

# The isotropy constants of the Schatten classes are bounded

by H. König, M. Meyer and A. Pajor \*

## Abstract

We prove the result announced in the title.

## Introduction

A well known conjecture states that there exists a constant  $\gamma > 0$ , such that for every integer  $d \geq 1$ , and every centrally symmetric convex body  $C$  in  $\mathbb{R}^d$ , there exists an hyperplane  $H$  of  $\mathbb{R}^d$  such that if  $|\cdot|$  denotes the volume in  $\mathbb{R}^d$  and in  $H$ ,

$$|H \cap C| \geq \gamma |C|^{\frac{d-1}{d}} .$$

This problem is equivalent to the boundedness of the isotropy constant  $L_C$  of  $C$  defined by

$$L_C^2 = \min \left\{ \frac{\int_C \|x\|_{\mathcal{E}}^2 dx}{d|C|^{1+\frac{2}{d}}}; \mathcal{E} \text{ ellipsoid, } |\mathcal{E}| = v_d \right\}$$

where  $v_d$  denotes the volume of the Euclidean unit ball. It was proved by J. Bourgain ([Bo]) that one has always  $L_C \leq d^{\frac{1}{4}} \ln(d)$ , and several authors showed that the conjecture is true for various classes of convex bodies  $\mathcal{C}$ , in the sense that there is a constant  $\gamma_{\mathcal{C}} > 0$  such that  $L_C \leq \gamma_{\mathcal{C}}$  for every  $C \in \mathcal{C}$  (see [Ba], [J], [M-P]). The unit ball  $B(S_p^n)$  of the Schatten trace class  $S_p^n$  does not belong to any of these classes (for  $1 \leq p < 2$ ), and it was a natural candidate for a counter-example. It was shown by S. Dar (see [D]) that in that case  $L_C \leq c\sqrt{\ln(n)}$ ; we prove here:

**Theorem.** *There exists a constant  $c > 0$  such that for all  $n \geq 1$  and  $p \in [1, +\infty]$ ,  $L_{B(S_p^n)} \leq c$ .*

Although some computations are different, the method of proof of the theorem relies on the same method in the real and in the complex case. This paper is organized as follows : after some notation and some calculations of volumes, we prove the theorem, and state some remarks.

## Notations

The spaces  $\mathbb{R}^d$  and  $\mathbb{C}^d$  are equipped with their canonical Euclidean structure. The spaces  $\mathcal{M}_n(\mathbb{R})$  and  $\mathcal{M}_n(\mathbb{C})$  of  $n \times n$  matrices with real or complex entries are equipped with the associated Euclidean structure defined by  $\|T\|_2^2 = \text{Trace}(T^*T)$  for any  $T \in \mathcal{M}_n$ , embedded in  $\mathbb{R}^{n^2}$  or  $\mathbb{R}^{2n^2}$  endowed with the Lebesgue measure denoted by  $dT$ . For any Borel set  $A \subset \mathbb{R}^d$ , we denote by  $|A|$  its Lebesgue measure in its affine hull. We say that two positive

---

\* The authors were supported by Procope 1997/98

quantities  $A$  and  $B$  are equivalent, and we write  $A \simeq B$ , if the ratio  $A/B$  is bounded from above and below by two positive universal constants (independent of all parameters).

For  $1 \leq p < +\infty$ , let

$$B_p^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n; \|x\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} \leq 1\},$$

and

$$B_\infty^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n; \|x\|_\infty := \sup_{i=1, \dots, n} |x_i| \leq 1\}.$$

We denote by  $v_n$  the volume of the Euclidean unit ball  $B_2^n$ .

