Arch. Math., Vol. 55, 82–93 (1990)

0003-889X/90/5501-0082 $ 3.90/0

On the Blaschke-Santaló inequality

By

Mathieu Meyer and Alain Pajor

1. Introduction. Let K be a convex body in the n-dimensional Euclidean space $E (= E^n)$. For z in int (K), the interior of K, let K^z be the polar body of K with respect to z; denoting by $\langle , \rangle$ the scalar product, we have:

$$K^z = \{y + z \mid y \in E^n, \langle y, x - z\rangle \leqq 1 \text{ for every } x \in K\}.$$

The *volumic product* $p(K)$ of the body K is defined by

$$p(K) = \inf \{|K|\,|K^z|, z \in \operatorname{int}(K)\}$$

where $|\;|$ denotes a volume measure on E.

It should be noticed that $p(K)$ is affine invariant, that is, for every affine isomorphism $T: E \to E$, we have $p(TK) = p(K)$.

As it is well-known (see Sections 2 and 3 below), this infimum is reached for a unique point $z = s(K)$, sometimes called the *Santaló point* of K.

It was proved by Blaschke ([2], 1923) for $n = 2, 3$ that $p(K) \leqq p(B)$, where B denotes the Euclidean ball in E^n; this result was extended by Santaló ([16], 1948) for all values of n, with some restrictive hypothesis on the smoothness of the boundary of K (see [17]); however, it was shown later, by some rather technical arguments, that these assumptions can be dropped. The inequality

$$p(K) \leqq p(B)$$

is the so called Blaschke-Santaló inequality. In 1981, Saint Raymond ([15]) gave a simple proof of this inequality, in the special case when K is centrally symmetric; namely he proved that in that case, $p(K)$ is maximal if and only if K is an ellipsoid; some years later, Petty ([11], 1985) characterized in the same way the case of equality for general convex bodies.

Our aim in this work is to give a more general result than the Blaschke-Santaló inequality, by a rather simple proof, using Steiner symmetrization; namely we prove the following:

Theorem. *Let K be a convex body in E^n and let $H = \{x \in E^n \mid \langle x, u\rangle = a\}$ be an affine hyperplane ($u \in E^n \backslash 0$, $a \in \mathbb{R}$), such that* $\operatorname{int}(K) \cap H \neq \emptyset$. *Then there exists* $z \in \operatorname{int}(K) \cap H$ *such that*

$$|K|\,|K^z| \leqq p(B)/4\lambda(1 - \lambda)$$

where $\lambda \in]0, 1[$ *is defined by*

$$|\{x \in K | \langle x, u \rangle \geqq a\}| = \lambda |K|.$$

This paper is organized in the following way; in Section 2, we give a short proof of the Blaschke-Santaló inequality for centrally symmetric bodies; in Section 3 we state and prove some technical lemmas and in Section 4, we prove the theorem quoted before and study the case of equality in the Blaschke-Santaló inequality.

Finally, let us mention the problem of giving a lower bound to $p(K)$; it was proved by Mahler ([8], 1909) for $n = 2$ that $p(K) \geqq 8$ and by Bourgain and Milman ([3], 1987) that there exist some $c > 0$ such that for every n and every convex body K of E^n,

$$p(K) \geqq c^n p(B).$$

However the problem of finding the exact value of

$$\inf\{p(K) | K \text{ convex body in } E^n\}$$

is still open for $n \geqq 3$.

The infimum on special classes of convex bodies, of E^n was obtained ([15], [12]) and the extremal case characterized on these classes ([13], [14], [9], [5]).

2. The centrally symmetric case. When K is centrally symmetric, it is easy to verify (see Section 3 below) that the Santaló point of K is the center of symmetry. Let H be an affine hyperplane and denote by P the orthogonal projection onto H.

Then the Steiner symmetral $S(K, H)$ of K about H is defined by

$$S(K, H) = \left\{\frac{x_1 - x_2}{2} + y \,\middle|\, x_1, x_2 \in K, y \in H, Px_1 = Px_2 = y\right\}.$$

It is clear that $S(K, H)$ is also a convex body such that

- $S(K, H)$ is symmetric about H (that is $2Px - x \in S(K, H)$ whenever $x \in S(K, H)$).
- $|S(K, H)| = |K|$.
- If K is centrally symmetric (about z), then $S(K, H)$ is also centrally symmetric (about Pz).

