RATIOS OF VOLUMES AND FACTORIZATION THROUGH ℓ_∞

Y. GORDON* M. MEYER A. PAJOR

Abstract

We develop a method effective in the computation of projection constants $\lambda(X)$ for n-dimensional normed, as well as quasi-normed spaces X. We show that for every centrally symmetric compact body K in $\mathbb{R}^n$ there exist a parallelotope P and a zonoid Z such that $Z \subset K \subset P$, and $\Big(\operatorname{vol}_n(P)/\operatorname{vol}_n(Z)\Big)^{1/n} \leq \lambda(X)$, where $\lambda(X)$ is the projection constant of the quasi normed space $X = (\mathbb{R}^n, \|.\|)$ having K as its unit ball. Thus, it follows easily that the projection constant of the quasi-normed space ℓ_p^n $(0 < p < 1)$ is equivalent to $n^{1/p-1/2}$. This improves the estimate due to T. Peck. The method also yields easily the projection constants of the quasi-normed Schatten classes s_p^n $(0 < p < 1)$, consisting of operators on ℓ_2^n, and other examples, including classical examples of normed spaces.

In addition, we establish relations between various other known parameters associated with the geometry of normed spaces, such as volume-ratio, the ℓ_∞ norm of operators between quasi-normed spaces, and others. We prove that if X is a normed space of dimension n then the unit ball B of X and B^* of X^*, contain zonoids Z_1 and Z_2 respectively, such that $\left(\frac{\operatorname{vol}_n(B)\operatorname{vol}_n(B^*)}{\operatorname{vol}_n(Z_1)\operatorname{vol}_n(Z_2)}\right)^{1/n} \leq c\, gl_2(X)$, where c is an absolute constant and $gl_2(X)$ is the Gordon-Lewis parameter of X.

We show also that if K is the convex hull of m points in $\mathbb{R}^n$, $n \leq m$, then there is a simplex Δ inside K satisfying the inequality $\big(\operatorname{vol}_n(K)\big)^{1/n} \leq c\sqrt{\ln \frac{2m}{n}}\,\big(\operatorname{vol}_n(\Delta)\big)^{1/n}$.

The authors wish to thank E. D. Gluskin for valuable discussions and suggestions in the preparation of this article.

* Partially supported by the fund for the Promotion of Research at the Technion, and the V.P.R. grant.

Introduction

The projection constant $\lambda(X)$ of a finite dimensional normed space X is often difficult to compute, but it plays an important role in the classical and in the local theory of Banach spaces. We extend its definition in a natural way so as to include the class of quasi-normed spaces as well, and we present a new method for getting a lower bound for $\lambda(X)$ in terms of ratios of volumes. This bound allows us for example to to obtain easily the right asymptotic estimate for $\lambda(\ell_p^n)$ in the case $0 < p < 1$ and dispenses of the logarithmic factor in the estimate obtained by Peck (**[Pe]**) who used some involved probabilistic method. The method applies also for the Schatten classes s_p^n $(0 < p \leq 1)$ of operators on ℓ_2^n.

Given a centrally symmetric body K in $\mathbb{R}^n$ we can endow $\mathbb{R}^n$ with the quasi-norm defined by

$$\|x\| = \inf\{a > 0;\ x \in aK\}\,,$$

and set $E = (\mathbb{R}^n, \|.\|)$ to be the n-dimensional quasi-normed space with K as its unit ball. Let $B = B_X$ be the unit ball of a given Banach space X. We define the volume ratio $\mathrm{vr}(E, X)$, sometimes denoted also by $\mathrm{vr}(K, B)$, to be

$$\mathrm{vr}(E, X) = \inf \Big(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(T(B))}\Big)^{1/n}$$

where the infimum ranges over all onto linear maps $T : X \to \mathbb{R}^n$ satisfying $T(B) \subset K$. We define the external volume ratio $\mathrm{evr}(E, X)$, denoted sometimes also by $\mathrm{evr}(K, B)$, to be

$$\mathrm{evr}(E, X) = \inf \Big(\frac{\mathrm{vol}_n(T(B))}{\mathrm{vol}_n(K)}\Big)^{1/n}$$

where the infimum ranges over all onto linear maps $T : X \to \mathbb{R}^n$ such that $T(B) \supset K$. For $0 < p \leq \infty$, let ℓ_n^p be the space $\mathbb{R}^n$ equipped with the quasi-norm

$$\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$$

and let B_p^n be its unit ball:

$$B_p^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n\,;\ \sum_{i=1}^n |x_i|^p \leq 1\}\,.$$

Let $Q_n = [-1, 1]^n = B_\infty^n$ be the unit cube of $\mathbb{R}^n$ and $\mathbf{C}_n = B_1^n$ be its polar body:

$$\mathbf{C}_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n;\ \sum_{i=1}^n |x_i| \leq 1\}\,.$$

The ratio $\mathrm{evr}(K, B_\infty^n) = \mathrm{evr}(K, Q_n)$, known also as the *cubic ratio* of K, was studied by various authors (see **[B1]**, **[Ge]**, **[PS]**) in relation with the classical *volume ratio*

$\mathrm{vr}(K, B_n^2)$. The *zonoid ratio*, which is in our notation $\mathrm{vr}(K, B_{\ell_\infty})$, was also studied in [**B1**].

We prove that if K is the unit ball of a quasi-normed space X, then

$$\mathrm{evr}(K, Qatdt322044_n)\,\mathrm{vr}(K, B_{\ell_\infty}) \le \lambda(X)$$

geometrically this means that there exist a parallelotope P and a zonoid Z such that $Z \subset K \subset P$, and $\left(\frac{vol_n(P)}{vol_n(Z)}\right)^{1/n} \le \lambda(X)$, and we study various relations between the above mentioned quantities and other parameters associated with centrally symmetric, and not necessarily convex, bodies K.

Notation.

If I is a finite set , we shall denote by $|I|$ its cardinality, and by $\mathcal{M}_{n,m}$ the set of all matrices $A = [a_{ij}]_{i=1,\dots,n,\, j=1,\dots,m}$ with real entries consisting of n rows and m columns. For $I \subset \{1, 2, \dots, n\}$ and $J \subset \{1, 2, \dots, m\}$, let $A_{IJ} = [a_{ij}]_{i\in I,\, j\in J}$. We will make use of the following Cauchy-Binet formula: If $A \in \mathcal{M}_{n,m}$ and $B \in \mathcal{M}_{m,n}$, and if $N = \{1, \dots, n\}$ and $M = \{1, \dots, m\}$, $1 \le n \le m$, then

$$\det(AB) = \sum_{I\subset M, |I|=n} \det(A_{NI}) \det(B_{IN}) \,.$$

Let $v_n = \pi^{\frac{n}{2}}/\Gamma(1 + \frac{n}{2})$ denote the volume of the Euclidean ball B_2^n of $\mathbb{R}^n$; then $v_n^{1/n} \sim \sqrt{\frac{2\pi e}{n}}$. If K is a centrally symmetric body (not necessarily convex) in $\mathbb{R}^n$, we note that by our definition

$$\mathrm{vr}(K, \ell_1^n) = \min\left\{\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(\mathbf{C})}\right)^{1/n};\ \mathbf{C} \subset K \text{ is the symmetric convex hull of } n \text{ points}\right\}.$$

and

$$\mathrm{vr}(K, B_{\ell_\infty}) = \min\left\{\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(Z)}\right)^{1/n};\ Z \subset K \text{ is a zonoid}\right\}.$$

