JOURNAL OF FUNCTIONAL ANALYSIS 87, 273–293 (1989)

Volume Ratio and Other s-Numbers of Operators Related to Local Properties of Banach Spaces

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Communicated by the Editors

Received December 1, 1987; revised June 30, 1988

Volume ratio numbers are studied and inequalities relating them to other s-numbers of operators are established. Characterizations of spaces of weak cotype 2 in terms of s-numbers and of operator ideal norms are obtained.

INTRODUCTION

For many years strong interactions have existed between the Banach space theory and the theory of operator ideals. More recently we observe forming of mutual relationships between the local theory of Banach spaces and, so-called, s-numbers of operators. This paper stems from the latter trend.

The development and importance of volume techniques which can be observed at present in the local theory of Banach spaces (cf., e.g., [B-M, M-P1, M-P2, M-Sch, P3, S-T]) motivate the new concept of the volume ratio numbers. We study these numbers in Section 1, combining methods from the operator ideals theory with investigations of convex bodies in $\mathbb{R}^n$. In particular we establish various inequalities relating volume ratio numbers to other s-numbers of operators, notably the Gelfand and entropy numbers.

A recent "meeting" point of the theory of operator ideals and of the local theory of Banach spaces is the concept of "weak properties," suggested in [P2]. Among them, the most important seems to be the notion of a space of weak cotype 2 (cf. [M-P1]), due to its deep connections with the classical Dvoretzky theorem. In Section 2 we give a number of characterizations

* Research partially supported by NSERC Operating Grant.

0022-1236/89 $3.00

of spaces of weak cotype 2 in terms of s-numbers and of operator ideal norms.

Finally, in Section 3 we study inequalities between individual entropy numbers and Gelfand and Kolmogonov numbers of operators. The result requires additional assumptions on the domain and the range space; still, it is an essential generalization of previously known theorems. Combining this result with a general theorem of König–Milman, we obtain, in our context, a duality theorem for entropy numbers.

0. Notation and Preliminaries

Throughout the paper we use standard definitions and notations of Banach space theory as can be found in [L-T]. For any Banach space X we denote by B_X its closed unit ball. By l_p^n we denote $\mathbb{R}^n$ with the norm $\|x\|_p = (\sum_{j=1}^n |x_j|^p)^{1/p}$, for $x = (x_j) \in \mathbb{R}^n$ and $1 \leqslant p < \infty$, and by B_p^n we denote the closed unit ball in l_p^n.

Let us recall the definitions of some s-numbers. The reader should refer to [Pie] for more information on these notions. Let $u: X \to Y$ be an operator between two Banach spaces and n be a positive integer. The nth approximation number $a_n(u)$ is defined by

$$a_n(u) = \inf\{\|u - v\| \mid v: X \to Y, \operatorname{rank}(v) < n\}.$$

The nth Kolmogorov number $d_n(u)$ is defined by

$$d_n(u) = \inf_{Z \subset Y} \sup_{x \in B_X} \inf_{y \in Z} \|ux - y\|,$$

where the infimum on Z runs over all subspaces Z of Y with $\dim Z < n$. The dual concept is that of so-called Gelfand numbers defined by

$$c_n(u) = \inf\{\|u|_Z\| \mid Z \subset X, \operatorname{codim} Z < n\}.$$

We have $c_k(u) = d_k(u^*)$, for arbitrary $u: X \to Y$. If H is a Hilbert space then for any operator $u: H \to Y$, we have $c_k(u) = a_k(u)$ and for any operator $w: X \to H$, $d_k(w) = a_k(w)$.

All these numbers express some finite dimensional approximations for the operator u. Geometrically they may be viewed as different widths of the ball $u(B_X)$ as a subset of Y. The degree of compactness of an operator u may be measured by means of the well known concept of the metric entropy. In the framework of operators between Banach spaces it is more convenient to work with the entropy numbers. For $u: X \to Y$, the nth entropy number $e_n(u)$ is the infimum of all positive ε such that there exist $y_1, ..., y_{2^{n-1}}$ in Y such that

$$u(B_X) \subset \bigcup_{j=1}^{2^{n-1}} (y_j + \varepsilon B_Y).$$

The following weak-type inequality due to Carl [C1] relates entropy numbers to other s-numbers. Let $\{s_n\}$ denote any of the sequences $\{c_n\}$, $\{d_n\}$, $\{a_n\}$. For every $\alpha > 0$ there exists C_α such that for arbitrary operator $u: X \to Y$ we have