For more details about Steiner symmetrization, see [7]. Our first lemma says that the volume product of a centrally symmetric convex body increases by Steiner symmetrization about hyperplanes through its center of symmetry.

Lemma 1. *Let K be a centrally symmetric body with center z and H be an affine hyperplane through z, then*

$$|(S(K, H))^z| \geqq |K^z| \quad \textit{and} \quad p(S(K, H)) \geqq p(K).$$

Proof. By the preceding remarks, we have only to prove the first inequality. After an affine transformation, it may be supposed that $z = 0$ and that $H = \{(x_i)_{i=1}^{i=n} | x_n = 0\}$ is the hyperplane of symmetrization. We identify $E (= \mathbb{R}^n)$ with $H \times \mathbb{R}$ and denote $K_1 = S(K, H)$.

Let PK be the orthogonal projection of K onto H; then

$$K_1 = \{(X, x) \mid X \in PK, x = (x_1 - x_2)/2, (X, x_i) \in K, i = 1, 2\}$$

$$K^0 = \{(Y, y) \in H \times \mathbb{R} \mid \langle X, Y \rangle + xy \leqq 1, \text{ for } X \in PK \text{ and } x \text{ such that } (X, x) \in K\}$$

$$K_1^0 = \{(Y, y) \in H \times \mathbb{R} \mid \langle X, Y \rangle + y \cdot (x_2 - x_1)/2 \leqq 1, \text{for } X \in PK \text{ and } x_i \text{ such that } (X, x_i) \in K, i = 1, 2\}.$$

For a subset A of $E (= \mathbb{R}^n = H \times \mathbb{R})$ and $y \in \mathbb{R}$, denote $A(y) = \{Y \in H \mid (Y, y) \in A\}$. Addition meaning here Minkowski sum, we have then

$$\frac{K^0(y) + K^0(-y)}{2} \subset K_1^0(y), \quad \text{for every} \quad y \in \mathbb{R}.$$

Observe that since K is centrally symmetric we have for every $y \in \mathbb{R}$, $K^0(y) = -K^0(-y)$; it follows from the Brunn-Minkowski theorem applied in H that

$$(*) \qquad |K_1^0(y)| \geqq \left| \frac{K^0(y) - K^0(y)}{2} \right| \geqq |K^0(y)|$$

and by integration

$$|K_1^0| = \int |K_1^0(y)| \, dy \geqq \int |K^0(y)| \, dy = |K^0|. \quad \square$$

P r o o f o f t h e S a n t a l ó ' s i n e q u a l i t y f o r c e n t r a l l y s y m m e t r i c b o d i e s. As it is well-known, there exists a sequence (K_n) of centrally symmetric convex bodies, converging in the sense of Hausdorff to λB (where $|\lambda B| = |K|$) and such that $K_0 = K$ and K_n is a Steiner symmetral of K_{n-1}, for $n \geqq 1$. By the lemma, the sequence $(p(K_n))$ is increasing and by continuity it converges to $p(\lambda B) = p(B)$. Thus $p(K) = p(K_0) \leqq p(B)$. $\square$

3. Preliminary results. Let K be a convex body in E.

We define a function $f: \operatorname{int}(K) \to \mathbb{R}_+$, by

$$f(z) = |K^z| = v_n \int_{S^{n-1}} \left(\max_{x \in K} \langle y, x - z \rangle \right)^{-n} d\sigma(y)$$

where $v_n = |B|$ denotes the volume of the Euclidean ball in $E (= \mathbb{R}^n)$ and σ the rotation invariant probability measure on the sphere S^{n-1}.

It is clear that

(i) $\lim \{f(z); z \text{ approaches the boundary of } K\} = +\infty$,

(ii) f is strictly convex and differentiable on $\operatorname{int}(K)$.