The cubic ratio of K is

$$\mathrm{evr}(K, Q_n) = \min\left\{\left(\frac{\mathrm{vol}_n(P)}{\mathrm{vol}_n(K)}\right)^{1/n};\ P \text{ is a parallelotope containing } K\right\},$$

and the classical volume ratio of K is

$$\mathrm{vr}(K, B_2^n) = \min\left\{\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(D)}\right)^{1/n};\ D \subset K \text{ is an ellipsoid}\right\}.$$

Since B_2^n is a zonoid and ℓ_1^n has uniformly bounded volume ratio, it is clear that

$$\mathrm{vr}(E, \ell_\infty) \le \mathrm{vr}(E, \ell_2^n) \le \mathrm{vr}(E, \ell_1^n)\,\mathrm{vr}(\ell_1^n, \ell_2^n) \sim \sqrt{\frac{2e}{\pi}}\,\mathrm{vr}(E, \ell_1^n)\,.$$

Ratios of Volumes and factorization through ℓ_∞

The following proposition is essentially known ([**B1**], [**Ge**], [**PS**]).

Proposition 1. *Let K be a centrally symmetric convex body in $\mathbb{R}^n$ and K^o be its polar body with respect to the ordinary scalar product denoted by $<,>$. Suppose that $u_i \in R^n$, $< u_i, u_i >= 1$ and $c_i > 0$, $i = 1, \dots, m$ satisfy*

$$\sum_{i=1}^{m} c_i < u_i, x > u_i = x$$

for every $x \in \mathbb{R}^n$. Then

(i) *If $u_i \in K^o$, $i = 1 \dots, m$, there exists a parallelotope P such that $K \subset P$ and $\big(\mathrm{vol}_n(P)\big)^{1/n} \le \sqrt{e}\big(\mathrm{vol}_n(Q_n)\big)^{1/n} = 2\sqrt{e}$.*

(ii) *If $u_i \in K$, $i = 1 \dots, m$, there exists a cross-polytope $\mathbf{C}$ such that $C \subset K$ and $\big(\mathrm{vol}_n(\mathbf{C})\big)^{1/n} \ge \frac{1}{\sqrt{e}}\big(\mathrm{vol}_n(\mathbf{C}_n)\big)^{1/n} \sim 2\frac{\sqrt{e}}{n}$.*

Proof: If $C \in \mathcal{M}_{n,m}$ is the matrix whose columns are the coordinates of the vectors $\sqrt{c_i}\, u_i$, $1 \le i \le m$, in the canonical basis of $\mathbb{R}^n$, we have $C^*C = I_n$, where I_n denotes the identity on $\mathbb{R}^n$ and thus $\sum_{i=1}^m c_i = n$. It follows from the Cauchy-Binet identity that

$$1 = \sum_{I \subset \{1,\dots,m\},\, |I|=n} \big(\prod_{i \in I} c_i\big)\big(\det(u_i)_{i\in I}\big)^2$$

$$\le \binom{m}{n} \max_{I \subset \{1,\dots,m\},\, |I|=n} \big(\det(u_i)_{i\in I}\big)^2 \Big(\frac{\sum_{|I|=n}\prod_{i\in I} c_i}{\binom{m}{n}}\Big)$$

Now, since $\sum_{i=1}^m c_i = n$, we get by Newton's inequality

$$1 \le \binom{m}{n}\Big(\frac{n}{m}\Big)^n \max_{I \subset \{1,\dots,m\},\, |I|=n} \big(\det(u_i)_{i\in I}\big)^2 .$$

It follows that

$$\max_{I \subset \{1,\dots,m\},\, |I|=n} |\det_{i\in I}(u_i)|^{1/n} \ge \frac{1}{\sqrt{e}} .$$

In case (i) for some $I \subset \{1, \dots, m\}$, $|I| = n$, the parallelotope $P = \{x \in \mathbb{R}^n;\ | < x, u_i > | \le 1 \text{ for every } i \in I\}$ satisfies the required properties. Case (ii) follows from (i) by replacing K^o with K and taking $\mathbf{C} = P^o$. $\square$

The following results relate $\mathrm{vr}(K, B_2^n)$ to $\mathrm{evr}(K, Q_n)$. It is a direct consequence of the preceding proposition.

Corollary 2. *(**[B2]**,**[Ge]**,**[PS]**) Let K be a convex symmetric convex body, then*

$$\mathrm{vr}(Q_n, B_2^n) \le \mathrm{evr}(K, Q_n)\, \mathrm{vr}(K, B_2^n) \le \sqrt{e}\ \mathrm{vr}(Q_n, B_2^n)$$

where $\mathrm{vr}(Q_n, B_2^n) = \frac{2}{v_n^{1/n}} \sim \sqrt{\frac{2n}{\pi e}}$.

Proof: The left-hand side inequality follows from the definition. For the right hand-side, we may suppose that the Euclidean ball is the maximal volume ellipsoid inside K, that is the John ellipsoid of K; then by **[J]**, both assumptions (i), (ii) of Proposition 1 are satisfied. The result follows. $\square$

The next corollary is an improvement of an estimate due independently to many authors (**[BF]**,**[BP]**, **[C]**, **[Gl]**, and B. Maurey in **[Pi1]**).

Corollary 3. *There exists a constant $c > 0$ such that if $x_1, \ldots, x_m \in \mathbb{R}^n$, $m \geq n$ and $K = \mathrm{conv}(\pm x_i, 1 \leq i \leq m)$, then*

$$\big(\mathrm{vol}_n(K)\big)^{1/n} \leq c\sqrt{\ln\left(\frac{2m}{n}\right)} \max_{I\subset\{1,\ldots,m\},|I|=n} \Big(\mathrm{vol}_n\big(\mathrm{conv}(\pm x_i, i \in I)\big)\Big)^{1/n}.$$

Proof: Let $A \in \mathcal{M}_{n,n}$ be a matrix such that the minimal volume ellipsoid containing the body $K' = AK$ is the Euclidean ball. Then again the assumptions of Proposition 1,(ii) are satisfied. Thus there exists a cross-polytope $\mathbf{C} \subset K'$ such that $\big(\mathrm{vol}_n(\mathbf{C})\big)^{1/n} \geq \frac{1}{\sqrt{e}}(\big(\mathrm{vol}_n(\mathbf{C}_n)\big)^{1/n}$. It follows that

$$\Big(\frac{\mathrm{vol}_n(K)}{\max_{y_1,\ldots,y_n\in K}\mathrm{vol}_n\big(\mathrm{conv}(\pm y_i, 1 \leq i \leq n)\big)}\Big)^{1/n}$$

$$= \Big(\frac{\mathrm{vol}_n(K')}{\max_{z_1,\ldots,z_n\in K'}\mathrm{vol}_n\big(\mathrm{conv}(\pm z_i, 1 \leq i \leq n)\big)}\Big)^{1/n}$$

$$\leq \Big(\frac{\mathrm{vol}_n(K')}{\mathrm{vol}_n(\mathbf{C})}\Big)^{1/n} \leq \sqrt{e}\Big(\frac{\mathrm{vol}_n(K')}{\mathrm{vol}_n(\mathbf{C}_n)}\Big)^{1/n}.$$