$$\sup_n n^\alpha e_n(u) \leqslant C_\alpha \sup_n n^\alpha s_n(u). \tag{0.1}$$

We shall also use a geometric notion of covering numbers. If K_1 and K_2 are subsets of $\mathbb{R}^n$ then $N(K_1, K_2)$ denotes the minimal cardinality of a subset $S \subset \mathbb{R}^n$ such that

$$K_1 \subset \bigcup_{x \in S} (x + K_2).$$

We shall often use related easy observations. If $K_1 \subset \mathbb{R}^n$ is a compact convex body and $x \in \mathbb{R}^n$ then

$$(x + B_2^n) \cap K_1 \subset z + B_2^n, \qquad \text{for some } z \in K_1. \tag{0.2}$$

If $K_2 \subset \mathbb{R}^n$ is another compact convex body and $x \in \mathbb{R}^n$ then

$$(x + K_2) \cap K_1 \subset z + 2K_2, \qquad \text{for some } z \in K_1. \tag{0.3}$$

Let $(\cdot, \cdot)$ denote the usual inner product on $\mathbb{R}^n$ and let $\text{Vol}(\cdot)$ denote the Lebesgue measure on $\mathbb{R}^n$. For a centrally symmetric compact convex body $K \subset \mathbb{R}^n$ define the polar K^0 by

$$K^0 = \{x \in \mathbb{R}^n \mid |(x, y)| \leqslant 1 \text{ for all } y \in K\}.$$

The following important inequalities relate $\text{Vol}(K)$ and $\text{Vol}(K^0)$. There exists a constant $c > 0$ such that for every n and every contrally symmetric compact convex body $K \subset \mathbb{R}^n$ one has

$$c^n \leqslant \frac{\text{Vol}(K)\,\text{Vol}(K^0)}{(\text{Vol}(B_2^n))^2} \leqslant 1. \tag{0.4}$$

The upper estimate is the classical inequality due to Santaló [S]. The lower estimate was recently proved by Bourgain and Milman [B–M].

Given an operator $u: l_2^n \to X$, let

$$l(u) = \left(\int \|ux\|^2 \, d\gamma_n(x) \right)^{1/2},$$

where γ_n is the canonical Gaussian probability measure on $\mathbb{R}^n$. The dual norm l^* is defined for operators $w: X \to l_2^n$ by

$$l^*(w) = \sup\{ |\text{trace } wu| \mid u: l_2^n \to X, \, l(u) \leqslant 1 \}.$$

For the notion of the K-convexity we refer the reader to [P1] (cf. also [M-Sch, T2]). The well known equivalent definition ([F-T], cf. also [T2]) says that a Banach space is K-convex if and only if there is a constant C such that for every n and every operator $w: X \to l_2^n$, one has

$$l(w^*) \leqslant C l^*(w). \tag{0.5}$$

Moreover, the smallest constant C for which (0.5) holds is equal to the (Gaussian) K-convexity constant X, $K(X)$.

For the definition and basic properties of the absolutely 2-summing norm, $\pi_2(\cdot)$, we refer the reader to [Pie] or [T2].

The l-norm is intimately related to geometric and local properties of Banach spaces. In the context of operator ideals, its connections with entropy numbers and Gelfand numbers are particularly useful. The classical Sudakov minoration theorem [Su] can be stated as follows. For every operator $u: X \to l_2$ one has

$$\sup_n \sqrt{n}\, e_n(u) \leqslant c \sup\{ l(u^*|_E) \mid E \subset l_2, \dim E < \infty \}, \tag{0.6}$$

where c is a universal constant.

The inequality due to the authors [P-T1] (cf. also [P-T3]), which improves the result of Milman [M2], states that for every operator $u: X \to l_2$ one has

$$\sup_n \sqrt{n}\, c_n(u) \leqslant c \sup\{ l(u^*|_E) \mid E \subset l_2, \dim E < \infty \}, \tag{0.7}$$

where c is a universal constant.

1. Volume Ratio Numbers and Their Relation to Gelfand and Entropy Numbers.

The importance of volume techniques in the local theory of Banach spaces and the close connection of volume to the metric entropy suggest the concept of "volume ratio" numbers. In all geometric and probabilistic applications known up to now, it is natural to consider only operators taking values in a Hilbert space, as we do here.