For any  $n \times n$  matrix  $T \in \mathcal{M}_n(\mathbb{R})$  or  $\mathcal{M}_n(\mathbb{C})$  we denote by  $s(T) = (s_1(T), \dots, s_n(T))$  the decreasing arrangement of the singular values of  $T$  and for any  $1 \leq p < +\infty$ , we define

$$\sigma_p(T) = \|s(T)\|_p = (\sum_{i=1}^n |s_i(T)|^p)^{1/p}$$

and for  $p = +\infty$ ,

$$\sigma_\infty(T) = \|s(T)\|_\infty = \sup_{1 \leq i \leq n} s_i(T).$$

Observe that  $\sigma_2(T) = \|T\|_2$  is the Hilbert-Schmidt norm of  $T$  and that  $\sigma_\infty(T) = \|T\|_{\ell_2^n \rightarrow \ell_2^n}$ . Let  $S_p^n$  be the Schatten trace class of matrices on the  $n$ -dimensional complex Euclidean space equipped with the norm  $\sigma_p$ ; we denote

$$B_{\mathbb{R}}(S_p^n) = \{T \in \mathcal{M}_n(\mathbb{R}); \sigma_p(T) \leq 1\} \text{ and } B_{\mathbb{C}}(S_p^n) = \{T \in \mathcal{M}_n(\mathbb{C}); \sigma_p(T) \leq 1\}.$$

More generally, let us consider a *unitarily invariant* norm on  $\mathcal{M}_n(\mathbb{R})$  or  $\mathcal{M}_n(\mathbb{C})$ , that is a norm  $\mathcal{N}$  on  $\mathcal{M}_n(\mathbb{R})$  or  $\mathcal{M}_n(\mathbb{C})$  which satisfies  $\mathcal{N}(USV) = \mathcal{N}(S)$  for any  $S \in \mathcal{M}_n(\mathbb{R})$  (resp.  $\mathcal{M}_n(\mathbb{C})$ ) and any real (resp. complex) isometries  $U, V$  on  $\mathbb{R}^n$  (resp.  $\mathbb{C}^n$ ) with the Euclidean norm. It is well known that one can associate to such a norm  $\mathcal{N}$  on  $\mathcal{M}_n$  a 1-symmetric norm  $\tau$  on  $\mathbb{R}^n$ , such that

$$\mathcal{N}(T) = \tau(s_1(T), \dots, s_n(T)) = \tau(s(T)) \text{ for every } T \in \mathcal{M}_n$$

and  $\tau(x_1, \dots, x_n) = \mathcal{N}(X)$ , where  $X$  is the diagonal matrix with entries  $(x_1, \dots, x_n)$  on the diagonal (a norm  $\tau$  on  $\mathbb{R}^n$  is called *1-symmetric* if for any  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$  and any permutation  $\pi$  of  $\{1, \dots, n\}$ ,  $\tau(\varepsilon_1 x_{\pi(1)}, \dots, \varepsilon_n x_{\pi(n)}) = \tau(x_1, \dots, x_n)$ ). We denote by  $B_{\mathbb{R}}(\mathcal{N})$  the unit ball of  $(\mathcal{M}_n(\mathbb{R}), \mathcal{N})$ , by  $B_{\mathbb{C}}(\mathcal{N})$  the unit ball of  $(\mathcal{M}_n(\mathbb{C}), \mathcal{N})$  and by  $B_\tau$  the unit ball of  $(\mathbb{R}^n, \tau)$ .

We say that a function  $F: \mathbb{R}^n \rightarrow \mathbb{R}$  is *symmetric* if for any permutation  $\pi$  on  $\{1, \dots, n\}$ , one has

$$F(x_1, \dots, x_n) = F(x_{\pi(1)}, \dots, x_{\pi(n)}) .$$

**Lemma 1.** For any symmetric continuous function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  and any unitarily invariant norm  $\mathcal{N}$  on  $\mathcal{M}_n(\mathbb{R})$ , one has

$$\int_{T \in B_{\mathbb{R}(\mathcal{N})}} F(s_1(T), \dots, s_n(T)) dT = c_n \int_{x \in B_\tau} F(x) f_n(x) dx$$

where  $c_n = n! 4^{-n} (\prod_{k=1}^n v_k)^2$  and  $f_n(x) = f_n(x_1, \dots, x_n) = \prod_{1 \leq j < i \leq n} |x_i^2 - x_j^2|$ .

