JOURNAL OF FUNCTIONAL ANALYSIS **80**, 109–123 (1988)

Sections of the Unit Ball of l_p^n

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Communicated by Paul Malliavin

Received December 1986

Let $B_p^n = \{(x_i) \in \mathbb{R}^n; \sum_{i=1}^{i=n} |x_i|^p \leqslant 1\}$, $1 \leqslant p \leqslant +\infty$, and let E^k be a k-dimensional subspace of $\mathbb{R}^n$; it is proved that if $p \geqslant 2$ (resp. $\leqslant$) then $\text{vol}(B_p^n \cap E^k) \geqslant \text{vol}(B_p^k)$ (resp. $\leqslant$). We give some applications to linear forms.

INTRODUCTION

In [17], J. D. Vaaler proved that the volume of sections of the cube $[-\frac{1}{2}, \frac{1}{2}]^n$ by k-dimensional subspaces of $\mathbb{R}^n$ is always bigger than 1. Our aim here is to extend this result to some other bodies of $\mathbb{R}^n$. Let $B_p^n = \{x \in \mathbb{R}^n; \sum_{i=1}^{i=n} |x_i|^p \leqslant 1\}$, $1 \leqslant p \leqslant +\infty$. We prove the following result:

THEOREM. *For every k-dimensional subspace E^k of $\mathbb{R}^n$ and for every $1 \leqslant q \leqslant p \leqslant +\infty$, we have*

$$\text{vol}(B_p^n \cap E^k)/\text{vol}(B_p^k) \geqslant \text{vol}(B_q^n \cap E^k)/\text{vol}(B_q^k).$$

Applying this result to $q = 2$, we get a lower estimate of sections of B_p^n, for $p \geqslant 2$, and to $p = 2$, an upper estimate of sections of B_q^n, $1 \leqslant q \leqslant 2$.

In view of applications to number theory, it is often useful to have lower estimates of the volumes of sections of symmetric bodies; the preceding theorem provides them for B_p^n, $p \geqslant 2$; for $p = 1$, some other arguments are necessary.

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0022-1236/88 $3.00

As in [17], the method of proofs uses logarithmically concave measures on $\mathbb{R}^n$; in the first section we recall some facts about these measures; the second section is devoted to the proof of the theorem quoted before and to some variations around this result, with more general balls. In the third section, various applications, to number theory and to Banach spaces theory, are given and we answer partially a conjecture due to J. D. Vaaler.

I. Definitions and Preliminary Results

Definitions and Notations

All measures μ, ν, ... will be *positive* Radon measures on $\mathbb{R}^n$; λ^n will denote Lebesgue measure on $\mathbb{R}^n$. We say that μ is *log-concave* if for any two compact subsets A and B of $\mathbb{R}^n$, and $0<\theta<1$, we have

$$\mu(\theta A+(1-\theta)B)\geqslant \mu(A)^\theta \mu(B)^{1-\theta},$$

where $\theta A+(1-\theta)B=\{\theta a+(1-\theta)b,\ a\in A,\ b\in B\}$ is the Minkowski sum. Let $f\colon \mathbb{R}^n\to\mathbb{R}$ be a non-negative function; we say that f is log-concave if $\log f$ is a concave function on $\{f>0\}$.

Let μ, ν be two measures; μ is said to be *more peaked* than ν, which we denote by $\mu\succ\nu$, if for any compact convex centrally symmetric subset C of $\mathbb{R}^n$, we have

$$\mu(C)\geqslant \nu(C).$$

From now, "symmetric" will mean here centrally symmetric.

For $1\leqslant p\leqslant +\infty$, and $x=(x_1,...,x_n)\in\mathbb{R}^n$, denote

$$\|x\|_p=(|x_1|^p+\cdots+|x_n|^p)^{1/p} \qquad (=\sup|x_i|,\ \text{if } p=+\infty).$$

Then $B_p^n=\{x\in\mathbb{R}^n;\ \|x\|_p\leqslant 1\}$ is the unit ball of the normed space l_p^n; let us recall that

$$\operatorname{vol}(B_p^n)=\lambda^n(B_p^n)=[2\Gamma(1+1/p)]^n[\Gamma(1+n/p)]^{-1}.$$

Let $m_1,...,m_n$ be non-null positive integers and $N=m_1+\cdots+m_n$; for $x\in\mathbb{R}^N$, let P^ix be the canonical projection from $\mathbb{R}^N$ onto $\mathbb{R}^{m_i}$, $1\leqslant i\leqslant n$; we define a norm on $\mathbb{R}^N$, setting

$$\|x\|_p^{m_1,...,m_n}=\left(\sum_{i=1}^{i=n}\|P^ix\|_2^p\right)^{1/p}.$$

We shall denote $B_p^{m_1,...,m_n}$ the unit ball of this norm. Finally, for $p\geqslant 1$, let μ_p^n be the probability measure $f_p^n\cdot\lambda^n$ on $\mathbb{R}^n$, where

$$f_p^n(x)=\exp(-\|\alpha_p x\|_p^p), \qquad \alpha_p=2\Gamma(1+1/p),$$

and let $\nu_p^{m_1, \ldots, m_n}$ be the probability measure $\varphi_p^{m_1, \ldots, m_n} \cdot \lambda^N$ on $\mathbb{R}^N$, where

$$\varphi_p^{m_1, \ldots, m_n}(x) = \exp\left(- \sum_{i=1}^{i=n} \|\alpha_p^{m_i} P^i x\|_2^p \right)$$

and $\alpha_p^m = [\mathrm{vol}(B_2^m)\, \Gamma(1 + m/p)]^{1/m}$.

