On computing M^*

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1 Preliminaries and Notation

1.1 Notation

By $|\cdot|$ and $\langle\cdot,\cdot\rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^n$. The canonical basis of $\mathbb{R}^n$ we denote by $e_1,\dots,e_n$. By $\|\cdot\|_p$, $0<p\le\infty$, we denote the ℓ_p-norm, i.e.

$$\|x\|_p = \left(\sum_{i\ge 1}|x_i|^p\right)^{1/p} \quad \text{for} \quad p<\infty \quad \text{and} \quad \|x\|_\infty = \sup_{i\ge 1}|x_i|.$$

In particular, $\|\cdot\|_2 = |\cdot|$. As usual, $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$, and the unit ball of ℓ_p^n is denoted by B_p^n.

Given sequence $\{a_i\}_{i=1}^n$, by $\{a_i^*\}_{i=1}^n$ we denote the non-increasing rearrangement of $\{|a_i|\}_{i=1}^n$. We will also use weak ℓ_p-norm, $\|\cdot\|_{p\infty}$, $0<p<\infty$, defined by

$$\|x\|_{p\infty} = \inf\left\{\lambda \mid x_k^* \le \lambda/k^{1/p} \text{ for every } k\le n\right\}.$$

The unit ball of this norm is denoted by $B_{p\infty}$. Thus

$$B_{p\infty} = \left\{x\in\mathbb{R}^n \mid x_k^* \le k^{-1/p} \text{ for every } k\le n\right\}$$

Given points $x_1,\dots,x_k$ in $\mathbb{R}^n$ we denote their convex hull by $\operatorname{conv}\{x_i\}_{i\le k}$ and their absolute convex hull by $\operatorname{abs\,conv}\{x_i\}_{i\le k} = \operatorname{conv}\{\pm x_i\}_{i\le k}$. Similarly, the convex hull of a set $A\subset\mathbb{R}^n$ is denoted by $\operatorname{conv} A$ and absolute convex hull of A is denoted by $\operatorname{abs\,conv} A$ $(=\operatorname{conv}\{A\cup -A\})$.

Let $K \subset \mathbb{R}^n$ be a compact convex body with non-empty interior (below we consider only such convex bodies) such that $0 \in K$. We denote by K° the polar of K, i.e.

$$K^\circ = \{x \mid \langle x, y \rangle \leq 1 \text{ for every } y \in K\}.$$

We will also use that if E is a linear subspace of $\mathbb{R}^n$ then the polar of $K \cap E$ (taken in E) is

$$(K \cap E)^\circ = P_E K^\circ,$$

where P_E is the orthogonal projection onto E. Note also that $K^{\circ\circ} = K$ and $\left(B_p^n\right)^\circ = B_q^n$, where $1/p + 1/q = 1$.

By g_1, g_2, ... we always denote independent $N(0,1)$ Gaussian random variables.

Given two functions (or quantities) F and G we write

$$F \approx G$$

if there are two absolute positive constants c and C such that

$$cF \leq G \leq CF.$$

1.2 Some results on expectation of Gaussian variables

Here we quote some known results on expectation of Gaussian variables.

The following Lemma is well known (and can be shown by direct calculations, see e.g. []).

Lemma 1.1 *Let $1 \leq k \leq n/2$. Let g_i, $i \leq n$, be independent $N(0,1)$ Gaussian random variables. Then*

$$c \sqrt{\ln \frac{n}{k}} \leq \mathbb{E}\ g_k^* \leq C \sqrt{\ln \frac{n}{k}},$$

where c and C are absolute positive constants. In particular,

$$c_1\ k \sqrt{\ln \frac{n}{k}} \leq \mathbb{E} \sum_{i=1}^{k} g_i^* \leq C_1\ k \sqrt{\ln \frac{n}{k}},$$

where c and C are absolute positive constants.

The next Lemma can be shown by direct calculations (see Example 10 in [**?**]).

Lemma 1.2 *Let $1 \le k \le n$. Let g_i, $i \le n$, be independent $N(0,1)$ Gaussian random variables. Then for $k \ge n/2$ one has*

$$\sqrt{\frac{\pi}{2}}\ \frac{n+1-k}{n+1} \le \mathbb{E}\ g_k^* \le \sqrt{2\pi}\ \frac{n+1-k}{n+1}.$$

We will also use the following Lemma, which is a particular case of Example 16 in [**?**].