Let F be an affine subspace of E such that $\operatorname{int}(K) \cap F \neq \emptyset$, and define

$$F^{\perp} = \{x \in E \mid \langle x, y - y' \rangle = 0 \text{ for every } y, y' \in F\}.$$

It follows from (i) and (ii) that f has a unique critical point $z(K, F)$ on $\operatorname{int}(K) \cap F$, and that this critical point is also a strict minimum of f on $\operatorname{int}(K) \cap F$. This allows to say that $z = z(K, F)$ is characterized by the identities

$$z \in \operatorname{int}(K) \cap F, \quad \operatorname{grad} f(z) \in F^{\perp}.$$

By the dominated convergence theorem, we have for $z \in \operatorname{int}(K)$

$$\operatorname{grad} f(z) = n\, v_n \int_{S^{n-1}} \vec{y} \left(\max_{x \in K} \langle y, x - z \rangle \right)^{-(n+1)} d\sigma(y).$$

The following lemma summarizes the preceding facts.

Lemma 2. *Let K be a convex body in $\mathbb{R}^n$ and F be an affine subspace of $\mathbb{R}^n$ such that* $\operatorname{int}(K) \cap F \neq \emptyset$. *Then there exists a unique point $z = z(K, F) \in \operatorname{int}(K) \cap F$ such that one of the following equivalent properties holds.*

(a) $|K^z| = \min\{|K^x|;\, x \in \operatorname{int}(K) \cap F\}$;

(b) $\displaystyle\int_{S^{n-1}} \frac{\vec{y}}{\left(\max_{x \in K} \langle y, x - z \rangle \right)^{n+1}} d\sigma(y) \in F^{\perp}$;

(c) $\displaystyle\int_{K^z - z} \vec{y}\, dy \in F^{\perp}$.

If $F = E$, then $z(K, F)$ is the Santaló point $s(K)$ of K; in that case, Lemma 2 gives the well-known fact that $s(K)$ is the unique point s in $\operatorname{int}(K)$ which is the center of mass of the body K^s.

Dealing with Steiner symmetrization, we need some properties of stability which are given in three lemmas. The proof of the first two ones is very easy.

Before stating them, let us recall that if K is a convex body and $z \in \operatorname{int}(K)$, then $z \in \operatorname{int}(K^z)$ and, by the bipolar theorem, we have $(K^z)^z = K$. We shall say that two affine hyperplanes are orthogonal if their respective orthogonal directions are orthogonal, and when speaking of symmetry about an affine hyperplane H, we shall mean always orthogonal symmetry.

Lemma 3. *Let H_1 and H_2 be two orthogonal hyperplanes in $\mathbb{R}^n$; if a body K is symmetric about H_1, then the body $S(K, H_2)$ is symmetric about both H_1 and H_2.*

Lemma 4. *Suppose that a convex body K is symmetric about some hyperplane H through $z \in \operatorname{int}(K) \cap H$; then K^z is also symmetric about H.*

Lemma 5. *Let K be a convex body, F an affine subspace of $\mathbb{R}^n$ such that* $\operatorname{int}(K) \cap F \neq \emptyset$ *and $z = z(K, F)$. Let H be an affine hyperplane such that $F \subset H$ and let L be the convex body defined by $L^z = S(K^z, H)$. Then we have*

$$z(L, F) = z = z(K, F).$$

Proof. It may be supposed that $z = z(K, F) = 0$, that $H = \{x_n = 0\}$ and that $F = \cap(\{x_i = 0\}; i = p+1, \ldots, n)$ for some p, $1 \leqq p \leqq n-1$. By Lemma 2, we have

$$\int_{K^0} \vec{y}\, dy \in F^{\perp}.$$

Denoting for $x = (x_i)_{i=1}^{i=n-1} \in \mathbb{R}^{n-1}$, $K^0(x) = \{x_n \,|\, (x_i)_{i=1}^{i=n} \in K^0\}$, it follows that

$$\int_{\mathbb{R}^{n-1}} x_i\, |K^0(x)|\, dx_1 \cdots dx_{n-1} = 0 \quad \text{for} \quad 1 \leqq i \leqq p.$$

By the definition of Steiner symmetrization, we have $|K^0(x)| = |L^0(x)|$ for every $x \in \mathbb{R}^{n-1}$; we get thus

$$\int_{\mathbb{R}^{n-1}} x_i\, |L^0(x)|\, dx_1 \ldots dx_{n-1} = 0 \quad \text{for} \quad 1 \leqq i \leqq p$$

which conversely gives

$$\int_{L^0} \vec{y}\, dy \in F^{\perp}$$

that is by Lemma 2, $z(L, F) = 0 = z(K, F)$. □

For sake of completeness, we give a simple proof of the following lemma (see [10]), which is a particular case of a result due to K. Ball [1].