Observe that if $m_1 = \cdots = m_n = 1$, then $N = n$ and $\nu_p^{m_1, \ldots, m_n} = \mu_p^n$.

Log-Concave Measures

We shall now recall some well-known results about log-concave measures; a very simple proof of the following one, due independently to Borell and Prekopa [2, 12], can be found in a paper of Rinott [14].

I.1. THEOREM (Borell and Prekopa). *Let μ be a positive measure on $\mathbb{R}^n$, with density f with respect to λ^n; then μ is log-concave if and only if f has a log-concave version with respect to λ^n.*

The following corollary is an immediate consequence of Theorem I.1.

I.2. COROLLARY. *Let μ_i, $1 \leqslant i \leqslant n$, be log-concave measures on $\mathbb{R}^{m_i}$, with density f_i with respect to λ^{m_i}; then if $N = m_1 + \cdots + m_n$, the measure $\mu = \otimes_{i=1}^{i=n} \mu_i$ is log-concave on $\mathbb{R}^N$.*

In fact, Theorem I.1 and its corollary are true without any density assumption with respect to λ^n; but we shall only need these weak forms of the results of Borell and Prekopa. The natural links between log-concavity and peakedness will be explained in Theorem I.5; before proving it, we state two simple lemmas.

I.3. LEMMA. *Let μ, ν be two positive measures on $\mathbb{R}^n$; then $\mu \succ \nu$ if and only if $\int f\, d\mu \geqslant \int f\, d\nu$ for any log-concave symmetric positive upper semi-continuous function on $\mathbb{R}^n$.*

I.4. LEMMA. *Let μ be a log-concave symmetric measure on $\mathbb{R}^{m_1}$ and let A be a compact convex symmetric subset of $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$; define $g\colon \mathbb{R}^{m_2} \to \mathbb{R}$ by*

$$g(y) = \mu(\{x \in \mathbb{R}^{m_1};\ (x, y) \in A\}).$$

Then g is a log-concave symmetric positive upper semi-continuous function on $\mathbb{R}^{m_2}$.

We are now ready to give a simple proof of Theorem I.5, which is a particular case of a result due to Kanter [7].

I.5. THEOREM (Kanter). *Let μ_i, ν_i be log-concave symmetric measures on $\mathbb{R}^{m_i}$, $1 \leqslant i \leqslant n$, with density with respect to λ^{m_i} and suppose that $\mu_i \succ \nu_i$ on $\mathbb{R}^{m_i}$, $1 \leqslant i \leqslant n$. Then $\otimes_{i=1}^{i=n} \mu_i \succ \otimes_{i=1}^{i=n} \nu_i$ on $\mathbb{R}^N$, $N = m_1 + \cdots + m_n$.*

Proof. By Corollary I.2, we may use induction, and have only to work for $n=2$. Let A be a compact convex symmetric subset of $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$. Using Fubini's theorem, we have

$$(\mu_1 \otimes \mu_2)(A) = \int \mu_1(\{x, (x, y) \in A\})\, d\mu_2(y).$$

By Lemma I.4 applied to μ_1 and Lemma I.3 applied to (μ_2, ν_2) on $\mathbb{R}^{m_2}$, we get

$$(\mu_1 \otimes \mu_2)(A) \geqslant \int \mu_1(\{x, (x, y) \in A\})\, d\nu_2(y) = (\mu_1 \otimes \nu_2)(A).$$

Reversing the argument, Theorem I.5 is proved. ∎

Let us give now the application of Theorem I.5 that we shall use in the sequel:

I.6. PROPOSITION. *With the previous notations, for $1 \leqslant q \leqslant p \leqslant +\infty$, we have*

$$\nu_p^{m_1, \ldots, m_n} \succ \nu_q^{m_1, \ldots, m_n} \qquad \text{on} \quad \mathbb{R}^N$$

and in particular, $\mu_p^n \succ \mu_q^n$ on $\mathbb{R}^n$.

Proof. In view of Theorem I.1 and Corollary I.2, it is clear that our measures are log-concave and symmetric; by Theorem I.5, we have thus only to show that

$$\nu_p^m \succ \nu_q^m, \qquad \text{for} \quad 1 \leqslant q \leqslant p \leqslant +\infty \text{ and } m \geqslant 1.$$

Since these measures have density, it suffices to test $\nu_p^m(C) \geqslant \nu_q^m(C)$ for symmetric bodies C of $\mathbb{R}^m$. Let C be such a body and $\rho(x) = \inf\{\lambda > 0; x \in \lambda C\}$ be the norm associated to C on $\mathbb{R}^m$.

Using polar coordinates, we get

$$\begin{aligned} \nu_p^m(C) &= \int_C \exp(-\|\alpha_p^m y\|_2^p)\, d\lambda^m(y) \\ &= m \operatorname{vol}(B_2^m) \int_{S_{m-1}} \left(\int_0^{1/\rho(\omega)} \exp(-(\alpha_p^m r)^p)\, r^{m-1}\, dr \right) d\sigma_{m-1}(\omega), \end{aligned}$$

where σ_{m-1} is the normalized rotation invariant measure on the sphere S_{m-1}.