Lemma 1.3 *Let $1 \le q \le \ln(2n)$. Let $1 \le k \le n/2$. Let g_i, $i \le n$, be independent $N(0,1)$ Gaussian random variables. Then*

$$k\ \left(c\ \max\left\{\ln\frac{n}{k}, q\right\}\right)^{q/2} \le \mathbb{E}\ \sum_{i=1}^{k} (g_i^*)^q \le k\ \left(C\ \max\left\{\ln\frac{n}{k}, q\right\}\right)^{q/2},$$

where c and C are absolute positive constants.

Now we provide two corollaries of Lemma **??** and Lemma **??**, which will be used below.

Corollary 1.4 *Let $1 \le q \le \ln(2n)$. Let $1 \le k \le n$. Let g_i, $i \le n$, be independent $N(0,1)$ Gaussian random variables. Then*

$$c\ k^{1/q}\ \sqrt{q + \ln\frac{2n}{k}} \le \mathbb{E}\left(\sum_{i=1}^{k} (g_i^*)^q\right)^{1/q} \le C\ k^{1/q}\ \sqrt{q + \ln\frac{2n}{k}}$$

where c and C are absolute positive constants.

Remark. In particular it implies the following well known estimate

$$c\ n^{1/q}\ \sqrt{q} \le \mathbb{E}\left(\sum_{i=1}^{n} |g_i|^q\right)^{1/q} \le C\ n^{1/q}\ \sqrt{q}$$

for $1 \le q \le \ln(2n)$. Note also that if $q \ge \ln(2k)$ then

$$\left(\sum_{i=1}^{k} (g_i^*)^q\right)^{1/q} \approx \max_{1\le i\le n} |g_i|,$$

hence

$$\mathbb{E}\left(\sum_{i=1}^{k}(g_i^*)^q\right)^{1/q} \approx \sqrt{\ln(2n)}.$$

Proof of Corollary ??. Without loss of generality we can assume that $k \le n/2$. The upper bound follows immediately from Lemma **??** and comparison between first and q-th moments.

To obtain the lower bound note that by Lemma **??** for every $m \le k$ we have

$$\mathbb{E}\left(\sum_{i=1}^{k}(g_i^*)^q\right)^{1/q} \ge \mathbb{E}\left(\sum_{i=1}^{m}(g_i^*)^q\right)^{1/q} \ge m^{1/q-1}\ \mathbb{E}\sum_{i=1}^{m} g_i^* \ge c\ m^{1/q}\ \sqrt{\ln\frac{3n}{m}},$$

where $c>0$ is an absolute constant. Choosing $m=[1+k/e^q]$ we obtain the desired result. □

Corollary 1.5 *Let* $1 \le q \le \ln(2n)$. *Let* g_i, $i \le n$, *be independent* $N(0,1)$ *Gaussian random variables. There are absolute positive constants* $c_1 < 1$ *and* c_2 *such that for every* $k \le c_1^q n$ *one has*

$$\mathbb{E}\left(\sum_{i=k+1}^{n}(g_i^*)^q\right)^{1/q} \ge c_2\ \sqrt{q}\ n^{1/q}.$$

Remark. Note that

$$\mathbb{E}\left(\sum_{i=k+1}^{n}(g_i^*)^q\right)^{1/q} \le \mathbb{E}\left(\sum_{i=1}^{n}(g_i^*)^q\right)^{1/q} \le C\ \sqrt{q}\ n^{1/q},$$

where C is an absolute constant.

