Covering numbers and "low M^*-estimate" for quasi-convex bodies. *

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Abstract

This article gives estimates on covering numbers and diameters of random proportional sections and projections of quasi-convex bodies in $\mathbb{R}^n$. These results were known for the convex case and played an essential role in development of the theory. Because duality relations can not be applied in the quasi-convex setting, new ingredients were introduced that give new understanding for the convex case as well.

1. Introduction and notation.

Let $|\cdot|$ be on $\mathbb{R}^n$. Let D be an ellipsoid associated with this norm. Denote $A = \sqrt{\frac{n}{k}} \int\limits_{S^{n-1}} \sqrt{\sum_{i=1}^k x_i^2}\, d\sigma(x)$, where σ is the normalized rotation invariant measure on the euclidean sphere S^{n-1}. Then $A = A(n,k) < 1$ and $A \longrightarrow 1$ as $n, k \longrightarrow \infty$. For any star-body K in $\mathbb{R}^n$ define $M_K = \int\limits_{S^{n-1}} \| x \| \, d\sigma(x)$, where $\| x \|$ is the gauge of K. Let M_K^* be M_{K^0}, where K^0 is the polar of K. For any subsets K_1, K_2 of $\mathbb{R}^n$ denote by $N(K_1, K_2)$ the smallest number N such that there are N points $y_1, ..., y_N$ in K_1 such that

$$K_1 \subset \bigcup_{i=1}^{N} (y_i + K_2).$$

Recall that a body K is called quasi-convex if there is a constant c such that $K + K \subset cK$, and given a $p \in (0,1]$ a body K is called p-convex if for any $\lambda, \mu > 0$ satisfying $\lambda^p + \mu^p = 1$ and any points $x, y \in K$ the point $\lambda x + \mu y$ belongs to

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K. Note that for the gauge $\| \cdot \| = \| \cdot \|_K$ associated with the the quasi-convex (p-convex) body K the following inequality holds for any $x, y \in \mathbb{R}^n$

$$\| x + y \| \leq C \max\{\| x \|, \| y \|\} \quad (\| x + y \|^p \leq \| x \|^p + \| y \|^p).$$

In particular, every p-convex body K is also quasi-convex one and $K + K \subset 2^{1/p} K$. A more delicate result is that for every quasi-convex body K ($K + K \subset cK$) there exists a q-convex body K_0 such that $K \subset K_0 \subset 2cK$, where $2^{1/q} = 2c$. This is Aoki-Rolewicz theorem ([KPR], [R], see also [K], p.47). In this note by a body we always mean a compact star-body, i.e. a body K satisfying $tK \subset K$ for all $t \in [0, 1]$.

Let us remind of the so-called "low M^*-estimate" result.

Theorem 1 *Let $\lambda > 0$ and n be large enough. Let K be a centrally-symmetric convex body in $\mathbb{R}^n$ and $\| \cdot \|$ be the gauge of K. Then there exists a subspace E of $(\mathbb{R}^n, \| \cdot \|)$ such that $\dim E = [\lambda n]$ and for any $x \in E$ the following inequality holds*

$$\| x \| \geq \frac{f(\lambda)}{M_K^*} |x|$$

for some function $f(\lambda)$, $0 < \lambda < 1$.

Remark. Inequality of this type was first proved in [M1] with very poor dependence on λ and then improved in [M2] to $f(\lambda) = C(1 - \lambda)$. It was later shown ([PT]), that one can take $f(\lambda) = C\sqrt{1 - \lambda}$ (for different proofs see [M3] and [G]).

By duality this theorem is equivalent to the following theorem.

Theorem 1' *Let $\lambda > 0$ and n be large enough. For every centrally-symmetric convex body K in $\mathbb{R}^n$ there exists an orthogonal projection P of rank $[\lambda n]$ such that*

$$PD \subset \frac{M_K}{f(\lambda)} PK.$$

In this note we will extend both theorems to quasi-convex, not necessary central-symmetric bodies. Because duality arguments can not be applied to a non-convex body these two theorems become different statements. Also "M_K^*" should be substituted by an appropriate quantity not involving duality. Note that by avoiding the use of convexity assumption we in fact simplified proof also for a convex case.

