entropy numbers of the convex hull we have the asymptotically optimal estimate

$$e_n(\mathrm{co}(A)) \leq c_{\rho,\alpha,p} n^{-(1-\frac{1}{p})-\alpha} \quad for \quad n = 1, 2, \dots \;.$$

This is equivalent to say that if the covering number is of power type

$$N(A;\varepsilon) \leq c_0\, \varepsilon^{-\frac{1}{\alpha}} \quad for \quad \varepsilon \downarrow 0,$$

then

$$\ln N(A;\varepsilon) \leq c_1\, \varepsilon^{-\frac{1}{1-\frac{1}{p}+\alpha}}\,.$$

Proof. Similarly as in the proof to Proposition 5.1 we get by Proposition 3.4 the desired estimate. The result is asymptotically optimal. Indeed, for

$$A = \{n^{-\alpha} e_n : n \in I\!N\} \subset \ell_p, \quad 1 < p \leq 2,$$

we have $\varepsilon_n(A) \leq n^{-\alpha}$ and

$$c_0 n^{-(1-\frac{1}{p})-\alpha} \leq e_n(\mathrm{co}(A)) \leq c_1 n^{-(1-\frac{1}{p})-\alpha}$$

for $n = 1, 2, \dots$ with positive constants $c_o, c_1 > 0$ (cf. [C3]). □

An alternative but direct proof of the above result can be given using a method similar to that in [BP].

Proposition 6.2 *Let A be a precompact subset of the unit ball of a Banach space X of type p, $1 < p \leq 2$, and let us assume that*

$$\varepsilon_n(A) \leq (\ln(n+1))^{-\alpha} \quad for \quad n = 1, 2, \dots,$$

for some $\alpha > 1 - \frac{1}{p}$. Then there exists a constant $c(\alpha, p)$, depending only on α and on p, such that for every $n \geq 2$,

$$e_n(\mathrm{co}(A)) \leq c(\alpha,p) n^{-1+\frac{1}{p}} (\ln n)^{-\alpha+1-\frac{1}{p}}.$$

To prove this proposition, we first study a special case.

Lemma 6.3 *For every integer $N \geq 1$ and every finite subset A of the unit ball of a Banach space X of type p, $1 < p \leq 2$, with* $\mathrm{card}(A) \leq N$ *and satisfying*

$$\varepsilon_n(A) \leq (\ln(n+1))^{-\alpha} \quad \textit{for} \quad n = 1, 2, \ldots,$$

for some $\alpha > 1 - \frac{1}{p}$, there exists a constant $C_{\alpha,p}$ depending only on α and p, such that

$$e_N(\mathrm{co}(A)) \leq C_{\alpha,p} N^{-1+\frac{1}{p}} (\ln N)^{-\alpha}.$$

Proof. We proceed by induction on N. Let $K > 1$ be a universal constant which we shall define later and let $C_1 = K^{1-\frac{1}{p}} (\ln K)^{\alpha}$. Since A is in the unit ball of X, we have

$$e_N(\mathrm{co}(A)) \leq C_1 N^{-1+\frac{1}{p}} (\ln N)^{-\alpha}$$

whenever $\mathrm{card}(A) \leq N \leq K$.

Now let us assume that $N > K$. Let C be a constant to be defined later and $m = [\frac{N}{C}] + 1$. There exists a $2\varepsilon_m(A)$-net Γ_1 of A, such that $\Gamma_1 \subset A$ and $\mathrm{card}(\Gamma_1) \leq m$. For any $t \in A$ let us choose $\zeta(t)$ an element of Γ_1 such that $\|\zeta(t) - t\| \leq 2\varepsilon_m(A)$ and set

$$\Gamma_2 = \{t - \zeta(t) : t \in A\}.$$

If $C \leq \sqrt{K} < \sqrt{N}$, then

$$\begin{aligned} \sup_{t \in \Gamma_2} \|t\| &\leq 2(\ln m)^{-\alpha} \leq 2(\ln([\frac{N}{C}] + 1))^{-\alpha} \\ &\leq 2(\ln(\frac{N}{C}))^{-\alpha} \leq (\ln\ N - \ln C)^{-\alpha} \leq 2^{\alpha+1} (\ln\ N)^{-\alpha}. \end{aligned}$$