Proof of Corollary ??. By the remark after Corollary **??** there exists an absolute constant $c_2>0$ such that

$$\left(\sum_{i=1}^{n}|g_i|^q\right)^{1/q} \ge 2c_2 n^{1/q}\ \sqrt{q}.$$

Therefore, since $q \geq 1$, using Corollary **??**, we obtain

$$\begin{aligned} \mathbb{E}\left(\sum_{i=k+1}^{n} (g_i^*)^q\right)^{1/q} &\geq \mathbb{E}\left(\sum_{i=1}^{n} |g_i|^q\right)^{1/q} - \mathbb{E}\left(\sum_{i=1}^{k} (g_i^*)^q\right)^{1/q} \\ &\geq 2c_2 n^{1/q}\sqrt{q} - C\ k^{1/q}\sqrt{q + \ln\frac{2n}{k}}, \end{aligned}$$

where C is an absolute positive constant. Since the function $f(x) = x^{2/q}(q + \ln(2n/x))$ is increasing on $[0, n]$, we observe that for $k \leq c_1^q n$ one has

$$\begin{aligned} \mathbb{E}\left(\sum_{i=k+1}^{n} (g_i^*)^q\right)^{1/q} &\geq 2c_2 n^{1/q}\sqrt{q} - C\ c_1\ n^{1/q}\sqrt{q\ln(2e/c_1)} \\ &= (2c_2 - c_1 C\ln(2e/c_1))\, n^{1/q}\sqrt{q}. \end{aligned}$$

Choosing $c_1 > 0$ such that $c_1 C \ln(2e/c_1) \leq c_2$, we obtain the desired result. □

1.3 Interpolation results

The two following lemmas are well-known (see e.g. []).

Lemma 1.6 *Let $1 \leq q_0 < q_1 < \infty$. Let r be such that $1/r = 1/q_0 - 1/q_1$. Let $1 \leq t \leq n^{1/r}$. Let $K = \mathrm{conv}\,\{B_{q_0}, \frac{1}{t}B_{q_1}\}$. Then for every $x \in \mathbb{R}^n$ one has*

$$c\left(\left(\sum_{i\leq t^r}(x_i^*)^{q_0}\right)^{1/q_0} + t\left(\sum_{i>t^r}(x_i^*)^{q_1}\right)^{1/q_1}\right) \leq \|x\|_K$$

$$\leq \left(\left(\sum_{i\leq t^r}(x_i^*)^{q_0}\right)^{1/q_0} + t\left(\sum_{i>t^r}(x_i^*)^{q_1}\right)^{1/q_1}\right)$$

where c is an absolute positive constants. Moreover, if $q_1 = \infty$ then the lemma holds with $r = q_0$ and $x^*_{[t^r+1]}$ instead of second summand.

Lemma 1.7 *There is an absolute constant $c > 0$ such that the following holds. Let $0 < p \leq 1$. Let $\gamma = 1/(1/p - 1/2)$ and $n^{-1/\gamma} < \rho < 1$. Let $K = B_{p\infty} \cap \rho B_2^n$. Then*

$$c \left(\rho \ |||x||| + \sum_{i>m} x_i^* \ i^{-1/p} \right) \leq \sup_{y \in K} \langle x, y \rangle \leq \rho \ |||x||| + \sum_{i>m} x_i^* \ i^{-1/p},$$

where $m = [1/\rho^\gamma]$ and

$$|||x||| = \left(\sum_{i \leq m} (x_i^*)^2 \right)^{1/2}.$$

Remark. Note that if $\rho \leq n^{-1/\gamma}$ then $K = \rho B_2^n$. Note also that if $p < 1$ then

$$\sum_{i>m} x_i^* \ i^{-1/p} \leq \frac{\sqrt{2} \ \rho}{1-p} \ |||x|||.$$

Indeed, since $x_i^* \leq |||x|||/\sqrt{m}$ for every $i \geq m$ and $m \leq 1/\rho^\gamma \leq m+1$, one has

$$\sum_{i>m} x_i^* \ i^{-1/p} \leq \frac{|||a|||}{\sqrt{m}} \sum_{i>m} i^{-1/p}$$

$$\leq |||x||| \ \sqrt{\frac{2}{m+1}} \left(\frac{1}{(m+1)^{1/p}} + \int_{m+1}^{\infty} x^{-1/p} \ dx \right) \leq \frac{\sqrt{2} \ a \ \rho}{1-p}.$$