For any finite-dimensional Hilbert space we denote by Vol($\cdot$) the natural Lebesgue measure. Let X be a Banach space and let $u: X \to l_2$. For $n = 1, 2, \ldots$, set

$$vr_n(u) = \sup\{\mathrm{Vol}(Pu(B_X))/\mathrm{Vol}(B_2^n)\}^{1/n}, \tag{1.1}$$

where the supremum is taken over all orthogonal projections $P: l_2 \to l_2$ with rank $P = n$.

This motion was introduced in [D]. In the context of the local theory of Banach spaces these numbers were first used in [M-P2], where some majoration problems for the l-norm were studied.

Let us observe some general properties of the volume ratio numbers. If $n > \mathrm{rank}\, u$, then $vr_n(u) = 0$. If $u: X \to l_2$ and $S: Y \to X$, $T: l_2 \to l_2$, then $vr_n(TuS) \leqslant \|T\|\, vr_n(u) \|S\|$. The next property is less trivial.

PROPOSITION 1.1. *Let* $u: X \to l_2$. *Then* $\|u\| = vr_1(u) \geqslant \cdots \geqslant vr_n(u) \geqslant \cdots \geqslant 0$.

Proof. We have

$$vr_1(u) = \sup\left\{\sup_{y \in u(B_X)} |(x, y)| \;\Big|\; \|x\|_2 \leqslant 1\right\} = \|u\|.$$

Let $n > 1$ and let $P: l_2 \to l_2$ be an orthogonal projection with rank $P = n$. Set $w = Pu$ and $K = w(B_X)$. If $V_k(K)$ denotes the mixed volume of K $(k = 0, \ldots, n)$ then, by Steiner–Minkowski formula [B-Z], we have

$$V_k(K) = \frac{\mathrm{Vol}(B_2^n)}{\mathrm{Vol}(B_2^k)} \int_{G_{n,k}} \mathrm{Vol}(P_E K)\, d\mu_k(E),$$

where $G_{n,k}$ is the Grassman manifold of all k-dimensional subspaces of $\mathbb{R}^n$ and μ_k is the normalized Haar measure on $G_{n,k}$ and, for $E \in G_{n,k}$, P_E denotes the orthogonal projection onto E (cf. [B-Z]). In particular, the Cauchy formula states that $nV_{n-1}(K)$ is the surface area of K. Therefore from the isoperimetric inequality in $\mathbb{R}^n$ we get

$$\begin{aligned} A_n(K) &= (V_n(K)/\mathrm{Vol}(B_2^n))^{1/n} \\ &\leqslant A_{n-1}(K) = (V_{n-1}(K)/\mathrm{Vol}(B_2^n))^{1/(n-1)}. \end{aligned}$$

Thus

$$\mathrm{Vol}(Pu(B_X)) = A_n(K) \leqslant A_{n-1}(K) \leqslant vr_{n-1}(u),$$

and taking supremum over P we conclude that $vr_n(u) \leqslant vr_{n-1}(u)$. ∎

Using the polar decomposition it is easy to see that any operator in a Hilbert space, $w: l_2 \to l_2$, satisfies

$$vr_n(w) = \left(\prod_{j=1}^{n} s_j(w) \right)^{1/n}, \tag{1.2}$$

where $s_j(w)$ are singular numbers of w.

If $1 \leqslant p \leqslant 2$, then for the identity operator $i_{p2}: l_p \to l_2$ we have

$$c^{-1} k^{1/2 - 1/p} \leqslant vr_k(i_{p2}) \leqslant ck^{1/2 - 1/p}, \tag{1.3}$$

where $c > 0$ is a universal constant. Observe first that, by Santaló inequality (0.4), $vr_k(i_{p2})$ is smaller than or equal to the expression

$$\sup \left\{ \left(\frac{\mathrm{Vol}(B_2^k)}{\mathrm{Vol}(B_{p^*} \cap F)} \right)^{1/k} \middle| F \subset l_2, \dim F = k \right\},$$

where $p^* = p/(p-1)$ for $p > 1$ and $p^* = \infty$ for $p = 1$, and $B_{p^*} = B_{l_{p^*}}$ denotes the unit ball in l_{p^*}. Now, the upper estimate easily follows from [V], for $p = 1$, and from [Me-Pa], for $1 < p < 2$. The lower estimate is obvious, as $vr_k(i_{p2}) \geqslant (\mathrm{Vol}(B_p^k)/\mathrm{Vol}(B_2^k))^{1/k}$.