If moreover  $F$  is positively homogeneous of degree  $k$ , then for every  $p > 0$ , one has

$$\int_{T \in B_{\mathbb{R}(\mathcal{N})}} F(s_1(T), \dots, s_n(T)) dT = \frac{c_n}{\Gamma(1 + \frac{n^2+k}{p})} \int_{\mathbb{R}^n} F(x) e^{-\tau(x)^p} f_n(x) dx.$$

**Proof:** Since every  $T \in \mathcal{M}_n(\mathbb{R})$  can be written  $T = UDV$ , where  $D$  is a diagonal matrix with non-negative entries and  $U, V \in \mathcal{O}(n)$ , the change of variables discussed and explained in [S-R] works, and we get the first formula, with  $c_n = n! 4^{-n} (\prod_{k=1}^n v_k)^2$ . The second one comes from Fubini's theorem and the following

$$\begin{aligned} \int_{\mathbb{R}^n} F(x) e^{-\tau(x)^p} f_n(x) dx &= \int_{\mathbb{R}^n} F(x) f_n(x) \left( \int_{\tau(x)^p}^{+\infty} e^{-t} dt \right) dx \\ &= \int_0^{+\infty} e^{-t} \left( \int_{\{x; \tau(x) \leq t^{1/p}\}} F(x) f_n(x) dx \right) dt = \Gamma\left(1 + \frac{n^2+k}{p}\right) \int_{B_\tau} F(x) f_n(x) dx \end{aligned}$$

since  $f_n(x)$  is homogeneous of degree  $n(n-1)$ .  $\square$

In the complex case, one also uses the change of variables in [S-R] to get the following

**Lemma 2.** For any symmetric continuous function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  and any unitarily invariant norm  $\mathcal{N}$  on  $\mathcal{M}_n(\mathbb{C})$ , one has

$$\int_{T \in B_{\mathbb{C}(\mathcal{N})}} F(s_1(T), \dots, s_n(T)) dT = d_n \int_{x \in B_\tau} F(x) g_n(x) dx,$$

where  $d_n = \frac{n!}{\pi^n} (\prod_{k=1}^n v_{2k})^2$  and  $g_n(x) = g_n(x_1, \dots, x_n) = \prod_{i=1}^n |x_i| \prod_{1 \leq j < i \leq n} |x_i^2 - x_j^2|^2$ .

If moreover  $F$  is positively homogeneous of degree  $k$ , then for every  $p > 0$ , one has

$$\int_{T \in B_{\mathbb{C}(\mathcal{N})}} F(s_1(T), \dots, s_n(T)) dT = \frac{d_n}{\Gamma(1 + \frac{2n^2+k}{p})} \int_{\mathbb{R}^n} F(x) e^{-\tau(x)^p} g_n(x) dx.$$

### The volume of the unit balls of unitarily invariant normed classes of matrices

Let now  $u = \sum_{i=1}^n e_i$ , where  $e_1, e_2, \dots, e_n$  is the canonical basis of  $\mathbb{R}^n$ ; modulo a scaling, we shall suppose from now on that  $\tau(e_i) = 1$ ,  $1 \leq i \leq n$ . As a particular case of Lozanovskii's theorem ([L]), one has

$$\frac{1}{\tau(u)} B_\infty^n \subset B_\tau \subset \frac{n}{\tau(u)} B_1^n \quad \text{and thus} \quad \frac{1}{\tau(u)} B_{\mathbb{R}}(S_\infty^n) \subset B_{\mathbb{R}}(\mathcal{N}) \subset \frac{n}{\tau(u)} B_{\mathbb{R}}(S_1^n),$$

and these inclusions are also true in the complex case. It follows that

$$|B_\tau|^{\frac{1}{n}} \simeq \frac{1}{\tau(u)} \quad , \quad \frac{1}{\tau(u)} |B_{\mathbb{R}}(S_\infty^n)|^{\frac{1}{n^2}} \leq |B_{\mathbb{R}}(\mathcal{N})|^{\frac{1}{n^2}} \leq \frac{n}{\tau(u)} |B_{\mathbb{R}}(S_1^n)|^{\frac{1}{n^2}}$$

and

$$\frac{1}{\tau(u)} |B_{\mathbb{C}}(S_\infty^n)|^{\frac{1}{2n^2}} \leq |B_{\mathbb{C}}(\mathcal{N})|^{\frac{1}{2n^2}} \leq \frac{n}{\tau(u)} |B_{\mathbb{C}}(S_1^n)|^{\frac{1}{2n^2}} .$$