2 Computing M^* for interpolated body.

Theorem 2.1 *There are positive constants c, $c_1 < 1$ and C such that the following statement holds. Let g_i, $i \leq n$, be independent $N(0,1)$ Gaussian random variables. Let $1 \leq q_0 < q_1 \leq \infty$. Let r be such that $1/r = 1/q_0 - 1/q_1$. Let $1 \leq t \leq c_1^{q_1/r} n^{1/r}$. Let $K = \mathrm{conv}\,\{B_{q_0}, \frac{1}{t} B_{q_1}\}$. Then*

(i) if $q_0 \geq \ln(2n)$ we have

$$c \ \sqrt{\ln(2n)} \leq \mathbb{E} \left\| \sum_{i=1}^n g_i e_i \right\|_K \leq C \ \sqrt{\ln(2n)};$$

(ii) if $q_0 < \ln(2n) \leq q_1$ we have

$$c \ t \ \sqrt{q_0 + \ln(2n/t^{q_0})} \leq \mathbb{E} \left\| \sum_{i=1}^n g_i e_i \right\|_K \leq C \ t \ \sqrt{q_0 + \ln(2n/t^{q_0})};$$

(iii) *if* $q_1 < \ln(2n)$ *and* $t > c_1^{q_1/r} n^{1/r}$ *we have*

$$c \ \sqrt{q_0} \ n^{1/q_0} \leq \mathbb{E} \left\| \sum_{i=1}^{n} g_i e_i \right\|_K \leq C \ c_1^{-q_1/r} \ \sqrt{q_0} \ n^{1/q_0}$$

(iii) *if* $q_1 < \ln(2n)$ *and* $t \leq c_1^{q_1/r} n^{1/r}$ *we have*

$$c \ \left(t^{r/q_0} \ \sqrt{q_0 + \ln(2n/t^r)} + t \ \sqrt{q_1} \ n^{1/q_1} \right) \leq \mathbb{E} \left\| \sum_{i=1}^{n} g_i e_i \right\|_K$$

$$\leq C \ \left(t^{r/q_0} \ \sqrt{q_0 + \ln(2n/t^r)} + t \ \sqrt{q_1} \ n^{1/q_1} \right).$$

Proof:
(i) In this case $e^{-1} B_\infty^n \subset K \subset B_\infty^n$ and the estimate is known.
(ii) In this case $e^{-1} B_\infty^n \subset B_{q_1}^n \subset B_\infty^n$. Therefore, without loss of generality we assume that $q_1 = \infty$, which implies $r = q_0$. By Lemmas **??**, **??** we have

$$\mathbb{E} \ g_{t^r}^* \leq \sqrt{\ln \frac{2n}{t^r}},$$

and, by Lemma **??**,

$$\mathbb{E} \ \left(\sum_{i \leq t^r} (g_i^*)^r \right)^{1/r} \approx t \sqrt{r + \ln \frac{2n}{t^r}} \geq \sqrt{\ln \frac{2n}{t^r}}.$$

Using Lemma **??**, we obtain the desired result.
(iii) Since $B_{q_1}^n \subset n^{1/r} B_{q_0}^n$, in this case, we have $B_{q_0}^n \subset K \subset c_1^{-q_1/r} B_{q_0}^n$ and the estimate is known (cf. Remark after Corollary **??**).
(iv) This case is an immediate consequence of Lemma **??** and Corollaries **??**, **??**. □

Theorem 2.2 *There are positive constants* c *and* C *such that the following statement holds. Let* g_i*,* $i \leq n$*, be independent* $N(0,1)$ *Gaussian random variables. Let* $0 < p \leq 1$*. Let* $\gamma = 1/(1/p - 1/2)$ *and* $n^{-1/\gamma} < \rho < 1$*. Let* $K = B_{p\infty} \cap \rho B_2^n$*. Then*

(i) *for* $p < 1$

$$c \ \rho^{2\frac{1-p}{2-p}} \ \sqrt{\ln (2n\rho^\gamma)} \leq \mathbb{E} \sup_{y \in K} \left\langle \sum_{i=1}^{n} g_i e_i, \ y \right\rangle \leq \frac{C}{1-p} \ \rho^{2\frac{1-p}{2-p}} \ \sqrt{\ln (2n\rho^\gamma)};$$