But since $K' \subset B_2^n$ is the convex hull of $Ax_1, \ldots, Ax_m$, it follows from [**BP**], [**BF**], [**CS**], [**Gl**] or [**Pi1**] that for some constant $d > 0$, independent of n and m, we have

$$\big(\mathrm{vol}_n(K')\big)^{1/n} \leq d\sqrt{\ln\left(\frac{2m}{n}\right)}\,\big(\mathrm{vol}_n(\mathbf{C}_n)\big)^{1/n}.$$

Combining the preceding inequalities, we get our estimate. □

Remark. If the convex body K is not supposed to be centrally symmetric, then Proposition 1 can be generalized in both cases if we replace the parallelotope P and the cross-polytope $\mathbf{C}$ by a simplex, and Q_n and $\mathbf{C}_n$ by the regular simplices circumscribed to B_2^n and inscribed in B_2^n. Observe also that Corollary 3 can be generalized as follows: if $K = \mathrm{conv}(x_1, \ldots, x_m)$, then

$$\big(\mathrm{vol}_n(K)\big)^{1/n} \leq d\sqrt{\ln\left(\frac{2m}{n}\right)}\ \max\{\big(\mathrm{vol}_n(\Delta)\big)^{1/n};\, \Delta \text{ simplex },\, \Delta \subset K\},$$

for some constant $d > 0$ independent of n, $m > n + 1$ and $x_1, \ldots, x_m \in \mathbb{R}^n$.

If E is a subspace of $\mathbb{R}^n$, then P_E will denote the orthogonal projection onto E.

Lemma 4. *Let B be a symmetric body in $\mathbb{R}^n$ (not necessarily convex). Let $1 \leq k \leq n \leq m$ and $T = VU$, with $T \in \mathcal{M}_{n,n}$, $V \in \mathcal{M}_{n,m}$ and $U \in \mathcal{M}_{m,n}$, $\mathrm{rank}(T) = k$ and $UB \subset \|U\|Q_m$, where $\|U\| > 0$. Then*

$$\frac{\mathrm{vol}_k(TB)}{\mathrm{vol}_k(P_{\ker(T)^\perp}B)} \leq \frac{k!}{4^k}\sqrt{\binom{n}{k}}\,\lambda_k(B^o)\|U\|^k \max_{\dim(E)=k}\mathrm{vol}_k(P_E V Q_m)$$

$$= \sqrt{\binom{n}{k}}\, \|U\|^k \max_{\dim(E)=k} \mathrm{vol}_k(P_E V Q_m)\Big(\min_{\dim(E)=k} \big(\mathrm{evr}(P_E B, Q_k)^k \,\mathrm{vol}_k(P_E B)\big)\Big)^{-1}.$$

where $B^o = \{y \in \mathbb{R}^n;\ < x, y > \le 1,$ *for all* $x \in B\}$ *and for a subset* C *of* $\mathbb{R}^n$, $\lambda_k(C) =: \max\{\mathrm{vol}_k\big(\mathrm{conv}(\pm x_1, \dots, \pm x_k)\big); x_1, \dots, x_k \in C\}$.

Proof: By standard linear algebra methods, we have

$$\mathrm{vol}_k(TB) = \Big(\sum_{|I|=|J|=k} \big(\det(T_{IJ})\big)^2\Big)^{1/2} \ \mathrm{vol}_k(P_{\ker(T)^\perp} B)\,.$$

For $I, J \subset \{1, \dots, n\}$ with $|I| = |J| = k$ define $t_{IJ} = \det(T_{IJ})$ and similarly for u_{IJ} and v_{IJ}. Then

$$t_{IJ} = \sum_{K\subset\{1,\dots,m\},|K|=k} v_{IK}u_{KJ}\,,$$

so that

$$\sum_{|I|=|J|=k} t_{IJ}^2 = \sum_{|I|=|J|=k} \Big(\sum_{|K|=|L|=k} v_{IK}u_{KJ}v_{IL}u_{LJ}\Big)$$

$$= \sum_I \Big(\sum_{K,L} v_{IK}v_{IL} \big(\sum_J u_{KJ}u_{LJ}\big)\Big) \le \big(\max_K \sum_J u_{KJ}^2\big) \ \Big(\sum_I \big(\sum_K |v_{IK}|\big)^2\Big)$$

For $I \subset \{1, \dots, n\}$, $\mathrm{card}\,(I) = k$, let $\Pi_I : \mathbb{R}^n \to \mathbb{R}^I$ denote the orthogonal projection; we have

$$\sum_{|K|=k} |v_{IK}| = 2^{-k}\ \mathrm{vol}_k(\Pi_I V Q_m)\,.$$

Indeed, $Q_m = \sum_{j=1}^m [-e_j, e_j]$, hence $\Pi_I V Q_m = \sum_{j=1}^m [-\Pi_I v_j, \Pi_I v_j] \subset \Pi_I(\mathbb{R}^n) = \mathbb{R}^k$, where $\{v_j\}_{j=1}^m$ denote the columns of the matrix V; and now it is well known that if $Z = \sum_{j=1}^m [-z_j, z_j]$ is a zonotope in $\mathbb{R}^k$ then

$$\mathrm{vol}_k(Z) = 2^k \sum_{J\subset\{1,\dots,m\},|J|=k} |\det(z_j; j \in J)|\,.$$

It follows that

$$\sum_I \big(\sum_K |v_{IK}|\big)^2 \le \binom{n}{k} \big(2^{-k} \max_{\dim(E)=k} \mathrm{vol}_k(P_E V Q_m)\big)^2\,.$$

Observe also that since $U(B) \subset \|U\|\, Q_m$, the rows $U_1, \dots, U_m$ of the matrix U are vectors of $\mathbb{R}^n$ which satisfy $U_i \in \|U\| B^o$ for $1 \le i \le m$. We have then for $K \subset \{1, \dots, m\}$, $|K| = k$,

$$\Big(\sum_{J\subset\{1,\dots,n\},|J|=k} u_{KJ}^2\Big)^{1/2} = \frac{k!}{2^k}\ \mathrm{vol}_k\big(\mathrm{conv}(\pm U_i;\ i \in K)\big)\,.$$

