ABSTRACT. Given a finite-dimensional Banach space $E$ and a Euclidean norm on $E$, we study relations between the norm and the Euclidean norm on subspaces of $E$ of small codimension. Then for an operator taking values in a Hilbert space, we deduce an inequality for entropy numbers of the operator and its dual.

In this note we study the following problem: given an $n$-dimensional Banach space $E$ and a Euclidean norm $\| \cdot \|_2$ on $E$ and $0 < \lambda < 1$, find a subspace $F$ of $E$ with $\dim F \geq \lambda n$ such that

$$\|x\|_2 \leq M_*(f(1 - \lambda)\|x\| \text{ for } x \in F.$$  

Here $M_*$ denotes the Levy mean of the dual norm of $E$ (see the notation below).

This problem was considered by V. Milman, who proved in [18] that estimate (*) holds for a certain exponential function $f$. The estimate was improved later in [10] to $f(1 - \lambda) \leq K/(1 - \lambda)$, where $K$ is a universal constant. The latter result turned out to be important for various applications (cf. [1, 15, 11, 19]).

The main result of this note proves (*) with the function $f(1 - \lambda) \leq K/\sqrt{1 - \lambda}$. This estimate, besides being optimal (up to a logarithmic factor), can be used to compare entropy numbers of an operator and its dual for operators taking values in a Hilbert space.

Let us recall some notation.

Let $E$ be an $n$-dimensional Banach space; i.e., $E = (\mathbb{R}^n, \| \cdot \|)$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^n$, and let $\| \cdot \|$ be the associated Euclidean norm on $\mathbb{R}^n$ defined by $\|x\| = \sqrt{\langle x, x \rangle}$, for $x \in \mathbb{R}^n$.

Let $B_E$ be the closed unit ball in $E$. Set

$$\|x\|_* = \sup\{\|x, y\| : y \in B_E\} \text{ for } x \in \mathbb{R}^n.$$  

Clearly, $(\mathbb{R}^n, \| \cdot \|_*)$ can be identified with the dual space $E^*$. Let $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$, and let $\mu$ be the normalized rotation invariant measure on $S$.

Define the Levy means $M$ and $M_*$ by

$$M = \left(\int_S \|x\|^2 d\mu\right)^{1/2}, \quad M_* = \left(\int_S \|x\|^2_* d\mu\right)^{1/2}.$$  

We shall employ a similar notation in a context of symmetric convex bodies. For a closed symmetric convex body $V \subset \mathbb{R}^n$, by $V^*$ we denote the dual body defined by $V^* = \{x \in \mathbb{R}^n : \|x, y\| \leq 1 \text{ for all } y \in V\}$.  

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Note that if $\| \cdot \|_V$ is the norm associated to $V$ (i.e., $V = \{ x \in \mathbb{R}^n \mid \|x\|_V \leq 1 \}$), then the dual norm $\| \cdot \|_{V^*} = (\| \cdot \|_V)^*$, as defined by (1), is associated to $V^*$. We set $E_V = (\mathbb{R}^n, \| \cdot \|_V)$ and $E_{V^*} = (\mathbb{R}^n, \| \cdot \|_{V^*})$, and we denote by $M_V$ and $M_{V^*}$ the corresponding Levy means.

The main result of this note is

**Theorem 1.** Let $E = (\mathbb{R}^n, \| \cdot \|)$ and let $\| \cdot \|$ be a Euclidean norm on $\mathbb{R}^n$. For every $0 < \lambda < 1$ there exists a subspace $F \subset \mathbb{R}^n$ with $\dim F > \lambda n$ such that

$$\|x\| < K M_*(1 - \lambda)^{-1/2} \|x\| \quad \text{for } x \in F,$$

where $K$ is a universal constant.

In particular, we have a corollary for finite-dimensional subspaces of a Banach space of cotype 2. It improves earlier results from [4, 10, 18] (cf. also [11]).

**Corollary 1.** Let $X$ be a Banach space of cotype 2 and let $E \subset X$ be an $n$-dimensional subspace. For every $0 < \lambda < 1$ there exists $F \subset E$ with $m = \dim F > \lambda n$ such that

$$d(F, l_2^n) \leq K C_2(X)(1 - \lambda)^{-1/2} \log[C_2(X)(1 - \lambda)^{-1/2} + 1],$$

where $C_2(X)$ is the Gaussian cotype 2 constant of $X$ and $K$ is a universal constant.