The problem reduces now to show that for $1 \leqslant q \leqslant p \leqslant +\infty$,

$$\exp(-|\alpha_p^m t|^p)\, t^{m-1} \cdot dt \succ \exp(-|\alpha_q^m t|^q)\, t^{m-1} \cdot dt.$$

Let $f(x) = \int_0^x [\exp(-|\alpha_p^m t|^p) - \exp(-|\alpha_q^m t|^q)]\, t^{m-1}\, dt$. Then $f(0) = f(+\infty) = 0$ and f' has a unique zero and is positive near 0. It is now easy to conclude.

II. Sections of B_p^n

Notations

Let E^k be a k-dimensional subspace of $\mathbb{R}^n$, $1 \leqslant k \leqslant n$, and let F^{n-k} be its orthogonal subspace, for the canonical euclidean product $\langle , \rangle$ on $\mathbb{R}^n$. Let $u^1, \ldots, u^{n-k}$ be an orthonormal basis of F^{n-k} and define, for $\varepsilon > 0$,

$$E^k(\varepsilon) = \{x \in \mathbb{R}^n;\ |\langle x, u^j \rangle| \leqslant \varepsilon,\ 1 \leqslant j \leqslant n-k\}.$$

Then for any convex body B of $\mathbb{R}^n$, with the origin as interior point, we have

$$\mathrm{vol}(E^k \cap B) = \lim_{\varepsilon \to 0} (2\varepsilon)^{k-n}\, \mathrm{vol}(E^k(\varepsilon) \cap B). \tag{1}$$

The following lemma gives another expression of $\mathrm{vol}(E^k \cap B)$:

II.1. Lemma. *Let E^k be a k-dimensional subspace of $\mathbb{R}^n$, $1 \leqslant k \leqslant n$, and let B be a symmetric body of $\mathbb{R}^n$, with associated norm $\|\ \|$ ($B = \{x;\ \|x\| \leqslant 1\}$); then for any $0 < p \leqslant +\infty$, we have*

$$\Gamma\left(1 + \frac{k}{p}\right) \cdot \mathrm{vol}(E^k \cap B) = \lim_{\varepsilon \to 0} (2\varepsilon)^{k-n} \int_{E^k(\varepsilon)} \exp(-\|x\|^p)\, d\lambda^n(x). \tag{2}$$

Proof. For $\varepsilon > 0$, let $g(\varepsilon) = (2\varepsilon)^{k-n} \int_{E^k(\varepsilon)} \exp(-\|x\|^p)\, d\lambda^n(x)$. Then $g(\varepsilon) = (2\varepsilon)^{k-n} \int_{E^k(\varepsilon)} (\int_{\|x\|^p}^{+\infty} \exp(-t)\, dt)\, d\lambda^n(x)$ and by Fubini's theorem and a change of variables,

$$g(\varepsilon) = \int_0^{+\infty} [(2\varepsilon t^{-1/p})^{k-n}\, \mathrm{vol}(E^k(\varepsilon t^{-1/p}) \cap B)\, e^{-t}]\, t^{k/p}\, dt. \tag{3}$$

By the Brunn–Minkowski theorem, for every $x \in F^{n-k}$, we have $\mathrm{vol}((E^k + x) \cap B) \leqslant \mathrm{vol}(E^k \cap B)$. Hence for every $\eta > 0$, we have by integration

$$(2\eta)^{k-n} \mathrm{vol}(E^k(\eta) \cap B) \leqslant \mathrm{vol}(E^k \cap B).$$

Now (2) follows from (1) and (3), using dominated convergence. ∎

II.2. THEOREM. *Let E^k be a k-dimensional subspace of $\mathbb{R}^n$, $1 \leqslant k \leqslant n$, and define h: $[1, +\infty] \to \mathbb{R}_+$ by*

$$h(p) = \text{vol}(E^k \cap B_p^n)/\text{vol}(B_p^k).$$

Then the function h is increasing. In particular,

$$\text{vol}(E^k \cap B_p^n) \leqslant \text{vol}(B_p^k), \qquad \text{for} \quad 1 \leqslant p \leqslant 2$$
$$\text{vol}(E^k \cap B_p^n) \geqslant \text{vol}(B_p^k), \qquad \text{for} \quad 2 \leqslant p \leqslant +\infty.$$

Proof. Applying Lemma II.1 to B_p^n, $1 \leqslant p \leqslant +\infty$, we get

$$\Gamma\left(1 + \frac{k}{p}\right) \text{vol}(E^k \cap B_p^n) = \lim_{\varepsilon \to 0} (2\varepsilon)^{k-n} \int_{E^k(\varepsilon)} \exp[-(|x_1|^p + \cdots + |x_n|^p)] \, dx,$$

which gives after change of variables,

$$\text{vol}(E^k \cap B_p^n) = \alpha_p^k [\Gamma(1 + k/p)]^{-1} \lim_{\eta \to 0} (2\eta)^{k-n} \mu_p^n(E^k(\eta)).$$

Observing that $\text{vol}(B_p^k) = \alpha_p^k [\Gamma(1 + k/p)]^{-1}$, we get thus

$$\text{vol}(E^k \cap B_p^n)/\text{vol}(B_p^k) = \lim_{\eta \to 0} (2\eta)^{k-n} \mu_p^n(E^k(\eta)). \tag{4}$$