Now Theorem 1.7, for $K \geq 6$ and $r = [\frac{N}{2}]$ gives

$$\begin{aligned} e_r(\mathrm{co}(\Gamma_2)) &\leq c_p \tau_p(X) \left(\frac{\ln(\frac{N}{r} + 1)}{r} \right)^{1-\frac{1}{p}} \sup_{t \in \Gamma_2} \|t\| \leq \\ &\leq c_p \tau_p(X) \left(\frac{\ln(\frac{N}{[\frac{N}{2}]} + 1)}{[\frac{N}{2}]} \right)^{1-\frac{1}{p}} \frac{2^{\alpha+1}}{(\ln N)^{\alpha}} \leq \\ &\leq c_p \tau_p(X) \left(\frac{\ln(\frac{2N}{\frac{N}{3}})}{\frac{N}{3}} \right)^{1-\frac{1}{p}} \frac{2^{\alpha+1}}{(\ln N)^{\alpha}} \leq \\ &\leq c_p \tau_p(X) 2^{\alpha+1} (3(\ln 6))^{1-\frac{1}{p}} N^{-1+\frac{1}{p}} (\ln N)^{-\alpha}, \end{aligned} \tag{16}$$

where c_p and $\tau_p(X)$ are the constants appearing in Theorem 1.7.

On the other hand, since $\mathrm{card}(\Gamma_1) \le m < N$ the induction hypothesis implies that

$$e_m(\mathrm{co}(\Gamma_1)) \le C_{\alpha,p} m^{-1+\frac{1}{p}} (\ln m)^{\alpha}.$$

For $C > 6$, we have $r = [\frac{N}{2}] > \frac{N}{2} - 1 \ge m$ and by a well known inequality concerning entropy numbers (see [BP] Lemma 3), we obtain

$$\begin{aligned}
e_r(\mathrm{co}(\Gamma_1)) &\le 8 \cdot 2^{-\frac{r}{m}} e_m(\mathrm{co}(\Gamma_1)) \le C_{\alpha,p}\, 8 \cdot 2^{-\frac{r}{m}} m^{-1+\frac{1}{p}} (\ln m)^{-\alpha} \\
&\le C_{\alpha,p}\, 8 \cdot 2^{-\frac{r}{m}} ([\frac{N}{C}] + 1)^{-1+\frac{1}{p}} (\ln([\frac{N}{C}] + 1))^{-\alpha} \\
&\le C_{\alpha,p}\, 8 \cdot 2^{-\frac{r}{m}} (\frac{N}{C})^{-1+\frac{1}{p}} (\ln(\frac{N}{C}))^{-\alpha} \\
&\le C_{\alpha,p}\, 8 \cdot 2^{-\frac{r}{m}} C^{1-\frac{1}{p}}\, 2^{\alpha} N^{-1+\frac{1}{p}} (\ln N)^{-\alpha}.
\end{aligned}$$

For $N > C$, we have

$$\frac{r}{m} = \frac{[\frac{N}{2}]}{[\frac{N}{C}] + 1} \ge \frac{\frac{N}{2} - 1}{\frac{N}{C} + 1} = C\left(\frac{\frac{1}{2} - \frac{1}{N}}{1 + \frac{C}{N}}\right) \ge C\frac{\frac{1}{2} - \frac{1}{6}}{2} = \frac{C}{6},$$

so that

(17) $$e_r(\mathrm{co}(\Gamma_1)) \le C_{\alpha,p}\, 8C^{1-\frac{1}{p}}\, 2^{-\frac{C}{6}}\, 2^{\alpha} N^{-1+\frac{1}{p}} (\ln N)^{-\alpha}.$$

So by summing (16) and (17) we obtain

$$\begin{aligned}
e_N(\mathrm{co}(A)) &\le e_r(\mathrm{co}(\Gamma_1)) + e_r(\mathrm{co}(\Gamma_2)) \\
&\le \left(8C_{\alpha,p}\, 2^{\alpha} \frac{C^{1-\frac{1}{p}}}{2^{\frac{C}{6}}} + c_p \tau_p(X) 2^{\alpha+1} (3\ln 6)^{1-\frac{1}{p}}\right) N^{-1+\frac{1}{p}} (\ln N)^{-\alpha}.
\end{aligned}$$

Now we are in position to define the constants. We choose C an integer big enough so that $C \ge 6$ and $8 \cdot 2^{\alpha} C^{1-\frac{1}{p}} 2^{-\frac{C}{6}} < \frac{1}{2}$ and we take $K = C^2$; hence (16) and (17) are satisfied.
Then the induction proof is running provided

$$C_{\alpha,p} = \max\{C_1; 2c_p \tau_p(X) 2^{\alpha+1} (3\ln 6)^{1-\frac{1}{p}}\}.$$