Since by [S-R],

$$n |B_{\mathbb{R}}(S_1^n)|^{\frac{1}{n^2}} \simeq |B_{\mathbb{R}}(S_\infty^n)|^{\frac{1}{n^2}} \simeq \frac{1}{\sqrt{n}} \quad \text{and} \quad n |B_{\mathbb{C}}(S_1^n)|^{\frac{1}{2n^2}} \simeq |B_{\mathbb{C}}(S_\infty^n)|^{\frac{1}{2n^2}} \simeq \frac{1}{\sqrt{n}}$$

we get

$$(1) \quad |B_{\mathbb{R}}(\mathcal{N})|^{\frac{1}{n^2}} \simeq |B_{\mathbb{C}}(\mathcal{N})|^{\frac{1}{2n^2}} \simeq \frac{1}{\tau(u)\sqrt{n}} ,$$

and

$$(2) \quad |B_{\mathbb{R}}(S_p^n)|^{\frac{1}{n^2}} \simeq |B_{\mathbb{C}}(S_p^n)|^{\frac{1}{2n^2}} \simeq n^{-\frac{1}{2}-\frac{1}{p}} \quad , \quad \text{for } 1 \leq p \leq +\infty .$$

### The isotropy constants

We say that a centrally symmetric body  $C$  in  $\mathbb{R}^d$  is *isotropic* if for some  $c > 0$  one has

$$\int_C \langle x, y \rangle^2 dx = c \|y\|_2^2 \quad \text{for all } y \in \mathbb{R}^d .$$

The *isotropy constant*  $L_C > 0$  of  $C$  is then the positive number defined by

$$L_C^2 = \frac{\int_C \|x\|_2^2 dx}{d |C|^{\frac{1+\frac{2}{d}}{d}}} = |C|^{-\frac{2}{d}} \left( \frac{1}{|C|} \int_C \langle x, y \rangle^2 dx \right) \quad \text{for all } y \in \mathbb{R}^d \text{ such that } \|y\|_2 = 1 .$$

It is easy to see that the unit balls  $B_{\mathbb{R}}(\mathcal{N})$  and  $B_{\mathbb{C}}(\mathcal{N})$  of a unitarily invariant normed space of matrices (with norm  $\mathcal{N}$ ) are isotropic. Suppose that  $\tau$  is the 1-symmetric norm on  $\mathbb{R}^n$  associated to  $\mathcal{N}$ . Let  $L_{\mathbb{R}}(\mathcal{N})$  and  $L_{\mathbb{C}}(\mathcal{N})$  be the isotropy constant of  $B_{\mathbb{R}}(\mathcal{N})$  and of  $B_{\mathbb{C}}(\mathcal{N})$ . It follows from lemma 1 and (1) that

$$L_{\mathbb{R}}(\mathcal{N})^2 \simeq \frac{n\tau^2(u)}{n^2 |B_{\mathbb{R}}(\mathcal{N})|} \int_{B_{\mathbb{R}}(\mathcal{N})} \sigma_2^2(T) dT = \frac{\tau^2(u)}{n} \frac{\int_{B_\tau} \left( \sum_{i=1}^n x_i^2 \right) f_n(x) dx}{\int_{B_\tau} f_n(x) dx} .$$

Thus

$$(3) \quad L_{\mathbb{R}}(\mathcal{N})^2 \simeq \tau(u)^2 \frac{\int_{B_\tau} x_1^2 f_n(x) dx}{\int_{B_\tau} f_n(x) dx}$$

where

$$f_n(x) = f_n(x_1, \dots, x_n) = \prod_{1 \leq j < i \leq n} |x_i^2 - x_j^2|.$$

Similarly, using lemma 2 and (1), we have

$$(4) \quad L_{\mathbb{C}}(\mathcal{N})^2 \simeq \tau(u)^2 \frac{\int_{B_\tau} x_1^2 g_n(x) dx}{\int_{B_\tau} g_n(x) dx}$$

where

$$g_n(x) = g_n(x_1, \dots, x_n) = \prod_{i=1}^n |x_i| \prod_{1 \leq j < i \leq n} |x_i^2 - x_j^2|^2.$$

