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Entropy and asymptotic geometry of non-symmetric convex bodies

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Abstract. We extend to the general, not necessary centrally symmetric setting a number of basic results of Local Theory which were known before for centrally symmetric bodies and were using very essentially the symmetry in their proofs. The main additional tool is a study of volume behavior around the centroid of the body.

Entropie et géométrie asymptotique des convexes non symétriques

Résumé. On généralise au cadre non symétrique des résultats de la théorie local des espaces normés tels que le théorème du sous-espace du quotient et les propriétés des projections aléatoires. L'apport nouveau réside dans l'étude du comportement du volume quand on symétrise par rapport au centre de gravité.

Version française abrégée.

De nombreux résultats de la théorie locale des espaces normés expriment des propriétés géométriques asymptotiques des corps convexes symétriques. On peut formuler certains problèmes dans un cadre non symétrique. Certains résultats passent ainsi facilement au cadre non symétrique, mais ce n'est pas toujours le cas. De nombreux résultats de la théorie locale s'appuient en effet sur une estimation de Pisier sur la norme de la projection de Rademacher et une telle estimation dans le cas non symétrique est un problème ouvert (pour de récent progrès voir [11]). Par ailleurs, certains résultats ne passent pas au cadre non symétrique. On sait par exemple d'après un résultat de [1] que si P est un polytope symétrique de $\mathbb{R}^n$ avec $f(P)$ faces et $v(P)$ sommets, alors $\log f(P). \log v(P) \geq cn$

Note présentée par Mikhail GROMOV

où $c > 0$ est une constante numérique; mais cette inégalité n'est plus vraie dans le cas non symétrique, par exemple pour un simplexe.

On étudie le rôle du centre de gravité, d'un point de vue volumique, comme centre de symétrisation. On montre que si 0 est le centre de gravité d'un corps convexe K alors K et $K \cap (-K)$ ont à peu près le même volume. On s'intéresse aussi au calcul d'entropie dans le cas non symétrique. Pour comprendre la différence avec le cas symétrique, il suffit d'étudier le recouvrement de $2K$ par K (voir [10]). On donne une version générale du théorème de dualité de l'entropie de König et Milman ([3]).

On montre que les "projections aléatoires" d'un convexe sont à "volume ratio borné" (théorème 5) et on en déduit une version non symétrique du théorème du sous-espace du quotient de ([7]). Cela nous amène au sujet principal de cet article qui est l'étude des projections aléatoires d'un convexe placé dans certaines positions et à la construction d'un M-ellipsoïde pour un corps convexe arbitraire. On désigne par $|K|$ le volume de K et par $N(K, \mathcal{E})$ le "nombre de recouvrement" de K par $\mathcal{E}$. On montre que pour tout corps convexe K, dont 0 est le centre de gravité, il existe une transformation $T \in GL_n$ telle que $|T(K)| = |B_2^n|$ où B_2^n est la boule euclidienne unité et telle que si on pose $C = T(K)$,

$$N(C, B_2^n).N(B_2^n, C^\circ).N(C^\circ, B_2^n).N(B_2^n, C^\circ) < e^{cn}$$

où C° désigne le polaire de C et c une constante numérique indépendante de n et de C. Cette transformation définit, comme dans le cas symétrique, un M-ellipsoïde de K. On note co(X) l'enveloppe convexe de X. On montre le théorème suivant:

Théorème. *Soient K et K' des convexes compacts de $\mathbb{R}^n$ d'intérieurs non vides, dont 0 est le centre de gravité, et tels que B_2^n soit un M-ellipsoïde pour K et K'. Alors il existe des constantes universelles c, r_1 et r_2 telles que l'on ait*

$$(1/r_1)\, B_2^n \subset \mathrm{co}(K \cup T(K')) \ \mathit{et}\ \left(\frac{|\mathrm{co}(K \cup T(K'))|}{|(1/r_1)\, B_2^n|}\right)^{1/n} \leq c$$

$$\mathit{et}\ (1/r_1)\, B_2^n \subset L \cap V(L) \subset r_2\, B_2^n$$

pour des isométries "aléatoires" $T, V \in \mathcal{O}_n$, où $L = \mathrm{co}(K \cup T(K'))$.

