On the Euclidean sections of some Banach and operator spaces

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1 Introduction.

Following the studies of Milman and Schechman ([M-S1] [M-S2]) and of [G-G-M] and [G], we investigate here the "large" Euclidean sections of centrally symmetric convex bodies in $\mathbb{R}^n$, or equivalently, the Banach-Mazur distance of subspaces with "big dimension" of a finite dimensional normed space to an Euclidean space. We give first a general result about subspaces of a normed spaces which possesses a system of vectors satisfying a (C, s)-estimate (see the definition below), and apply these results to give sharp estimates of the distance to ℓ_2^k of k-dimensional subspace of ℓ_q^n, for $q > 2$. We treat then the same problem for subspaces of some normed spaces of operators from $\mathbb{R}^m$ to $\mathbb{R}^n$, and in particular of Schatten classes, for $q \geq 2$. These results are obtained mainly by the use of Gaussian operators ([G]), and so we obtain random subspaces.

Let E be a n-dimensional normed space. We say that a family $u_1, \ldots, u_N$ of vectors of E, with $N \leq n$, satisfies a (C, s)-*estimate* for $C > 0$ and $s > 0$, if for all $(t_i)_{i=1}^N \in \mathbb{R}^N$ and all $m = 1, \ldots, N$, one has

$$\frac{C}{m^{1/s}}\Big(\sum_{i=1}^{m}(t_i^\star)^2\Big)^{1/2} \leq \|\sum_{i=1}^{N} t_i u_i\| \leq \Big(\sum_{i=1}^{N} t_i^2\Big)^{1/2}, \tag{1}$$

where $(t_i^\star)_{i=1}^N$ denotes the decreasing rearrangement of the sequence $(|t_i|)_{i=1}^N$. By a result of Bourgain and Szarek [B-S], there exists a constant $C > 0$ such that for any n, any n-dimensional normed space contains a sequence $u_1, \ldots, u_N$, with $N \geq \frac{n}{2}$, satisfying a $(C, 2)$-estimate. We shall be interested here with $s \geq 2$. It is easy to see that for $q \geq 2$, ℓ_q^n satisfies a $(1, s)$-estimate, with $\frac{1}{s} = \frac{1}{2} - \frac{1}{q}$. It may be also observed that if we define $s' > 0$ by $\frac{1}{s'} = \frac{1}{s} - \frac{1}{\ln(n)}$, and if $(u_1, \ldots, u_N)$ satisfies a (C, s)-estimate, then it satisfies also a $(C/e, s')$-estimate ; so one can restrict the study to the case when

$s \leq \ln(n)$. Finally, we denote by $d(E,F)$ the Banach-Mazur distance between two normed spaces E and F.

Let us recall the following estimates for the norm of Gaussian operators : if E is a Banach space and $(v_j)_{j=1}^N \in E$, we define a Gaussian operator $G_\omega : \ell_2^k \to E$ by

$$G_\omega = \sum_{i=1}^{k} \sum_{j=1}^{N} g_{ij}(\omega) e_i \otimes v_j : \ell_2^k \to E,$$

where $(e_1, \dots, e_k)$ denotes the canonical basis of ℓ_2^k and g_{ij} are pairwise independent real Gaussian random variables for $1 \leq i \leq k$, $1 \leq j \leq N$. We have the following inequalities [G] :

$$\mathbb{E}\|\sum_{j=1}^{N} g_j v_j\| - a_k \sup_{\sum_{1\leq j\leq N} t_j^2 = 1} \|\sum_{j=1}^{N} t_j v_j\| \leq \mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\| \tag{2}$$

and

$$\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\| \leq \mathbb{E}\|\sum_{j=1}^{N} g_j v_j\| + a_k \sup_{\sum_{1\leq j\leq N} t_j^2 = 1} \|\sum_{j=1}^{N} t_j v_j\| \tag{3}$$

with

$$a_k = \sqrt{2}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} \leq \sqrt{k}.$$

2 Euclidean sections of Banach spaces.

The main result of this part is

Theorem 1 *Let E be a n-dimensional normed space, and for $n \geq N \geq n/2$, let $(u_i)_{i=1}^N \in E$ satisfy a (C,s)-estimate for $s > 2$ and $C > 0$. Let q satisfy $\frac{1}{s} = \frac{1}{2} - \frac{1}{q}$. Then for some universal constants c_i, d_i, $1 \leq i \leq 3$, and for all integers k, $1 \leq k \leq N$, there exists a k-dimensional subspace F^k of E such that*

(i) If $k \leq \frac{1}{4}\Big(\mathbb{E}\|\sum_{j=1}^{N} g_j u_j\|\Big)^2$, then $d(F^k, \ell_2^k) \leq 3$.