This may require an explanation: Let $\{z_i\}_{i=1}^k$ be k vectors in $\mathbb{R}^n$ and $C = \mathrm{conv}(\pm z_i; 1 \le i \le k)$. Denote by $Z \in \mathcal{M}_{k,n}$ the matrix with z_i's as rows. Then there is an orthogonal matrix $\Lambda \in \mathcal{M}_{n,n}$ such that $Z\Lambda = [W, 0]$, where $W \in \mathcal{M}_{k,k}$ is a the matrix with rows $\{w_i\}_{i=1}^k$ and 0 denotes the zero matrix in $\mathcal{M}_{k,n-k}$. Obviously, $WW^* = ZZ^*$, and we obtain that

$$\mathrm{vol}_k(C) = \mathrm{vol}_k\big(\mathrm{conv}(\pm w_i, 1 \le i \le k)\big) = \mathrm{vol}_k\big(W(\mathbf{C}_k)\big) = \frac{2^k}{k!}|\det(W)|$$

$$= \frac{2^k}{k!}\sqrt{\det(WW^*)} = \frac{2^k}{k!}\sqrt{\det(ZZ^*)} = \frac{2^k}{k!}\Big(\sum_{J\subset\{1,\dots,n\},|J|=k}\big(\det(Z_{KJ})\big)^2\Big)^{1/2},$$

where $K = \{1, \dots, k\}$.
It follows that

$$\big(\max_K \sum_J u_{KJ}^2\big)^{1/2} \le \frac{k!}{2^k}\|U\|^k \lambda_k(B^o).$$

Finally, we have by duality

$$\frac{k!}{2^k}\lambda_k(B^o) = 2^k\Big(\min_{\dim(E)=k}\big(\mathrm{evr}(P_E B, Q_k)^k \,\mathrm{vol}_k(P_E B)\big)\Big)^{-1}.$$

The lemma follows. $\square$

Lemma 5. *Let K be a symmetric body in $\mathbb{R}^n$ (not necessarily convex). Let $1 \le n \le m$ and $T = VU$, with $T \in \mathcal{M}_{n,n}$, $V \in \mathcal{M}_{n,m}$ and $U \in \mathcal{M}_{m,n}$, with $\mathrm{rank}(T) = n$ and $U(K) \subset \|U\|Q_m$, where $\|U\| > 0$. Then*

$$\mathrm{evr}(K, Q_n)\,|\det(T)|^{1/n} \le \|U\|\Big(\frac{\mathrm{vol}_n\big(V(Q_m)\big)}{\mathrm{vol}_n(K)}\Big)^{1/n}.$$

Proof: Apply Lemma 4 with $k = n$. $\square$

Let now E and F be two n-dimensional quasi-normed spaces with unit balls B_E and B_F respectively, and for $T \in L(E, F)$ define

$$\gamma_\infty(T) = \inf\big\{\|U\|\|V\|\big\}$$

where the infimum is taken over all the factorizations $T = VU$, $U \in L(E, \ell_\infty)$, $V \in L(\ell_\infty, F)$, and if B_∞ denotes the unit ball of ℓ_∞, $\|U\| = \inf\{a > 0; U(B_E) \subset aB_\infty\}$ and $\|V\| = \inf\{b > 0; V(B_\infty) \subset bB_F\}$.
For a centrally symmetric body K in $\mathbb{R}^n$, if E is the quasi-normed space such that $B_E = K$, we define the projection constant of E or, of K, to be

$$\lambda(K) = \lambda(E) = \gamma_\infty(I)$$

where $I : E \to E$ denotes the identity.

Theorem 6. *Let $T \in L(E,F)$, where $E = (\mathbb{R}^n, \| \|_E)$, and $F = (\mathbb{R}^n, \| \|_F)$ are n-dimensional quasi-normed spaces. Then,*

$$\mathrm{evr}(E, \ell_\infty^n)\ \mathrm{vr}(F, \ell_\infty) |\det T|^{1/n} \leq \gamma_\infty(T) \Big(\frac{\mathrm{vol}_n(B_F)}{\mathrm{vol}_n(B_E)} \Big)^{1/n}.$$

Proof: By Lemma 5, taking $K = B_E$, we obtain that for any factorization $T = VU$ through ℓ_∞^m

$$\mathrm{evr}(E, \ell_\infty^n)\, |\det(T)|^{1/n} \leq \|U\| \Big(\frac{\mathrm{vol}_n\big(V(Q_m)\big)}{\mathrm{vol}_n(B_E)} \Big)^{1/n}.$$

Thus, if Z is the zonoid $\|V\|^{-1} V(Q_m)$, then $Z \subset B_F$. □

Corollary 7. *If E is an n- dimensional quasi-normed space, with unit ball B_E, then*

$$\begin{aligned} \lambda(E) &\geq \mathrm{evr}(E, \ell_\infty^n)\ \mathrm{vr}(E, \ell_\infty) \\ &= \max \Big\{ \big(\tfrac{\mathrm{vol}_n(P)}{\mathrm{vol}_n(Z)} \big)^{1/n};\ Z \text{ zonoid, } P \text{ parallelotope, } Z \subset B_Z \subset P \Big\}. \end{aligned}$$

Remarks.

1) It is easy to prove that, under the hypothesis of the preceding corollary, we have

$$\mathrm{evr}(E, \ell_\infty^n) \leq \inf \Big\{ \|U\| \Big(\frac{\mathrm{vol}_n\big(V(B_\infty)\big)}{\mathrm{vol}_n(B_E)} \Big)^{1/n} \Big\}$$

where the infimum is taken over all linear operators $U : X \to \ell_\infty$ and $V : \ell_\infty \to X$ such that VU is the identity on $\mathbb{R}^n$.

2) By Lemma 5, if E, F are n-dimensional, then

$$\sup_{T: E \to F, T \neq 0} \frac{|\det(T)|^{1/n}}{\gamma_\infty(T)} \leq \frac{1}{\mathrm{evr}(E, \ell_\infty^n)\ \mathrm{vr}(F, \ell_\infty)} \Big(\frac{\mathrm{vol}_n(B_F)}{\mathrm{vol}_n(B_E)} \Big)^{1/n}.$$

In particular it follows that

$$\mathrm{evr}(E, \ell_\infty^n)\, \mathrm{evr}(F, \ell_\infty^n)\, \mathrm{vr}(E, \ell_\infty)\, \mathrm{vr}(F, \ell_\infty) \leq \inf_{T \in L(E,F)} \left\{ \gamma_\infty(T) \gamma_\infty(T^{-1}) \right\}.$$

Let us recall now some definitions; if X is a normed space, we define the following quantities:

1) The Gordon-Lewis constant (according to G. Pisier [**Pi1**]) $gl_2(X)$ is the least constant C such that for any operator $T \in L(X, l_2)$,

$$\gamma_1(T) \leq C \pi_1(T).$$

2) The Gordon-Lewis constant $gl(X)$ of X (see [**GL**]), is the least constant C such that for any normed space Y and any operator $T \in L(X, Y)$,

$$\gamma_1(T) \leq C \pi_1(T).$$

3) The weak Gordon-Lewis constant (according to K. Ball [**B1**], a different definition was introduced in [**Pi3**]) $wrgl_2(X)$ of an n-dimensional space X is the least constant K such that for any $T \in L(X, l_2)$

$$\Big(\mathrm{vol}_n\big(T(B_X)\big)\Big)^{1/n} \leq \frac{2K}{n}\pi_1(T)\,.$$

(For the definition of the ideal norm $\pi_p(T)$ the reader may refer to the books [**Kö**], [**Pie**], [**Pi2**], [**Tj**]). It was proved in [**B2**] that for some constant $c > 0$, independent of X,

$$(*) \qquad wrgl_2(X) \leq c\,\min\{gl_2(X), \mathrm{vr}(X, B_2^{\dim(X)})\}\,.$$

and moreover, if X is finite dimensional,

$$wrgl_2(X) \sim \mathrm{vr}(X, \ell_\infty)$$

in the sense that there exist absolute constants c_1 and $c_2 > 0$, such that,

$$(**) \qquad c_1\ \mathrm{vr}(X, \ell_\infty) \leq wrgl_2(X) \leq c_2\ \mathrm{vr}(X, \ell_\infty)\,.$$

To see better how these numbers are related, let us define also the local unconditional constant of X, $\chi_u(X)$ ([**GL**]): this is the least constant C such that for any finite-dimensional subspace $F \subset X$ there exists a Banach space U with a finite unconditional basis constant $\chi(U)$, and operators $A \in L(F, U)$ and $B \in L(U, X)$ with $BA = i_F$ (the inclusion of F into X), and satisfying

$$\|A\|\|B\|\chi(U) \leq C\,.$$

Of course if E is finite-dimensional, $\chi_u(E) = \chi_u(E^*)$, and clearly subspaces of L_1, and quotients of L_∞, have finite l.u.st. constants. We see then, by results of [**GL**] and inequalities (*) and (**) that for some absolute constants c and $d > 0$, we have

$$(***) \qquad c \leq d\,wrgl_2(X) \leq gl_2(X) \leq gl(X) \leq \chi_u(X) \leq \chi(X) \leq \sqrt{\dim(X)}\,.$$

The following result improves inequality (*):

Corollary 8. *There is an absolute positive constant c such that for every finite dimensional normed space F, if F^* denotes the dual of F, we have*

$$\mathrm{vr}(F, \ell_\infty)\ \mathrm{vr}(F^*, \ell_\infty) \sim wrgl_2(F)\,wrgl_2(F^*) \leq c\ \min(gl_2(F), gl_2(F^*)).$$

Proof. Take E in Theorem 6 to be the space l_2^n, then $\mathrm{evr}(E, \ell_\infty^n) = \Big(\frac{2^n}{\mathrm{vol}_n(B_E)}\Big)^{1/n} \sim \sqrt{n}$. Now, multiplying the inequality of Theorem 6 by $\big(\mathrm{vol}_n(B_{F^*})\big)^{1/n}$, and using Santalò's inequality, we have that

$$wrgl_2(F)\big(\,\mathrm{vol}_n(T^*B_{F^*})\big)^{1/n} \leq \frac{c}{n}\gamma_\infty(T)\,.$$

The inequality

$$wrgl_2(F)\, wrgl_2(F^*) \leq c\, gl_2(F)$$

follows now immediately from the fact that $T^* \in L(F^*, l_2^n)$, and that $\gamma_\infty(T) = \gamma_1(T^*) \leq gl_2(F^*)\pi_1(T^*)$, and the definition of $wrgl_2(F^*)$. For the second inequality replace F by F^*. We use then (**). $\square$

Remarks: a) In particular, since $gl_2(F^*) \leq \sqrt{\dim(F)}$ and $\lambda(F^*) \leq \sqrt{\dim(F)}$, it follows from Corollaries 7 and 8 that if $wrgl_2(F^*) \sim \sqrt{\dim(F)}$, then $\mathrm{evr}(F^*, \ell_\infty^{\dim(F)}) \sim 1$, i.e. there are a cross-polytope C contained in B_F (see the comments before Proposition 10) such that $\left(\frac{\mathrm{vol}_n(B_F)}{\mathrm{vol}_n(C)}\right)^{1/n} \sim 1$; in 'volume sense' B_F is equivalent to a cross-polytope; moreover both $\lambda(F^*)$, $gl_2(F^*)$ and $gl_2(F)$ are then asymptotically equivalent to $\sqrt{\dim(F)}$.

b) If $gl_2(F) \sim 1$, which happens for example when F is 'well' complemented in a Banach lattice, then both B_F and B_{F^*} are in 'volume sense' equivalent to zonoids.

As we shall see, the estimate given by Corollary 7 for $\lambda(K)$ can provide a good information about its real value; however, we ignore whether it is a sharp estimate. The weaker estimate: $\mathrm{evr}(F, \ell_\infty^{\dim(F)}) \leq \lambda(F)$ is not sharp, as it is shown by the following example.

Example: There is a n-dimensional subspace F of ℓ_∞^{2n} such that

$$\lambda(F) \sim \sqrt{n} \text{ and } \mathrm{evr}(F, \ell_\infty^n) \leq \sqrt{2}e,$$

(for another example of the same type, see **[B1]**). In order to show this we first prove:

a) Let E be a n-dimensional subspace of ℓ_∞^m with B_E as unit ball. Then

$$\mathrm{evr}(B_E, Q_n) \leq \left(\sqrt{\binom{m}{n}}\right)^{1/n} \leq \sqrt{\frac{me}{n}}.$$

Let P_E be the orthogonal projection of $\mathbb{R}^m$ onto E. Then B_E can be described as follows:

$$B_E = \{x \in E;\ |< x, P_E e_i >| \leq 1, \text{ for } 1 \leq i \leq m\}.$$

Let $u_1, \ldots, u_n$ be an orthonormal basis of E, and set $P_E = \sum_{i=1}^m \sum_{j=1}^n \mu_{ij} e_i \otimes u_j$. Denote by $M \in \mathcal{M}_{m,n}$ the matrix $(\mu_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ which represents P_E. Since P_E is an orthogonal projection, the matrix $M^*M \in \mathcal{M}_{n,n}$ is the identity on E. Therefore by the Cauchy-Binet formula,

$$1 = \det(M^*M) = \sum_{I \subset \{1,\ldots,m\}, |I|=n} \left(\det(M_{IN})\right)^2,$$

where $N = \{1, \ldots, n\}$. The matrix M_{IN} represents the operator $P_E|\,\mathrm{span}\{e_i, i \in I\}$ which maps e_i to $P_E(e_i) = \sum_{j=1}^n \mu_{ij} u_j$ for every $i \in I$. Hence, denoting by $\mathbf{C}^I$ the cross-polytope $\mathrm{conv}(\pm P_E(e_i), i \in I)$, we obtain

$$\mathrm{vol}_n(\mathbf{C}^I) = \frac{2^n}{n!} |\det(M_{IN})|.$$

It follows that

$$\sum_{|I|=n} \big(\frac{n!}{2^n}\ \mathrm{vol}_n(\mathbf{C}^I)\big)^2 = 1\,.$$

Therefore there exists $J \subset \{1,\ldots,m\}$, $|J| = n$ such that

$$\mathrm{vol}_n(\mathbf{C}^J) \geq \frac{2^n}{n!\sqrt{\binom{m}{n}}}\,.$$

Now the parallelotope $Q^J = \{x \in E;\ |< x, P_E e_i >| \leq 1 \text{ for } i \in J\}$ contains B_E, and moreover since $(Q^J)^o = \mathbf{C}^J$, we have that $\mathrm{vol}_n(Q^J)\,\mathrm{vol}_n(\mathbf{C}^J) = 4^n/n!$, from which it follows that

$$\big(\mathrm{vol}_n(Q^J)\big)^{1/n} \leq 2\,\Big(\sqrt{\binom{m}{n}}\Big)^{1/n} \leq 2\sqrt{\frac{me}{n}}\,.$$

