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SUBSPACES OF SMALL CODIMENSION OF FINITE-DIMENSIONAL BANACH SPACES

ALAIN PAJOR AND NICOLE TOMCZAK-JAEGERMANN[1]

ABSTRACT. Given a finite-dimensional Banach space E and a Euclidean norm on E, we study relations between the norm and the Euclidean norm on subspaces of E of small codimension. Then for an operator taking values in a Hilbert space, we deduce an inequality for entropy numbers of the operator and its dual.

In this note we study the following problem: given an n-dimensional Banach space E and a Euclidean norm $\|\cdot\|_2$ on E and $0 < \lambda < 1$, find a subspace F of E with $\dim F \geq \lambda n$ such that

$$(*) \qquad \|x\|_2 \leq M_* f(1-\lambda)\|x\| \quad \text{for } x \in F.$$

Here M_* denotes the Levy mean of the dual norm of E (see the notation below).

This problem was considered by V. Milman, who proved in [**18**] that estimate $(*)$ holds for a certain exponential function f. The estimate was improved later in [**10**] to $f(1-\lambda) \leq K/(1-\lambda)$, where K is a universal constant. The latter result turned out to be important for various applications (cf. [**1, 15, 11, 19**]).

The main result of this note proves $(*)$ with the function $f(1-\lambda) \leq K/\sqrt{1-\lambda}$. This estimate, besides being optimal (up to a logarithmic factor), can be used to compare entropy numbers of an operator and its dual for operators taking values in a Hilbert space.

Let us recall some notation.

Let E be an n-dimensional Banach space; i.e., $E = (R^n, \|\cdot\|)$. Let $[\cdot,\cdot]$ be an inner product on R^n, and let $|||\cdot|||$ be the associated Euclidean norm on R^n defined by $|||x||| = [x,x]^{1/2}$, for $x \in R^n$.

Let B_E be the closed unit ball in E. Set

$$(1) \qquad \|x\|_* = \sup\{|[x,y]|\,|y \in B_E\} \quad \text{for } x \in R^n.$$

Clearly, $(R^n, \|\cdot\|_*)$ can be identified with the dual space E^*. Let $S = \{x \in R^n|\,|||x||| = 1\}$, and let μ be the normalized rotation invariant measure on S. Define the Levy means M and M_* by

$$M = \left(\int_S \|x\|^2\,d\mu\right)^{1/2}, \quad M_* = \left(\int_S \|x\|_*^2\,d\mu\right)^{1/2}.$$

We shall employ a similar notation in a context of symmetric convex bodies. For a closed symmetric convex body $V \subset R^n$, by V^* we denote the dual body defined by $V^* = \{x \in R^n|\,|[x,y]| \leq 1 \text{ for all } y \in V\}$.

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Note that if $\|\cdot\|_V$ is the norm associated to V (i.e., $V = \{x \in R^n | \|x\|_V \le 1\}$, then the dual norm $\|\cdot\|_{V^*} = (\|\cdot\|_V)_*$, as defined by (1), is associated to V^*. We set $E_V = (R^n, \|\cdot\|_V)$ and $E_{V^*} = (R^n, \|\cdot\|_{V^*})$, and we denote by M_V and M_{V^*} the corresponding Levy means.

The main result of this note is

THEOREM 1. *Let $E = (R^n, \|\cdot\|)$ and let $|||\cdot|||$ be a Euclidean norm on R^n. For every $0 < \lambda < 1$ there exists a subspace $F \subset R^n$ with $\dim F > \lambda n$ such that*

$$|||x||| \le KM_*(1-\lambda)^{-1/2}\|x\| \quad \text{for } x \in F,$$

where K is a universal constant.

In particular, we have a corollary for finite-dimensional subspaces of a Banach space of cotype 2. It improves earlier results from [**4**, **10**, **18**] (cf. also [**11**]).

COROLLARY 1. *Let X be a Banach space of cotype 2 and let $E \subset X$ be an n-dimensional subspace. For every $0 < \lambda < 1$ there exists $F \subset E$ with $m = \dim F > \lambda n$ such that*

$$d(F, l_2^m) \le KC_2(X)(1-\lambda)^{-1/2}\log[C_2(X)(1-\lambda)^{-1/2}+1],$$

where $C_2(X)$ is the Gaussian cotype 2 constant of X and K is a universal constant.