□

Proof of proposition 6.2: Let q be a positive integer to be defined later. Set $r = [\log_2 n - 1]$, for every integer i, $1 \le i \le r$ there exist a $2\varepsilon_{n^{2^{iq}}}(A)$-net A_i of A such that $A_i \subset A$ and $\mathrm{card}(A_i) \le n^{2^{iq}}$.
To each $t \in A$ let us associate $\zeta_i(t)$ an element of A_i such that

$$\|\zeta_i(t) - t\| \le 2\varepsilon_{n^{2^{iq}}}(A).$$

We set

$$B_0 = A_0, \quad B_i = \{\zeta_i(t) - \zeta_{i-1}(t) : t \in A\}, \ 1 \le i \le r,$$

and

$$C = \{t - \zeta_r(t) : t \in A\}.$$

Then we have

$$e_{2n}(\mathrm{co}(A)) \le e_{2n}(\sum_{i=0}^{r} \mathrm{co}(B_i)) + e_1(\mathrm{co}(C))\,.$$

Let us observe that

$$\begin{aligned} \sup_{x\in C} \|x\| &= \sup_{t\in A} \|t - \zeta_r(t)\| \le 2\varepsilon_{n^{2^{rq}}}(A) \\ &\le 2(\ln n^{2^{-2q+q\log_2 n}})^{-\alpha} = 2 \cdot 2^{2q\alpha}(n^q \ln n)^{-\alpha}. \end{aligned}$$

For $q > \frac{p-1}{p\alpha}$, this implies that

$$e_1(\mathrm{co}(C)) = \sup_{x\in C}\|x\| \le \frac{2\cdot 2^{2q\alpha}}{n^{q\alpha}(\ln n)^{\alpha}} \le \frac{2\cdot 2^{2q\alpha}}{n^{1-\frac{1}{p}}(\ln n)^{\alpha-\frac{1}{p}}}\,. \tag{18}$$

We also have

$$e_{2n}(\sum_{i=0}^{r} \mathrm{co}(B_i)) \le e_n(\mathrm{co}(B_0)) + e_n(\sum_{i=1}^{r} \mathrm{co}(B_i))\,.$$

Now by Lemma 6.3

$$e_n(\mathrm{co}(B_0)) \le C_{\alpha,p}\, n^{-1+\frac{1}{p}}(\ln n)^{-\alpha} \le C_{\alpha,p}\, n^{-1+\frac{1}{p}}(\ln n)^{-\alpha+1-\frac{1}{p}}\,. \tag{19}$$

Also by a well known inequality concerning entropy numbers, we have

$$e_n(\sum_{i=1}^{r}(\mathrm{co}(B_i)) \le \sum_{i=1}^{r} e_{[\frac{n}{2^i}]}(\mathrm{co}(B_i)).$$

Since by Theorem 1.7

$$\begin{aligned} e_{[\frac{n}{2^i}]}(\mathrm{co}(B_i)) &\le c_p\tau_p(X)\left(\frac{\ln(\frac{n^{2^{iq}}}{[\frac{n}{2^i}]}+1)}{[\frac{n}{2^i}]}\right)^{1-\frac{1}{p}} \cdot 4\varepsilon_{n^{2^{(i-1)q}}}(A) \\ &\le c_p\tau_p(X)\frac{(\ln(2n^{2^{iq}}))^{1-\frac{1}{p}}}{(\frac{n}{2^i}-1)^{1-\frac{1}{p}}} \cdot 4\varepsilon_{n^{2^{(i-1)q}}}(A) \\ &\le 4.2^{q\alpha}c_p\tau_p(X)\frac{(\ln(2n^{2^{iq}}))^{1-\frac{1}{p}}}{(\frac{n}{2^i}-1)^{1-\frac{1}{p}}}2^{-iq\alpha}(\ln n)^{-\alpha} \\ &\le 4.2^{q\alpha}c_p\tau_p(X)\frac{(\ln(n^{2^{iq+1}}))^{1-\frac{1}{p}}}{(\frac{n}{2^{i+1}})^{1-\frac{1}{p}}}2^{-iq\alpha}(\ln n)^{-\alpha} \\ &\le 4.2^{q\alpha}c_p\tau_p(X)2^{(i+1)(1-\frac{1}{p})}2^{(iq+1)(1-\frac{1}{p})}2^{-iq\alpha}n^{-1+\frac{1}{p}}(\ln n)^{1-\frac{1}{p}-\alpha} \\ &\le 2^{4+q\alpha}c_p\tau_p(X)2^{i(1-\frac{1}{p}+q(1-\frac{1}{p}-\alpha))}n^{-1+\frac{1}{p}}(\ln n)^{1-\frac{1}{p}-\alpha}\,, \end{aligned}$$