(ii) If $\frac{1}{4}\Big(\mathbb{E}\|\sum_{j=1}^{N} g_j u_j\|\Big)^2 \leq k \leq c_1 q e^{-q} n$, then $d(F^k, \ell_2^k) \leq \dfrac{d_1\sqrt{k}}{C\sqrt{q}\, n^{1/q}}$.

(iii) If $c_1 q e^{-q} n \leq k \leq c_2 n$, $d(F^k, \ell_2^k) \leq \dfrac{d_2\, k^{1/2-1/q}}{C\,\ln(1+n/k)}$.

(iv) If $c_2 n \leq k \leq N$, *then* $d(F^k, \ell_2^k) \leq d_3\, k^{1/s}$.

Moreover the spaces F^k, $1 \leq k \leq N$, *can be chosen randomly with high probability as subspaces of the linear span of* $(u_i)_{i=1}^N$.

Proof:

Let $U = \text{span}\,\{u_1, \ldots, u_N\}$; we define a Gaussian operator $G_\omega : \ell_2^k \to U$ by

$$G_\omega = \sum_{i=1}^{k} \sum_{j=1}^{N} g_{i,j}(\omega) e_i \otimes u_j.$$

Observe that $\sup_{\sum_{j=1}^N t_j^2 = 1} \| \sum_{j=1}^{N} t_j u_j \| \leq 1$. Applying (2) and (3), we get

- $\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\| \geq \mathbb{E}\| \sum_{j=1}^{N} g_j u_j \| - a_k \sup_{\sum_{1\leq j\leq N} t_j^2 = 1} \| \sum_{j=1}^{N} t_j u_j \|$

- $\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\| \leq \mathbb{E}\| \sum_{j=1}^{N} g_j u_j \| + a_k \sup_{\sum_{1\leq j\leq N} t_j^2 = 1} \| \sum_{j=1}^{N} t_j u_j \|$

We distinguish now between the different values of k, $1 \leq k \leq m$:

1. If $k \leq \frac{1}{4}(\mathbb{E}\| \sum_{j=1}^{N} g_j u_j \|)^2$, then

$$\frac{\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\|}{\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|} \leq \left(1 + \frac{a_k}{\mathbb{E}\| \sum_{j=1}^{N} g_j u_j \|}\right) \Big/ \left(1 - \frac{a_k}{\mathbb{E}\| \sum_{j=1}^{N} g_j u_j \|}\right) \leq 3.$$

So, there exists ω_0 such that $\dim(\text{Im}\, G_{\omega_0}) = k$ and

$$\frac{\sup_{|x|_2=1} \|G_{\omega_0}(x)\|}{\inf_{|x|_2=1} \|G_{\omega_0}(x)\|} \leq 3.$$

Let $F^k = \text{Im}\, G_{\omega_0}$; then $\dim F^k = k$, $d(F^k, \ell_2^k) \leq 3$ and case (i) is proved (it is the classical Dvoretzky's theorem).

2. In the other cases, one has $k \geq \frac{1}{4}(\mathbb{E}\| \sum_{j=1}^{N} g_j u_j \|)^2$ so that

$$\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\| \leq 3\sqrt{k}.$$

For $1 \le m \le N$, in order to get a better lower bound for $\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|$, we define a new norm $\|y\|_{(m)}$ on U. For all $y \in U$, $y = \sum_{j=1}^N y_j u_j$, let

$$\|y\|_{(m)} = \|\sum_{j=1}^N y_j u_j\|_{(m)} = \frac{C}{m^{1/s}} (\sum_{i=1}^m (y_i^\star)^2)^{1/2}.$$

It is clear from (1) that $\|G_\omega(x)\| \ge \|G_\omega(x)\|_{(m)}$. By inequality (2) applied to $G_\omega : \ell_2^k \to (U, \|\cdot\|_{(m)})$, we get

$$\begin{aligned}
\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\| &\ge \mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|_{(m)} \\
&\ge \mathbb{E}\|\sum_{j=1}^N g_j u_j\|_{(m)} - a_k \sup_{\sum_{1\le j\le N} t_j^2 = 1} \|\sum_{j=1}^N t_j u_j\|_{(m)} \\
&\ge \frac{1}{m^{1/s}}\Big(C\,\mathbb{E}\Big(\sum_{i=1}^m (g_i^\star)^2\Big)^{1/2} - \sqrt{k}\Big). \\
&\ge m^{1/q}\Big(Cc\sqrt{\ln(1+\frac{N}{m})} - \sqrt{\frac{k}{m}}\Big),
\end{aligned}$$

the last inequality following from classical estimates of $\mathbb{E}\Big(\sum_{i=1}^m (g_i^\star)^2\Big)^{1/2}$ (see for instance [Gl]).