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1. Notation. Through the paper we consider the space $\mathbb{R}^n$ being equipped with its canonical Euclidean scalar product $(\,.\,,\,.\,)$ and the corresponding norm $|\,.\,|$. Its unit ball is denoted by B_2^n. The volume of a measurable subset A of $\mathbb{R}^n$ is denoted by $|A|$. We denote by v_n the volume of B_2^n. The standard Gaussian measure on $\mathbb{R}^n$ with density $e^{-|x|^2/2}/(2\pi)^{n/2}$ is denoted by γ_n.

We denote by $\mathrm{co}(A)$ the convex hull of a subset A in $\mathbb{R}^n$. For any two subsets A and B of $\mathbb{R}^n$, and any scalars $\lambda, \mu \in \mathbb{R}$, $\lambda A + \mu B = \{\lambda x + \mu y\,;\, x \in A, y \in B\}$ denotes the Minkowski sum. Let K be convex body in $\mathbb{R}^n$ with 0 in its interior, its polar K° is defined as usual by $K^\circ = \{x \in \mathbb{R}^n \,:\, (x, y) \leq 1 \text{ for every } y \in K\}$.

2. The role of the barycenter. The first result explain the role of the centroid as center of symmetrization.

Theorem 1. *Let K and L be two convex compact subsets of $\mathbb{R}^n$.*
1) If K and L have the same centroid with respect to the measure γ_n, then

$$\gamma_n(K) \times \gamma_n(L) \leq \gamma_n\Big(\frac{K+L}{\sqrt{2}}\Big) \times \gamma_n\big(\sqrt{2}(K \cap (-L)\big).$$

In particular if 0 is the Gaussian barycenter of K, then

$$\gamma_n(K)^2 \leq \gamma_n\Big(\sqrt{2}K\Big) \times \gamma_n\big(\sqrt{2}(K \cap (-K)\big).$$

2) If K and L have the same centroid, then

$$|K| \times |L| \leq |K+L| \times |K \cap (-L)|. \tag{1}$$

In particular if 0 is the centroid of K, then

$$|K \cap (-K)| \geq 2^{-n}\,|K|, \tag{2}$$

and

$$\frac{|K-K|}{|K \cap (-K)|} \leq 8^n.$$

The proof uses the following simple lemma:

Lemma 2. *Let μ be a probability on $\mathbb{R}^n$ and $\psi \in L^1(\mu)$ be a non-negative log-concave function with $\int \psi\, d\mu > 0$. Then*

$$\int \psi\, d\mu \leq \psi\left(\int x \frac{\psi(x)}{\int \psi\, d\mu}\, d\mu(x)\right).$$

Remarks. 1. Inequality (1) is well known for symmetric convex bodies (see [9]).
2. Let $E \oplus F = \mathbb{R}^n$ be an orthogonal decomposition of $\mathbb{R}^n$ and denote by $P_F(K)$ the orthogonal projection of K onto F. Applying lemma 2 we obtain that

$$|K| \leq |P_F(K)| \times |K \cap (x_0 + E)|$$

where $x_0 = P_F(\frac{1}{|K|}\int_K z dz)$. In particular if 0 is the centroid of K, we get a result of Spingarn [12].
3. It is easy to see that $\max_{x\in K} |(2x-K) \cap K|/|K| \geq 2^{-n}$ (see [2] where this ratio is called Kovner-Besicovitch measure of symmetry). Concerning the inequality (2), the best previously known estimate for $|K \cap (-K)|/|K|$ which is referred in [2] when 0 is the centroid of K, was of the order of $1/n^n$.