Now, since for any $\eta > 0$, $E^k(\eta)$ is a closed convex symmetric subset of $\mathbb{R}^n$, we can conclude, invoking Proposition I.6. Notice that $h(2) = 1$ to complete the proof. ∎

Using measures $\nu_p^{m_1, \ldots, m_n}$ instead of μ_p^n, we have also:

II.3. THEOREM. *Let $m_1, \ldots, m_n$ be strictly positive integers and $a_1, \ldots, a_n > 0$ and define*

$$B_p^n(m_i, a_i) = \left\{ x \in \mathbb{R}^{m_1 \times \cdots \times m_n};\ \sum_{i=1}^{i=n} \|a_i P^i x\|_2^p \leqslant 1 \right\}.$$

Then, for any k-dimensional subspace E^k of $\mathbb{R}^n$, the function φ: $[1, +\infty] \to \mathbb{R}_+$ defined by

$$\varphi(p) = \text{vol}(E^k \cap B_p^n(m_i, a_i \alpha_p^{m_i}))[\Gamma(1 + k/p)]^{-1}$$

is increasing.

Let $T: \mathbb{R}^k \to \mathbb{R}^n$ be a linear operator with rank k; then for any compact subset K of $\mathbb{R}^n$, we have

$$\operatorname{vol}(T^{-1}(K)) = [\det(T^*T)]^{-1/2} \operatorname{vol}(R(T) \cap K),$$

where $R(T)$ denotes the range of T and $T^*: \mathbb{R}^n \to \mathbb{R}^k$ is the adjoint of T. In particular, if $B_2^n(b_i) = \{x \in \mathbb{R}^n; \sum_{i=1}^{i=n} b_i^2 x_i^2 \leqslant 1\}$, we have

$$\operatorname{vol}(R(T) \cap B_2^n(b_i)) = [\det(T^*T)/\det(T^*D^2T)]^{1/2} \operatorname{vol}(B_2^k),$$

where D is the diagonal $[n \times n]$ matrix, with $(b_1, ..., b_n)$ on the diagonal. Thus we get operator versions of Theorems II.2 and II.3.

II.4. Corollary. *Let $T: \mathbb{R}^k \to \mathbb{R}^n$ be a linear rank k operator; then $\tilde{h}(p) = \operatorname{vol}(T^{-1}(B_p^n))/\operatorname{vol}(B_p^k)$ is an increasing function on $[1, +\infty]$; in particular,*

$$\text{for} \quad p \geqslant 2, \qquad \operatorname{vol}(T^{-1}(B_p^n)) \geqslant \det(T^*T)^{-1/2} \operatorname{vol}(B_p^k)$$
$$\text{for} \quad p \leqslant 2, \qquad \operatorname{vol}(T^{-1}(B_p^n)) \leqslant \det(T^*T)^{-1/2} \operatorname{vol}(B_p^k).$$

II.5. Corollary. *Let $T: \mathbb{R}^k \to \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$ be a linear rank k operator and let*

$$B_p^n(m_i, a_i) = \left\{x \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}; \sum_{i=1}^{i=n} \|a_i P_i x\|_2^p \leqslant 1\right\}.$$

Then if $p \geqslant 2$ (resp. $1 \leqslant p \leqslant 2$),

$$\operatorname{vol}(T^{-1}(B_p^n(m_i, a_i))) \underset{(\text{resp.} \leqslant)}{\geqslant} \frac{\Gamma(k/2+1)}{\Gamma(k/p+1)} [\det(T^*D^2T)]^{-1/2},$$

where D is the diagonal operator: $\mathbb{R}^N \to \mathbb{R}^N$, $N = m_1 + \cdots + m_n$, such that $DP^i x = a_i \alpha_2^{m_i} (\alpha_p^{m_i})^{-1} P^i x$, $1 \leqslant i \leqslant n$.

Remarks. (a) For $p = +\infty$, the inequality $\operatorname{vol}(E^k \cap B_\infty^n) \geqslant \operatorname{vol}(B_\infty^k)$ was proved first by Hadwiger [5] (see also Hensley [6]) when $k = n-1$, and generalized by Vaaler [17], for any value of k.

(b) If E is an m-dimensional normed space, let us denote, for $1 \leqslant p \leqslant +\infty$, $l_p^n(E)$ the space $\mathbb{R}^m \times \cdots \times \mathbb{R}^m$ (n times) with the norm associated to the unit ball $B_p^n(E)$, where

$$B_p^n(E) = \left\{y \in E \times \cdots \times E \ (n \text{ times});\ y = (y_1, ..., y_n), \sum_{i=1}^{i=n} \|y_i\|^p \leqslant 1\right\}.$$

If $E = l_2^m$, the estimates of Theorem II.3 may be read as follows: for $p \geqslant 2$ (resp. $1 \leqslant p \leqslant 2$)

$$\mathrm{vol}(B_p^n(l_2^m) \cap E^k) \underset{(\text{resp. } \leqslant)}{\geqslant} \mathrm{vol}(B_p^k)(\mathrm{vol}(B_2^m)/\mathrm{vol}(B_p^m))^{k/m}.$$

In the particular case when $k = rm$, r integer, $1 \leqslant r \leqslant n$, these bounds are attained when $E^k = l_p^r(l_2^m)$, naturally embedded in $l_p^n(l_2^m)$.