### The isotropy constant of $B_{\mathbb{R}}(S_p^n)$ and $B_{\mathbb{C}}(S_p^n)$

Let us denote by  $L_{\mathbb{R}}(n, p) = L_{B_{\mathbb{R}}(S_p^n)}^2$  and  $L_{\mathbb{C}}(n, p) = L_{B_{\mathbb{C}}(S_p^n)}^2$ . Using lemma 1 and estimates (2) and (3), we get

$$L_{\mathbb{R}}(n, p) \simeq n^{-1+\frac{2}{p}} \frac{\int_{B_p^n \cap (\mathbb{R}_+)^n} (\sum_{i=1}^n x_i^2) f_n(x) dx}{\int_{B_p^n \cap (\mathbb{R}_+)^n} f_n(x) dx} \simeq n^{-1+\frac{2}{p}} \frac{\Gamma(1 + \frac{n^2}{p})}{\Gamma(1 + \frac{n^2+2}{p})} \frac{M_p(\sum_{i=1}^n x_i^2)}{M_p(1)},$$

where  $M_p$  is the measure with density

$$f_{n,p}(x_1, \dots, x_n) = 1_{\{x_1 \geq 0, \dots, x_n \geq 0\}} f_n(x) e^{-\sum_{i=1}^n x_i^p}$$

with respect to the Lebesgue measure on  $\mathbb{R}^n$ .

Thus, since

$$\frac{\Gamma(1 + \frac{n^2}{p})}{\Gamma(1 + \frac{n^2+2}{p})} \simeq n^{-\frac{4}{p}},$$

we have

$$(5) \quad L_{\mathbb{R}}(n, p) \simeq n^{-\frac{2}{p}} \frac{M_p(x_1^2)}{M_p(1)}.$$

Similarly, by lemma 2, (2) and (4), we get

$$(6) \quad L_{\mathbb{C}}(n, p) \simeq n^{-\frac{2}{p}} \frac{N_p(x_1^2)}{N_p(1)}.$$

where  $N_p$  is the measure with density

$$g_{n,p}(x_1, \dots, x_n) = 1_{\{x_1 \geq 0, \dots, x_n \geq 0\}} g_n(x) e^{-\sum_{i=1}^n x_i^p}$$

with respect to the Lebesgue measure on  $\mathbb{R}^n$ .

### Proof of the theorem.

We shall use a method similar to the one developed by K. Aomoto ([**A**]) to study Jacobi polynomials associated to Selberg integrals (see [**M**], p. 343-347). This method yields the following lemma, which, when applied in the real case with  $b = 0$  and  $c = 1$  and in the complex case with  $b = 1$  and  $c = 2$ , proves the theorem by using (5) and (6).

**Lemma 3.** For  $b, c > 0$ ,  $p \geq 1$ , and  $n \geq 1$ , let  $M_{n,p,b,c}$  be the measure with density

$$f_{n,p,b,c}(x_1, \dots, x_n) = 1_{\{x_1 \geq 0, \dots, x_n \geq 0\}} \prod_{i=1}^n x_i^b \prod_{1 \leq j < i \leq n} |x_i^2 - x_j^2|^c e^{-\sum_{i=1}^n x_i^p}$$

with respect to the Lebesgue measure on  $\mathbb{R}^n$ . Then there exists a constant  $C(b, c)$ , such that for all  $n \geq 1$  and all  $p \geq 1$

$$\frac{M_{n,p,b,c}(x_1^2)}{M_{n,p,b,c}(1)} \leq C(b, c) n^{\frac{2}{p}}.$$

**Proof:** For simplification, we shall denote  $M = M_{n,p,b,c}$  and  $f = f_{n,p,b,c}$ . For fixed  $x_2 \geq 0, \dots, x_n \geq 0$  let  $x = (x_1, \dots, x_n)$ , and for  $a > 0$ , define  $\phi = \phi_{p,n,a} : [0, +\infty[ \rightarrow \mathbb{R}$ , by