3. Duality of entropy. Let A and B be two subsets of $\mathbb{R}^n$, the covering number $N(A,B)$ is defined as usual as $N(A,B) = \min\big\{{}^\sharp\Lambda : \Lambda \subset \mathbb{R}^n, A \subset \Lambda + B\big\}$. It is a long standing and fascinating problem to understand as presicely as possible the duality of covering numbers (or, shortly, entropy). Theorem 1 allows us to symmetrize the bodies without significant changes on the volumes and to extend to a non-symmetric setting the result by König and Milman [3].

Theorem 3. *There exist a constant $c_1 > 0$ such that for any integer $n \geq 1$ and any convex compact subsets A and B of $\mathbb{R}^n$ with 0 as centroid, we have*

$$\frac{1}{c_1} N(B^\circ, A^\circ)^{1/n} \leq N(A,B)^{1/n} \leq c_1 N(B^\circ, A^\circ)^{1/n}.$$

4. Random projections. We extend in this section to the general convex setting, the QS-Theorem of [7], stating that for any normed space, in a correctly choosen Euclidean structure, random quotient of subspaces of the space of proportional dimension are close to Euclidean spaces. Surprisingly the Theory of Normed Spaces is not needed and results are true for arbitrary convex bodies.

We denote by $\mathbb{P}$rob the rotation invariant probability measure on the orthogonal group $\mathcal{O}_n$. We will use the same notation to denote the rotation invariant probability measure on the Grassmann manifold $\mathcal{G}_{n,k}$, thought as the set of orthogonal projection of rank k.

Lemma 4. *Let $1 \leq k \leq n$ and $\xi \in [0,1[$. Let $x \in \mathbb{R}^n$ and P be an orthogonal projection on $\mathbb{R}^n$ of rank k. Then we have,*

$$\mathbb{P}\mathrm{rob}[T \in \mathcal{O}_n : |PTx| > \xi |x|\,] < \left(e\,(1-\xi^2)\cdot \frac{n}{n-k}\right)^{\frac{n-k}{2}}.$$

Fix $c \in]0,1[$. We say that a property in $\mathbb{R}^n$ is satisfied for "random orthogonal projection" of rank k, if the set of rank k projections satisfying the property has a probability larger than $1-c^n$ in $\mathcal{G}_{n,k}$. We show that random projection of a convex set has "bounded volume ratio".

Theorem 5. *Let $1 \leq k < n$ and $\lambda = k/n$. Let C be a convex compact subset of $\mathbb{R}^n$ with non-empty interior. There exists an affine transformation T such that in the position where $K = T(C)$, K has 0 as centroid and random rank k orthogonal projections satisfy*

$$P(B_2^n) \subset P(K \cap -K) \subset P(K) \text{ and } \left(\frac{|P(K)|}{|P(B_2^n)|}\right)^{1/k} \leq r(\lambda) = e^{\frac{c}{1-\lambda}}$$

where c is numerical constant.

We conclude this section with the non-symmetric statement for the QS-theorem [7]. It can be obtained from the previous theorem and volume ratio approach of Szarek and Szarek-Tomczak-Jaegermann (see [13]).

Theorem 6. *Let $1 \leq k < n$ and $\lambda = k/n$. Let K be a convex compact subset of $\mathbb{R}^n$ with non-empty interior and 0 as centroid. There exists a projection P from $\mathbb{R}^n$ onto a subspace F of $\mathbb{R}^n$ and a subspace E of F and an ellipsoid $\mathcal{E}$ in E such that $dim(E) = k$ and for some $c(\lambda)$ depending only on λ,*

$$\mathcal{E} \subset P(K) \cap E \subset c(\lambda)\,\mathcal{E}.$$

5. M-ellipsoids; existence. We consider in this section ellipsoids exclusively centered at 0. Let $\sigma > 0$ and let K be a convex compact subset of $\mathbb{R}^n$ with 0 in its interior. We say that an ellipsoid $\mathcal{E}$ of $\mathbb{R}^n$ is an M-ellipsoid of K with constant σ, or shortly an M-ellipsoid of K, if setting $\lambda = (|K|/|\mathcal{E}|)^{1/n}$ in order that $|K| = |\lambda\,\mathcal{E}|$, we have $N(K, \lambda\,\mathcal{E}) \leq e^{\sigma n}$.