**Sketch of the Proof.** It can be shown, using results of [15 and 16] (cf. e.g. [1, 4, 10, 11, 14]), that the Euclidean norm $\| \cdot \|$ induced on $E$ by the ellipsoid of maximal volume contained in $B_E$ satisfies $\|x\| < \|x\|$ for $x \in E$ and $M_* \leq K' C_2(X) \log[d(E, l_2^n) + 1]$. Combining this fact either with Milman’s iteration procedure (cf., e.g., [1, 4, 10]) or with an argument from [10] (proof of Theorem 1 and Remark after Theorem 8), yields the result. □

The estimate obtained in Theorem 1 and Corollary 1 is, up to a logarithmic factor, the best possible. If $F \subset l_1^n$ is an $m$-dimensional subspace (with $m > \lambda n$), considering a projection $P: l_1^n \to l_1^n$ with ker $P = F$ and $\gamma_2(P) \leq \sqrt{n - m}$, one can easily show that $d(F, l_2^n) \geq c(1 - \lambda)^{-1/2}$, where $c > 0$ is an absolute constant.

Theorem 1 can be used to obtain some new estimates between various $s$-numbers of operators from a Banach space $X$ into $l_2$. For this let us recall some more notation (cf., e.g., [12]).

For any compact metric space $(T, d)$, we denote by $N(T, d, \varepsilon)$ the smallest number of open balls of radius $\varepsilon$ which cover $T$. If $T$ is the unit ball in an $n$-dimensional Banach space then $N(T, \| \cdot \|, \varepsilon) \leq (1 + 2/\varepsilon)^n$ (cf., e.g., [6, Lemma 2.4]). Let $X$ and $Y$ be Banach spaces and let $B_X$ denote the closed unit ball in $X$. For a compact operator $u: X \to Y$ between Banach spaces we define the $k$th entropy number of $u$ by

$$e_k(u) = \inf\{ \varepsilon > 0 \mid N(u(B_X), \| \cdot \|_Y, \varepsilon) \leq 2^k \}.$$

Moreover, we define the $k$th Gelfand number of $u$ by

$$c_k(u) = \inf\{ \|u\| \mid \|Z\|_X, \text{codim } Z < k \}$$

and the $k$th Kolmogorov number by

$$d_k(u) = \inf_{L \subset V} \sup_{\dim L < K} \inf_{x \in B_X, y \in L} \|ux - y\|.$$
It is well known (cf., e.g., [12]) that

\( c_k(u) = d_k(u^*) \).

For the operator \( u: l_2^n \to X \), we define

\[
    l(u) = \left( \int_{R^n} \|ux\|^2 d\gamma_n(x) \right)^{1/2},
\]

where \( \gamma_n \) denotes the canonical (normalized) Gaussian measure on \( R^n \) (cf., e.g., [6]). For any bounded operator \( u: l_2 \to X \) we set

\[
    l(u) = \sup\{l(uv) | v: l_2 \to l_2, v \leq 1, n \in N\}.
\]

If \( \dim E = n \) and \( u: l_2^n \to E \) is one-to-one, then \( l(u) = \sqrt{n} M_V \), where \( V = u^{-1}(B_E) \).

Now we have

**Proposition 1.** Let \( X \) be a Banach space and let \( u: X \to l_2 \) be a compact operator. Then

\[
    \sup_k \sqrt{k} e_k(u) \leq Kl(u^*),
\]

where \( K \) is a universal constant.

**Proof.** Approximating a given operator by a finite-rank operator and replacing \( u \) by \( \tilde{u}: X/\ker u \to u(X) \), we may assume that \( u: X \to l_2^n \) is one-to-one. Define \( \|x\| = \|u^{-1}x\|_X \) for \( x \in R^n \). The conclusion follows from Theorem 1 applied for \( E = (R^n, \| \cdot \|), \| \cdot \| \) the \( l_2 \)-norm on \( R^n \) and \( \lambda = 1 - k/n \). □

The next theorem compares entropy numbers of operators \( u \) and \( u^* \) for \( u: X \to l_2 \). Let us recall that it is still an open problem whether for every compact operator \( u: X \to Y \), \( e_k(u) \) and \( e_k(u^*) \) are asymptotically of the same order, as \( k \to \infty \). For more details on this and related problems and results, cf. [2, 3, 8].

**Theorem 2.** Let \( X \) be a Banach space and let \( u: X \to l_2 \) be a compact operator. Then

\[
    \sup_k \sqrt{k} e_k(u^*) \leq K \sum_{k=1}^{\infty} \frac{e_k(u)}{\sqrt{k}},
\]

where \( K \) is a universal constant.