Lemma 6. [1]. *Let $f, g, h: \mathbb{R}_+ \to \mathbb{R}_+$ be three functions vanishing outside some closed intervals containing the origin, on which they are continuous and suppose that*

$$(*) \qquad f\left(\frac{2xy}{x+y}\right) \geqq g(x)^{\frac{y}{x+y}}\, h(y)^{\frac{x}{x+y}} \quad \textit{for every} \quad x, y > 0.$$

Then one has

$$\frac{1}{\int_0^{+\infty} f(t)\,dt} \leqq \frac{1}{2}\left(\frac{1}{\int_0^{+\infty} g(t)\,dt} + \frac{1}{\int_0^{+\infty} h(t)\,dt}\right).$$

Proof. Let $A = \int_0^{+\infty} f(t)\,dt$, $B = \int_0^{+\infty} g(t)\,dt$, $C = \int_0^{+\infty} h(t)\,dt$; by continuity one can define differentiable functions x, y: $[0, 1] \to \mathbb{R}_+$ such that

$$Bu = \int_0^{x(u)} g(t)\,dt,$$

$$Cu = \int_0^{y(u)} h(t)\,dt.$$

We have $B = x'(u)\, g(x(u))$, $C = y'(u)\, h(y(u))$ for every $u \in [0, 1]$. Thus if we set

$$t = \frac{2\,x(u)\,y(u)}{x(u) + y(u)},$$

we get:

$$A \geqq 2 \int_0^1 f\left(\frac{2xy}{x+y}\right)\left[\frac{Cx^2}{h(y)} + \frac{By^2}{g(x)}\right] \quad (x+y)^{-2}\, du.$$

But we have

$$(**) \qquad \alpha r + (1-\alpha)\, s \geqq r^\alpha s^{1-\alpha}, \quad 0 \leqq \alpha \leqq 1, \quad r, s \geqq 0.$$

By (*) and (**) applied to $\alpha = \frac{x}{x+y}$, $r = \frac{Cx}{h(y)}$, $s = \frac{By}{g(x)}$ we get

$$A \geqq 2 \int_0^1 (Cx)^{x/x+y} (By)^{y/x+y} (x+y)^{-1}\, dy.$$

Now by (**) applied to $\alpha = \frac{x}{x+y}$, $r = 1/Cx$, $s = 1/By$ we get

$$A \geqq 2 \Big/ \left(\frac{1}{C} + \frac{1}{B}\right), \text{ which gives the result.} \quad \square$$

4. The Blaschke-Santaló inequality. Let K be a convex body and H an affine hyperplane separating the Euclidean space E into two half-spaces D_+ and D_-; let $0 < \lambda < 1$; we shall say that H is *λ-separating for K* if

$$|D_+ \cap K| \cdot |D_- \cap K| = \lambda(1-\lambda)\, |K|^2$$

and when $\lambda = 1/2$, we shall say that H is *medial for K* (we have then $|D_+ \cap K| = |D_- \cap K| = |K|/2$).

It is easy to see that for every direction $u \in S^{n-1}$ and every λ, $0 < \lambda < 1$, there exists at least one (and in fact two if $\lambda \neq 1/2$) affine hyperplane H, orthogonal to u and λ-separating for K; it is then clear that $\operatorname{int}(K) \cap H \neq \emptyset$.

The following lemma generalizes Lemma 1 to general convex bodies.

Lemma 7. *Let K be a convex body, H an affine hyperplane and $z \in \operatorname{int}(K) \cap H$; let λ, $0 < \lambda < 1$ such that H is λ-separating for K^z. Then if K_1 is the Steiner symmetral of K with respect to H, we have*

$$|K_1^z| \geqq 4\lambda(1-\lambda)\, |K^z|.$$

P r o o f. We can suppose that $H = \{x_n = 0\}$ and $z = 0$; let us use then the same notations as in Lemma 1.