It is proved in [4] (see also [5] and [8] for simplified proofs) that there exists a universal constant such that for every n, every n-dimensional symmetric convex body has an M-ellipsoid with respect to this constant. An important interest of such ellipsoids, is that they give reverse Brunn-Minkowski inequalities. Many interesting properties of centrally symmetric convex bodies and corresponding normed spaces were revealed using M-ellipsoids. We refer to a survey [6]. In this section we build M-ellipsoid for arbitrary non-symmetric convex body and we show in this and the next sections that many results known in the symmetric case can be translated to the general non-symmetric case.

We will not take care below of numerical constants, we write for two positive numbers that $a \sim b$ if the ratio is bounded by two universal constants. Similarly we write $a \lesssim b$, meaning that $a \leq c\,b$ where $c > 0$ is a universal constant.

Lemma 7. *Let K and L be two convex compact subsets of $\mathbb{R}^n$ with non-empty interior and with 0 as centroid. Let $c > 0$, the following properties are equivalent:*
1) $N(K, L)^{1/n} \lesssim c$; *2)* $|K - L|^{1/n} \lesssim c\,|L|^{1/n}$; *3)* $|K \cap L|^{1/n} \gtrsim c^{-1}\,|K|^{1/n}$
4) $N(L^\circ, K^\circ)^{1/n} \lesssim c$; *5)* $|L^\circ - K^\circ|^{1/n} \lesssim c\,|K^\circ|^{1/n}$; *6)* $|L^\circ \cap K^\circ|^{1/n} \gtrsim c^{-1}\,|L^\circ|^{1/n}$.

Remark. The previous lemma gives equivalent characterization of M-ellipsoid. Let K with 0 as centroid and let $\mathcal{E}$ be an ellipsoid such that $|K| = |\mathcal{E}|$. Then

$$\begin{aligned} N(K, \mathcal{E})^{1/n} &\sim N(\mathcal{E}, K)^{1/n} \sim (|K - \mathcal{E}|/|\mathcal{E}|)^{1/n} \sim (|K \cap \mathcal{E}|/|K|)^{1/n} \\ &\sim N(\mathcal{E}^\circ, K^\circ)^{1/n} \sim (|\mathcal{E}^\circ - K^\circ|/|K^\circ|)^{1/n} \sim (|\mathcal{E}^\circ \cap K^\circ|/|\mathcal{E}^\circ|)^{1/n}. \end{aligned}$$

We see that if $\mathcal{E}$ is an M-ellipsoid for K then $\mathcal{E}^\circ$ is an M-ellipsoid for K°.

The Lemma is used to show the following

Theorem 8. *There exists a constant σ such that for any convex compact subset K of $\mathbb{R}^n$ with non-empty interior and 0 as centroid, then there exists an ellipsoid $\mathcal{E}$ of $\mathbb{R}^n$ such that*

$$|K| = |\mathcal{E}| \text{ and } N(K, \mathcal{E}) \leq e^{\sigma n}\,.$$

Proof: Following a result from [4], there exists an M-ellipsoid $\mathcal{E}$ for the centrally symmetric body $K - K$ associated to some universal constant σ. Let $|\mathcal{E}| = |K - K|$ so that from Lemma 7 we have $N(\mathcal{E}, K - K)^{1/n} \lesssim e^\sigma$. Let $\lambda = (|K|/|\mathcal{E}|)^{1/n}$ then $\lambda \leq 1$, therefore $N(\lambda\,\mathcal{E}, K)^{1/n} \leq N(\mathcal{E}, K)^{1/n} \leq N(\mathcal{E}, K - K)^{1/n} N(K - K, K)^{1/n} \lesssim e^\sigma$. Using Lemma 7, we conclude that $N(K, \lambda\,\mathcal{E})^{1/n} \lesssim e^\sigma$.