(c) The estimates of Corollary II.5 are easy to calculate when D is a multiple of the identity. However, if it is not the case, one may use the inequalities

$$A_{N-k+1} \cdots A_N \leqslant (\det(T^*D^2T)/\det(T^*T))^{1/2} \leqslant A_1 \cdots A_k$$

if $A_1 \geqslant A_2 \geqslant \cdots \geqslant A_N > 0$ is the decreasing rearrangement of diagonal terms of the matrix D.

In fact, formula (4) of Theorem II.2 may be read as follows: Let E^k be a k-dimensional subspace of $\mathbb{R}^n$, with equation $\sum_{i=1}^{i=n} a_i^j x_i = 0$, $1 \leqslant j \leqslant n-k$, where the $u^j = (a_1^j, ..., a_n^j)$, $1 \leqslant j \leqslant n-k$, are an orthonormal basis of the orthogonal subspace F^{n-k} of E^k.

Let $(X_1, ..., X_n)$ be n independant random variables with distribution μ_p^1 (that is, with density $\exp(-|\alpha_p x|^p)$ for dx). Define

$$S_j = \sum_{i=1}^{i=n} a_i^j X_i, \qquad 1 \leqslant j \leqslant n-k.$$

If S is the random variable $(S_1, ..., S_{n-k})$ with value in $\mathbb{R}^{n-k}$ and if $f_p^n(E^k)$ is the value at $(0, ..., 0)$ of the density of S with respect to λ^{n-k}, formula (4) says

$$f_p^n(E^k) = \mathrm{vol}(E^k \cap B_p^n)/\mathrm{vol}(B_p^k). \tag{5}$$

Let now g_p be the characteristic function of μ_p^1, $1 \leqslant p \leqslant +\infty$; then $|g_p| \leqslant 1$, $g_p \in L^2(dt)$ and for $1 \leqslant p \leqslant 2$, g_p is a positive function in $L^1(dt)$.

Let $G_p^S \colon \mathbb{R}^{n-k} \to \mathbb{R}$ be the characteristic function of S; we get from the independence of $X_1, ..., X_n$ that

$$G_p^S(t_1, ..., t_{n-k}) = \prod_{i=1}^{i=n} g_p\left(\sum_{j=1}^{j=n-k} a_i^j t_j\right).$$

For $1 \leqslant p \leqslant 2$, we have $G_p^S \in L^1(\lambda^{n-k})$: as a matter of fact, since the (u^j), $1 \leqslant j \leqslant n-k$, are linearly independent, there exists a subset I of $\{1, ..., n\}$, $\mathrm{card}(I) = n-k$ such that

$$A_I = \det(a_i^j, 1 \leqslant j \leqslant n-k, i \in I) \neq 0$$

and since $0 \leqslant g_P \leqslant 1$, we get

$$0 \leqslant G_p^S(t_1, ..., t_{n-k}) \leqslant \prod_{i \in I} g_p\left(\sum_{j=1}^{j=n-k} a_i^j t_j\right)$$

and thus

$$\int |G_p^S| \, d\lambda^{n-k} \leqslant (A_I)^{-1}(2\pi)^{n-k}.$$

We may thus apply, for $1 \leqslant p \leqslant 2$, the inversion theorem for Fourier transforms to get from (5):

II.6. Lemma. *For any k-dimensional subspace E^k of $\mathbb{R}^n$ and for any $1 \leqslant p \leqslant 2$, we have with the preceding notations*

$$\operatorname{vol}(E^k \cap B_p^n)/\operatorname{vol}(B_p^k) = (2\pi)^{k-n} \int_{\mathbb{R}^{n-k}} g_p\left(\sum_{j=1}^{j=n-k} a_i^j t_j\right) dt_1 \cdots dt_{n-k}.$$

In the particular case when $k = n-1$ and $p = 1$,

$$\operatorname{vol}(E^{n-1} \cap B_1^n) = \left(\pi^{-1} \int_{\mathbb{R}} \prod_{i=1}^{i=n} (1 + a_i^2 t^2)^{-1} \, dt\right) \operatorname{vol}(B_1^{n-1})$$

if E^{n-1} has equation $\sum_{i=1}^{i=n} a_i x_i = 0$, with $\sum_{i=1}^{i=n} a_i^2 = 1$, and integrating we get

$$\operatorname{vol}(E^{n-1} \cap B_1^n) = \frac{2^{n-1}}{(n-1)!} \sum_{i=1}^{i=n} \frac{|a_i|^{2n-3}}{\prod_{j \neq i}(a_i^2 - a_j^2)}. \tag{6}$$

Observing that

$$\left[1 + \frac{t^2}{n}\right]^{-n} \leqslant \prod_{i=1}^{i=n} [1 + a_i^2 t^2]^{-1} \leqslant (1 + t^2)^{-1}$$

we get thus:

II.7. Proposition. *For any hyperplane H of $\mathbb{R}^n$ with equation $a_1 x_1 + \cdots + a_n x_n = 0$, the following inequality holds:*

$$\frac{\sqrt{n}}{4^{n-1}} \binom{2n-2}{n-1} \frac{2^{n-1}}{(n-1)!} \leqslant \operatorname{vol}(H \cap B_1^n) \leqslant \frac{2^{n-1}}{(n-1)!}$$

with equality on the left if and only if all the $|a_i|$ are equal, on the right if and only if only one of the a_i is different from 0.