Let $x, y > 0$, $X \in K^0(x)$ and $Y \in K^0(-y)$; then for every $Z \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$ such that $(Z, z) \in K$ we have:

$$\langle X, Z\rangle + xz \leqq 1$$

$$\langle Y, Z\rangle - yz \leqq 1.$$

Now if $T = \frac{y}{x+y} X + \frac{x}{x+y} Y$, we get

$$\langle T, Z\rangle + \frac{2xy}{x+y}\left(\frac{z_1 - z_2}{2}\right) \leqq 1$$

for every $Z \in \mathbb{R}^{n-1}$ and $z_1, z_2 \in \mathbb{R}$ such that $(Z, z_i) \in K$, $i = 1, 2$.

This means that

$$T \in K_1^0\left(\frac{2xy}{x+y}\right).$$

Using Minkowski sum in $\mathbb{R}^{n-1}$, we get thus for every $x, y > 0$.

$$\frac{y}{x+y} K^0(x) + \frac{x}{x+y} K^0(-y) \subset K_1^0\left(\frac{2xy}{x+y}\right).$$

By the Brunn-Minkowski theorem in $\mathbb{R}^{n-1}$, denoting

$$f(z) = |K_1^0(z)|, \quad g(x) = |K^0(x)|, \quad h(y) = |K^0(-y)|,$$

we get formula (*) of Lemma 6.

Now, since by Lemma 3, K_1^0 is symmetric about H, we have $\int_0^{+\infty} f(z)\,dz = |K_1^0|/2$, and since H is λ-separating for K^0, we have $\left(\int_0^{+\infty} g(x)\,dx\right)\left(\int_0^{+\infty} h(y)\,dy\right) = \lambda(1-\lambda)\,|K^0|^2$. It is clear that the functions $f, g, h\colon [0, +\infty[\to \mathbb{R}_+$ satisfy the hypothesis of Lemma 6. Since by definition of g and h, one has $\int_0^{+\infty} g(x)\,dx + \int_0^{+\infty} h(y)\,dy = |K^0|$, we thus get

$$\frac{2}{|K_1^0|} \leqq \frac{1}{2}\left(\frac{1}{\int_0^{+\infty} g(x)\,dx} + \frac{1}{\int_0^{+\infty} h(y)\,dy}\right) = \frac{1}{2}\,\frac{1}{\lambda(1-\lambda)\,|K^0|}$$

which gives the result. □

Theorem. *Let K be a convex body in E, H be an affine hyperplane such that $\operatorname{int}(K) \cap H \neq \emptyset$ and suppose that H is λ-separating for K for some λ, $0 < \lambda < 1$. Then there exists $z \in \operatorname{int}(K) \cap H$ such that*

$$|K|\,|K^z| \leqq v_n^2/4\,\lambda(1-\lambda) = p(B)/4\,\lambda(1-\lambda).$$

Proof. We proceed by n successive Steiner symmetrizations until we get a centrally symmetric body.

Let $u_1 \in S^{n-1}$, u_1 orthogonal to $H = H_1$ and let $(u_i)_{i=2}^{i=n} \subset S^{n-1}$ such that $(u_1, \ldots, u_n)$ form an orthonormal basis for E. Let $z_1 = z(K, H_1)$, with the notations of Lemma 2, and define a body K_1 by the identity

$$K_1^{z_1} = S(K^{z_1}, H_1).$$

Then $|K_1^{z_1}| = |K^{z_1}|$. By Lemma 4, K_1 is symmetric about H_1 and by Lemma 7, applied to K^{z_1}, $z = z_1$ and $H = H_1$, λ-separating for $K = (K^{z_1})^{z_1}$, we get

$$|K_1| \geqq 4\lambda(1-\lambda)|K| \quad \text{and thus} \quad |K_1||K_1^{z_1}| \geqq 4\lambda(1-\lambda)|K||K^{z_1}|.$$

Choose now the hyperplane H_2, orthogonal to u_2, and medial for K_1 and define

$$z_2 = z(K_1, H_1 \cap H_2).$$

Since by Lemma 5 we have $z_1 = z(K, H_1) = z(K_1, H_1)$, we get

$$\begin{aligned} |K_1^{z_2}| &= \min\{|K_1^z|; z \in H_1 \cap H_2\} \\ &\geqq \min\{|K_1^z|; z \in H_1\} = |K_1^{z(K,H_1)}| = |K_1^{z_1}|. \end{aligned}$$