$$\phi(x_1) = x_1^a f(x).$$

Suppose that  $x_2 < x_3 < \dots < x_n$  and that  $x_1 \in ]x_m, x_{m+1}[$  for some  $2 \leq m \leq n$ ; then

$$\phi(x_1) = g(x_2, \dots, x_n) x_1^{a+b} e^{-x_1^p} \prod_{i=1}^m (x_1^2 - x_i^2)^c \prod_{j=m+1}^n (x_j^2 - x_1^2)^c$$

where

$$g(x_2, \dots, x_n) = e^{-\sum_{i=2}^n x_i^p} \prod_{i=2}^n x_i^b \prod_{2 \leq j < i \leq n} |x_i^2 - x_j^2|^c$$

and  $\phi$  has a derivative on the intervals  $]0, x_2[, ]x_2, x_3[, \dots, ]x_{n-1}, x_n[, ]x_n, +\infty[$ , which is given, for  $x_1 \in ]x_m, x_{m+1}[$ , by

$$\begin{aligned} \frac{\partial \phi}{\partial x_1}(x_1) &= f(x) \left( (a+b)x_1^{a-1} - px_1^{p+a-1} + 2cx_1^{a+1} \left( \sum_{i=2}^m \frac{1}{x_1^2 - x_i^2} - \sum_{j=m+1}^n \frac{1}{x_j^2 - x_1^2} \right) \right) \\ &= f(x) \left( (a+b)x_1^{a-1} - px_1^{p+a-1} + 2cx_1^{a+1} \left( \sum_{i=2}^n \frac{1}{x_1^2 - x_i^2} \right) \right). \end{aligned}$$

For  $a > 0$ ,  $\frac{\partial \phi}{\partial x_1}$  is continuous on all the open intervals  $]x_{i-1}, x_i[$  or  $]0, x_2[$  and  $]x_n, +\infty[$ , and has integrable singularities at their endpoints; it follows that for almost all  $(x_2, \dots, x_n)$  with respect to the Lebesgue measure on  $\mathbb{R}^{n-1}$ ,

$$\int_0^{+\infty} \frac{\partial \phi}{\partial x_1}(x_1) dx_1 = \lim_{x_1 \rightarrow +\infty} \phi(x_1) - \phi(0) = 0$$

and thus, integrating with respect to  $(x_2, \dots, x_n) \in ([0, +\infty[)^{n-1}$ , we get

$$(a+b)M(x_1^{a-1}) - pM(x_1^{a+p-1}) + cM\left(2x_1^{a+1} \sum_{i=2}^n \frac{1}{x_1^2 - x_i^2}\right) = 0.$$

By symmetry and linearity, we have for  $i = 2, \dots, n$ ,

$$M\left(\frac{x_1^{a+1}}{x_1^2 - x_i^2}\right) = M\left(\frac{x_1^{a+1}}{x_1^2 - x_2^2}\right) = M\left(\frac{x_2^{a+1}}{x_2^2 - x_1^2}\right) = -M\left(\frac{x_2^{a+1}}{x_1^2 - x_2^2}\right),$$

so that

$$cM\left(2x_1^{a+1} \sum_{i=2}^n \frac{1}{x_1^2 - x_i^2}\right) = c(n-1)M\left(\frac{2x_1^{a+1}}{x_1^2 - x_2^2}\right) = c(n-1)M\left(\frac{x_1^{a+1} - x_2^{a+1}}{x_1^2 - x_2^2}\right).$$

It follows that

$$(a+b)M(x_1^{a-1}) - pM(x_1^{a+p-1}) + c(n-1)M\left(\frac{x_1^{a+1} - x_2^{a+1}}{x_1^2 - x_2^2}\right) = 0.$$

Now, for  $1 \leq a \leq 3$ , one has for all  $x_1, x_2 > 0$ ,

$$\frac{a+1}{4}(x_1^{a-1} + x_2^{a-1}) \leq \frac{x_1^{a+1} - x_2^{a+1}}{x_1^2 - x_2^2} \leq x_1^{a-1} + x_2^{a-1},$$

so that

$$\frac{(a+1)}{2}M(x_1^{a-1}) \leq M\left(\frac{x_1^{a+1} - x_2^{a+1}}{x_1^2 - x_2^2}\right) \leq 2M(x_1^{a-1})$$

and

$$\left(\frac{c(n-1)(a+1) + 2a + 2b}{2p}\right)M(x_1^{a-1}) \leq M(x_1^{a+p-1}) \leq \frac{a+b+2c(n-1)}{p}M(x_1^{a-1}).$$