Remarks. (1) We conjecture that for $1<p<2$ the function $x \to g_p(\sqrt{x})$ is log-convex on $\mathbb{R}_+$; if it is true, we would have also

$$\mathrm{vol}(H_0 \cap B_p^n) \leqslant \mathrm{vol}(H \cap B_p^n), \qquad \text{for } 1<p<2,$$

where H_0 is the hyperplane $x_1 + \cdots + x_n = 0$.

(2) It can be proved by induction that if $a_1, ..., a_n$ are real numbers,

$$\mathrm{vol}\left(B_1^n \cap \left\{x \in \mathbb{R}^n; \sum_{i=1}^{i=n} a_i x_i \leqslant t\right\}\right) = \begin{cases} \dfrac{2^{n-1}}{n!}\left[\displaystyle\sum_{i=1}^{i=n} \frac{|a_i|^{n-2}}{\prod_{j\neq i}(a_i^2 - a_j^2)}(|a_i| + t)_+^n\right] & \text{if} \quad t \leqslant 0 \\ \dfrac{2^{n-1}}{n!}\left[2 - \displaystyle\sum_{i=1}^{i=n} \frac{|a_i|^{n-2}}{\prod_{j\neq i}(a_i^2 - a_j^2)}(|a_i| - t)_+^n\right] & \text{if} \quad t \geqslant 0, \end{cases}$$

where for $x \in \mathbb{R}$, $x_+ = \max(x, 0)$.

Derivating this equality, we get again formula (6) when supposing that $a_1^2 + \cdots + a_n^2 = 1$.

(3) Formula (6) may be interpreted in terms of generalized Vandermonde determinants: If H is the hyperplane,

$$a_1 x_1 + \cdots + a_n x_n = 0.$$

For $k \in \mathbb{N}$, let $A^k = (|a_1|^k, ..., |a_n|^k) \in \mathbb{R}^n$. Then

$$\mathrm{vol}(H \cap B_1^n) = \frac{2^{n-1}}{(n-1)!} \|A^2\|_2 \frac{\det(A^{2n-3}, A^{2(n-2)}, A^{2(n-3)}, ..., A^2, A^0)}{\det(A^{2(n-1)}, A^{2(n-2)}, A^{2(n-3)}, ..., A^2, A^0)}.$$

III. Applications

A. *Applications to Number Theory*

Using Minkowski's theorem for convex bodies, we get from Corollary II.4:

III.1. Theorem. *Let (L_j), $j = 1, ..., n$, be a ssystem of n real linear forms with k variables, and let T be the corresponding $[n \times k]$ matrix. Then if $det(T^*T) \neq 0$ and $p \geqslant 2$, there exists a point $x = (x_1, ..., x_k)$ in $\mathbb{Z}^k \setminus \{0\}$ such that*

$$\left(\sum_{j=1}^{j=n} |L_j(x)|^p\right)^{1/p} \leqslant \Gamma\left(1+\frac{k}{p}\right)\Gamma\left(1+\frac{1}{p}\right)^{-k} [\det(T^*T)]^{1/2}.$$

With the aid of Corollary II.5, a more complicated version of this result may be given, as in [17], if we have to solve a linear system with complex conjugate linear forms; by Van der Corput's extension of Minkowski's theorem, one can get also a system of m distinct points $\pm x^i$ in $\mathbb{Z}^k \setminus \{0\}$, $1 \leqslant i \leqslant m$, with the Tx^i bounded in l_p^n-norm.

Finally, using the arguments of Bombieri and Vaaler ([1], see also [4]), we get the following l_p^n-version of their Siegel's lemma:

III.2. THEOREM. *Let A be a $[k \times n]$ matrix, with rank k and entries in $\mathbb{Z}$, $1 \leqslant k < n$, and let $p \geqslant 2$; then the system $Ax = 0$ admits $(n-k)$ linearly independent solutions $x_l = (x_l^1, ..., x_l^n)$ in $\mathbb{Z}^n$, $1 \leqslant l \leqslant n-k$, such that*

$$\prod_{1 \leqslant l \leqslant n-k} \left(\sum_{i=1}^{i=n} |x_l^i|^p \right)^{1/p} \leqslant D^{-1} \Gamma\left(1 + \left(\frac{n-k}{p}\right)\right) \Gamma\left(1 + \frac{1}{p}\right)^{k-n} [\det(AA^*)]^{1/2},$$

where D denotes the GCD of $[k \times k]$ determinants extracted from A.

Remark. Using Theorem III.5, at the end of this section, one can give also a version of Theorems III.1 and III.2 for $p = 1$.

B. *Volume Ratio Numbers*

Let X be a Banach space and B_X its unit ball. For any operator $u: X \to l_2$ from X into the Hilbert space l_2, define the kth *volume ratio number*, $k \geqslant 1$, by

$$\mathrm{vr}_k(u) = \mathrm{Sup}\{\mathrm{vol}(P^k u(B_X))/\mathrm{vol}(B_2^k))^{1/k}\},$$

where the supremum is taken over all rank k orthogonal projections P_k on l_2. These numbers were introduced by V. D. Milman and G. Pisier in [9] and studied in [11] for their connection with entropy.