We define now a new convex body K_2 by the identity

$$|K_1^{z_2}| = S(K_1^{z_2}, H_2).$$

By Lemmas 3 and 4, K_2 is symmetric about both, H_1 and H_2. Since H_2 is medial for K_1, we get by Lemma 7 applied to $K_1^{z_1}$, $z = z_2$ and $H = H_2$ that

$$|K_2| \geqq |K_1|.$$

Moreover, we have

$$|K_2^{z_2}| = |S(K_1^{z_2}, H_2)| = |K_1^{z_2}| \geqq |K_1^{z_1}|.$$

It follows that

$$|K_2||K_2^{z_2}| \geqq |K_1||K_1^{z_1}|.$$

We continue this procedure by choosing hyperplanes $H_2, \ldots, H_n$, points $z_2, \ldots, z_n$, and defining convex bodies $K_2, \ldots, K_n$ such that for $2 \leqq i \leqq n$, we have

(i) H_i is medial for K_{i-1} and orthogonal to u_i;
(ii) $z_i = z(K_{i-1}, H_1 \cap H_2 \cap \ldots \cap H_i)$;
(iii) $K_i^{z_i} = S(K_{i-1}^{z_i}, H_i)$.

We prove then by induction that $(|K_i||K_i^{z_i}|)$ is an increasing sequence, for $2 \leqq i \leqq n$. From (ii) (iii) and Lemma 5, we have

$$z_i = z(K_{i-1}, H_1 \cap \ldots \cap H_i) = z(K_i, H_1 \cap \ldots \cap H_i).$$

Choosing H_{i+1}, z_{i+1}, K_{i+1} according to (i) (ii) (iii), we get thus

$$\begin{aligned} |K_{i+1}^{z_{i+1}}| &= |S(K_i^{z_{i+1}}, H_{i+1})| = |K_i^{z_{i+1}}| \\ &= \inf\{|K_i^z|; z \in H_1 \cap \ldots \cap H_i \cap H_{i+1}\} \\ &\geqq \inf\{|K_i^z|; z \in H_1 \cap \ldots \cap H_i\} = |K_i^{z(K_i, H_1 \cap \ldots \cap H_n)}| = |K_i^{z_i}|. \end{aligned}$$

Now, Lemma 7 applied to $K_i^{z_i}$, $z = z_{i+1}$ and H_{i+1}, medial for $K_{i+1} = (K_i^{z_{i+1}})^{z_{i+1}}$, gives:

$$|K_{i+1}| \geqq |K_i|.$$

Thus

$$4\lambda(1-\lambda)|K||K^{z_1}| \leqq |K_1||K_1^{z_1}| \leqq |K_2||K_2^{z_2}| \leqq \ldots \leqq |K_n||K_n^{z_n}|$$

from Lemmas 3 and 4 we get that K_i is symmetric about H_j, $1 \leqq j \leqq i$. Thus K_n is symmetric about H_i, $1 \leqq i \leqq n$. It follows that the point $z_n (= H_1 \cap H_2 \cap \ldots \cap H_n)$ is a center of symmetry for K_n; one has now to apply the Blaschke-Santaló inequality for centrally symmetric convex bodies (Section 2) to get the result.

Corollary (Blaschke, Santaló, Saint Raymond, Petty). *Let K be a convex body in the Euclidean space E; then $p(K) \leqq p(B)$, with equality if and only if K is an ellipsoid.*

Proof. The inequality $p(K) \leqq p(B)$ follows from the preceding theorem applied to some hyperplane H medial for K. By the affine invariance, we have $p(\mathscr{E}) = p(B)$ for any ellipsoid $\mathscr{E}$ of E.