For  $a = 1$ , we get

$$(7) \quad \frac{M(x_1^p)}{M(1)} = \frac{1+b+c(n-1)}{p},$$

and for  $a = p+1$ ,  $0 < p \leq 2$ ,

$$(8) \quad \frac{c(n-1)(p+2) + 2(p+1) + 2b}{2p} \leq \frac{M(x_1^{2p})}{M(x_1^p)} \leq \frac{p+1+b+2c(n-1)}{p}.$$

If  $1 \leq p \leq 2$ , by Hölder inequality applied with  $\frac{1}{2} = \frac{p-1}{p} + \frac{2-p}{2p}$ , we have

$$M(x_1^2)^{\frac{1}{2}} \leq M(x_1^p)^{\frac{p-1}{p}} M(x_1^{2p})^{\frac{2-p}{2p}},$$

and thus

$$\frac{M(x_1^2)}{M(1)} \leq \left(\frac{M(x_1^p)}{M(1)}\right)^{\frac{2p-2}{p}} \left(\frac{M(x_1^{2p})}{M(1)}\right)^{\frac{2-p}{p}} = \frac{M(x_1^p)}{M(1)} \left(\frac{M(x_1^{2p})}{M(x_1^p)}\right)^{\frac{2-p}{p}}.$$

By (7) and (8), one gets

$$\frac{M(x_1^2)}{M(1)} \leq \frac{1+b+c(n-1)}{p} \left( \frac{p+1+b+2c(n-1)}{p} \right)^{\frac{2-p}{p}} \leq C(b,c) n^{\frac{2}{p}},$$

with  $C(b,c) = (3+b+2c)^2$ .

For  $p \geq 2$ , one gets the same result using

$$\frac{M(x_1^2)}{M(1)} \leq \left( \frac{M(x_1^p)}{M(1)} \right)^{\frac{2}{p}} = \left( \frac{1+b+c(n-1)}{p} \right)^{\frac{2}{p}}. \quad \square$$

## Remarks

1) Actually, the boundedness of  $L_{\mathbb{R}}(n,p)$  and of  $L_{\mathbb{C}}(n,p)$  for  $p \geq 2$  are well known, because the polar body  $B(S_p^n)$  of  $B(S_p^n)$  has bounded "volume ratio" (see [P]). Also, it follows directly from Hölder's inequality and some computation of volumes ; for instance, we have

$$\begin{aligned} L_{\mathbb{R}}(n,p)^2 &\simeq n^{\frac{2}{p}} \frac{\int_{B_p^n} x_1^2 f_n(x) dx}{\int_{B_p^n} f_n(x) dx} \\ &\leq n^{\frac{2}{p}} \left( \frac{\int_{B_p^n} x_1^p f_n(x) dx}{\int_{B_p^n} f_n(x) dx} \right)^{\frac{2}{p}} = n^{\frac{2}{p}} \left( \frac{\int_{B(S_p^n)} \sigma_p^p(T) dT}{n |S_p^n|} \right)^{\frac{2}{p}} = \left( \frac{n}{n+p} \right)^{\frac{2}{p}} \simeq 1. \end{aligned}$$

2) We conjecture that  $L_{\mathbb{R}}(\mathcal{N})$  and  $L_{\mathbb{C}}(\mathcal{N})$  are bounded independently of the unitarily invariant norm  $\mathcal{N}$  on  $\mathcal{M}_n$ . In fact, one can expect that for  $p \geq 1$ , the numbers

$$\left( \frac{1}{|B_{\mathbb{R}}(\mathcal{N})|} \int_{B_{\mathbb{R}}(\mathcal{N})} \frac{\sigma_p^p(T)}{n} dT \right)^{\frac{1}{p}}$$

are all equivalent, with constants independent of  $n$  and  $\mathcal{N}$ , and perhaps depending on  $p$ , when  $p \rightarrow +\infty$  (we normalize here the trace).