we have

$$e_n\left(\sum_{i=1}^{r}(\mathrm{co}(B_i)\right) \le \sum_{i=1}^{r} e_{[\frac{n}{2^i}]}(\mathrm{co}(B_i)) \le \frac{2^{4+q\alpha}c_p\tau_p(X)}{n^{1-\frac{1}{p}}(\ln n)^{-1+\frac{1}{p}+\alpha}}\sum_{i=1}^{r}2^{i(1-\frac{1}{p}+q(1-\frac{1}{p}-\alpha))}.$$

Now, let us take $q > \frac{p-1}{\alpha p-p+1}$. Since $\alpha > 1-\frac{1}{p}$, we have $1-\frac{1}{p}+q(1-\frac{1}{p}-\alpha) < 0$. Therefore,

$$\sum_{i=1}^{r} 2^{i(1-\frac{1}{p}+q(1-\frac{1}{p}-\alpha))} \le C_1\,,$$

for some positive constant $C_1 = C_1(p,q,\alpha)$, which in turn implies that

$$(20) \qquad e_n\left(\sum_{i=1}^{r}\mathrm{co}(B_i)\right) \le \frac{C_1 2^{4+q\alpha}c_p\tau_p(X)}{n^{1-\frac{1}{p}}(\ln n)^{\alpha-1+\frac{1}{p}}}\,.$$

The assertion of the proposition follows by summing (18), (19) and (20). □

Proposition 6.4 *Let A be a precompact subset of the unit ball of a Banach space X of type p, $1 < p \le 2$, such that for some $0 < \alpha < 1-\frac{1}{p}$,*

$$\varepsilon_n(A) \le (\ln(n+1))^{-\alpha} \quad \textit{for} \quad n = 1,2,\ldots.$$

Then there exists a constant $c(\alpha,p)$, depending only on α and on p, such that

$$e_n(\mathrm{co}(A)) \le c(\alpha,p)n^{-\alpha} \quad \textit{for} \quad n = 1,2,\ldots.$$

Proof. We define an operator $u : \ell_1(A) \to X$ by $u(e_t) = t$ for $t \in A$ and we prove that for all $n \ge 1$,

$$e_n(u) \le c(\alpha,p)\,n^{-\alpha}.$$

For any integer $i \ge 0$, let A_i be a 2^{-i}-net of A with $\mathrm{card}(A_i) = N(A;2^{-i}) \le \exp(2^{i/\alpha})$. For every $t \in A$ and every integer $i \ge 0$, let $\zeta_i(t)$ be an element of A_i such that $\|t-\zeta_i(t)\| \le 2^{-i}$. For every integer $i \ge 0$, denote by $u_i : \ell_1(A) \to X$, the operator associated to A_i in the sense that $u_i(e_t) = \zeta_i(t)$ for all $t \in A$.
Let $N \in \mathbb{N}$, since $u = u_N + u - u_N$, we have

$$(21) \qquad k^{1-\frac{1}{p}}e_k(u) \le k^{1-\frac{1}{p}}e_k(u_N) + k^{1-\frac{1}{p}}2^{-N}\,.$$

Put $U_0 = u_0$ and $U_i = u_i - u_{i-1}$, $i = 1,\ldots,N$, so that $u_N = \sum_{i=0}^{N} U_i$. Since $\sup_{k\ge1} k^{1-\frac{1}{p}}e_k(\,\cdot\,)$ is a s-norm with $s = 1/(2-\frac{1}{p})$, ([BP], lemma 4), we have :

$$(\sup_{k\ge1} k^{1-\frac{1}{p}}e_k(u_N))^{\frac{1}{2-\frac{1}{p}}} \le \sum_{i=0}^{N}(\sup_{k\ge1} k^{1-\frac{1}{p}}e_k(U_i))^{\frac{1}{2-\frac{1}{p}}}.$$