We pass now to the discussion of relationship between volume ratio numbers and Gelfand and entropy numbers. Most estimates are in a form of so-called weak-type inequalities. Some of them are proved under the additional assumption of K-convexity. The main result in this direction states.

THEOREM 1.2. *Let X be a Banach space and let $u: X \to l_2$ be an operator. Then*

$$vr_k(u) \leqslant \min[2e_k(u), 4e_k(u^*)], \qquad \textit{for} \quad k = 1, 2, ... \tag{1.4}$$

If X is K-convex then

$$\sqrt{k}\, c_k(u) \leqslant CK(X) \sum_{j \geqslant Ak} vr_j(u)/\sqrt{j}, \qquad \textit{for } k = 1, 2, ..., \tag{1.5}$$

where $C > 1$ is a universal constant and $A = (2CK(X))^{-2}$.

Proof. Let $P: l_2 \to l_2$ be an orthogonal projection with rank $P = k$. Let $H = P(l_2)$ and $B_H = B_2 \cap H$. Set $v = Pu$. Fix $\varepsilon > e_{k+1}(v)$. By the definition and (0.2), there exist $x_1, ..., x_{2^k}$ in $v(B_X)$ such that $v(B_X) \subset \bigcup_{j=1}^{2^k} (x_j + \varepsilon B_2)$. Thus

$$v(B_X) \subset \bigcup_{j=1}^{2^k} (x_j + \varepsilon B_H).$$

Since $\mathrm{Vol}(B_H) = \mathrm{Vol}(B_2^k)$, comparing volumes we get

$$\mathrm{Vol}(v(B_X))/\mathrm{Vol}(B_2^k) \leqslant (2\varepsilon)^k.$$

This immediately implies $vr_k(u) \leqslant 2e_{k+1}(v) \leqslant 2e_k(u)$.

Now let $E = u^*(H)$. By $w: H \to E$ denote the restriction of u^*. Let $\varepsilon > e_{k+1}(u^*)$. By the definition and (0.3) there exist $x_1,..., x_{2^k}$ in $w(B_H)$ such that $w(B_H) \subset \bigcup_{j=1}^{2^k} (x_j + 2\varepsilon B_{X^*})$, hence also

$$w(B_H) \subset \bigcup_{j=1}^{2^k} (x_j + 2\varepsilon B_E).$$

Thus

$$\mathrm{Vol}(B_H)/\mathrm{Vol}(w^{-1}(B_E)) \leqslant (4\varepsilon)^k.$$

Observe that the polar $(w^{-1}(B_E))^0$ coincides with $Pu(B_X)$. By Santaló inequality we get $\mathrm{Vol}(Pu(B_X))/\mathrm{Vol}(B_2^k) \leqslant (4\varepsilon)^k$. This implies $vr_k(u) \leqslant 4e_{k+1}(u^*) \leqslant 4e_4(u^*)$.

Proof of (1.5) requires some additional results.

LEMMA 1.3. *Let X be a Banach space and let $v: l_2 \to X$, $u: X \to l_2$ be operators. Then,*

$$vr_k(uv) \leqslant 2e_k(v)\, vr_k(u), \qquad \textit{for} \quad k = 1, 2,....$$

Proof. Using polar decomposition and (1.2) pick orthogonal projections P and Q of rank k such that

$$vr_k(uv) = (\mathrm{Vol}(PuvQ(B_2))/\mathrm{Vol}(B_2^k))^{1/k}.$$

Let $\varepsilon > e_{k+1}(vQ)$. Then there exist $x_1,..., x_{2^k}$ in X such that

$$vQ(B_2) \subset \bigcup_{j=1}^{2^k} (x_j + \varepsilon B_X).$$

Thus

$$PuvQ(B_2) \subset \bigcup_{j=1}^{2^k} (Pux_j + \varepsilon Pu(B_X)).$$

Hence $\mathrm{Vol}(PuvQ(B_2)) \leqslant 2^k\varepsilon^k\, \mathrm{Vol}(Pu(B_X)) \leqslant 2^k\varepsilon^k[vr_k(u)]^k\, \mathrm{Vol}(B_2^k)$. So $vr_k(uv) \leqslant 2e_{k+1}(vQ)\, vr_k(u) \leqslant 2e_k(v)\, vr_k(u)$, concluding the proof of the lemma. ∎

LEMMA 1.4. *Let X be a Banach space and let $u: X \to l_2^n$ be an operator. Then*

$$l^*(u) \leqslant c \sum_{j=1}^{n} vr_j(u)/\sqrt{j},$$

where c is a universal constant.

