Essential Uniqueness of an M-Ellipsoid of a Given Convex Body

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Summary. We show that different M-ellipsoids associated with a convex body, that are introduced in Asymptotic Convexity, are "essentially" unique, in the sense that they admit equivalent large dimensional sections.

1 Introduction

This is a short note, just an observation, but curious and important enough to be broadly known. So we decided to write it down.

We use the following standard notation. The space $\mathbb{R}^n$ is equipped with the canonical Euclidean scalar product $(\,.\,,\,.\,)$. Its unit ball is denoted by D.

Let K be a convex body in $\mathbb{R}^n$ such that 0 is in its interior. Its polar K° is defined as usual by

$$K^\circ = \left\{ x \in \mathbb{R}^n \,:\, (x, y) \le 1 \text{ for every } y \in K \right\}.$$

Let A and B be two subsets of $\mathbb{R}^n$. The covering number $N(A, B)$ is defined as usual as

$$N(A, B) = \min \left\{ {}^\sharp\Lambda : \Lambda \subset \mathbb{R}^n, A \subset \Lambda + B \right\}$$

where $\Lambda + B = \{x + y \,:\, x \in \Lambda, y \in B\}$.

The volume of a measurable subset A of $\mathbb{R}^n$ is denoted by $|A|$. Let $\sigma > 0$ and let K be a convex compact subset of $\mathbb{R}^n$ with 0 in its interior. We say that an ellipsoid $\mathcal{E}$ of $\mathbb{R}^n$ is an M-ellipsoid of K with constant σ, or shortly an M-ellipsoid of K, if, setting $\lambda = (|K|/|\mathcal{E}|)^{1/n}$ in order that $|K| = |\lambda\,\mathcal{E}|$, we have

$$N(K, \lambda\,\mathcal{E}) \le e^{\sigma n}.$$

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We assume below in the paper that ellipsoids are centered at 0.

It is proved in [M1] (see also [M2] and [Pi] for simplified proofs) that there exists a universal constant such that for every n, every n-dimensional symmetric convex body has an M-ellipsoid with respect to this constant. Many interesting properties of centrally symmetric convex bodies and corresponding normed spaces were revealed using M-ellipsoids. The existence of an M-ellipsoid was established even for non centrally symmetric bodies in [M-P1]. We refer to the survey in [G-M].

It is proved in [M-P1] that if $\mathcal{E}$ is an M-ellipsoid of $K \subset \mathbb{R}^n$ with constant σ, and if 0 is the barycenter of K, then there exists $\sigma_1 \geq \sigma$ depending only on σ such that

$$N(\lambda\,\mathcal{E}, K) \leq e^{\sigma_1 n}\ , N\big(K^\circ, (\lambda\,\mathcal{E})^\circ\big) \leq e^{\sigma_1 n} \text{ and } N\big((\lambda\,\mathcal{E})^\circ, K^\circ\big) \leq e^{\sigma_1 n}.$$

Of course, we cannot expect the uniqueness of such an ellipsoid and since the definition is of isomorphic type, it should at least depend on the parameter σ. At the same time, one can construct different M-ellipsoids, having deep and non-trivial additional properties. We say that an ellipsoid $\mathcal{E}$ is a regular M-ellipsoid of order α with constant σ for a convex body $K \subset \mathbb{R}^n$, if $|\mathcal{E}| = |K|$ and for any $t > 0$,

$$N(K, t\mathcal{E}) \leq e^{\sigma n / t^\alpha} \text{ and } N(\mathcal{E}, tK) \leq e^{\sigma n / t^\alpha}. \tag{1}$$

It is proved in [Pi] that for any $0 < \alpha < 2$, there exists $\sigma(\alpha) = O\big((2-\alpha)^{1/2}\big)$ as $\alpha \to 2$, such that every centrally symmetric convex body K has an ellipsoid $\mathcal{E}_\alpha$ which is a regular M-ellipsoid of order α with constant $\sigma(\alpha)$ for K. Note that the covering numbers in (1) do not depend on the choice of the center. The previous assumption on the barycenter of K was needed because this statement used polarity.

These ellipsoids depend on α and the purpose of this note is to show that surprisingly, in some "essential" way, these M-ellipsoids are unique. This question arises from consideration of different M-ellipsoids in [M-P2].

2 Essential Uniqueness of M-Ellipsoids

We begin by a definition of essential uniqueness.