But it follows from [**V**] that $\mathrm{vol}_n(B_E) \geq \mathrm{vol}_n(Q_n) = 2^n$. Therefore we have

$$\mathrm{evr}(B_E, Q_n) \leq \sqrt{\frac{me}{n}}\,.$$

b) Under the same hypothesis as in a), if $i : \ell_2^m \to \ell_\infty^m$ denotes the identity map, and if i_E denotes its restriction to E, we have

$$\|i_E\| \geq \sqrt{\frac{n}{m}}\,.$$

In fact, if we set $u_i = \sum_{j=1}^m u_{ij} e_j$, $1 \leq i \leq n$, then

$$i_E = \sum_{i=1}^n u_i \otimes u_i = \sum_{i=1}^n \sum_{j=1}^m u_{ij} u_i \otimes e_j\,,$$

so that

$$\|i_E\|^2 = \max_{a_1^2+\ldots+a_n^2\leq 1} \Big(\max_{j=1,\ldots,m} \big(\sum_{i=1}^n a_i u_{ij}\big)^2\Big) = \max_{j=1,\ldots,m} \Big(\max_{a_1^2+\ldots+a_n^2\leq 1} \big(\sum_{i=1}^n a_i u_{ij}\big)^2\Big)$$

$$= \max_{j=1,\ldots,m} \sum_{i=1}^n u_{ij}^2 \geq \frac{1}{m} \sum_{j=1}^n \sum_{i=1}^m u_{ij}^2 = \frac{n}{m}\,.$$

c) Let now F be a n-dimensional subspace of ℓ_∞^m and let $q_F : \ell_\infty^m \to \ell_\infty^m/F$ be the quotient mapping. If P is any linear projection of ℓ_∞^m onto F , set $E = i^{-1}(\ker P)$ and $Q = I_\infty - P$, where I_∞ denotes the identity mapping on ℓ_∞^m. If we set

$$r_Q(q_F y) = Qy \text{ for every } y \in \ell_\infty^m\,,$$

we get a mapping $r_Q : \ell_\infty^m/F \to \ker P$ such that $r_Q\, q_F$ is the identity on $\ker P$, hence $i_E = r_Q\, q_F\, i_E$, and moreover $\|r_Q\| = \|Q\| = \|I_\infty - P\|$, so that

$$\|i_E\| \leq \|I_\infty - P\|\, \|q_F\, i_E\|.$$

Let now $m = 2n$; then it follows from a result of Kashin ([**K1**]) that for some constant $c > 0$, independent of n, there exists an n-dimensional subspace F of ℓ_∞^{2n} satisfying with the previous notation

$$\|q_F\, i\| \leq \frac{c}{\sqrt{n}}.$$

Applying b), we get

$$\frac{1}{\sqrt{2}} = \sqrt{\frac{n}{2n}} \leq \|i_E\| \leq \|I_\infty - P\| \|q_F\, i_E\| \leq \|I_\infty - P\| \|q_F\, i\| \leq (1 + \|P\|)\frac{c}{\sqrt{n}}$$

for every projection $P : \ell_\infty^m \to F$, with $i^{-1}(\ker P) = E$. It follows that

$$\lambda(F) \geq \frac{\sqrt{n}}{c\sqrt{2}} - 1\,,$$

but by a) applied to F, we have

$$\mathrm{evr}(B_F, Q_n) \leq \sqrt{2e}\,.$$

Theorem 9. *For every $0 < p \leq 1$ there exists a constant $c(p) > 0$ such that for every integer n, we have*

$$c(p)\, n^{\frac{1}{p}-\frac{1}{2}} \leq \mathrm{evr}(\ell_p^n, \ell_\infty^n) \leq \lambda(\ell_p^n) \leq\, n^{\frac{1}{p}-\frac{1}{2}}\,.$$

Proof: Observe that a parallelotope contains the unit ball B_p^n of ℓ_p^n, $0 < p \leq 1$, if and only if it contains $C_n = B_1^n$. Therefore, since $\mathrm{vr}(C_n, B_2^n)$ is bounded and hence from Corollary 2, $\mathrm{evr}(C_n, Q_n) \sim \sqrt{n}$, it follows from Corollary 7

$$\lambda(\ell_p^n) \geq \mathrm{evr}(B_p^n, Q_n) = \mathrm{evr}(C_n, Q_n)\Big(\frac{\mathrm{vol}_n(C_n)}{\mathrm{vol}_n(B_p^n)}\Big)^{1/n} \geq c(p)\,\sqrt{n}\, n^{-1+\frac{1}{p}} = c(p)\, n^{\frac{1}{p}-\frac{1}{2}}\,.$$

The upper estimate is trivial, since the distance between ℓ_p^n and ℓ_1^n is $n^{1/p-1}$, and $\lambda(\ell_1^n) < \sqrt{n}\,$. □

Remark. The preceding lower estimate for $\lambda(\ell_p^n)$ has been obtained in [**Pe**], with an extra multiplicative $\ln(n)$, using a much more involved proof.
We also observe that easy calculation of $\mathrm{evr}(\ell_p^n, Q_n)$ yields the known asymptotic estimates of the projection constants for all values of $p \geq 1$.

For $0 < p \leq +\infty$, let s_p^n be the n^2-dimensional space of all real $[n\times n]$ matrices A equipped with the quasi-norm

$$\|A\|_p = (\sum_{i=1}^{n} \lambda_i^p)^{1/p}\,,$$

where $(\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of $(A^*A)^{1/2}$, and let S_p^n be the unit ball of s_p^n.