Proof. Let $v: l_2^n \to X$ with $l(v) \leqslant 1$. Then, by Lemma 1.3,

$$\begin{aligned}|\text{trace}\, uv| &\leqslant \sum_{j=1}^{n} s_j(uv) \leqslant \sum_{j=1}^{n} vr_j(uv) \\ &\leqslant 2 \sup_k \sqrt{k}\, e_k(v) \sum_{j=1}^{n} vr_j(u)/\sqrt{j}.\end{aligned}$$

By Sudakov's inequality (0.6) we get

$$|\text{trace}\, uv| \leqslant 2c \sum_{j=1}^{n} vr_j(u)/\sqrt{j},$$

and the proof is completed, by the definition of $l^*(u)$. ∎

Returning to the proof of (1.5), Lemma 1.4 and (0.5) imply that for any finite-dimensional subspace $E \subset l_2$ and the orthogonal projection $P: l_2 \to E$ we have

$$\begin{aligned}l(u^*|_E) &\leqslant K(X)\, l^*(Pu) \\ &\leqslant C'K(X) \sum_{j=1}^{\infty} vr_j(u)/\sqrt{j}.\end{aligned} \tag{1.6}$$

Therefore, by (0.7),

$$\sqrt{k}\, c_k(u) \leqslant CK(X) \sum_{j=1}^{\infty} vr_j(u)/\sqrt{j}. \tag{1.7}$$

Now we use the proof from [P-T2], inspired by the renorming argument from [M-P1]. For the reader's convenience we sketch it here. Fix $\rho > 0$ to be defined later. For $x \in X$ let $\|x\|_\rho = \max(\|x\|, \rho^{-1} \|ux\|_2)$. Let $X_\rho = (X, \|\cdot\|_\rho)$ and let $i: X_\rho \to X$ denote the natural identity operator. It was shown in [P-T2] that

$$c_k(u) = \inf\{\rho > 0 \mid \rho > c_k(ui: X_\rho \to l_2)\}.$$

The last formula and (1.7) imply

$$c_k(u) \leqslant \inf\left\{\rho > 0 \mid \rho\sqrt{k} > CK(X_\rho) \sum_{j=1}^{\infty} vr_j(ui)/\sqrt{j}\right\}.$$

Set $A = (4\sqrt{2}\,CK(X))^{-2}$. Since $vr_j(ui) \leqslant \|ui\| \leqslant \rho$, we have $\sum_{j \leqslant Ak} vr_j(ui)/\sqrt{j} \leqslant 2\rho k^{1/2} A^{1/2} = (2\sqrt{2}\,CK(X))^{-1} \rho k^{1/2}$. Since $\|i\| \leqslant 1$, then $vr_j(ui) \leqslant vr_j(u)$. Moreover it is easy to check from (0.5) that $K(X_\rho) \leqslant \sqrt{2}K(X)$ and thus the infimum in the last inequality is smaller than or equal to

$$\inf\left\{\rho > 0 \mid \rho\sqrt{k} > 2\sqrt{2}\,CK(X) \sum_{j > Ak} vr_j(u)/\sqrt{j}\right\}.$$

This shows (1.5), with the constant $C' = 2\sqrt{2}C$ and $A = (2C'K(X))^{-2}$, and completes the proof of the theorem. ∎

Remark. Estimate (1.6) can be viewed as a strengthening, in the K-convex case, of Dudley's inequality [D]

$$l(u^*|_E) \leqslant c \sum_{j=1}^{\infty} e_j(u)/\sqrt{j},$$

valid for arbitrary operator $u: X \to l_2$ and any finite-dimensional subspace E of l_2. Let us note that a recent result on duality of entropy numbers of operators taking values in a Hilbert space [T1] yields that for an arbitrary operator $v: l_2 \to Y$ one has

$$l(v) \leqslant c \sum_{j=1}^{\infty} e_j(v)/\sqrt{j}. \tag{1.8}$$

In the case when Y is K-convex, (1.8) follows directly from (1.6) and (1.4) (the constant c should be then replaced by $cK(Y)$). It would be interesting to decide whether (1.6) holds without the K-convexity assumption and to get a direct proof of (1.8).