- If $k \le cqe^{-q}n$, we choose $m = Ne^{-q}$. Since $N \ge n/2$, we get

$$\frac{\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\|}{\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|} \le \frac{c\sqrt{k}}{C\sqrt{q}\, n^{1/q}}$$

and we conclude like in **1.**.

- If $cqe^{-q}n \le k \le cn$, we choose $m = k$. We have then

$$\frac{\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\|}{\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|} \le \frac{c\,k^{1/s}}{C\,\ln(1+n/k)}$$

and as before, we get (iii).

- If $cn \le k \le N$, then by the definition of the (C,s)-estimate, one has $d(U, \ell_2^N) \le N^{1/s}$; thus every k-dimensional subspace F^k of U satisfies

$$d(F^k, \ell_2^k) \le N^{1/s} \le n^{1/s} \le \Big(\frac{k}{c}\Big)^{1/s}. \qquad \square$$

Remark
Using inequality (1), it is easy to prove that

$$\mathbb{E}\|\sum_{j=1}^{N} g_j u_j\| \geq c\,C\,\sqrt{q}\,n^{1/q}.$$

Indeed, by (1), for all $m \in \{1,\dots,N\}$, we have

$$\begin{aligned}\mathbb{E}\|\sum_{j=1}^{N} g_j u_j\| &\geq \frac{C}{m^{1/s}}\,\mathbb{E}\Big(\sum_{i=1}^{m}(g_i^{\star})^2\Big)^{1/2}\\ &\geq c'\,C\,m^{1/q}\sqrt{\ln(1+\frac{N}{m})},\end{aligned}$$

and we choose $m = Ne^{-q}$ (recall that $N \geq n/2$).

As a corollary, we get more precise estimates in the particular case of $E = \ell_q^n$.

Corollary 2 *For some universal constant $c_i, d_i > 0$, $1 \leq i \leq 3$, for all $n \geq 1$, and all integer $k = 1,\dots,n$, there exists a k-dimensional subspace F^k of ℓ_q^n with $q \geq 2$, such that*

(i) If $k \leq c_1\,q\,n^{2/q}$, then $d(F^k, \ell_2^k) \leq 3$.

(ii) If $c_1\,q\,n^{2/q} \leq k \leq c_2 q e^{-q} n$, then $d(F^k, \ell_2^k) \leq \dfrac{d_1\sqrt{k}}{\sqrt{q}\,n^{1/q}}$.

(iii) If $c_2 q e^{-q} n \leq k \leq c_3 n$, then $d(F^k, \ell_2^k) \leq \dfrac{d_2\,k^{1/2-1/q}}{\ln(1+n/k)}$.

(iv) If $c_3 n \leq k \leq n$, then $d(F^k, \ell_2^k) \leq d_3\,k^{1/2-1/q}$.

Moreover, the spaces F^k can be choosen randomly with high probability in ℓ_q^n.

Proof:
Let $(e_1,\dots,e_n)$ be the canonical basis of ℓ_q^n ; then for all $t_1,\dots,t_n$ and for all $m = 1,\dots,n$,

$$\begin{aligned}\Big(\sum_{i=1}^{n}|t_i|^q\Big)^{1/q} = |\sum_{i=1}^{n} t_i e_i|_q &\geq |\sum_{i=1}^{m} t_i^{\star} e_i|_q = \Big(\sum_{i=1}^{m}(t_i^{\star})^q\Big)^{1/q}\\ &\geq \frac{1}{m^{\frac{1}{2}-\frac{1}{q}}}\Big(\sum_{i=1}^{m}(t_i^{\star})^2\Big)^{1/2},\end{aligned}$$

using Hölder's inequality. Since $q \geq 2$, $(e_1,\dots,e_n)$ satisfies a $(1,s)$-estimate, with $\dfrac{1}{s} = \dfrac{1}{2} - \dfrac{1}{q}$. It is clear from the preceeding remark that

$$\mathbb{E}||\sum_{j=1}^{n} g_j e_j||_q \sim c\sqrt{q}\,n^{1/q}.$$

Then we apply Theorem 1 to get random subspaces in the whole space ℓ_q^n.
□

Remarks :
1. As it is proved in [C-P], the result of Corollary 2 is optimal up to absolute constant. We include here a short proof of this optimality :
Let $T : \ell_2^k \to \ell_q^n$ a linear operator such that for all $x \in \ell_2^k$,

$$|x|_2 \le |Tx|_q \le d\,|x|_2.$$

Now we write

$$\begin{aligned} 1 &= \int_{S^{k-1}} |x|_2 d\sigma_{k-1}(x) \\ &\le \int_{S^{k-1}} |Tx|_q d\sigma_{k-1}(x) = \int_{S^{k-1}} \Big(\sum_{i=1}^n |\langle x, T^\star(e_i)\rangle|^q\Big)^{1/q} d\sigma_{k-1}(x) \\ &= \frac{1}{a_k}\mathbb{E}\Big(\sum_{i=1}^n |\langle G, T^\star(e_i)\rangle|^q\Big)^{1/q}, \end{aligned}$$

