ON SANTALÓ'S INEQUALITY

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Let K be a centrally symmetric convex body in a finite dimensional Euclidean space E and let K^0 be its polar body:

$$K^0 = \{y \in E | \langle x, y \rangle \leq 1 \quad \text{for every} \quad x \in K\}$$

where $\langle \ , \ \rangle$ denotes the scalar product on E. Let $| \cdot |$ denote a volume measure on E and set

$$p(K) = |K| \cdot |K^0| \ .$$

We recall that $p(K)$ is affine invariant, i.e., for any linear isomorphism $T : E \to E, p(T(K)) = p(K)$. It was proved by Blaschke [B] for dimension E less than or equal to 3 and by Santalo [S] for larger dimensions that

$$p(K) \leq p(\mathcal{D}) = |\mathcal{D}|^2$$

where $\mathcal{D} = \{x \in E | \langle x, x \rangle = 1\}$ is the Euclidean ball.

We give here a proof of this inequality using the following lemma

Lemma. *Let K_1 be the Steiner symmetral of K about some hyperplane through 0, then*

$$|K_1^0| \geq |K^0| \quad \text{and} \quad p(K_1) \geq p(K) \ .$$

Proof: Since Steiner symmetrization preserves volume, one has only to prove the first inequality. Changing coordinates we may suppose that $H = \{(x_i)_{i=1}^{i=n} | x_n = 0\}$ is the hyperplane of symmetrization and identify $E (= \mathbb{R}^n)$ to $H \times \mathbb{R}$. Let PK be the orthogonal projection of K onto H; then

$$K_1 = \{(X, x) | X \in PK \quad , \quad x = (x_1 - x_2)/2 \quad , \quad (X, x_i) \in K \ , \ i = 1, 2\}$$

$$K^0 = \{(Y,y) \in H \times \mathbb{R} | \langle X,Y\rangle + xy \le 1, \quad \text{for } X \in PK \text{ and } x \text{ such that } (X,x) \in K\}$$

$$K_1^0 = \left\{(Y,y) \in H \times \mathbb{R} | \langle X,Y\rangle + \frac{(x_2 - x_1)}{2} y \le 1 \right.$$
$$\left. \text{for } X \in PK \text{ and } x_i \text{ such that } (X,x_i) \in K \ , \ i = 1,2 \right\} .$$

For a subset A of $\mathbb{R}^n (= H \times \mathbb{R})$ and $y \in \mathbb{R}$, denote $A(y) = \{Y \in H | (Y,y) \in A\}$. Addition meaning here Minkowski sum, we have then

$$\frac{K^0(y) + K^0(-y)}{2} \subset K_1^0(y) \quad \text{for every} \quad Y \in \mathbb{R} .$$

Observe that since K is centrally symmetric, we have $K^0(-y) = -K^0(y)$ for every $y \in \mathbb{R}$, it follows from Brunn- Minkowski theorem in H that

$$(*) \qquad |K_1^0(y)| \ge \left| \frac{K^0(y) - K^0(y)}{2} \right| \ge |K^0(y)|$$

and by integration

$$|K_1^0| = \int |K_1^0(y)|\, dy \ge \int |K^0(y)| dy = |K^0| \qquad \square$$

Proof of Santaló's inequality. As it is well known, there exists a sequence (K_n) of centrally symmetric convex bodies converging in the sense of Hausdorff to $\lambda\mathcal{D}$ (where $|\lambda\mathcal{D}| = |K|$) and such that $K_0 = K$ and K_n is a Steiner symmetral of K_{n-1} for $n \ge 1$. By the lemma, the sequence $(p(K_n))$ is increasing and by continuity it converges to $p(\lambda\mathcal{D}) = p(\mathcal{D})$. Thus $p(K) = p(K_0) \le p(\mathcal{D})$. $\square$

As proved by Saint-Raymond [SR], the equality case in Santaló's inequality for centrally symmetric bodies occurs if and only if K is an ellipsoid. From (*) and the equality case in Brunn-Minkowski theorem, the equality $p(K) = p(\mathcal{D})$ implies that every $(n-1)$- dimensional cross-section of K^0 has a center of symmetry; it follows then from a theorem of Brunn ([BR], see [SR]) that for $\dim E = n \ge 3$, K^0 and thus K are ellipsoids.

Actually, for any convex body K, defining the affine invariance

$$p(K) = \inf\{|K|\,|K^z| | z \in E\}$$

where K^z is the polar body of K with respect to z, Santaló's inequality is still valid [S]. In this situation Petty [P] showed that equality characterizes also ellipsoids. These results are consequences of the following one [MP], that we announce here:

Theorem. *Let K be a convex body in E and H be an hyperplane separating K into two parts with respective volume $\lambda|K|$ and $(1-\lambda)|K|$, for some λ, $0 < \lambda < 1$. Then there exists $z \in K \cap H$ such that*

$$|K|\,|K^z| \le p(\mathcal{D})/4\lambda(1-\lambda) \cdot$$

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