COROLLARY 1.5. *Let X be a K-convex Banach space and let $u: X \to l_2$ be an operator. Let $\alpha > \frac{1}{2}$. Then*

$$\sup_k k^\alpha c_k(u) \leqslant CK(X)^{2\alpha}\left(\alpha - \frac{1}{2}\right)^{-1} \sup_k k^\alpha vr_k(u), \tag{1.9}$$

where C is a universal constant. Furthermore, if rank $u = n$ *then, for every* $1 \leqslant k \leqslant n$,

$$c_k(u) \leqslant CK(X)^{2\alpha} \left(\alpha - \frac{1}{2}\right)^{-1} (n/k)^{\alpha}\, vr_{[Ak]}(u), \tag{1.10}$$

where C is a universal constant and $A = (2CK(X))^{-2}$.

Proof. By (1.5) we have, for $k = 1, 2, \ldots$,

$$\begin{aligned}
\sqrt{k}\, c_k(u) &\leqslant CK(X) \sum_{j \geqslant Ak} vr_j(u)/\sqrt{j} \\
&\leqslant CK(X) \sup_{m \geqslant Ak} m^{\alpha}\, vr_m(u) \sum_{j \geqslant Ak} j^{-\alpha - 1/2} \\
&\leqslant CK(X)(Ak)^{1/2 - \alpha} \left(\alpha - \frac{1}{2}\right)^{-1} \sup_{m \geqslant Ak} m^{\alpha} vr_m(u) \\
&\leqslant C' K(X)^{2\alpha} \left(\alpha - \frac{1}{2}\right)^{-1} k^{1/2 - \alpha} \sup_{m \geqslant Ak} m^{\alpha}\, vr_m(u).
\end{aligned} \tag{1.11}$$

So (1.9) follows. If rank $u = n$ then $vr_m(u) = 0$ for $m > n$. Therefore

$$\sup_{m \geqslant Ak} m^{\alpha}\, vr_m(u) \leqslant n^{\alpha}\, vr_{[Ak]}(u).$$

Thus (1.10) follows immediately from (1.11). ∎

Remarks. 1. Let us observe that combining the result from [P-T2] and the duality result from [T1] it follows that if $\alpha > \frac{1}{2}$ then for arbitrary operator $u\colon X \to l_2$ one has

$$\sup_k k^{\alpha} c_k(u) \leqslant \left(\alpha - \frac{1}{2}\right)^{-1} \min\left[\sup_k k^{\alpha} e_k(u), \sup_k k^{\alpha} e_k(u^*)\right]. \tag{1.12}$$

2. Using Milman's theorem on quotients of subspaces [M1] it can be shown that for an arbitrary operator $u\colon X \to l_2^n$,

$$c_{2k}(u) \leqslant C(n/k) \log(1 + n/k)\, vr_k(u). \tag{1.13}$$

Other relations concerning volume ratio numbers can be found in the forthcoming book of Pisier [P3, Chap. 9].

The upper estimate for Gelfand numbers given in (1.9) should be complemented by the lower estimate which follows directly from (1.4) and

Carl's inequality (0.1). For every $\beta > 0$ there exists C_β such that for every operator $u: X \to l_2$ we have

$$\sup_k k^\beta vr_k(u) \leqslant C \sup_k k^\beta e_k(u) \leqslant C_\beta \sup_k k^\beta c_k(u). \tag{1.14}$$

Applying Carl's inequality for both u and u^* we get, by (1.9), the following.