The entropy of the operator U_i is the entropy of $\mathrm{co}(\{\zeta_i(t)-\zeta_{i-1}(t) \,:\, t\in A\})$, which is the absolute convex hull of at most $\exp(2^{i/\alpha})+\exp(2^{i-1/\alpha})$ points of X of norm less than $2^{-i}+2^{-(i-1)} \le 2^{-i+2}$. Applying Theorem 1.7, we get

$$e_k(U_i) \le C 2^{-i+2} \left(\frac{\ln \frac{2\exp(2^{\frac{i}{\alpha}})}{k}}{k} \right)^{1-\frac{1}{p}} \le C 2^{-i+2} \left(\frac{\ln \exp(2^{\frac{i}{\alpha}+1})}{k} \right)^{1-\frac{1}{p}}$$

where C depends only on p and on the type p constant of X. Using the fact that $0<\alpha<1-\frac{1}{p}$, we arrive at

$$\begin{aligned}
\sum_{i=0}^{N} \left(\sup_{k\ge 1} k^{1-\frac{1}{p}} e_k(U_i) \right)^{\frac{1}{2-\frac{1}{p}}} &\le \sum_{i=0}^{N} \left(C \sup_{k\ge 1} k^{1-\frac{1}{p}} \left(\frac{\ln\exp(2^{\frac{i}{\alpha}+1})}{k} \right)^{1-\frac{1}{p}} 2^{-i+2} \right)^{\frac{1}{2-\frac{1}{p}}} \le \\
&\le \sum_{i=0}^{N} \left(C \left(\ln\exp(2^{\frac{i}{\alpha}+1})\right)^{1-\frac{1}{p}} 2^{-i+2} \right)^{\frac{1}{2-\frac{1}{p}}} \le \\
&\le \left(C 2^{3-\frac{1}{p}} \right)^{1/(2-\frac{1}{p})} \sum_{i=0}^{N} 2^{\left[i(\frac{1}{\alpha}(1-\frac{1}{p})-1)\right][1/(2-\frac{1}{p})]} \le \\
&\le c_{\alpha,p}\, 2^{\left[N(\frac{1}{\alpha}(1-\frac{1}{p})-1)\right][1/(2-\frac{1}{p})]} ,
\end{aligned}$$

where $c_{\alpha,p}$ is a positive constant depending only on α and p.
This implies that for all $k\ge 1$,

$$k^{1-\frac{1}{p}} e_k(u_N) \le (c_{\alpha,p})^{2-\frac{1}{p}}\; 2^{N(\frac{1}{\alpha}(1-\frac{1}{p})-1)}. \tag{22}$$

Formulae (21) and (22) give that for every $N\in \mathbb{N}$ and every integer $k\ge 1$,

$$k^{1-\frac{1}{p}} e_k(u) \le (c_{\alpha,p})^{2-\frac{1}{p}}\; 2^{N(\frac{1}{\alpha}(1-\frac{1}{p})-1)} + k^{1-\frac{1}{p}} 2^{-N}.$$

We choose $N=[\alpha \log_2 k+1]$ which gives the assertion of the proposition. □

Remark. The assertion of the Proposition 6.1, 6.2, 6.3 and 6.4 remain valid if we replace type p by weak type p in the case $1<p<2$ (cf. section 1). What happens for Banach spaces of weak type 2? What happens in the remaining case $\alpha = 1-\frac{1}{p}$? Is the estimate of Proposition 6.4 still valid for $\alpha = 1-\frac{1}{p}$?

Corollary 6.5 *Let $A\subset L_q$, $1<q<\infty$, be a precompact subset of of the unit ball of an L_q space with the property that there are $\rho,\alpha>0$ such that*

$$\varepsilon_n(A) \le \rho n^{-\alpha} \quad \textit{for} \quad n=1,2,\dots \ .$$

Then we have the asymptotically optimal estimate

$$e_n(\mathrm{co}(A)) \le c_{\rho,\alpha,q}\, n^{\min\{1-\frac{1}{q};\frac{1}{2}\}-\alpha} \quad \textit{for} \quad n=1,2,\dots,$$

where $c_{\rho,\alpha,q}$ depends only on ρ,α and q.

Proof. Since L_q, $1 < q < \infty$, is of type $\min\{q; 2\}$ the estimate follows immediately from Proposition 6.1. The optimality for $1 < q \leq 2$ is already given in the proof to Proposition 6.1. In the case $2 < q < \infty$ we take the set

$$A = \{n^{-\alpha} x_n : n \in I\!N\} \subset \ell_q,$$

where the x_n are the normed one vectors of the Littleword matrix (cf. [C3]). Then we have $\varepsilon_n(A) \leq n^{-\alpha}$ and

$$c_0 n^{-\frac{1}{2}-\alpha} \leq e_n(\mathrm{co}(A)) \leq c_1 n^{-\frac{1}{2}-\alpha} \quad \text{for} \quad n = 1, 2, \ldots,$$

with positive constants $c_0, c_1 > 0$ which may depend on q. □

References

[BP] K. Ball and A. Pajor, The entropy of convex bodies with "few" extreme points, London Math. Soc. Lecture Notes Series. **158** (1990) 25-32.