where G is a gaussian vector of $\mathbb{R}^k$. Since $\langle G, T^\star(e_i)\rangle$ is $\mathcal{N}\Big(0, |T^\star(e_i)|_2\Big)$ and by Hölder inequality, we get

$$\mathbb{E}\Big(\sum_{i=1}^n |\langle G, T^\star(e_i)\rangle|^q\Big)^{1/q} \le \Big(\sum_{i=1}^n \mathbb{E}|\langle G, T^\star(e_i)\rangle|^q\Big)^{1/q} \le n^{1/q}\gamma(q) \sup_{1\le i\le n} |T^\star(e_i)|_2,$$

where $\gamma(q)$ is the moment of order q of a gaussian $\mathcal{N}(0,1)$-variable. Moreover $|T^\star(e_i)|_2 \le \|T^\star\|\,|e_i|_{q'} = d$, so that we get a universal constant $c > 0$ such that,

$$\sqrt{k} \le c\,d\,n^{1/q}\sqrt{q}.$$

2. A constructive proof of a single subspace of ℓ_q^n satisfying the desired conclusion is given in [G-J2].
3. In fact by [L], the inequality $d(F^k, \ell_2^k) \le k^{1/2-1/q}$ is always true.

3 The case of operator spaces

Let τ be a 1-symmetric norm on $\mathbb{R}^n$. It is well known that for $m \ge n$, one defines a norm $\|\cdot\|_\tau$ on the mn-dimensional vector space $\mathcal{M}_{m\times n}(\mathbb{R})$ of all $[m \times n]$-matrices with real entries by setting

$$\|M\|_\tau = \tau(s_1(M), \dots, s_n(M)) \text{ for all } M \in \mathcal{M}_{m\times n}(\mathbb{R})$$

where the $s_i(M)$, $1 \leq i \leq n$, are the eigenvalues of $\sqrt{M^*M}$. If for some $q \geq 1$

$$\tau(x) = \tau(x_1, \ldots, x_n) = (\sum_{i=1}^{n} |x_i|^q)^{\frac{1}{q}} = |x|_q$$

we get the so called Schatten class $S_q(m \times n)$ with the norm

$$\|T\|_q = |(s_i(T))_{i=1}^n|_q.$$

Theorem 3 *Let τ be a 1-symmetric norm on $\mathbb{R}^n$ and $\|\cdot\|_\tau$ the norm on $\mathcal{M}_{m\times n}(\mathbb{R})$ associated to τ. Let d_τ be the Banach-Mazur distance between $(\mathbb{R}^n, \tau)$ and ℓ_2^n. Then for some universal constant $c > 0$, and for every integer k, $1 \leq k \leq nm$, there exists a k-dimensional subspace F^k of $(\mathcal{M}_{m\times n}(\mathbb{R}), \|\cdot\|_\tau)$ such that*

(i) If $k \leq \frac{1}{4}(\mathbb{E}\|G\|_\tau)^2$, then $d(F^k, \ell_2^k) \leq 3$.

(ii) If $\frac{1}{4}(\mathbb{E}\|G\|_\tau)^2 \leq k \leq nm$, then $d(F^k, \ell_2^k) \leq 1 + c\, d_\tau \sqrt{\frac{k}{nm}}$.

Proof:
Since τ is a 1-symmetric norm, we can assume that $\frac{1}{d_\tau}|x|_2 \leq \tau(x) \leq |x|_2$. Then for all $T \in \mathcal{M}_{m\times n}(\mathbb{R})$ one has

$$\frac{1}{d_\tau}\|T\|_2 \leq \|T\|_\tau = \tau(s_1(T), \ldots, s_n(T)) \leq \|T\|_2 \tag{4}$$

where $\|T\|_2 = (\mathrm{tr}(\mathrm{T}^\star\mathrm{T}))^{1/2}$ is the Hilbert-Schmidt norm. For $1 \leq p \leq m$ and $1 \leq q \leq n$, let E_{pq} be the canonical basis of $\mathcal{M}_{m\times n}(\mathbb{R})$ (with entries $(E_{pq})_{ij} = \delta_{ip}\delta_{qj}$). Let $G_\omega : \ell_2^k \to (\mathcal{M}_{m\times n}(\mathbb{R}), \|\cdot\|_\tau)$ be the Gaussian operator defined by

$$G_\omega = \sum_{l=1}^{k} \sum_{\substack{1\leq p\leq m\\ 1\leq q\leq n}} g_{lpq}(\omega)\ e_l \otimes E_{pq}.$$