COROLLARY 1.6. *Let X be a K-convex Banach space and let $u: X \to l_2$ be an operator. Let $\alpha > \frac{1}{2}$. Then*

$$\max\left[\sup_k k^\alpha e_k(u), \sup_k k^\alpha e_k(u^*)\right] \leqslant CK(X)^{2\alpha}\left(\alpha - \frac{1}{2}\right)^{-1} \sup_k k^\alpha vr_k(u).$$

Weak-type estimates from Corollaries 1.5 and 1.6 and (1.12) are "close to optimal"; indeed, analogous estimates do not hold for $\alpha < \frac{1}{2}$. This can be seen comparing the order of growth of volume ratio, entropy, and Gelfand numbers of the identity operator $i_{p2}: l_p^n \to l_2^n$, for $1 \leqslant p < 2$. The entropy numbers $e_k(i_{p2})$ and $e_k(i_{p2}^*)$ were studied in [Sch], where their order of growth was calculated. We have, up to universal constants,

$$e_k(i_{p2}) \underset{c}{\sim} e_k(i_{p2}^*) \underset{c}{\sim} \begin{cases} 1 & \text{for } k \leqslant \log n \\ (\log(1+n/k)/k)^{1/p-1/2} & \text{for } \log n \leqslant k \leqslant n \\ 2^{-k/n} n^{1/2-1/p} & \text{for } n \leqslant k. \end{cases} \tag{1.15}$$

The Gelfand numbers $c_k(i_{p2})$ were studied in [G1, G2, G-G] (cf. also [P-T3]). For $1 < p \leqslant 2$ we have, with constants which depend on p,

$$c_k(i_{p2}) \underset{c_p}{\sim} \min(1, n^{1/p^*} k^{-1/2}), \tag{1.16}$$

while for $p = 1$ we have

$$c_k(i_{12}) \underset{c}{\sim} \min(1, (\log(1+n/k)/k)^{1/2}). \tag{1.17}$$

2. SPACES OF WEAK COTYPE 2

The notion of spaces of weak cotype 2 was recently introduced and studied in [M-P1]. In particular, finite-dimensional subspaces of a space of weak cotype 2 were shown to have nearly Euclidean sections of large dimensions and volume ratios uniformly bounded. In this section we give characterizations of spaces of weak cotype 2 in terms of s-numbers and of operator ideal norms.

Let us recall that a Banach space X is said to be of weak cotype 2, if there is a constant C such that for all n and all operators $u: l_2^n \mapsto X$,

$$\sup_k \sqrt{k}\, a_k(u) \leqslant Cl(u). \tag{2.1}$$

The smallest constant C is called the weak cotype 2 constant of X and is denoted by $wC_2(X)$.

The following result was proved in [M-P1, Theorems 1 and 8].

PROPOSITION 2.1. *Let X be a Banach space of weak cotype 2. For every positive integer n and every subspace E of X with* $\dim E = n$ *and every operator $u: E \to l_2^n$ we have*

(i) $\sqrt{k}\, c_k(u) \leqslant a(n/k)\, C\pi_2(u)$, *for every* $1 \leqslant k \leqslant n$,

(ii) $e_n(u) \leqslant C\pi_2(u)/\sqrt{n}$,

(iii) $e_n(u^*) \leqslant C\pi_2(u)/\sqrt{n}$,

where $C \leqslant a(wC_2(X))^2 \log(1 + wC_2(X))$ *and* $a > 1$ *is a universal constant.*

Remark. In fact, (iii) is not stated explicitly in [M-P1], but it follows from (i) just the same way as (ii) does, using Carl's result (0.1) and the fact that $d_k(u^*) = c_k(u)$.

We shall also use the following result from [C2].

LEMMA 2.2. *Let X and Y be Banach spaces and let $u: X \to Y$ be an operator. Let $\{s_i\}$ stand either for Gelfand numbers $\{c_i\}$ or for Kolmogorov numbers $\{d_i\}$. Then, for every positive integer n we have*

$$\prod_1^n s_i(u) \leqslant \sup\{\det[(y_i^*, ux_j)] \mid \{x_j\}_1^n \subset B_X,\ \{y_i^*\}_1^n \subset B_{Y^*}\}.$$

For the sake of completeness let us sketch the proof for Gelfand numbers. The case of Kolmogorov numbers follows by duality.