Comparing these numbers for  $p = 1$  and  $p = 2$ , this would imply that

$$\begin{aligned} L_{\mathbb{R}}(\mathcal{N}) &= \frac{1}{\sqrt{n} |B_{\mathbb{R}}(\mathcal{N})|^{\frac{1}{n^2}}} \left( \frac{1}{|B_{\mathbb{R}}(\mathcal{N})|} \int_{B_{\mathbb{R}}(\mathcal{N})} \frac{\sigma_2^2(T)}{n} dT \right)^{\frac{1}{2}} \\ &\simeq \frac{1}{\sqrt{n} |B_{\mathbb{R}}(\mathcal{N})|^{\frac{1}{n^2}}} \left( \frac{1}{|B_{\mathbb{R}}(\mathcal{N})|} \int_{B_{\mathbb{R}}(\mathcal{N})} \frac{\sigma_1(T)}{n} dT \right). \end{aligned}$$

Since  $\frac{\sigma_1(T)}{n} \leq \frac{\mathcal{N}(T)}{\tau(u)}$  and  $|B_{\mathbb{R}}(\mathcal{N})|^{\frac{1}{n^2}} \simeq \frac{1}{\tau(u)\sqrt{n}}$ , we would get

$$L_{\mathbb{R}}(\mathcal{N}) \simeq \frac{1}{\tau(u)\sqrt{n} |B_{\mathbb{R}}(\mathcal{N})|^{\frac{1}{n^2}}} \left( \frac{1}{|B_{\mathbb{R}}(\mathcal{N})|} \int_{B_{\mathbb{R}}(\mathcal{N})} \mathcal{N}(T) dT \right) \simeq 1,$$



and similarly for  $B_{\mathbb{C}}(\mathcal{N})$ .

## References

- [A] A. Aomoto: Jacobi polynomials associated with Selberg integrals, *SIAM J. Math. Anal.* **18** (1987), 545-549.
- [Ba] K. Ball: Normed spaces with a weak Gordon-Lewis property, *Lectures notes in Mathematics* **1376**, Springer, Berlin, 1991, 251-263.
- [Bo] J. Bourgain: On the distribution of polynomials on high-dimensional convex sets, *Lecture Notes in Mathematics* **1469**, Springer, Berlin, 1991, 127-137.
- [D] S. Dar: Remarks on Bourgain's problem on slicing of convex bodies, *Operator Theory, Advances and Applications* **77**, Birkhäuser, Basel, 1995, 61-66.
- [J] M. Junge: On the hyperplane conjecture for quotient spaces of  $L_p$ , *Forum Math.* **6** (1994), 617-635.
- [L] G. Ja. Lozanovskii: On some Banach lattices, *Siberian J. Math.* **10** (1969), 419-431.
- [M] M. L. Mehta: *Random matrices* (2nd edition), Academic press, Boston, 1991.
- [M-P] V. D. Milman and A. Pajor: Isotropic position and inertia ellipsoids and zonoids of the unit ball of a  $n$  dimensional normed space, *Lectures Notes in Mathematics* **1376**, Springer, Berlin, 1989, 107-131.
- [P] G. Pisier: *The volume of convex bodies and Banach space geometry*, Cambridge University Press, Cambridge, 1989.
- [S-R] J. Saint-Raymond: Le volume des idéaux d'opérateurs classiques, *Studia Math.* **80** (1984), 63-75.

Hermann KÖNIG

*Mathematisches Seminar Universitaet Kiel*

*Ludewig-Meyn-Str. 4*

*D-24098 KIEL*

*GERMANY*

*E-mail* : hkoenig@math.uni-kiel.de

Mathieu MEYER *and* Alain PAJOR

*Equipe d'Analyse et de Mathématiques Appliquées, Université de Marne-la-Vallée,*  
*5, boulevard Descartes*

*Champs-sur-Marne*

*77454 Marne-la-Vallée cedex 2, FRANCE*

*E-mail* : meyer@math.univ-mlv.fr *and* : pajor@math.univ-mlv.fr