where $e_1, \ldots, e_k$ is the canonical basis of ℓ_2^k and the g_{lpq}, $1 \leq l \leq k$, $1 \leq p \leq m$, $1 \leq q \leq n$, are pairwise independent normalized Gaussian variables. By inequalities (2) and (3), we have

$$\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\|_\tau \leq \mathbb{E}\,\|G\|_\tau + a_k \sup\{\|T\|_\tau\ ;\ T \in \mathcal{M}_{m\times n}(\mathbb{R}),\ \|T\|_2 = 1\}$$

and

$$\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|_\tau \geq \mathbb{E}\,\|G\|_\tau - a_k \sup\{\|T\|_\tau\ ;\ T \in \mathcal{M}_{m\times n}(\mathbb{R}),\ \|T\|_2 = 1\}$$

where G is a matrix with pairwise independent normalized real Gaussian entries in $\mathcal{M}_{m\times n}(\mathbb{R})$.

It is clear that $\sup\{\|T\|_\tau\ ;\ T\in\mathcal{M}_{m\times n}(\mathbb{R}),\ \|T\|_2=1\}=1$. We distinguish now three cases.

1. If $\mathbb{E}\,\|G\|_\tau \geq 2a_k$, we have

$$\mathbb{E}\sup_{|x|_2=1}\|G_\omega(x)\|_\tau \Big/ \mathbb{E}\inf_{|x|_2=1}\|G_\omega(x)\|_\tau \leq \frac{1+a_k/\mathbb{E}\,\|G\|_\tau}{1-a_k/\mathbb{E}\,\|G\|_\tau} \leq 3.$$

2. If $\mathbb{E}\,\|G\|_\tau \leq 2a_k \leq \frac{\sqrt{nm}}{2}$, then by condition (4) and inequality (2) with $G_\omega : \ell_2^k \to (\mathcal{M}_{m\times n}(\mathbb{R}), \|\cdot\|_2)$, we get

$$\mathbb{E}\inf_{|x|_2=1}\|G_\omega(x)\|_\tau \geq \frac{1}{d_\tau}\,\mathbb{E}\inf_{|x|_2=1}\|G_\omega(x)\|_2 \geq \frac{1}{d_\tau}\,(\mathbb{E}\|G\|_2 - a_k).$$

It is well known that $\mathbb{E}\|G\|_2 \geq \dfrac{\sqrt{nm}}{2}$, so that

$$\mathbb{E}\inf_{|x|_2=1}\|G_\omega(x)\|_\tau \geq \frac{1}{d_\tau}(\frac{\sqrt{nm}}{2} - a_k).$$

Since $\mathbb{E}\,\|G\|_\tau \leq 2a_k \leq n/2$, we deduce that

$$\mathbb{E}\sup_{|x|_2=1}\|G_\omega(x)\|_\tau \Big/ \mathbb{E}\inf_{|x|_2=1}\|G_\omega(x)\|_\tau \leq \frac{3a_k}{\frac{1}{d_\tau}\frac{\sqrt{nm}}{4}} \leq \frac{12d_\tau\sqrt{k}}{\sqrt{nm}}.$$

3. If $a_k \geq \frac{\sqrt{nm}}{4}$, we know from condition (4) that for all subspaces F^k of $(\mathcal{M}_{m\times n}(\mathbb{R}), \tau)$ with $\dim F^k = k$, one has $d(F^k, \ell_2^k) \leq d_\tau$. This concludes the proof of the theorem because $a_k \sim \sqrt{k}$. □

As a consequence of the preceeding theorem, we get

Corollary 4 *Let $q \geq 2$ and let $S_q(m\times n)$ be the Schatten class. Assume that for some fixed $r > 1$, one has $m = rn$. Then for some universal constant $c > 0$, and for every integer k, $1 \leq k \leq nm$, there exists a k-dimensional subspace F^k of $S_q(m\times n)$ such that*

$$d(F^k, \ell_2^k) \leq 1 + \frac{c}{\sqrt{r}}\, n^{-1/q}\sqrt{\frac{k}{n}}\quad .$$

Proof:
It is well known that for $q \geq \ln(n)$ the norm on $S_q(m\times n)$ is equivalent up to universal constant to the norm on $S_\infty(m\times n)$; so we reduce to the case when $2\leq q\leq \ln(n)$. We have $\tau(x)=(\sum\limits_{i=1}^{n}|x_i|^q)^{1/q}$ so that $d_\tau = n^{\frac{1}{2}-\frac{1}{q}}$. We need to compute $\mathbb{E}\|G\|_q$ for a Gaussian matrix. It is well known that

$$a_m - a_n \leq \mathbb{E}\min_{1\leq i\leq n} s_i(G) \leq \mathbb{E}\sup_{1\leq i\leq n} s_i(G)\leq a_m + a_n,$$

with

$$a_k = \sqrt{2}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\leq \sqrt{k},$$