(ii) for $p = 1$

$$c \ \left(\ln\left(2n\rho^2\right)\right)^{3/2} \le \mathbb{E} \sup_{y\in K} \left\langle \sum_{i=1}^n g_i e_i,\, y \right\rangle \le C \ \left(\ln\left(2n\rho^2\right)\right)^{3/2}.$$

Proof: Let

$$|||x||| = \left(\sum_{i\le m} (x_i^*)^2\right)^{1/2},$$

where $m = [1/\rho^\gamma]$. By Corollary **??**

$$\mathbb{E}\ |||\sum_{i=1}^n g_i e_i||| \approx \sqrt{m\ \ln\frac{2n}{m}}.$$

By Lemma **??** (Remark that follows) we have for $p < 1$

$$c\ \rho\ \sqrt{m\ \ln\frac{2n}{m}} \le \mathbb{E} \sup_{y\in K} \left\langle \sum_{i=1}^n g_i e_i,\, y \right\rangle \le \frac{C\rho}{1-p}\ \ \sqrt{m\ \ln\frac{2n}{m}},$$

where c and C are absolute positive constants.

Now let $p = 1$. Then $m = [1/\rho^2]$ and, hence,

$$\rho\, \mathbb{E}\ |||\sum_{i=1}^n g_i e_i||| \approx \sqrt{\ln\frac{2n}{m}}.$$

Using Lemmas **??**, **??**, we observe that there are absolute constants C_0, C_1, such that

(i) for every $m > n/2$,

$$\mathbb{E}\sum_{i>m} g_i^*\ i^{-1} \approx \sum_{i>m} \frac{1}{i}\ \frac{n-i+1}{n} \approx \frac{(n-m)(n+1-m)}{n^2};$$

(ii) for every $n/4 < m < n/2$

$$C_0 \le \mathbb{E}\sum_{i>m} g_i^*\ i^{-1} \le C_1,$$

(iii) for every $m \le n/4$,

$$\mathbb{E}\sum_{m<i\le n/2} g_i^*\ i^{-1} \approx \sum_{m<i\le n/2} \frac{1}{i}\ \sqrt{\ln\frac{2n}{i}} \approx \left(\ln\frac{2n}{m}\right)^{3/2}.$$

By Lemma **??** it implies that

$$\mathbb{E}\sup_{y\in K}\left\langle \sum_{i=1}^{n} g_i e_i,\, y\right\rangle \approx \left(\ln\frac{2n}{m}\right)^{3/2},$$

which completes the proof. □

3 Addition

Let K, L be two centrally symmetric convex bodies in R^n, and $\{e_i\}_{i=1}^n$ be a 1-symmetric basis for both norms $\|.\|_K, \|.\|_L$. Given $t > 0$, denote by $K_t := K \cap tL$, then $K_t^0 := \operatorname{conv}\left(K^0 \cup \frac{L^0}{t}\right)$ is the polar ball of K_t. It is clear that for all $x \in \mathbb{R}^n$, $\|x\|_{K_t^0} = \inf_{x=y+z}\left(\|y\|_{K^0} + t\,\|z\|_{L^0}\right)$, and therefore for all $0 \le T \le n$

$$\left\|\sum_{i=1}^{n} x_i e_i\right\|_{K_t^0} \le \left\|\sum_{i=1}^{T} x_i^* e_i\right\|_{K^0} + t\left\|\sum_{i=1+T}^{n} x_i^* e_i\right\|_{L^0}$$

hence

$$\mathbb{E}\left\|\sum_{i=1}^{n} g_i e_i\right\|_{K_t^0} \le \mathbb{E}\left\|\sum_{i=1}^{T} g_i^* e_i\right\|_{K^0} + t\mathbb{E}\left\|\sum_{i=1+T}^{n} g_i^* e_i\right\|_{L^0}.$$

We would like to consider the tightness of these inequalities.