6. M-ellipsoids; application to global regularity. In this section, we use technique which provides existence of M-ellipsoids, to study global properties of convex sets. The main Theorem 9 was known in the symmetric setting. But the fact that it can be translated for general convex bodies is adding a new flavor to the theory.

Theorem 9. *Let K and K' be convex compact subsets of $\mathbb{R}^n$ with non-empty interior and 0 as centroid and such that B_2^n is an M-ellipsoid for K and K'. Then there are positive universal constants c, r_1 and r_2 such that the relations*

$$(1/r_1)\, B_2^n \subset \mathrm{co}(K \cup T(K')) \text{ and } \left(\frac{|\mathrm{co}(K \cup T(K'))|}{|(1/r_1)\, B_2^n|}\right)^{1/n} \leq c$$

$$\text{and } (1/r_1)\, B_2^n \subset L \cap V(L) \subset r_2\, B_2^n$$

are satisfied for random rotations $T, V \in \mathcal{O}_n$, where $L = \mathrm{co}(K \cup T(K'))$.

The method used to prove Theorem 9 may be applyied in a more general context. For example, we derive the following

Theorem 10. *Let K be a convex compact subsets of $\mathbb{R}^n$ with non-empty interior, 0 as centroid and such that B_2^n is an M-ellipsoid for K. Let $0 < \varepsilon < 1$ and set $k = [\varepsilon n]$, then there exist k unit vectors $u_1, \ldots, u_k$ such that denoting by $H_i^- = \{x\,;\,(x, u_i) \leq 1/\sqrt{n}\}$, $i = 1 \ldots, k$, we have*

$$K \cap \left(\cap_{i \leq k} H_i^-\right) \subset C(\varepsilon) B_2^n$$

for some function $C(\varepsilon) > 0$ depending only on ε.

References

[1] Figiel T., Lindenstrauss J., Milman V., The dimension of almost spherical sections of convex bodies, Acta. Math. 139 (1977), 53-94.

[2] Grünbaum B., Measures of symmetry for convex sets, in Convexity Proc. Symposia in Pure Math. vol 7 (1963) (Amer. Math. Soc. Providence, RI) 233-270.

[3] König H., Milman V., On the covering numbers of convex bodies, Lectures Notes in Mathematics (Springer Verlag), vol. 1267 (1987), 82-95.

[4] Milman V., An inverse form of Brunn-Minkowski inequality with applications to local theory of normed spaces, C. R. Acad Sci. série I, 302 (1986), 25-28.

[5] Milman V., Isomorphic symmetrizations and geometric inequalities, GAFA Seminar Notes, Springer Lectures Notes 1317, (1988), 107-131.

[6] Milman V., Surprising geometric phenomena in high-dimentional convexity theory, to appear in Proceeding of ECM-96.

[7] Milman V., Almost Euclidean quotient spaces of subspaces of a finite dimensional normed space, Proc. Amer. Math. Soc. 94, (1985), 445-449.

[8] Pisier G., The Volume of Convex Bodies and Banach Space Geometry, Cambridge University Press 1989.

[9] Rogers C. A., Shephard G. C., The difference body of a convex body, Arch. Math. 8 (1957), 220-233.

[10] Rogers C. A., Zong C., Covering convex bodies by translates of convex bodies, Mathematika 44 (1997), 215-218.

[11] Rudelson M., Distances between non-symmetric convex bodies and MM*-estimate, preprint.

[12] Spingarn J. E., An inequality for sections and projections of a convex set, Proc. Am. Math. Soc. 118 (1993), 1219-1224.

[13] Tomczak-Jaegermann N., Banach-Mazur Distances and Finite Dimensional Operator Ideals, Pitman (1988).