(see [H-T] for the more general case of gaussian matrices with operator entries). Then

$$n^{1/q}(a_m - a_n)\leq \mathbb{E}\|G\|_q \leq n^{1/q}(a_m+a_n),$$

and we apply Theorem 3. □

Remark : Using the same idea as for ℓ_q^n, we can prove the optimality of this Corollary.
Let $\Theta : \ell_2^k \to S_q(m\times n)$ an operator such that for all $x\in \ell_2^k$,

$$|x|_2 \leq \|\Theta x\|_q \leq d\,|x|_2.$$

Now we write

$$\begin{aligned} 1 &= \int_{S^{k-1}} |x|_2 d\sigma_{k-1}(x)\\ &\leq \int_{S^{k-1}} \|\Theta x\|_q d\sigma_{k-1}(x)\\ &\leq n^{1/q}\int_{S^{k-1}} \|\Theta x\|_\infty d\sigma_{k-1}(x). \end{aligned}$$

If T_i denotes the matrix $\Theta(e_i)$ and $G=(g_1,\dots,g_k)$ is a gaussian vector in $\mathbb{R}^k$, we have

$$1\leq \frac{n^{1/q}}{a_k}\mathbb{E}\|\sum_{i=1}^k g_iT_i\|_\infty.$$

But

$$\|\sum_{i=1}^k g_iT_i\|_\infty = \sup_{|x|_2=1,\, x\in\mathbb{R}^m}\ \sup_{|y|_2=1,\, y\in\mathbb{R}^n}\sum_{i=1}^k g_i\langle T_ix,y\rangle.$$

Let $h_1,\dots,h_m$, $h'_1,\dots,h'_n$ be nm independent gaussian variables and define the two gaussian process :

$$X_{x,y}=\sum_{i=1}^k g_i\langle T_ix,y\rangle \quad\text{and}\quad Y_{x,y}=\sqrt{2}\,d\left(\sum_{i=1}^m h_ix_i+\sum_{i=1}^n h'_iy_i\right).$$

By definition of T, one has

$$\|\sum_{i=1}^k \alpha_i T_i\|_\infty \leq d\left(\sum_{i=1}^k \alpha_i^2\right)^{1/2}.$$

If we choose $\alpha_i = \langle T_i x, y\rangle$, $1 \leq i \leq k$, we get for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$,

$$\left(\sum_{i=1}^k |\langle T_i x, y\rangle|^2\right)^{1/2} \leq d\,|x|_2|y|_2.$$

We conclude that for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, $|x|_2 = 1$ and $|y|_2 = 1$,

$$\begin{aligned}
\mathbb{E}|X_{x,y} - X_{x',y'}|^2 &= \sum_{i=1}^k (|\langle T_i x, y - y'\rangle + \langle x - x', T_i^\star y'\rangle|)^2 \\
&\leq 2\sum_{i=1}^k (|\langle T_i x, y - y'\rangle|^2 + |\langle x - x', T_i^\star y'\rangle|^2 \\
&\leq 2\,d^2(|y - y'|_2^2 + |x - x'|_2^2) = \mathbb{E}|Y_{x,y} - Y_{x',y'}|^2.
\end{aligned}$$

Then by Slepian's lemma, we obtain

$$\mathbb{E}\sup_{|x|_2=1}\sup_{|y|_2=1} X_{x,y} \leq \mathbb{E}\sup_{|x|_2=1}\sup_{|y|_2=1} Y_{x,y}$$

and since $\mathbb{E}\sup_{|x|_2=1}\sup_{|y|_2=1} Y_{x,y} = \sqrt{2}\,d(a_m + a_n)$, we get a universal constant $c > 0$ such that

$$\sqrt{k} \leq c\,d\,(\sqrt{r} + 1)\,n^{1/2+1/q}.$$

If a subspace of $S_q(m \times n)$ with dimension k is at distance $d \leq \dfrac{c}{\sqrt{r}}\,n^{-1/q}\sqrt{\dfrac{k'}{n}}$ then $k \leq ck'$ and it proves the optimality of corollary 4.

4 Volume ratios with respect to quotients of subspaces of L_q

In this section we introduce volume ratios of random $k-$dimensional subspaces F of an $n-$dimensional normed space X with respect to the class of all $k-$dimensional subspaces of quotients of ℓ_q, $2 \leq q \leq \infty$. This volume ratio yields among other things, in the case $q = 2$, a lower bound for the distance $d(F, \ell_2^k)$ for random subspaces F of X.