SKETCH OF THE PROOF. It can be shown, using results of [**15** and **16**] (cf. e.g. [**1**, **4**, **10**, **11**, **14**]), that the Euclidean norm $|||\cdot|||$ induced on E by the ellipsoid of maximal volume contained in B_E satisfies $\|x\| \le |||x|||$ for $x \in E$ and $M_* \le K'C_2(X)\log[d(E, l_2^n)+1]$. Combining this fact either with Milman's iteration procedure (cf., e.g., [**1**, **4**, **10**]) or with an argument from [**10**] (proof of Theorem 1 and Remark after Theorem 8), yields the result. □

The estimate obtained in Theorem 1 and Corollary 1 is, up to a logarithmic factor, the best possible. If $F \subset l_1^n$ is an m-dimensional subspace (with $m > \lambda n$), considering a projection $P: l_1^n \to l_1^n$ with $\ker P = F$ and $\gamma_2(P) \le \sqrt{n-m}$, one can easily show that $d(F, l_2^m) \ge c(1-\lambda)^{-1/2}$, where $c > 0$ is an absolute constant.

Theorem 1 can be used to obtain some new estimates between various s-numbers of operators from a Banach space X into l_2. For this let us recall some more notation (cf., e.g., [**12**]).

For any compact metric space (T, d), we denote by $N(T, d, \varepsilon)$ the smallest number of open balls of radius ε which cover T. If T is the unit ball in an n-dimensional Banach space then $N(T, \|\cdot\|, \varepsilon) \le (1+2/\varepsilon)^n$ (cf., e.g., [**6**, Lemma 2.4]). Let X and Y be Banach spaces and let B_X denote the closed unit ball in X. For a compact operator $u: X \to Y$ between Banach spaces we define the kth entropy number of n by

$$e_k(u) = \inf\{\varepsilon > 0 | N(u(B_X),\ \|\cdot\|_Y, \varepsilon) \le 2^k\}.$$

Moreover, we define the kth Gelfand number of u by

$$c_k(u) = \inf\{\|u|_Z\| \,|Z \subset X,\ \operatorname{codim} Z < k\}$$

and the kth Kolmogorov number by

$$d_k(u) = \inf_{\substack{L \subset Y \\ \dim L < K}} \sup_{x \in B_X} \inf_{y \in L} \|ux - y\|.$$

It is well known (cf., e.g., [**12**]) that

$$c_k(u) = d_k(u^*). \tag{2}$$

For the operator $u: l_2^n \to X$, we define

$$l(u) = \left(\int_{R^n} \|ux\|^2 \, d\gamma_n(x) \right)^{1/2},$$

where γ_n denotes the canonical (normalized) Gaussian measure on R^n (cf., e.g., [**6**]). For any bounded operator $u: l_2 \to X$ we set

$$l(u) = \sup\{l(uv) | v: l_2^n \to l_2,\ v \leq 1,\ n \in \mathbf{N}\}.$$

If $\dim E = n$ and $u: l_2^n \to E$ is one-to-one, then $l(u) = \sqrt{n} M_V$, where $V = u^{-1}(B_E)$.
Now we have

PROPOSITION 1. *Let X be a Banach space and let $u: X \to l_2$ be a compact operator. Then*

$$\sup_k \sqrt{k} c_k(u) \leq K l(u^*),$$

where K is a universal constant.

PROOF. Approximating a given operator by a finite-rank operator and replacing u by $\bar{u}: X/\ker u \to u(X)$, we may assume that $u: X \to l_2^n$ is one-to-one. Define $\|x\| = \|u^{-1}x\|_X$ for $x \in R^n$. The conclusion follows from Theorem 1 applied for $E = (R^n, \|\cdot\|)$, $|||\cdot|||$ the l_2-norm on R^n and $\lambda = 1 - k/n$. □

The next theorem compares entropy numbers of operators u and u^* for $u: X \to l_2$. Let us recall that it is still an open problem whether for every compact operator $u: X \to Y$, $e_k(u)$ and $e_k(u^*)$ are asymptotically of the same order, as $k \to \infty$. For more details on this and related problems and results, cf. [**2**, **3**, **8**].

THEOREM 2. *Let X be a Banach space and let $u: X \to l_2$ be a compact operator. Then*

$$\sup_k \sqrt{k} e_k(u^*) \leq K \sum_{k=1}^{\infty} \frac{e_k(u)}{\sqrt{k}},$$

where K is a universal constant.