Suppose now that $p(K) = p(B)$ and let s be the Santaló point of K; by the theorem, every hyperplane which is medial for K contains a point $z(H)$ in H such that

$$|K||K^{z(H)}| \leqq p(B).$$

We have thus

$$p(K) = |K||K^s| \leqq |K||K^{z(H)}| \leqq p(B)$$

and by the uniqueness of the minimum of $f(z)$ on $\operatorname{int}(K)$ (see the beginning of Section 3), we get

$$z(H) = s, \quad \text{for every medial hyperplane } H \text{ for } K.$$

It follows that s belongs to every medial hyperplane for K. Thus, since for every direction $u \in S^{n-1}$ there exists a medial hyperplane for K orthogonal to u, every hyperplane through s is medial for K. It follows now from the Funk-Hecke theorem (as applied in [4]) that s is a center of symmetry for K. We have now to refer to the simple and elegant proof of the case of equality $p(K) = p(B)$ for centrally symmetric bodies, given by Saint Raymond [15] to get that K is an ellipsoid (see the appendix).

Remark. The volume product $p(K)$ may be equivalent to $p(B)$ and yet the body K be far from any ellipsoid as is shown by the following example communicated to us by K. Ball. Take t such that $t \log n \geqq 1$, define $t_n = \left(\frac{t \log n}{n}\right)^{1/2}$ and $K_n = [-t_n, t_n]^n \cap B$. An elementary computation shows that $p(K_n) \geqq \left(1 - \frac{c}{n^{t/4-1}}\right) \cdot p(B)$ for some universal constant $c > 0$. On the other hand it is clear that for any ellipsoid $\mathscr{E}$ such that $\mathscr{E} \subset K_n \subset \lambda \mathscr{E}$ then $\lambda \geqq 1/t_n$.

Appendix. We show that for a centrally symmetric body K, if $p(K) = p(B)$, then K is an ellipsoid.

Suppose that $p(K) = p(B)$; then there is equality in Lemma 1, for any hyperplane H through the center of symmetry; using the notations of this lemma, it follows from the equality case in the Brunn-Minkowski theorem that

(i) $K^0(y)$ is a translate of $K^0(-y) = -K^0(y)$,

(ii) $K_1^0(y) = \dfrac{K^0(y) + K^0(-y)}{2}$,

for every y such that $K^0(y) \neq \emptyset$.

Property (i) says that $K^0(y)$ has a center of symmetry, say $Z(y)$.

Thus if $p(K) = p(B)$, every cross-section of K^0 by an affine hyperplane is centrally symmetric. If $n \geqq 3$, this gives, by a theorem of Brunn (see [6]) that K^0 and thus K is an ellipsoid.

However, instead of using this deep result of Brunn, and to solve the problem also in dimension 2, a more precise version of the equality case in Lemma 1 can be given. The following lemma is implicit in [15].

Lemma 8. *Under the hypothesis of Lemma* 1, *the following are equivalent*

(a) $|S(K, H)^0| = |K^0|$.

(b) *Every cross section of* K^0 *by an hyperplane parallel to* H *has a center of symmetry, and these centers of symmetry are in line.*

(c) *The centers of all the cross-sections of* K *by lines orthogonal to* H *lie in a hyperplane.*

P r o o f. By the bipolar theorem (b) and (c) are equivalent; they clearly imply (a); on the other side, to verify that (a) implies (c) we first reduce to the 2-dimensional case.

Indeed, with the preceding notations, it is enough to prove that for every $X \in PK$, we have for some α depending on X,

$$\langle X, Z(y) \rangle = \alpha y, \quad \text{for every} \quad y \in I = \{z; K^0(z) \neq \emptyset\}.$$

Fix thus $X \in PK$, $X \neq 0$, and define $\varphi, \psi: I \to \mathbb{R}$ by

$$\varphi(y) = \max\{\langle X, Y \rangle; Y \in K^0(y)\},$$

$$\psi(y) = \max\{\langle X, Y \rangle; Y \in K_1^0(y)\}.$$

By (i) and (ii), we have for every $y \in I$.

(iii) $\dfrac{\varphi(y) - \varphi(-y)}{2} = \langle Z(y), X \rangle$,

(iv) $\psi(y) = \dfrac{\varphi(y) + \varphi(-y)}{2}$.