Let us consider the following concept of volume ratios introduced in [G-J1, G-J2]. Given a $n-$dimensional Banach space $X = (\mathbb{R}^n, ||.||)$ with unit ball

B_X, and a finite or infinite dimensional Banach space Z with unit ball B_Z, we define the volume ratios

$$\mathrm{vr}\,(X,Z) := \inf\left\{ \left(\frac{\mathrm{vol(B_X)}}{\mathrm{vol(T(B_Z))}}\right)^{1/n} ;\, T(B_Z)\subset B_X \right\},$$

$$\mathrm{vr}\,(X,S(Z)) := \inf\left\{ \left(\frac{\mathrm{vol(B_X)}}{\mathrm{vol(T(B_F))}}\right)^{1/n} ;\, F\subset Z,\, \dim F = n,\, T(B_F)\subset B_X \right\},$$

$$\mathrm{vr}\,(X,S_p) :=\ \mathrm{vr}\,(X,S(\ell_p)),$$

and

$$\mathrm{vr}\,(X,SQ(\ell_p)) := \inf_{Q \text{ quotient of } \ell_p} \mathrm{vr}\,(X,S(Q)).$$

As in [G-J2] the n-th volume number of an operator $T: X\to Y$ is defined by

$$v_n(T) = \sup\left\{ \left(\frac{\mathrm{vol(T(B_E))}}{\mathrm{vol(B_F)}}\right)^{1/n} ; E\subset X, T(E)\subset F\subset Y, \dim E = \dim F = n \right\}$$

We shall also need the definition of the $p-$nuclear norm of an operator $T: X\to Y$ between two finite dimensional Banach spaces, which is defined by

$$\nu_p(T) = \inf\{||A_N||\,||\sigma_N||\,||B_N||; T = B_N\sigma_N A_N,\ N\geq 1\}$$

where $A_N : X\to \ell_\infty^N$, $\sigma_N : \ell_\infty^N \to \ell_p^N$ is a diagonal operator, $B_N : \ell_p^N \to Y$.

Theorem 5 *Let $X = (\mathbb{R}^n, ||.||)$ be a $n-$dimensional normed space, $\{b_i, b_i^*\}_{i=1}^n$ be a biorthogonal basis for X and $J = \sum_{j=1}^n e_j^*\otimes b_j : \mathbb{R}^n \to X$. For all $u\in\mathcal{O}_n$, define $u_k : \mathbb{R}^k\to\mathbb{R}^n$ by $u_k(e_j) = u(e_j)$ for all $1\leq j\leq k$ and A_u by $A_u = J\circ u_k : \ell_2^k \to X$.*

Then for some universal constant $c>0$ and for all $2\leq q\leq\infty$ the $k-$dimensional random subspace $F_u = A_u(\ell_2^k)\subset X$ satisfies

$$\mathbb{E}_u\ \mathrm{vr}\,(F_u, SQ(\ell_q)) \geq \frac{c\sqrt{k}}{\left(\sqrt{q}+\dfrac{\sqrt{k}}{n^{1/q}}\right)\max_{1\leq i\leq n}||b_i^*||\,\mathbb{E}||\sum_{i=1}^n g_i\, b_i||}$$

where $\mathbb{E}_u$ denotes the expectation with respect to the Haar measure on $\mathcal{O}_n$.

Proof:
For $u \in \mathcal{O}_n$, we define also $B_u : X \to \ell_2^k$ by $B_u = u_k^\star \circ J^{-1}$ where $u_k^\star : \mathbb{R}^n \to \mathbb{R}^k$ is the adjoint of u_k. Clearly $B_u A_u = id_{\ell_2^k}$.

Claim: Let q' be the conjugate of q, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$, then

$$\mathbb{E}_u \nu_{q'}(B_u : X \to \ell_2^k) \leq c\sqrt{n}\left(\sqrt{q} + \frac{\sqrt{k}}{n^{1/q}}\right) \max_{1 \leq j \leq n} \|b_j^*\|. \tag{5}$$

To show this procceed as in the definition of the $q-$nuclear ideal norm to factor $B_u|_{X \to \ell_2^k} = u_k^*|_{\ell_{q'}^n \to \ell_2^k}\, I\, J^{-1}$ where $I : \ell_\infty^n \to \ell_{q'}^n = \sum_{i=1}^n e_i \otimes e_i$ is the identity map on $\mathbb{R}^n$, and $J^{-1} = \sum_{i=1}^n b_i^* \otimes e_i : X \to \ell_\infty^n$. Then clearly

$$\nu_{q'}(B_u|_{X \to \ell_2^k}) \leq \|J^{-1}\|\, \|I\|\, \|u_k^*|_{\ell_{q'}^n \to \ell_2^k}\| = \max_{1 \leq i \leq n} \|b_i^*\|\, n^{1/q' - 1/2} \|u_k^*|_{\ell_{q'}^n \to \ell_2^k}\|.$$