Definition. *Let c be a positive function on $(0,1)$. Two ellipsoids $\mathcal{E}_1$ and $\mathcal{E}_2$ in $\mathbb{R}^n$ are called* essentially *c-equivalent if for every $0 < \varepsilon < 1$, there is a subspace $E \subset \mathbb{R}^n$ of dimension larger than $(1-\varepsilon)n$, such that*

$$\frac{1}{c(\varepsilon)}\ \mathcal{E}_2 \cap E \subset \mathcal{E}_1 \cap E \subset c(\varepsilon)\, \mathcal{E}_2 \cap E. \tag{2}$$

As we will actually compare two ellipsoids in a given n-dimensional space, we put "$c(\varepsilon)$" in the definition. However, in Asymptotic Convexity, one compares two families of ellipsoids, say $\mathcal{E}_1(n)$ and $\mathcal{E}_2(n)$, $n \in \mathbb{N}$, and call these families *essentially* equivalent if for every $0 < \varepsilon < 1$, there exists $c(\varepsilon) > 0$ such that for every integer n, there is a subspace $E(\varepsilon) \subset \mathbb{R}^n$ of dimension larger than $(1-\varepsilon)n$ such that

$$\frac{1}{c(\varepsilon)}\,\mathcal{E}_2(n) \cap E(\varepsilon) \subset \mathcal{E}_1(n) \cap E(\varepsilon) \subset c(\varepsilon)\,\mathcal{E}_2(n) \cap E(\varepsilon).$$

We can now state our result on the uniqueness of an M-ellipsoid.

Theorem. *Let $K \subset \mathbb{R}^n$ be any convex compact body. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two M-ellipsoids of K, with the same volume $|K| = |\mathcal{E}_1| = |\mathcal{E}_2|$ and the same constant σ. Then there exists a function $c > 0$ on $(0,1)$, depending only on σ, such that $\mathcal{E}_1$ and $\mathcal{E}_2$ are essentially c-equivalent.*

Proof. In the proof, we use notation $\sigma_1, \sigma_2, ..$ to denote positive numbers depending only on the parameter σ associated with the M-ellipsoids $\mathcal{E}_1$ and $\mathcal{E}_2$ given above. As noticed previously, there exists $\sigma_1 > 0$ such that

$$N(K, \mathcal{E}_i) \le e^{\sigma_1 n} \text{ and } N(\mathcal{E}_i, K) \le e^{\sigma_1 n},\ i = 1, 2.$$

Therefore,

$$N(\mathcal{E}_1, \mathcal{E}_2) \le N(\mathcal{E}_1, K)\,.\,N(K, \mathcal{E}_2) \le e^{\sigma_2 n}$$

for some new constant σ_2. This means that $\mathcal{E}_2$ is an M-ellipsoid of $\mathcal{E}_1$, with constant σ_2.

It is a well-known property of an M-ellipsoid $\mathcal{E}$ of a symmetric convex body C with constant σ and same volume as C, that

$$|C \cap \mathcal{E}| \ge \sigma_3^n\,|C|$$

where $\sigma_3 > 0$ depends only on σ (see [M2]). Applying this property to the pair $\mathcal{E}_1$ and $\mathcal{E}_2$, we have

$$|\mathcal{E}_1 \cap \mathcal{E}_2| \ge \sigma_3^n\,|K|.$$

At the same time, the intersection of two ellipsoids is $\sqrt{2}$-equivalent to an ellipsoid, which means that there exists an ellipsoid $\mathcal{E}$ such that

$$\mathcal{E} \subset \mathcal{E}_1 \cap \mathcal{E}_2 \subset \sqrt{2}\,\mathcal{E}.$$

Thus $N(\mathcal{E}_i, \mathcal{E}) \le e^{\sigma_4 n}$, $i = 1, 2$, for some new constant σ_4. We conclude that

$$\mathcal{E} \subset \mathcal{E}_i \text{ and } N(\mathcal{E}_i, \mathcal{E}) \le e^{\sigma_4 n}, \quad i = 1, 2.$$

It is now standard and trivial to show that for any $\varepsilon > 0$, there are subspaces $E_i(\varepsilon)$, $i = 1, 2$, of dimension larger than $(1 - (\varepsilon/2))n$ such that

$$\mathcal{E} \cap E_i(\varepsilon) \subset \mathcal{E}_i \cap E_i(\varepsilon) \subset e^{\sigma_5/\varepsilon}\, \mathcal{E} \cap E_i(\varepsilon)$$

for some new constant σ_5. Indeed, let $\mathbb{R}^n$ be equipped with the Euclidean structure associated with $\mathcal{E}$. Then for each $i = 1, 2$, $E_i(\varepsilon)$ is spanned by the $(1-(\varepsilon/2))n$ axes of $\mathcal{E}_i$ corresponding to the $(1-(\varepsilon/2))n$ smallest eigenvalues. Now let $E(\varepsilon) = E_1(\varepsilon) \cap E_2(\varepsilon)$. It has dimension larger than $(1-\varepsilon)n$ and (2) is satisfied with $c(\varepsilon) = e^{\sigma_5/\varepsilon}$, which shows that $\mathcal{E}_1$ and $\mathcal{E}_2$ are essentially c-equivalent. □