Theorem 3.1 *Let L, K be two centrally symmetric convex bodies in $\mathbb{R}^n$ admitting the same 1-symmetric canonical basis $\{e_i\}_{i=1}^n$ for both norms. Let $t > 0$, and for $1 \le m \le n$ define $f(m) := \max_{\|x\|_{L^0}=1}\|\sum_{i=1}^m x_i e_i\|_{K^0}$. Let $1 \le T \le n$ and*

$$C(T,t) := 2 + 2\max\left\{ t\left\|\sum_{i=[(T+1)/2]}^{[(n+1)/2]} \frac{e_i}{\left\|\sum_{j=1}^{i} e_j\right\|_{K^0}}\right\|_{L^0},\ \frac{f([\frac{T+1}{2}])}{t}\right\}$$

Let $T(t)$ be that integer T which solves the equation

$$t^2 = f\left([(T+1)/2]\right)\left\|\sum_{i=[(T+1)/2}^{[(n+1)/2]} \frac{e_i}{\left\|\sum_{j=1}^{i} e_j\right\|_{K^0}}\right\|_{L^0}^{-1}$$

and assume that the solution $T(t)$ is in $[1,n]$.
Then, setting $C(t) = C(T(t),t)$ we have

$$C(t) = 2 + 2\sqrt{f\left([(T(t)+1)/2]\right) \cdot \left\|\sum_{i=[(T(t)+1)/2}^{[(n+1)/2]} \frac{e_i}{\left\|\sum_{j=1}^{i} e_j\right\|_{K^0}}\right\|_{L^0}}$$

moreover $C(t) = \min_{1\le T\le n} C(T,t)$, and for all $x \in \mathbb{R}^n$

$$\|x\|_{K_t^0} \le \left\|\sum_{i=1}^{T(t)} x_i^* e_i\right\|_{K^0} + t\left\|\sum_{i=T(t)+1}^{n} x_i^* e_i\right\|_{L^0} \le C(t)\,\|x\|_{K_t^0}.$$

In particular

$$\mathbb{E}\left\|\sum_{i=1}^{n} g_i e_i\right\|_{K_t^0} \le \mathbb{E}\left\|\sum_{i=1}^{T(t)} g_i^* e_i\right\|_{K^0} + t\mathbb{E}\left\|\sum_{i=1+T(t)}^{n} g_i^* e_i\right\|_{L^0} \le C(t)\mathbb{E}\left\|\sum_{i=1}^{n} g_i e_i\right\|_{K_t^0}.$$

For the proof we need the following Lemma

Lemma 3.2 *Let $x = y+z$, $1 \le k,l$ and $k+l \le n$. Then*

$$(y+z)_{k+l}^* \le (y+z)_{k+l-1}^* \le y_k^* + z_l^*.$$

Proof of Theorem ??: Let $x = y+z$ where y,z are chosen so that $\|x\|_{K_t^0} = \|y\|_{K^0} + t\|z\|_{L^0}$. Let $1 \le T \le n$, then we have using Lemma **??** and the symmetry of the basis

$$\left\|\sum_{i=1}^{T} x_i^* e_i\right\|_{K^0} = \left\|\sum_{2\le 2i\le T} x_{2i}^* e_{2i} + \sum_{1\le 2i-1\le T} x_{2i-1}^* e_{2i-1}\right\|_{K^0}$$

$$\le \left\|\sum_{2\le 2i\le T} (y_i^* + z_i^*) e_{2i} + \sum_{1\le 2i-1\le T} (y_i^* + z_i^*) e_{2i-1}\right\|_{K^0} \le 2A + 2B$$

where

$$A := \left\| \sum_{i=1}^{[(T+1)/2]} y_i^* e_i \right\|_{K^0} \leq \|y\|_{K^0}, \qquad B := \left\| \sum_{i=1}^{[(T+1)/2]} z_i^* e_i \right\|_{K^0}.$$

Similarly, setting

$$C := \left\| \sum_{i=[(T+1)/2}^{[(n+1)/2]} y_i^* e_i \right\|_{L^0}, \qquad D := \left\| \sum_{i=[(T+1)/2}^{[(n+1)/2]} z_i^* e_i \right\|_{L^0} \leq \|z\|_{L^0}$$

we have that

$$\left\| \sum_{T+1 \leq i \leq n} x_i^* e_i \right\|_{L^0} = \left\| \sum_{T+1 \leq 2i \leq n} x_{2i}^* e_{2i} + \sum_{T+1 \leq 2i-1 \leq n} x_{2i-1}^* e_{2i-1} \right\|_{L^0}$$

$$\leq \left\| \sum_{T+1 \leq 2i \leq n} (y_i^* + z_i^*) e_{2i} + \sum_{T+1 \leq 2i-1 \leq n} (y_i^* + z_i^*) e_{2i-1} \right\|_{L^0} \leq 2C + 2D.$$