2. Main results.

The following theorem is an extension of Theorem 1'.

Theorem 2 *Let $\lambda > 0$ and n be large enough ($n > c/(1-\lambda)^2$). For any p-convex body K in $\mathbb{R}^n$ there exists an orthogonal projection P of the rank $[\lambda n]$ such that*

$$PD \subset \frac{A_p M_K}{(1-\lambda)^{1+1/p}} PK,$$

where $A_p = const^{\frac{\ln(2/p)}{p}}$.

Remark. Also, this result is new for the convex non-symmetric case. To appreciate the strength of this inequality apply it to the standard simplex S inscribed in D. Then $M_S \approx \sqrt{n \cdot \log n}$ and therefore for every $\lambda < 1$ there are λn-dimensional projections containing euclidean ball of radius $\approx 1/\sqrt{n \cdot \log n}$. At the same time S contains only a ball of radius $1/n$.

The proof of Theorem 2 is based on the next three lemmas. The first one was proved by W.B.Johnson and J.Lindenstrauss in [JL]. The second one was proved in [PT] for centrally-symmetric convex bodies and is the dual form of Sudakov minoration theorem.

Lemma 1 *There is an absolute constant c such that if $\varepsilon > \sqrt{c/k}$ and $N \leq 2e^{\varepsilon^2 k/c}$, then for any set of points $y_1, ..., y_N \in \mathbb{R}^n$ and any orthogonal projection P of rank k*

$$\mu\left(\{U \in O_n \mid \forall j : A(1-\varepsilon)\sqrt{k/n}\ |y_j| \leq |PUy_j| \leq A(1+\varepsilon)\sqrt{k/n}\ |y_j|\}\right) > 0.$$

Lemma 2 *Let K be a body such that $K + K \subset aK$. Then*

$$N(D, tK) \leq 2e^{8n(aM_K/t)^2}.$$

Proof: M. Talagrand gave a direct simple proof of this lemma for a convex case ([LT], pp. 82-83). One can check that his proof does not use symmetry and convexity of the body and produces estimate $N(D, tB) \leq 2e^{2n(aM_B/t)^2}$ for every body B, such that $B - B \subset aB$.

Now for a body K, satisfying $K + K \subset aK$ denote $B = K \cap -K$.

Then $B - B \subset aB$ and $M_B \leq 2M_K$, since

$$\| x \|_B = \max(\| x \|_K, \| x \|_{-K}) \leq \| x \|_K + \| x \|_{-K}.$$

Thus

$$N(D, tK) \leq N(D, tB) \leq 2e^{2n(2aM_K/t)^2}.$$

□

Lemma 3 *Let B be a body, K be a p-convex body, $r \in (0,1)$, $\{x_i\} \subset rB$ and $B \subset \bigcup(x_i + K)$. Then $B \subset t_r K$, where $t_r = \frac{1}{(1-r^p)^{1/p}}$.*

Proof: Obviously $t_r = \max\{\| x \|_K \mid x \in B\}$. Since $B \subset \bigcup(x_i + K)$, for any point x in B there are points x_0 in rB and y in K such that $x = x_0 + y$. Then by maximality of t_r and p-convexity of K we have $t_r^p \leq r^p t_r^p + 1$. That proves the lemma. □

Remark. Somewhat similar argument was used by N. Kalton in dealing with p-convex sets.

Proof of Theorem 2:
Any p-convex body K satisfies $K + K \subset aK$ with $a = 2^{1/p}$. By Lemma 2 we have

$$N = N(D, tK) \leq 2 \cdot \exp\left(2^{1+2/p} n (M_K/t)^2\right),$$

i.e. there exist points $x_1, ..., x_N$ in D, such that

$$D \subset \bigcup_{i=1}^{N} (x_i + tK).$$

Denote $c_p = 2^{1+2/p}$. Let t and ε satisfy

$$c_p n \left(\frac{M_K}{t}\right)^2 \leq \frac{\varepsilon^2 k}{c}$$

and $\varepsilon > \sqrt{c/k}$ for c being the constant from Lemma 1.