The proof of the theorem is based upon two lemmas on entropy numbers. Lemma 1 is an operator version of results on Gaussian processes: the majorization theorem is due to Dudley [**5**], the minorization theorem to Sudakov [**13**]. A direct proof of Lemma 1 can be also found in [**9**].

LEMMA 1. *Let X be a Banach space and let $u: X \to l_2$ be a compact operator. Then*

(i) $l(u^*) \leq K \sum_{k=1}^{\infty} e_k(u)/\sqrt{k}$,
(ii) $\sup_k \sqrt{k} e_k(u) \leq K l(u^*)$,
where $K \geq 1$ is a universal constant.

The next result is due to Carl [**2**].

LEMMA 2. *Let $0 < p < \infty$. Let X and Y be Banach spaces and let $v: X \to Y$ be an operator. Then*

$$\sup_{k \leq n} k^{1/p} e_k(v) \leq b_p \sup_{k \leq n} k^{1/p} d_k(v) \quad \text{for } n = 1, 2, \dots,$$

where b_p depends only on p.

Now Theorem 2 follows immediately from Lemma 2, formula (2), Proposition 1 and Lemma 1(i).

Our proof of Theorem 1 is based upon a similar idea to [**7**]. It will be a consequence of the following technical result.

PROPOSITION 2. *There exists $0 < d < 2^{-7/2}$ such that the following is true. Let $|||\cdot|||$ be a Euclidean norm on R^n and let B_2 be the closed unit ball. Let $1 \leq j \leq n$. Let V be a closed symmetric convex body such that*

$$V \subset \bigcup_{z \in \Lambda} (z + d|||z|||W) \cup cB_2, \tag{3}$$

with a set $\Lambda \subset R^n \backslash \{0\}$ with $\Lambda \leq e^{d^2 j}$ and some closed symmetric convex body W such that $W \subset B_2$ and $M_{W^*} \leq \sqrt{j/n}$, and for some constant c. Then there exists a subspace $F \subset R^n$ with $\dim F \geq n - j$ such that

$$|||x||| \leq c\|x\|_V \quad \text{for } x \in F. \tag{4}$$

PROOF. Applying an affine transformation, if necessary, we may assume that $|||x||| = (\sum_{i=1}^n x_i^2)^{1/2}$ for $x \in R^n$. Denote by M_j the median of the function $\psi_j(x) = (\sum_{i=1}^j x_i^2)^{1/2}$ on S. It is well known (cf., e.g., [**6**]) that $M_j \geq a\sqrt{j}/n$, where $0 < a < \sqrt{2}$ is a universal constant. Set $d = 2^{-4}a$. Let $\{e_i\}$ be the standard unit vector basis in R^n. Let $P_j: R^n \to R^n$ denote the orthogonal projection onto the subspace spanned by the first j unit vectors.

We need two lemmas. In their formulation $O(n)$ denotes the group of isometries of l_2^n and $\mathcal{P}$ is the Haar measure on $O(n)$. For an operator $T: R^n \to R^n$, $\|T: E \to F\|$ denotes the norm of T as an operator from E into F.

LEMMA 3. *Let*

$$A_1 = \{U \in O(n) | \, \|P_j U: E_W \to l_2^n\| > 8M_j/a\}.$$

Then $\mathcal{P}(A_1) < \frac{1}{2}$.

LEMMA 4. *Let*

$$A_2 = \left\{U \in O(n) | \exists z \in \Lambda | \, |||P_j U z||| < \tfrac{3}{4} M_j |||z||| \right\}.$$

Then $\mathcal{P}(A_2) < \frac{1}{2}$.

Assuming the truth of Lemmas 3 and 4 we complete the proof as follows. Pick an isometry $U_0: l_2^n \to l_2^n$ such that

$$\|P_j U_0: E_W \to l_2^n\| \leq 8M_j/a, \qquad |||P_j U_0 z||| \geq (3M_j/4)|||z||| \quad \text{for all } z \in \Lambda.$$

Then

$$(\ker P_j U_0) \cap (z + d|||z|||W) = 0 \quad \text{for all } z \in \Lambda. \tag{5}$$

Indeed, if $y \in (\ker P_j U_0) \cap (z + d|||z|||W)$, then

$$\begin{aligned} 0 = |||P_j U_0 y||| &\geq |||P_j U_0 z||| - |||P_j U_0 (y - z)||| \\ &\geq (3M_j/4)|||z||| - (8M_j/a)\|y - z\|_W \geq (3M_j/4)|||z||| - (8M_j/a)d|||z||| > 0, \end{aligned}$$

giving a contradiction. Set $F = \ker P_j U_0$. From (5) and (3) it follows that $F \cap V \subset cB_2$, which is equivalent to (4). □