Theorem 10. *For every $0 < p \leq 1$, there exists a positive constant $a(p)$ such that*

$$a(p)\, n^{1/p} \leq \mathrm{evr}(S_p^n, Q_{n^2}) \leq \lambda(s_p^n) \leq n^{1/p}\,.$$

Proof: By Corollary 8 of [**S**], for some constant $d(p) > 0$, $\big(\mathrm{vol}_{n^2}(S_p^n)\big)^{1/n^2} \sim d(p) n^{-(\frac{1}{2}+\frac{1}{p})}$ (the proof of [**S**] considers only the case $p \geq 1$, but it is easily seen that it yields this estimate for $0 < p \leq 1$). As in Theorem 9, we have

$$\mathrm{evr}(S_p^n, Q_{n^2}) = \mathrm{evr}(S_1^n, Q_{n^2}) \Big(\frac{\mathrm{vol}_{n^2}(S_1^n)}{\mathrm{vol}_{n^2}(S_p^n)}\Big)^{1/n^2} \geq d(p)\, \mathrm{evr}(S_1^n, Q_{n^2}) n^{-1+\frac{1}{p}}\,.$$

But $S_1^n \subset S_2^n \subset \sqrt{n}\, S_1^n$ and $\mathrm{vr}(S_1^n, B_2^{n^2}) \leq \Big(\mathrm{vol}_{n^2}(S_1^n)/\,\mathrm{vol}_{n^2}(S_2^n)\Big)^{1/n^2} \sqrt{n} \leq c_1$. Hence, by Corollary 2, $\mathrm{evr}(S_1^n, Q_{n^2}) \geq c_2 n$, from which the lower estimates follows. For the upper estimate, observe that

$$\lambda(s_p^n) \leq \lambda(s_1^n)\, \mathrm{d}(s_p^n, s_1^n) \leq n.n^{\frac{1}{p}-1} = n^{\frac{1}{p}}$$

where $\mathrm{d}(.,.)$ denotes here the Banach-Mazur distance. $\square$

Remark

If K is a centrally symmetric convex body in $\mathbb{R}^n$, let

$$K^o = \{x \in \mathbb{R}^n;\ < x, y > \leq 1 \text{ for every } x \in K\}$$

be its polar body; Then for some absolute constant $c > 0$,

$$\frac{2}{\pi} \leq \frac{\mathrm{evr}(K, Q_n)}{\mathrm{vr}(K^o, C_n)} \leq c.$$

Indeed,

$$\frac{\mathrm{evr}(K, Q_n)}{\mathrm{vr}(K^o, C_n)} = \inf_{P \subset K} \sup_{\mathbf{C} \subset K^o} \Big(\frac{\mathrm{vol}_n(P)\,\mathrm{vol}_n(\mathbf{C})}{\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^o)}\Big)^{1/n}$$

$$\geq \inf_{P \supset K} \Big(\frac{\mathrm{vol}_n(P)\,\mathrm{vol}_n(P^o)}{\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^o)}\Big)^{1/n} = \Big(\frac{4^n}{n!\,\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^o)}\Big)^{1/n},$$

and by Santalò's inequality we get

$$\frac{\mathrm{evr}(K, Q_n)}{\mathrm{vr}(K^o, C_n)} \geq \Big(\frac{4^n}{n! v_n^2}\Big)^{1/n} \geq \frac{2}{\pi}\,.$$

On the other hand, by the inverse Santalò's inequality (see [**BM**] or [**Pi2**])

$$\frac{\mathrm{evr}(K, Q_n)}{\mathrm{vr}(K^o, C_n)} \leq \sup_{\mathbf{C} \subset K^o} \Big(\frac{\mathrm{vol}_n(\mathbf{C})\,\mathrm{vol}_n(\mathbf{C}^o)}{\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^o)}\Big)^{1/n} \leq c\,.$$

If we suppose that K or K^o is a zonoid, we have using **[R]**, or **[GMR]**, that

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^o) \geq \frac{4^n}{n!}\,,$$

so that

$$\frac{\mathrm{evr}(K, Q_n)}{\mathrm{vr}(K^o, C_n)} \leq 1\,.$$

Proposition 11. *Let Z be a zonoid in $\mathbb{R}^n$. Then* $\mathrm{cr}(Z, B_1^n) \geq \mathrm{vr}(Q_n, C_n) \geq \frac{\sqrt{n}}{e}$.

Proof: Let us observe first that

$$\mathrm{vr}(Q_n, C_n) = \Big(\frac{n!}{\max\{|\det(\theta_{ij},\ 1 \leq i, j \leq n)|;\ |\theta_{ij}| \leq 1\}}\Big)^{1/n} \geq \frac{\sqrt{n}}{e}.$$

Indeed, any cross-polytope $\mathbf{C} \subset Q_n$ has the form $\mathrm{conv}(\pm\sum_{j=1}^n \theta_{ij} e_j,\ 1 \leq i \leq n)$, for some choice of the $n \times n$ matrix $\Theta = (\theta_{ij})$ and clearly

$$\mathrm{vol}_n(\mathbf{C}) = \frac{2^n}{n!}|\det(\Theta)| = \frac{\mathrm{vol}_n(Q_n)}{n!}|\det(\Theta)|\,.$$

By Hadamard's inequality, $|\det(\Theta)| \leq \prod_{i=1}^n \Big(\sum_{j=1}^n {\theta_{ij}}^2\Big)^{1/2} \leq n^{n/2}$, hence

$$\mathrm{vr}(Q_n, C_n) \geq \Big(\frac{n!}{n^{n/2}}\Big)^{1/n} \geq \frac{\sqrt{n}}{e}\,.$$

Since a zonoid can be approximated by zonotopes in the Hausdorff metric, we may reduce to the latter case and suppose that $Z = \sum_{j=1}^m [-z_j, z_j]$, for some $z_j \in \mathbb{R}^n$, $1 \leq j \leq m$ and $n \leq m$.

Let $A \in \mathcal{M}_{n,m}$ be the matrix with the coordinates of z_j, $1 \leq j \leq m$ in the canonical basis of $\mathbb{R}^n$ as columns. If $x_1, \ldots, x_n$ are points in Z, then they have the form $x_i = \sum_{j=1}^m \theta_{ij} z_j$, with $\theta_{ij} \in [-1, 1]$, so letting $\mathbf{C} = \mathrm{conv}(\pm x_1, \ldots, \pm x_n) \subset Z$, and denoting by $L = [x_1, \ldots, x_n] \in \mathcal{M}_{n,n}$ the corresponding matrix, and by $\Theta \in \mathcal{M}_{m,n}$ the matrix with entries θ_{ji} in the i-th row and j-th column, for $1 \leq i \leq m$, $1 \leq j \leq n$, we have that $L = A\Theta$. By the Cauchy-Binet formula,

$$\det(L) = \sum_{I \subset \{1,\ldots,m\},\, |I|=n} \det(A_{NI}) \det(\Theta_{IN})\,.$$

But $\mathrm{vol}_n(Z) = 2^n\big(\sum_{I \subset \{1,\ldots,m\},\,|I|=n} |\det(A_{NI})|\big)$, and hence

$$2^n|\det(L)| \leq \mathrm{vol}_n(Z) \max_{|I|=n} |\det(\Theta_{IN})| \leq (n!)\frac{\mathrm{vol}_n(Z)}{\big(\mathrm{vr}(Q_n, C_n)\big)^n}\,.$$

Therefore

$$\Big(\frac{\mathrm{vol}_n(Z)}{\mathrm{vol}_n(\mathbf{C})}\Big)^{1/n} = \Big(\frac{\mathrm{vol}_n(Z)}{\frac{2^n}{n!}|\det(L)|}\Big)^{1/n} \geq \mathrm{vr}(Q_n, C_n) \geq \frac{\sqrt{n}}{e}. \quad \square$$

Remarks.