Applying Lemma 1 we obtain that there exist an orthogonal projection P of rank k such that

$$PD \subset \bigcup(Px_i + tPK) \quad \text{and} \quad |Px_i| \leq (1+\varepsilon)\sqrt{\frac{k}{n}}|x_i|.$$

Let $\lambda = k/n$. Denote $r = (1+\varepsilon)\sqrt{\lambda}$. Lemma 3 gives us

$$PD \subset tt_r PK \quad \text{for} \quad t = \frac{\sqrt{cc_p} M_K}{\varepsilon\sqrt{\lambda}} \quad \text{and} \quad \varepsilon^2 > \frac{c}{\lambda n}\ , \ r < 1.$$

Choose

$$\varepsilon = \frac{1-\sqrt{\lambda}}{2\sqrt{\lambda}}\ .$$

Then for n large enough we get

$$PD \subset \frac{A_p M_K}{(1-\lambda)^{1+1/p}} PK,$$

for $A_p = const^{\frac{\ln(2/p)}{p}}$. This completes the proof. $\square$

Theorem 2 can be formulated in the global form.

Theorem 2' *Let K be a p-convex body in $\mathbb{R}^n$. Then there is an orthogonal operator U such that*

$$D \subset A_p' M_K (K + UK)\,,$$

where $A_p' = const^{\frac{\ln(2/p)}{p}}$.

This theorem can be proved independently, but we show how it follows from Theorem 2.

Proof of Theorem 2': It follows from the proof of Theorem 2 that actually the measure of such projections is large. So we can choose two orthogonal subspaces E_1, E_2 of $\mathbb{R}^n$ such that $\dim E_1 = [n/2]$, $\dim E_2 = [(n+1)/2]$ and

$$P_i D \subset A_p'' M_K P_i K,$$

where P_i is the projection on the space E_i $(i = 1, 2)$. Denote $I = id_{\mathbb{R}^n} = P_1 + P_2$ and $U = P_1 - P_2$. So $P_1 = 1/2(I + U)$ and $P_2 = 1/2(I - U)$. Then U is an orthogonal operator and for any $x \in D$ we have

$$x = P_1 x + P_2 x \subset 1/2\ A_p'' M_K (I+U)K + 1/2\ A_p'' M_K (I-U)K \subset$$

$$\subset A_p'' M_K \frac{K+K}{2} + A_p'' M_K \frac{UK - UK}{2} = A_p' M_K (K + UK)\ .$$

That proves Theorem 2'. $\square$

Let us complement Lemma 2 by mentioning how covering number $N(K, tD)$ can be estimated. In the convex case this estimate is given by Sudakov inequality, using quantity M^*. More precisely, if K is a centrally-symmetric convex body, then

$$N(K, tD) \leq 2e^{cn(M_K^*/t)^2}\ .$$

Of course, using duality for a non-convex setting leads to a weak result, and we suggest below a substitution for quantity M^*.

For two quasi-convex bodies K, B define the following number

$$M(K,B) = \frac{1}{|K|} \int\limits_K \| x \|_B \, dx,$$

where $|K|$ is volume of K, and $\| x \|_B$ is the gauge of B. Such numbers are considered in [MP1], [MP2] and [BMMP].

Lemma 4 *Let K be p-convex body and B be a body. Assume $B - B \subset aB$. Then*

$$N(K,tB) \leq 2e^{(cn/p)(aM(K,B)/t)^p} \ ,$$

where c is an absolute constant.