Let $p \geqslant 1$ and $I_p: l_p^n \to l_2^n$ be the canonical embedding. The numbers $(\mathrm{vr}_k(I_p))$ are computed in [11] for $p = 1$. Theorem II.2 will now provide their order of magnitude for all p, $1 \leqslant p \leqslant 2$.

III.3. THEOREM. *Let $1 \leqslant p \leqslant 2$ and $1 \leqslant k \leqslant n$, then for some universal constants $0 < a < b$, we have*

$$a k^{1/2 - 1/p} \leqslant \mathrm{vr}_k(I_p: l_p^n \to l_2^n) \leqslant b\, k^{1/2 - 1/p}.$$

Proof. The lower estimate is clear since using canonical projection, we have

$$\mathrm{vr}_k(I_p) \geqslant (\mathrm{vol}(B_p^k)/\mathrm{vol}(B_2^k))^{1/k}.$$

To obtain the upper estimate, we use the Santaló inequality ([16], see also [15]). Indeed identifying the polar body of a projection of B_p^n as a section of B_q^n where $1/p+1/q=1$, the Santaló inequality yields to

$$\mathrm{vr}_k(I_p) \leqslant \mathrm{Sup}\{(\mathrm{vol}(B_2^k)/\mathrm{vol}(E^k \cap B_q^n))^{1/k}\},$$

where E^k runs over all k-dimensional subspaces of $\mathbb{R}^n$. Therefore, from Theorem II.2, we have

$$\mathrm{vr}_k(I_p) \leqslant (\mathrm{vol}(B_2^k)/\mathrm{vol}(B_q^k))^{1/k}.$$

Remark. The last result gives a limit of validity of Proposition 1 in [10] when comparing the numbers $(\mathrm{vr}_k(u))$ to entropy: the main point to observe in Theorem III.3 is that the order of growth of $\mathrm{vr}_k(I_p)$ does not depend on n for $1 \leqslant p \leqslant 2$.

C. *Lower Estimates for the Volume of Sections*

A symmetric body B of $\mathbb{R}^n$ is said to be *isotropic* if there is a number $L>0$ such that

$$\int_B x_i x_j \, d\lambda^n(x) = L^2 \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker symbol.

The body B is isotropic if and only if the covariance matrix of the random variable 1_B ($1_B(x)=1$ if $x \in B$, $=0$ if $x \notin B$) is a multiple of the identity. It is easily seen that for any symmetric body B of $\mathbb{R}^n$, there exists a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$, with $\det(T)=1$ and $T(B)$ isotropic. Clearly, because of symmetry, the balls B_p^n, $1 \leqslant p \leqslant +\infty$, are isotropic.

Let us state a conjecture due to Vaaler [17], in a slightly different form:

III.4. *Conjecture.* If B is an isotropic symmetric body of $\mathbb{R}^n$, then for any subspace $E \neq \{0\}$,

$$\mathrm{vol}(E \cap B)^{1/\dim(E)} \geqslant \mathrm{vol}(B)^{1/n},$$

where $\dim(E)$ denotes the dimension of E.

As a consequence of Theorem II.2, we get:

III.5. THEOREM. *The conjecture is true for B_p^n, $2 \leqslant p \leqslant +\infty$.*

Proof. For $p=+\infty$, it is Vaaler's result [17]; for $2 \leqslant p < +\infty$. Let E^k be a k-dimensional subspace of $\mathbb{R}^n$, $1 \leqslant k \leqslant n$; then by Theorem III.2, $\mathrm{vol}(E^k \cap B_p^n)^{1/k} \geqslant \mathrm{vol}(B_p^k)^{1/k} \geqslant \mathrm{vol}(B_p^n)^{1/n}$, by the log-convexity of $\Gamma(x)$. ∎

To prove that the conjecture is true for $p=1$, we shall need the following results.

III.6. LEMMA. *If E^k is a k-dimensional subspace of $\mathbb{R}^n$, then* $vol(B_1^n \cap E^k) \geqslant \binom{n}{k}^{-1/2} \operatorname{vol}(B_1^k)$.

Proof. Let $P^k\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto E^k. Then E^k, equipped with $\| \ \|_1$, is a k-dimensional normed space whose dual has $P^k(B_\infty^n)$ as unit ball. Since B_∞^n is a zonoid, $P_k(B_\infty^n)$ is also a zonoid; we may thus apply the inverse Santaló inequality for zonoids, due to Reisner [13],

$$\operatorname{vol}(E^k \cap B_1^n) \cdot \operatorname{vol}(P^k(B_\infty^n)) \geqslant \operatorname{vol}(B_1^k) \cdot \operatorname{vol}(B_\infty^k). \tag{7}$$

Now, by a result of McMullen [8], for any rank k linear operator $T\colon \mathbb{R}^n \to \mathbb{R}^k$, we have

$$\operatorname{vol}(T(B_\infty^n)) = \left(\sum_I |\det(T_I)|\right) \cdot \operatorname{vol}(B_\infty^k),$$

where I runs over cardinal k subsets of $\{1, ..., n\}$ and T_I is the square matrix $[k \times k]$ extracted from the matrix of T, with columns indexed by I. Using Cauchy–Schwarz inequality and Cauchy–Binet equality, we get

$$\operatorname{vol}(T(B_\infty^n)) \leqslant \binom{n}{k}^{1/2} \operatorname{vol}(B_\infty^k)\,[\det(T^*T)]^{1/2}.$$

Applying this inequality to $T = S \circ P^k$, where $S\colon \mathbb{R}^n \to \mathbb{R}^k$ is an isometry on E^k vanishing on the orthogonal subspace of E^k, we obtain

$$\operatorname{vol}(P^k(B_\infty^n)) \leqslant \binom{n}{k}^{1/2} \cdot \operatorname{vol}(B_\infty^k). \tag{8}$$

We conclude from (7) and (8). ∎

We are grateful to J. D. Vaaler for having pointed out to us McMullen's result and inequality (8).