The proof of the theorem is based upon two lemmas on entropy numbers. Lemma 1 is an operator version of results on Gaussian processes: the majorization theorem is due to Dudley [5], the minorization theorem to Sudakov [13]. A direct proof of Lemma 1 can be also found in [9].

**Lemma 1.** Let \( X \) be a Banach space and let \( u: X \to l_2 \) be a compact operator. Then

(i) \( l(u^*) \leq K \sum_{k=1}^{\infty} e_k(u)/\sqrt{k} \),

(ii) \( \sup_k \sqrt{k} e_k(u) \leq Kl(u^*) \),

where \( K \geq 1 \) is a universal constant.

The next result is due to Carl [2].
LEMMA 2. Let $0 < p < \infty$. Let $X$ and $Y$ be Banach spaces and let $v : X \to Y$ be an operator. Then

$$
\sup_{k \leq n} k^{1/p} e_k(v) \leq b_p \sup_{k \leq n} k^{1/p} d_k(v) \quad \text{for } n = 1, 2, \ldots,
$$

where $b_p$ depends only on $p$.

Now Theorem 2 follows immediately from Lemma 2, formula (2), Proposition 1 and Lemma 1(i).

Our proof of Theorem 1 is based upon a similar idea to [7]. It will be a consequence of the following technical result.

PROPOSITION 2. There exists $0 < d < 2^{-7/2}$ such that the following is true. Let $||| \cdot |||$ be a Euclidean norm on $R^n$ and let $B_2$ be the closed unit ball. Let $1 \leq j \leq n$. Let $V$ be a closed symmetric convex body such that

$$
V \subset \bigcup_{z \in \Lambda} (z + d|||z|||W) \cup cB_2,
$$

with a set $\Lambda \subset R^n \setminus \{0\}$ with $\Lambda \leq e^{d^2j}$ and some closed symmetric convex body $W$ such that $W \subset B_2$ and $M_{W}^j \leq \sqrt{j/n}$, and for some constant $c$. Then there exists a subspace $F \subset R^n$ with $\dim F \geq n - j$ such that

$$
|||x||| \leq c||x||_V \quad \text{for } x \in F.
$$

PROOF. Applying an affine transformation, if necessary, we may assume that $|||x||| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ for $x \in R^n$. Denote by $M_j$ the median of the function $\psi_j(x) = (\sum_{i=1}^{j} x_i^2)^{1/2}$ on $S$. It is well known (cf., e.g., [6]) that $M_j \geq a\sqrt{j/n}$, where $0 < a < \sqrt{2}$ is a universal constant. Set $d = 2^{-4}a$. Let $\{e_i\}$ be the standard unit vector basis in $R^n$. Let $P_j : R^n \to R^n$ denote the orthogonal projection onto the subspace spanned by the first $j$ unit vectors.

We need two lemmas. In their formulation $O(n)$ denotes the group of isometries of $l_2^n$ and $\mathcal{P}$ is the Haar measure on $O(n)$. For an operator $T : R^n \to R^n$, $\|T : E \to F\|$ denotes the norm of $T$ as an operator from $E$ into $F$.

LEMMA 3. Let

$$
A_1 = \{U \in O(n) | \|P_j U : E_W \to l_2^n\| > 8M_j/a\}.
$$

Then $\mathcal{P}(A_1) < \frac{1}{2}$.

LEMMA 4. Let

$$
A_2 = \{U \in O(n) | \exists z \in \Lambda | |||P_j Uz||| < \frac{3}{4}M_j |||z|||\}.
$$

Then $\mathcal{P}(A_2) < \frac{1}{2}$.

Assuming the truth of Lemmas 3 and 4 we complete the proof as follows. Pick an isometry $U_0 : l_2^n \to l_2^n$ such that

$$
\|P_j U_0 : E_W \to l_2^n\| \leq 8M_j/a, \quad |||P_j U_0 z||| \geq (3M_j/4)|||z||| \quad \text{for all } z \in \Lambda.
$$

Then

$$
(ker P_j U_0) \cap (z + d|||z|||W) = 0 \quad \text{for all } z \in \Lambda.
$$
Indeed, if \( y \in (\ker P_j U_0) \cap (z + d|||z|||W) \), then
\[
0 = |||P_j U_0 y||| \geq |||P_j U_0 z||| - |||P_j U_0 (y - z)||| \\
\geq (3M_j/4)|||z||| - (8M_j/a)|||y - z|||w \geq (3M_j/4)|||z||| - (8M_j/a)d|||z||| > 0,
\]
giving a contradiction. Set \( F = \ker P_j U_0 \). From (5) and (3) it follows that \( F \cap V \subset cB_2 \), which is equivalent to (4). \( \Box \)