Proof: We follow the idea of M.Talagrand of estimating covering numbers in case $K = D$ ([LT], pp. 82-83, see also [BLM] Proposition 4.2). Denote the gauge of K by $\| \cdot \|$ and the gauge of B by $| \cdot |_B$. Define the measure μ by following

$$d\mu = \frac{1}{A} e^{-\|x\|^p} dx, \ \text{where} \ A \ \text{ is chosen such that} \ \int\limits_{\mathbb{R}^n} d\mu = 1.$$

Let $L = \int\limits_{\mathbb{R}^n} |x|_B d\mu$. Then $\mu\{|x|_B \leq 2L\} \geq 1/2$. Let $x_1, x_2, ...$ be a maximal set of points in K such that $|x_i - x_j|_B \geq t$. So the sets $x_i + \frac{t}{a}B$ have mutually disjoint interiors. Let $y_i = \frac{ab}{t} x_i$ for some b. Then, by p-convexity of K and convexity of the function e^t, we have

$$\mu\{y_i + bB\} = \frac{1}{A} \int\limits_{bB} e^{-\|x + y_i\|^p} dx \geq \frac{1}{A} \int\limits_{bB} e^{(\|x\|^p + \|y_i\|^p)} dx =$$

$$= \frac{1}{A} e^{-\|y_i\|^p} \int\limits_{bB} e^{-\|x\|^p} dx \geq e^{-(ba/t)^p} \mu\{bB\}.$$

Choose $b = 2L$. Then $\mu\{bB\} \geq 1/2$ and, hence,

$$N(K,tB) \leq 2e^{(2aL/t)^p} \ .$$

Now compute L. First, the normalization constant A is equal

$$A = \int\limits_{\mathbb{R}^n} e^{-\|x\|^p} dx = \int\limits_{\mathbb{R}^n} \int\limits_{\|x\|}^{\infty} (-e^{-t^p})' dt dx = \int\limits_0^{\infty} pt^{p-1} e^{-t^p} \int\limits_{\|x\| \leq t} dx dt =$$

$$= \int\limits_{\|x\|\le 1} dx \int\limits_0^\infty p t^{p+n-1} e^{-t^p} dt = |K| \cdot \Gamma\left(1 + \frac{n}{p}\right),$$

where Γ is the gamma-function. The remaining integral is

$$\int\limits_{\mathbb{R}^n} |x|_B e^{-\|x\|^p} dx = \int\limits_{\mathbb{R}^n} |x|_B \int\limits_{\|x\|}^\infty (-e^{-t^p})' dt dx = \int\limits_0^\infty p t^{p-1} e^{-t^p} \int\limits_{\|x\|\le t} |x|_B dx dt =$$

$$= \int\limits_{\|x\|\le 1} |x|_B dx \int\limits_0^\infty p t^{p+n} e^{-t^p} dt = |K| \cdot M(K,B) \cdot \Gamma\left(1 + \frac{n+1}{p}\right) .$$

Using Stirling's formula we get

$$L \approx \left(\frac{n}{p}\right)^{1/p} M(K,B).$$

That proves the lemma. □

Remark. An analogous lemma for p-smooth ($1 \le p \le 2$) body K and convex centrally-symmetric body B was announced in [MP2]. Of course, the proof holds for all $p > 0$ and every quasi-convex centrally-symmetric body B. More precisely the following lemma holds.

Lemma 4' *Let K and B be bodies. Let $B - B \subset aB$ and assume that for some $p > 0$ there is a constant c_p which depends only on p and body K, such that*

$$\| x + y \|_K^p + \| x - y \|_K^p \le 2 \cdot (\| x \|_K^p + c_p \cdot \| y \|_K^p) \quad \textit{for all} \quad x, y \in \mathbb{R}^n.$$

Then

$$N(K, tB) \le 2e^{cn(c_p/p)(aM(K,B)/t)^p},$$

where c is an absolute constant.

Lemma 4' is an extension of Lemma 2 in the symmetric case. Indeed, since Euclidean space is a 2-smooth space, then in case $K = D$ being an ellipsoid, we have $c_2(D) = 1$. By direct computation, $M(D,B) = \frac{n}{n+1} M_B$. Thus,

$$N(D, tB) \le 2e^{(cn)(M_B/t)^2} .$$

Define the following characteristic of K,

$$\tilde{M}_K = \frac{1}{|K|} \int\limits_K |x| dx .$$

Lemma 4 shows that for p-convex body K

$$N(K, tD) \le 2e^{(cn/p)(2\tilde{M}_K/t)^p} .$$

The Theorem 3 follows from this estimate by arguments similar of that in [MP].