Define now two centrally symmetric convex bodies A and A_1 in $\mathbb{R}^2$ by

$$A = \{(s, x) \in \mathbb{R}^2; (sX, x) \in K\},$$

$$A_1 = \{(s, x) \in \mathbb{R}^2; (sX, x) \in K_1\}.$$

It is clear that A_1 is the Steiner symmetral of A about the second axis, and by the bipolar theorem we have,

$$A^0 = \{(t, y) \in \mathbb{R} \times I;\ -\varphi(-y) \leqq t \leqq \varphi(y)\},$$
$$A_1^0 = \{(t, y) \in \mathbb{R} \times I;\ -\psi(-y) \leqq t \leqq \psi(y)\}.$$

It follows from (iv) that A_1^0 is the Steiner symmetral of A^0 about the first axis, and thus that $|A_1^0| = |A^0|$. If the lemma is proved in dimension 2, then all the centers $\left(y, \frac{\varphi(y) - \varphi(-y)}{2}\right)$ of the chords of A^0 orthogonal to the first axis are in line; thus for some $\alpha \in \mathbb{R}$, we have by (iii),

$$\langle X, Z(y)\rangle = \frac{\varphi(y) - \varphi(-y)}{2} = \alpha y \quad \text{for every} \quad y \in I.$$

This concludes the reduction to the 2 dimensional case. We refer now to a lemma due to Saint Raymond ([15], Lemma 11).

Now if $p(K) = p(B)$, we get by the preceding lemma that (c) is true for any hyperplane H; as it is classical, using for instance the unicity of the ellipsoid of maximal volume contained in K, it implies that K is an ellipsoid.

References

[1] K. BALL, Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$. Studia Math. **88**, 68–84 (1988).
[2] W. BLASCHKE, Vorlesungen über Differentialgeometrie II. Berlin-Heidelberg-New York 1923.
[3] J. BOURGAIN and V. D. MILMAN, New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$. Invent. Math. **88**, 319–340 (1987).
[4] K. J. FALCONER, Applications of a result on spherical integration to the theory of convex sets. Amer. Math. Monthly **90**, 690–693 (1983).
[5] Y. GORDON, M. MEYER and S. REISNER, Zonoids with minimal volume product. Proc. Amer. Math. Soc. **104**, 273–276 (1988).
[6] P. M. GRUBER und J. HÖBINGER, Kennzeichnung von Ellipsoiden mit Anwendungen. Jahrbuch Überblicke Math. 9–29 (1976).
[7] K. LEICHTWEISS, Konvexe Mengen. Berlin-Heidelberg-New York 1980.
[8] K. MAHLER, Ein Übertragungsprinzip für konvexe Körper. Časopis Pěst. Math. Fys. **68**, 93–102 (1909).
[9] M. MEYER, Une caractérisation volumique de certains espaces normés. Israel J. Math. **55**, 317–326 (1986).
[10] V. D. MILMAN and A. PAJOR, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. GAFA (Israel Functional Analysis Seminar 87/88), LNM **1376**, 64–104, Berlin-Heidelberg-New York 1989.
[11] C. M. PETTY, Affine isoperimetric problems. Ann. New York Acad. Sci. **440**, 113–127 (1985).
[12] S. REISNER, Random polytopes and the volume product of symmetric convex bodies. Math. Scand. **57**, 386–392 (1985).
[13] S. REISNER, Zonoids with minimal volume product. Math. Z. **192**, 339–346 (1986).
[14] S. REISNER, Minimal volume product in Banach spaces with a 1-unconditional basis. J. London Math. Soc. **36**, 126–136 (1987).
[15] J. SAINT RAYMOND, Sur le volume des corps convexes symétriques. Séminaire d'Initiation à l'Analyse, 1980–1981, Université PARIS VI, Paris 1981.

[16] L. A. SANTALÓ, Un invariante afin para los cuerpos convexos del espacio de *n* dimensiones. Portugal. Math. **8**, 155–161 (1949).

[17] R. SCHNEIDER, Random polytopes generated by anisotropic hyperplanes. Bull. London Math. Soc. **14**, 549–553 (1982).

Eingegangen am 1. 12. 1988

Anschriften der Autoren:

Mathieu Meyer
Equipe d'Analyse
U.A. N° 754 au C.N.R.S.
Université Paris VI
Tour 46 – 4ème Etage
4, Place Jussieu
F-75252 – Paris Cedex 05

Alain Pajor
Université Paris VII
U.F.R. de Mathématiques
Tour 45–55 5ème Etage
2, Place Jussieu
F-75251 – Paris Cedex 05