PROOF OF LEMMA 3. Let $\tilde{\Lambda}$ be a $\frac{1}{2}$-net in $B_2 \cap [e_1, \ldots, e_j]$ with minimal cardinality. Then $|\tilde{\Lambda}| = N(B_2 \cap [e_1, \ldots, e_j], |||\cdot|||, \frac{1}{2}) \leq 5^j$. Thus

$$\begin{aligned} \mathcal{P}(A_1) &= \mathcal{P}\{U \in O(n) \mid \|U : l_2^j \to E_{W^*}\| > 8M_j/a\} \\ &\leq \mathcal{P}\{U \in O(n) \mid \exists z \in \tilde{\Lambda} \mid \|Uz\|_{W^*} > 4M_j/a\} \\ &\leq 5^j \mu\{y \in S \mid \|y\|_{W^*} > 4M_j/a\}. \end{aligned}$$

By the isoperimetric inequality (cf. [**6**, (2.6)], the measure of the latter set is smaller than or equal to

$$4\exp(-n(4M_j/a - M_{W^*})^2/2) \leq 4\exp(-9j/2).$$

Therefore

$$\mathcal{P}(A_1) \leq 4 \cdot 5^j \exp(-9j/2) < 1/2. \quad \square$$

Lemma 4 follows immediately from the isoperimetric inequality and the definition of d.

Now we are ready to prove Theorem 1.

PROOF OF THEOREM 1. Fix $0 < \lambda < 1$. Let j be the smaller integer larger than or equal to $(1 - \lambda)n$. Set $V = (d_1\sqrt{1-\lambda}/M_*)B_E$, where a constant $0 < d_1 < 1$ will be defined later, $W = (\sqrt{1-\lambda}/M_*)B_E \cap B_2$. Since $W \subset (\sqrt{1-\lambda}/M_*)B_E$, the dual bodies satisfy the opposite inclusion and $(M_*/\sqrt{1-\lambda})B_{E^*} \subset W^*$. Thus $M_{W^*} \leq (\sqrt{1-\lambda}/M_*)M_* \leq \sqrt{j/n}$. Notice also that $M_{V^*} = (d_1\sqrt{1-\lambda}/M_*)M_* = d_1\sqrt{1-\lambda}$.

Consider the compact metric space $(V, |||\cdot|||)$. Let Λ be a $1/2$-net in V with minimal cardinality. It follows from Lemma 1(ii), applied to the formal identity operator $u: (R^n, V) \to (R^n, |||\cdot|||)$ that

$$|\Lambda| = N(V, |||\cdot|||, \varepsilon) \leq 2^{4K^2 n M_{V^*}^2} \leq \exp(d^2 j),$$

where $K \geq 1$ is the constant from Lemma 1 and $d_1 = d/2K\sqrt{\log 2}$.

To show (3) fix $x \in V$. Pick $z \in \Lambda$ such that $|||x - z||| < 1/2$. If $|||x||| \geq 1/K\sqrt{\log 2} + 1/2d + 1/2$, then $|||z||| \geq 1/K\sqrt{\log 2} + 1/2d$. Therefore,

$$\begin{aligned} x - z &\in 2V = (2d_1\sqrt{1-\lambda}/M_*)B_E \subset d|||z|||(\sqrt{1-\lambda}/M_*)B_E, \\ x - z &\in \tfrac{1}{2}B_2 \subset d|||z|||B_2. \end{aligned}$$

Hence $x - z \in d|||z|||W$. This shows (3) with

$$c = 1/K\sqrt{\log 2} + 1/2d + 1/2.$$

Now Proposition 2 implies that there exists $F \subset R^n$ with $\dim F \geq n - j > \lambda n - 1$ such that

$$|||x||| \leq c\|x\|_V = K' M_* (1 - \lambda)^{-1/2}\|x\| \quad \text{for } x \in F,$$

where $K' = 2cK\sqrt{\log 2}/d$. □

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UNIVERSITÉ DES SCIENCES ET TECHNIQUES DE LILLE, U.E.R. DE MATHÉMATIQUES, 59655 VILLENEUVE D'ASCQ CEDEX, FRANCE

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA T6J 2G1, CANADA