Let $G = \sum_{i,j} g_{i,j} e_i \otimes e_j$ denote the Gaussian operator which maps $\ell_{q'}^n$ to ℓ_2^k ; we have by [B-G]

$$\mathbb{E}_u \|u_k^*|_{\ell_{q'}^n \to \ell_2^k}\| \leq \frac{c_0}{\sqrt{n}} \mathbb{E}\|G : \ell_{q'}^n \to \ell_2^k\| \leq \frac{c_1}{\sqrt{n}}(cn^{1/q}\sqrt{q} + \sqrt{k})$$

hence

$$\mathbb{E}_u \nu_{q'}(B_u|_{X \to \ell_2^k}) \leq c_0\, n^{1/2}(c\sqrt{q} + n^{-1/q}\sqrt{k}) \max_{1 \leq i \leq n} \|b_i^*\|$$

and (5) is proved.

Now recall that if $T : \ell_2^k \to X$ and $\text{rad}\,(T) =: \int_0^1 \|\sum_{i=1}^k r_i(t) T(e_i)\|_X \, dt$, then using the Marcus-Pisier inequality [B-G], [M-P]

$$\begin{aligned} \sqrt{n}\, \mathbb{E}_u \text{ rad}\,(A_u : \ell_2^k \to X) &= \sqrt{n}\, \mathbb{E}_u \int_0^1 \|\sum_{j=1}^k r_j(t) A_u(e_j)\| dt \\ &\leq c\, \mathbb{E} \int_0^1 \|\sum_{j=1}^k \sum_{i=1}^n r_j(t) g_{i,j} b_i\| dt \\ &= c\sqrt{k}\, \mathbb{E}\|\sum_{i=1}^n g_i b_i\|. \end{aligned}$$

Denote by $e_k(T)$ the $k-$th entropy number of an operator $T : Y \to X$, then by [C-P] one has

$$\mathbb{E}_u \sqrt{k} v_k(A_u) \leq 4\mathbb{E}_u \sqrt{k} e_k(A_u) \leq 4c_1 \mathbb{E}_u \text{ rad}\,(A_u) \leq \frac{c\sqrt{k}}{\sqrt{n}} \mathbb{E}\|\sum_{i=1}^n g_i b_i\|.$$

By [G-J2] Lemma 1.3, we have for any $k = 1, 2, \ldots, 2 \leq q \leq \infty$, and any operator T from a Banach space Z to ℓ_2

$$\frac{\sqrt{k}\, v_k(T)}{\nu_{q'}(T)} \leq c_0 \sup_{F \subset Z, \dim(F)=k} \text{vr}\,(F, SQ(\ell_q)).$$

Applying this to $B_u|_{F_u \to \ell_2^k}$ we have

$$\sqrt{k} v_k(B_u|_{F_u}) \leq c_0 \nu_{q'}(B_u)\ \text{vr}\,(F_u, SQ(\ell_q)).$$

Since $B_u A_u = id_{\ell_2^k}$, we have $1 = v_k(A_u^* A_u) = v_k(A_u) v_k(A_u^*|_{F_u})$.

Hence we obtain

$$1 \leq c_0 v_k(A_u)\, \frac{\nu_{q'}(B_u)}{\sqrt{k}}\ \text{vr}\,(F_u, SQ(\ell_q))$$

and taking the 3-rd root we get by Hölder inequality

$$\begin{aligned} 1 &\leq c_0 \mathbb{E}_u v_k(A_u)\, \mathbb{E}_u \left(\tfrac{\nu_{q'}(B_u)}{\sqrt{k}}\right) \mathbb{E}_u\ \text{vr}\,(F_u, SQ(\ell_q)) \\ &\leq \tfrac{c}{\sqrt{n}} \mathbb{E} \|\textstyle\sum_{i=1}^n g_i b_i\|\ \tfrac{c\sqrt{n}}{\sqrt{k}} (\sqrt{q} + \tfrac{\sqrt{k}}{n^{1/q}}) \max_i \|b_i^*\|\ \mathbb{E}_u\ \text{vr}\,(F_u, SQ(\ell_q)). \end{aligned}$$

This concludes the proof. $\square$

Remarks :

1. It was proved in [G-J2] that

$$\text{vr}\,(X, SQ(\ell_p)) \leq \text{vr}\,(X, S(\ell_p)) \leq c_o \sqrt{p + p'}\ \text{vr}\,(X, SQ(\ell_p))$$

with $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Estimates in the case $q = 2$ and $X = \ell_p^n$ or $X = S_p(m \times n)$ with $2 \leq p \leq \ln n$ which give optimal lower bound in expectation for random $k-$dimensional subspaces of X (in correlation with part 2 and 3 of this paper). (I have to write that and also for $q \geq 2$ as Yoram said us by mail)

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