III.7. LEMMA. *For every k-dimensional subspace E^k of $\mathbb{R}^n$, we have*

$$(n/\pi)^{(n-k)/2}\,\Gamma((n+k)/2)\,\Gamma(n)^{-1} \cdot \operatorname{vol}(B_1^k) \leqslant \operatorname{vol}(E^k \cap B_1^n) \leqslant \operatorname{vol}(B_1^k).$$

Proof. The inequality on the right comes from Theorem II.2. For $p=1$, we have $g_1(t) = (1 + (t^2/4))^{-1}$; we get from Lemma II.6 that

$$\operatorname{vol}(E^k \cap B_1^n) = \operatorname{vol}(B_1^k) \cdot \pi^{k-n} \cdot \int_{\mathbb{R}^{n-k}} \prod_{i=1}^{i=n} \left(1 + \left(\sum_{j=1}^{j=n-k} a_i^j t_j\right)^2\right)^{-1} dt_1 \cdots dt_{n-k}.$$

Since the (u^j), $1 \leqslant j \leqslant n-k$, are orthonormal, we get, comparing arithmetic and geometric means,

$$\prod_{i=1}^{i=n}\left(1+\left(\sum_{j=1}^{j=n-k} a_i^j t_j\right)^2\right)^{-1} \geqslant \left(1+\left(\sum_{j=1}^{j=n-k} t_j^2\right)\Big/ n\right)^{-n}$$

and thus

$$\operatorname{vol}(E^k \cap B_1^n) \geqslant \operatorname{vol}(B_1^k)\cdot \pi^{k-n}(n-k)\cdot \operatorname{vol}(B_2^{n-k}) \int_0^{+\infty} \frac{r^{n-k-1}}{(1+(r^2/n))^n}\, dr,$$

which gives the result. ∎

III.8. THEOREM. *The conjecture is true for* B_1^n.

Proof. By Lemma III.6, we have

$$\operatorname{vol}(B_1^n \cap E^k)/\operatorname{vol}(B_1^n)^{k/n} \geqslant (n!)^{k/n}(k!)^{-1}\binom{n}{k}^{-1/2}.$$

Using the decrease of the second derivative of $\log \Gamma(1+x)$, it is easy to conclude that for $1 \leqslant k \leqslant n/2$,

$$\operatorname{vol}(B_1^n \cap E^k) \geqslant \operatorname{vol}(B_1^n)^{k/n}.$$

For $n \geqslant k \geqslant n/2$, we use Lemma III.7 to get

$$\operatorname{vol}(B_1^n \cap E^k)/\operatorname{vol}(B_1^n)^{k/n} \geqslant \left(\frac{n}{\pi}\right)^{(n-k)/2} \Gamma\left(1+\frac{n+k}{2}\right)$$
$$\times [\Gamma(1+n)^{(n-k)/n}\,\Gamma(1+k)]^{-1}\frac{2n}{n+k}$$

and it is not difficult to prove that for $k \geqslant n/2$, the right-hand term of this inequality is bigger than 1. ∎

Remarks. (1) Observing that $n^{1/2-1/p} B_2^n \subset B_p^n$, $1 \leqslant p \leqslant 2$, we get

$$\operatorname{vol}(B_p^n \cap E^k) \geqslant n^{k(1/2-1/p)} \operatorname{vol}(B_2^k).$$

For $p = 1$, this inequality gives

$$\operatorname{vol}(B_1^n \cap E^k)/\operatorname{vol}(B_1^n)^{k/n} \geqslant (\pi/4n)^{k/2}\,\Gamma(1+n)^{k/n}\,\Gamma(1+k/2)^{-1}.$$

Thus, for any $\varepsilon > 0$ and any k such that $k/n \leqslant 2e/\pi - \varepsilon$, we have, for $n \geqslant n_0(\varepsilon)$,

$$\operatorname{vol}(B_1^n \cap E^k)/\operatorname{vol}(B_1^n)^{k/n} \geqslant 1.$$

But this estimate is only asymptotic, and does not give the preceding inequality for $n=4$, 6, 8, 10 and $k=n/2$.

(2) Inequality (8) in Lemma III.6 is nearly optimal: using integration over the Grassman manifold of the k-dimensional subspaces of $\mathbb{R}^n$, we get (see [3, p. 139]) that there exists always some orthogonal projection P^k with rank k such that

$$\operatorname{vol}(P^k(B^n_\infty)) \geqslant C\, n^{-1/4} \operatorname{vol}(B^k_\infty) \binom{n}{k}^{1/2}$$

for some constant C independent of k and n.

(3) If the conjecture that we made in Remark (1) after Proposition II.7 is true, we have a lower estimate of $\operatorname{vol}(B^n_p \cap E^k)$ in terms of $\int_0^{+\infty} r^{n-k-1}(g_p(r))^n\, dr$. Then it would be interesting to have an estimate of this last integral.

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