1) The estimate $\mathrm{vr}(Q_n, C_n) \geq \frac{(n!)^{1/n}}{\sqrt{n}}$ is sharp, in the case when $n = 2^k$, $k = 1, 2, \ldots$ (use Walsh matrix). For an upper estimate of $\mathrm{vr}(K, C_n)$, valid for every convex symmetric body K, observe that the quantity

$$\max_{x_1,\ldots,x_n \in K} \det(x_1, \ldots, x_n)$$

decreases under Steiner symmetrisation of K (see for instance **[M]**). It follows that

$$\mathrm{vr}(K, C_n) \leq \mathrm{vr}(B_2^n, C_n) = \left(\frac{v_n n!}{2^n}\right)^{1/n} \sim \sqrt{\frac{\pi n}{2e}}.$$

This estimate was proved in **[K2]**, up to a multiplicative constant.

2) Proposition 11 allows to give an easy geometric proof of the following result, which is also a consequence of the fact, originally due to Bourgain and Milman **[BM]**, that the set of all finite-dimensional subspaces $\{F\}$ of an infinite-dimensional normed space of cotype 2, have uniformly bounded volume ratios $\mathrm{vr}(F, \ell_2^{\dim(F)})$ (see also **[Pi2]**, **[Tj]**, and **[GK]** for the general quasi-normed case); this applies in particular for ℓ_1 which has cotype 2: In this case, we see that every zonoid Z in $\mathbb{R}^n$ satisfies $\mathrm{vr}(Z^o, B_2^n) \leq e\sqrt{\frac{\pi}{2}}$. Indeed, by Corollary 2, the remarks preceding Proposition 11 and Proposition 11 itself, we have successively

$$\mathrm{vr}(Z^o, B_2^n) \leq \sqrt{e}\sqrt{\frac{2n}{\pi e}}.\frac{1}{\mathrm{evr}(Z^o, Q_n)} \leq \sqrt{\frac{2n}{\pi}}.\frac{\pi/2}{\mathrm{vr}(Z, C_n)} \leq \sqrt{\frac{\pi n}{2}}.\frac{e}{\sqrt{n}} \leq e\sqrt{\frac{\pi}{2}}.$$

It was proved by K. Ball (**[B2]**), using more involved arguments, that if Z is a zonoid in $\mathbb{R}^n$, one has always

$$\mathrm{vr}(Z^o, B_2^n) \leq \mathrm{vr}(\mathbf{C}_n, B_2^n) \sim \sqrt{\frac{2e}{\pi}}.$$

It may be observed that finding the exact maximum of $\mathrm{evr}(K, Q_n)$ over all the centrally symmetric convex bodies K in $\mathbb{R}^n$ is still an open problem for $n \geq 3$ (see **[Ba]**, where it is solved for $n = 2$).

References.

[Ba] I. K. Babenko : Asymptotic volume of tori and geometry of convex bodies, Mat. Zametki **44** (1988), 177-190.

[B1] K. Ball : Normed spaces with a weak Gordon-Lewis property, in Functional Analysis, Springer Lectures Notes **1470**, 1991.

[B2] K. Ball : Volumes of sections of cubes and related problems, in Geometric aspects of Functional Analysis, Springer Lecture Notes **1376**, 1991.

[BP] K. Ball and A. Pajor: Convex bodies with few faces, Proc. . Math. Soc. **110** #1 (1990), 225-231.

[BF] I. Barany and Z. Furedi : Computing volume is difficult, Discrete Comput. Geom. **2** (1987), 319-326.

[BM] J. Bourgain and V.D. Milman : New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$. Invent. Math. **88** (1987), 319-340.

[C] B. Carl : Inequalities of Bernstein-Jackson type and the degree of compactness of operators in Banach spaces. Ann. Inst. Fourier Grenoble **35** (3) (1985), 79-118.

[Ge] S. Geiss : Antisymmetric Tensor Products of Absolutely p-summing Operators, J. Approximation Theory **68** (1992), 223-246.

[Gl] E. D. Gluskin : Extremal properties and orthogonal parallelepipeds and their applications to the geometry of Banach spaces, Math. USSR Sbornik **64** #1 (1989), 85-96.

[GK] Y. Gordon and N. J. Kalton : Local structure theory for quasi-normed spaces, Bull. Sci. Math. (to appear).

[GL] Y. Gordon, D. R. Lewis : Absolutely summing operators and local unconditional structures, Acta Math. **133** (1974), 27-48.

[GMR] Y. Gordon, M. Meyer and S. Reisner : Zonoids with minimal volume product. Proc. . Math. Soc. **104** (1988), 273-276.

[J] F. John : Extremum problems with inequalities as subsidiary conditions, Courant Anniversary volume, Interscience, New-York, 1948, 187-204.

[K1] B. Kashin : Sections of some finite dimensional sets and classes of smooth functions, Izv. Acad. Nauk. SSSR **41** (1977), 334-351.

[K2] B. Kashin : On parallelepipeds of minimal volume containing a convex body (Russian), Mat. Zametki **45** (1989), 134-135.

[Kö] H. König : Eigenvalue Distribution of Compact Operators, Operator Theory, Advances and Applications 16, Basel, Birkhäuser, 1986.

[M] A.M. Macbeath : An extremal property of the hypersphere, Proc. Cambridge Philos. Soc. **47** (1951), 245-247.

[Pe] N.T. Peck : A factorization constant for ℓ_p^n, $0 < p < 1$, Preprint.

[PS] A. Pelczynski and S. J. Szarek : On parallelepipeds of minimal volume containing a convex symmetric body in $\mathbb{R}^n$, Math. Proc. Camb. Phil. Soc., (1991), 109-125.

[Pie] A. Pietsch : Operators Ideals, Amsterdam, North-Holland Pub. Co., 1980.

[Pi1] G. Pisier : Remarque sur un résultat de B. Maurey, Séminaire d'Analyse Fonctionnelle 1980-81, Ecole Polytechnique, Paris.

[Pi2] G. Pisier : The Volume of Convex Bodies and Banach Space Geometry, Cambridge Tracts in Mathematics 94, Cambridge Univ. Press, 1989.

[Pi3] G. Pisier : Weak Hilbert spaces, Proc. London Math. Soc. **56** (1988), 547-579.

[R] S. Reisner : Random polytopes and the volume of product of symmetric convex bodies. Math. Scand. **57** (1985), 396-392.

[S] J. Saint-Raymond : Les volumes d'idéaux d'opérateurs classiques. Studia Math. **80** (1984), 63-75.

[Tj] N. Tomczak-Jaegermann : Banach-Mazur distances and finite-dimensional operator ideals, Pitman Monographs and surveys in pure and applied Mathematics, Longman Scientific & Technical, Wiley, New York 1989.

[V] G. Vaaler : A geometric inequality with applications to linear forms. Pacific J. of Math. **83** (1979), 543-553.

Y. Gordon
Department of Mathematics
Technion
Haifa, ISRAEL

M. Meyer
Equipe d'analyse, Université Paris VI
4 Place Jussieu
75252 Paris cedex 05, FRANCE

A. Pajor
Equipe de Mathématiques, Université de Marne-la-Vallée
2 rue de la Butte Verte
93160 Noisy-le-Grand, FRANCE