Corollary 1. *Let $K \subset \mathbb{R}^n$ be a convex body containing 0. Let $\mathcal{E}$ be an M-ellipsoid of K with constant σ and let $0 < \alpha < 2$. Then for every $\varepsilon > 0$, there exists a subspace $E \subset \mathbb{R}^n$ with dimension larger than $(1-\varepsilon)n$, such that for every $t > 0$, one has*

$$N\big(K \cap E, t\, c(\varepsilon)\, \mathcal{E} \cap E\big) \le e^{\sigma_1 n/t^\alpha}$$

where σ_1 and $c(\varepsilon)$ depend only on α and σ.

Proof. We first assume that K is centrally symmetric. Then as we recalled above, there exists a regular M-ellipsoid $\mathcal{E}_\alpha$ of K of order α and with same volume as $\mathcal{E}$. Since it is also an M-ellipsoid for K, it follows from the previous Theorem that $\mathcal{E}$ and $\mathcal{E}_\alpha$ are essentially equivalent. Therefore, using (1) and (2) there exists a subspace E with dimension larger than $(1-\varepsilon)n$, such that for every $t > 0$, we have

$$N\big(K \cap E, t\, c(\varepsilon)\, \mathcal{E} \cap E\big) \le N(K \cap E, t\, \mathcal{E}_\alpha \cap E) \le e^{\sigma_1 n/t^\alpha}$$

where σ_1 and $c(\varepsilon)$ depend only on α and σ. To get the right-hand side inequality above, observe that because $\mathcal{E}_\alpha$ is centrally symmetric, one has $N(K \cap E, t\, \mathcal{E}_\alpha \cap E) \le N(K, (t/2)\, \mathcal{E}_\alpha)$.

When K is not centrally symmetric the existence of a regular M-ellipsoid of K of order α is still an open problem. Let $K' = K - K$ be the difference body. Since we assume that $0 \in K$, it follows that $K \subset K'$. It is proved in [M-P1], that an M-ellipsoid for K is also an M-ellipsoid for K' and vice-versa, only the constants associated with them differ by a universal factor. Let $\mathcal{E}$ be an M-ellipsoid for K with same volume as K'. Recall that from [R-S], we have $|K| \le |K'| \le 4^n |K|$. Thus without loss of generality we can assume that $1 \le |K|/|\mathcal{E}| \le 4^n$. We now use the first part of the proof with K', which is centrally symmetric. Let $\mathcal{E}'_\alpha$ be a regular M-ellipsoid for K' of order α and with same volume as $\mathcal{E}$. These are M-ellipsoids for K'. From the above Theorem, they are essentially equivalent. Therefore, there exists a subspace E with dimension larger than $(1-\varepsilon)n$, such that for every $t > 0$, we have

$$N\big(K' \cap E, t\, c(\varepsilon)\, \mathcal{E} \cap E\big) \le N(K' \cap E, t\, \mathcal{E}'_\alpha \cap E) \le e^{\sigma'_1 n/t^\alpha}$$

where σ'_1 and $c'(\varepsilon)$ depend on α and σ. The result follows from $K \subset K'$. □

Corollary 2. *Let $K \subset \mathbb{R}^n$ be a convex body with 0 as barycenter. Let $\mathcal{E}_1$ be an M-ellipsoid of $K \cap (-K)$ and $\mathcal{E}_2$ be an M-ellipsoid of $K - K$, with the same volume $|K| = |\mathcal{E}_1| = |\mathcal{E}_2|$ and the same constant σ. Then there exists a function $c > 0$ on $(0,1)$, depending only on σ, such that $\mathcal{E}_1$ and $\mathcal{E}_2$ are essentially c-equivalent.*

Proof. It is proved in [M-P1] that when 0 is the barycenter of K, both ellipsoids $\mathcal{E}_1$ and $\mathcal{E}_2$ are M-ellipsoids of K. The result follows from the Theorem. □

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