[BPST] J. Bourgain, A. Pajor, S. Szarek and N. Tomczak-Jaegermann, On the duality problem for entropy numbers of operators, Geometric aspects of functional analysis (1987-88), LNM **1376** (1989), 50-63.

[C1] B. Carl, Entropy numbers, s-numbers and eigenvalue problems, J. Funct. Analysis **41** (1981), 290-306.

[C2] B. Carl, Inequalities of Bernstein-Jackson type and the degree of compactness of operators in Banach spaces, Ann. Inst. Fourier **35** (3) (1985), 79-118.

[C3] B. Carl, On a characterization of operators from ℓ_q into a Banach space of type p with some applications to eigenvalue problems, J. Funct. Anal **48** (1982), 394-407.

[CH] B. Carl and A. Hess, Estimates of covering numbers, J. Approx. Theory **65** (1991), 121-139.

[CHK] B. Carl, S. Heinrich and T. Kühn, s-numbers of integral operators with Hölder continuous kernels over metric compacta, J. Funct. Analysis **81** (1988), 54-73.

[CP] B. Carl and A. Pajor, Gelfand numbers of operators with values in Hilbert spaces, Inventiones Math. **94** (1988), 459-504.

[CS] B. Carl and I. Stephani, Entropy, Compactness and the Approximation of Operators, Cambridge University Press (1990).

[D] R.M. Dudley, Universal Donsker classes and metric entropy, Ann. Prob. **15** (1987), 1306-1326.

[DJ] M. Defant and M. Junge, Some estimates on entropy numbers, Israel J. Math. **84** (1993), 417-433.

[GG] A. Garnaev and E. Gluskin, On diameters of the Euclidean Sphere, Dokl. A. N. USSR. **277** (1984), 1048-1052.

[GKS] Y. Gordon, H. König and C. Schütt, Geometric and probabilistic estimates for entropy and approximation numbers of operators, J. Approx. Theory **49** (1987), 219-239.

[LT] J. Lindenstrauss and C. Tzafriri, Classical Banach spaces, vol. 1,2, Springer Verlag (1977/79)

[MP] V. Milman and G. Pisier, Banach spaces with a weak cotype 2 property, Israel J. Math. **54** (1986), 139-158.

[Pa] A. Pajor, Quotient volumique et espaces de Banach de type 2-faible, Israel J. Math. **57** (1987), 101-106.

[PT] A. Pajor and N. Tomczak-Jaegermann, Subspaces of small codimension of finite dimensional Banach spaces, Proc. Amer. Math. Soc. **97** (1986), 637-642.

[PTJ] A. Pajor and N. Tomczak-Jaegermann, Remarques sur les nombres d'entropie d'un opérateur et de son transposé, C. R. Acad. Sci. Paris **301** (1985), 743-746.

[P1] G. Pisier, Volume inequalities in the geometry of Banach spaces, Cambridge University Press (1989).

[P2] G. Pisier, Remarques sur un résultat non publié de B. Maurey, Séminaire d'Analyse Fonctionnelle, Ecole Polytechnique-Palaiseau, exposé 5 (1980/1981).

[RS] C. Richter and I. Stephani, Entropy and the approximation of bounded functions and operators, Arch. Math. (to appear)

[S] C. Schütt, Entropy numbers of diagonal operators between symmetric Banach spaces, J. Approx. Theory **40** (1984), 121-128.

[T] M. Talagrand, New Gaussian estimates for enlarged balls, Geometric and Functional Analysis, Vol. **3**, (1993), 502-526.

Bernd Carl
Fakultät für Mathematik und Informatik
Friedrich-Schiller-Universität Jena
07740 Jena, Germany
carl@minet.uni-jena.de

Ioanna Kyrezi
Equipe de Mathématiques
Université de Marne-la-Vallée
93166 Noisy-le-Grand Cedex, France
kyrezi@math.univ-mlv.fr

Alain Pajor
Equipe de Mathématiques
Université de Marne-la-Vallée
93166 Noisy-le-Grand Cedex, France
pajor@math.univ-mlv.fr