**Proof of Lemma 3.** Let \( \tilde{A} \) be a \( \frac{1}{2} \)-net in \( B_2 \cap [e_1, \ldots, e_j] \) with minimal cardinality. Then \( |\tilde{A}| = N(B_2 \cap [e_1, \ldots, e_j], ||| \cdot |||, \frac{1}{2}) \leq 5^j \). Thus
\[
P(A_1) = P\{U \in O(n) | \|U: l_2^j \to E_{W^*}\| > 8M_j/a\} \\
\leq P\{U \in O(n) \exists z \in \tilde{A} | \|Uz\|_{W^*} > 4M_j/a\} \\
\leq 5^j \mu\{y \in S | \|y\|_{W^*} > 4M_j/a\}.
\]
By the isoperimetric inequality (cf. [6, (2.6)]), the measure of the latter set is smaller than or equal to
\[
4 \exp(-n(4M_j/a - M_{W^*})^2/2) \leq 4 \exp(-9j/2).
\]
Therefore
\[
P(A_1) \leq 4 \cdot 5^j \exp(-9j/2) < 1/2. \quad \Box
\]

Lemma 4 follows immediately from the isoperimetric inequality and the definition of \( d \).

**Proof of Theorem 1.** Fix \( 0 < \lambda < 1 \). Let \( j \) be the smaller integer larger than or equal to \((1 - \lambda)n\). Set \( V = (d_1 \sqrt{1 - \lambda}/M_*)B_E \), where a constant \( 0 < d_1 < 1 \) will be defined later, \( W = (\sqrt{1 - \lambda}/M_*)B_E \cap B_2 \). Since \( W \subset (\sqrt{1 - \lambda}/M_*)B_E \), the dual bodies satisfy the opposite inclusion and \((M_*/\sqrt{1 - \lambda})B_{E^*} \subset W^* \). Thus \( M_{W^*} \leq (\sqrt{1 - \lambda}/M_*)M_* \leq \sqrt{j}/n \). Notice also that \( M_{W^*} = (d_1 \sqrt{1 - \lambda}/M_*)M_* = d_1 \sqrt{1 - \lambda} \).

Consider the compact metric space \((V, ||| \cdot |||)\). Let \( \Lambda \) be a \( 1/2 \)-net in \( V \) with minimal cardinality. It follows from Lemma 1(ii), applied to the formal identity operator \( u: (R^n, V) \to (R^n, ||| \cdot |||) \) that
\[
|\Lambda| = N(V, ||| \cdot |||, \varepsilon) \leq 2^{4K^2n M_*^2} \leq \exp(d^2j),
\]
where \( K \geq 1 \) is the constant from Lemma 1 and \( d_1 = d/2K \sqrt{\log 2} \).

To show (3) fix \( x \in V \). Pick \( z \in \Lambda \) such that \( |||x - z||| < 1/2 \). If \( |||x||| \geq 1/K \sqrt{\log 2} + 1/2d + 1/2 \), then \( |||z||| \geq 1/K \sqrt{\log 2} + 1/2d \). Therefore,
\[
x - z \in 2V = (2d_1 \sqrt{1 - \lambda}/M_*)B_E \subset d|||z|||((\sqrt{1 - \lambda}/M_*)B_E, \\
x - z \in \frac{1}{2} B_2 \subset d|||z|||B_2.
\]
Hence \( x - z \in d|||z|||W \). This shows (3) with
\[
c = 1/K \sqrt{\log 2} + 1/2d + 1/2.
\]

Now Proposition 2 implies that there exists \( F \subset R^n \) with \( \dim F \geq n - j > \lambda n - 1 \) such that
\[
|||x||| \leq c|||x|||_V = K'M_* (1 - \lambda)^{-1/2} |||x||| \quad \text{for } x \in F,
\]
where \( K' = 2cK \sqrt{\log 2}/d \). \( \Box \)
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UNIVERSITÉ DES SCIENCES ET TECHNIQUES DE LILLE, U.E.R. DE MATHÉMATIQUES, 59655 VILLENEUVE D’ASCQ CEDEX, FRANCE

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA T6J 2G1, CANADA