It follows that

$$\left\| \sum_{i=1}^{T} x_i^* e_i \right\|_{K^0} + t \left\| \sum_{i=T+1}^{n} x_i^* e_i \right\|_{L^0} \leq 2(A+B) + 2t(C+D).$$

Clearly $B \leq f\left([\frac{T+1}{2}]\right) \|z\|_{L^0}$. We now use the obvious inequality which is valid for all $1 \leq i \leq n$, $y_i^*. \left\| \sum_{j=1}^{i} e_j \right\|_{K^0} \leq \|y\|_{K^0}$, and it follows that

$$C \leq \left\| \sum_{i=[(T+1)/2]}^{[\frac{n+1}{2}]} \frac{e_i}{\left\| \sum_{j=1}^{i} e_j \right\|_{K^0}} \right\|_{L^0} .\|y\|_{K^0}.$$

Combining all the inequalities , we obtain that for all $1 \leq T \leq n$,

$$2(A+B)+2t(C+D) \leq 2\left\{1+t\left\|\sum_{i=[(T+1)/2}^{[\frac{n+1}{2}]}\frac{e_i}{\left\|\sum_{j=1}^{i}e_j\right\|_{K^0}}\right\|_{L^0}\right\}\|y\|_{K^0}$$

$$+2t\left\{1+\frac{f([\frac{T+1}{2}])}{t}\right\}\|z\|_{L^0} \leq C(T,t)\{\|y\|_K+t\|z\|_{L^0}\} = C(T,t)\|x\|_{K_t^0}$$

where

$$C(T,t) := 2+2\max\left\{t\left\|\sum_{i=[(T+1)/2}^{[(n+1)/2]}\frac{e_i}{\left\|\sum_{j=1}^{i}e_j\right\|_{K^0}}\right\|_{L^0}, \frac{f\left([\frac{T+1}{2}]\right)}{t}\right\},$$

and now we choose now $T(t)$ as in the statement of the theorem. It is easy to see that the function $f([\frac{T+1}{2}])$ is an increasing function for T in $[1,n]$ and that the function

$$\left\|\sum_{i=[(T+1)/2}^{[(n+1)/2]}\frac{e_i}{\left\|\sum_{j=1}^{i}e_j\right\|_{K^0}}\right\|_{L^0}$$

is a decreasing function of T, hence the value $C(t) := \min_{0\leq T\leq n} C(T,t)$ is achieved at a point $T=T(t)$ for which

$$t^2 = f\left([(T+1)/2]\right)\left\|\sum_{i=[(T+1)/2}^{[(n+1)/2]}\frac{e_i}{\left\|\sum_{j=1}^{i}e_j\right\|_{K^0}}\right\|_{L^0}^{-1},$$

recall that we assume $1\leq T(t)\leq n$. and we obtain

$$\|x\|_{K_t^0} \leq \left\|\sum_{i=1}^{T(t)}x_i^*e_i\right\|_{K^0}+t\left\|\sum_{i=T(t)+1}^{n}x_i^*e_i\right\|_{L^0} \leq C(t)\|x\|_{K_t^0}.$$

Remark: If t is such that $K\cap\frac{L}{t}$ is equal to K or to $\frac{L}{t}$, then this is a trivial uninteresting case, and will yield that the solution $T(t)$ will not be in the interval $[1,n]$.

A special interesting case is when $K = B_2^n$, which yields

$$t^2 = \max_{\|x\|_{L^0}=1} \left(\sum_{i=1}^{[(T+1)/2]} x_i^2 \right)^{1/2} \cdot \left\| \sum_{i=[(T+1)/2}^{[(n+1)/2]} \frac{e_i}{\sqrt{i}} \right\|_{L^0}^{-1}.$$

Taking $K = B_{q_0}^n, L = B_{q_1}^n$, the theorem provides the proof of Lemma 6 above for the appropriate choice of $T = T(t)$.

References

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