Theorem 3 *Let $\lambda > 0$ and n be large enough. Let K be a p-convex body in $\mathbb{R}^n$ and $\| \cdot \|$ be the gauge of K. Then there exists subspace E of $(\mathbb{R}^n, \| \cdot \|)$ such that* $\dim E = [\lambda n]$ *and for any $x \in E$ the following inequality holds*

$$\| x \| \ge \frac{(1-\lambda)^{1/2+1/p}}{a_p \tilde{M}_K} |x| ,$$

where a_p depends on p only (more precisely $a_p = const^{\frac{\ln(2/p)}{p}}$).

Proof: By Lemma 4 there are points $x_1, ..., x_N$ in K, such that $N < e^{c_p n (\tilde{M}_K/t)^p}$ and for any $x \in K$ there exists some x_i such that $|x - x_i| < t$. By Lemma 1 there exists an orthogonal projection P on a subspace of dimension δn such that for

$$c_p n \left(\frac{\tilde{M}_K}{t} \right)^p < \frac{\varepsilon^2 \delta n}{c} \quad \text{and} \quad \varepsilon > \sqrt{\frac{c}{\delta n}}$$

we have

$$b|x_i| = (1-\varepsilon) A \sqrt{\delta} |x_i| \le |P x_i| \le (1+\varepsilon) A \sqrt{\delta} |x_i|$$

for every x_i. Let $E = \mathrm{Ker} P$. Then $\dim E = \lambda n$, where $\lambda = 1 - \delta$. Take x in $K \bigcap E$. There is x_i such that $|x - x_i| < t$. Hence

$$|x| \le |x - x_i| + |x_i| \le t + \frac{|P x_i|}{b} = t + \frac{|P(x - x_i)|}{b} \le$$

$$\le t + \frac{|x - x_i|}{b} \le t(1 + \frac{1}{b}) \le \frac{const \cdot t}{(1-\varepsilon)\sqrt{\delta}} .$$

Therefore for n large enough and

$$t = \left(\frac{const \cdot c_p}{\varepsilon^2 \delta} \right)^{1/p} \tilde{M}_K$$

we get

$$\| x \| \ge \frac{const \cdot \varepsilon^2 (1-\varepsilon) \delta^{1/2+1/p}}{c_p^{1/p} \tilde{M}_K} |x| .$$

To obtain our result take ε, say, equal to $1/2$. □

As was noted in [MP2] in some cases $\tilde{M}_K << M^*$ and then Theorem 3 gives better estimate than Theorem 1 even for a convex body (in some range of λ). As an example, $K = B(l_1^n)$, $\tilde{M}_K \le c \cdot n^{-1/2}$, but $M_K^* \ge c \cdot n^{-1/2}(\log n)^{1/2}$ for some absolute constant c.

3. Additional remarks.

In fact, during the proof of Theorem 2 a more general fact was proved.

Fact. *Let D be an ellipsoid and K be a p-convex body. Let*

$$N(D,K) \le e^{\alpha n}.$$

Denote for an integer $1 \le k \le n$ the ratio $\lambda = k/n$. Then for some absolute constant c and

$$\gamma = c\sqrt{\alpha}, \quad k \in (\gamma^2 n, (1-2\gamma)^2 n)$$

there exists an orthogonal projection P of rank k such that

$$\left(p(1-\sqrt{\lambda})/2\right)^{1/p} PD \subset PK.$$

In terms of entropy numbers this means

$$\frac{\left(p(1-\sqrt{k/n})/2\right)^{1/p}}{e_k(D,K)} PD \subset PK,$$

where $e_k(D,K) = \inf\{\varepsilon > 0 \mid N(D, \varepsilon K) \le 2^{k-1}\}$.

It is worth to point out that Theorem 2 can be obtained from this results.

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