Proof. Fix $\varepsilon > 0$. Construct inductively sequences $\{x_i\}_1^n$ in B_X and $\{y_i^*\}_1^n$ in B_{Y^*} such that $(y_i^*, ux_i) \geqslant (1-\varepsilon)\, c_k(u)$ and $(y_i^*, ux_j) = 0$ for $j > i$. This can be done by picking $x_j \in \bigcap_{i<j} \ker u^* y_i^*$ such that $\|ux_j\| \geqslant (1-\varepsilon)\, c_j(u)$ and then choosing y_j^* in B_{Y^*} such that $(y_j^*, ux_j) = \|ux_j\|$. Clearly, the matrix $[(y_i^*, ux_j)]$ is triangular and so $(1-\varepsilon)^n \prod_1^n c_i(u) \leqslant \det[(y_i^*, ux_j)]$. ∎

We have the following characterization of spaces of weak cotype 2 in terms of s-numbers.

THEOREM 2.3. *Let X be a Banach space. The following conditions are equivalent.*

(i) *X is of weak cotype 2,*

(ii) $\exists C\, \forall v: X^* \to l_2\, \forall k \geqslant 1$

$$a_k(v) \leqslant C vr_k(v),$$

(iii) $\exists C\, \forall u: l_2 \to X\, \forall k \geqslant 1$

(a) $a_k(u) \leqslant C c_k(u)$,

(b) $a_k(u) \leqslant C e_k(u^*)$,

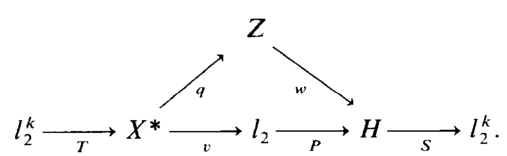
(iv) $\exists C\, \forall u: l_2 \to X$

$$\sup_k \sqrt{k}\, a_k(u) \leqslant C \sup_k \sqrt{k}\, d_k(u).$$

Proof. (i) $\Rightarrow$ (ii). Let $v: X^* \to l_2$ be an operator and let k be a positive integer. Let $\eta > 0$. Recall that $a_k(v) = d_k(v)$ and, by Lemma 2.2, pick $x_1, ..., x_k$ in B_{X^*} and $y_1^*, ..., y_k^*$ in B_{l_2} such that

$$a_k(v) \leqslant \left(\prod_1^k d_j(v) \right)^{1/k} \leqslant (1+\eta)(\det[(y_i^*, vx_j)])^{1/k}. \tag{2.2}$$

Clearly, we may assume that the determinant in (2.2) is non-zero. Set $H = \operatorname{span}(vx_1, ..., vx_k) \subset l_2$ and let $P: l_2 \to H$ be the orthogonal projection. Set $Z = X^*/\ker(Pv)$ and let $q: X^* \to Z$ be the quotient map and $w: Z \to H$ such that $wq = Pv$. Let $\{e_j\}_1^k$ be the unit vector basis in l_2^k. Define operators $T: l_2^k \to X^*$ by $Te_j = x_j$, for $j = 1, ..., k$ and $S: H \to l_2^k$ by $Sz = \sum_{i=1}^k (z, y_i^*)\, e_i$, for $z \in H$.



Set $U = SPvT$. The matrix $[(y_i^*, vx_j)]$ represents the operator U in the unit vector basis in l_2^k. Clearly,

$$(\det[(y_i^*, vx_j)])^{1/k} = (\det U)^{1/k} \leqslant |\det S|^{1/k}\, |\det(wqT)|^{1/k}. \tag{2.3}$$

To estimate the first factor observe that the norm of S, as an operator into

l_∞^k, satisfies $\|S: H \to l_\infty^k\| = \max \|y_i^*\| = 1$. Hence, $\pi_2(S) \leqslant \sqrt{k}\, \|S: H \to l_\infty^k\| = \sqrt{k}$. Thus

$$|\det S|^{1/k} = \left(\prod_{i=1}^{k} a_i(S)\right)^{1/k} \leqslant \left(\frac{1}{k}\sum_{i=1}^{k} a_i(S)^2\right)^{1/2}$$
$$\leqslant k^{-1/2}\pi_2(S) \leqslant 1.$$

To estimate the second factor, fix $\varepsilon > e_k(qT)$. There exist $z_1, ..., z_{2^{k-1}}$ in Z such that

$$qT(B_2^k) \subset \bigcup_{1}^{2^{k-1}} (z_i + \varepsilon B_Z).$$

Hence

$$wqT(B_2^k) \subset \bigcup_{1}^{2^{k-1}} (wz_i + \varepsilon w(B_Z)).$$

Comparing volumes, we get

$$|\det(wqT)|^{1/k} = (\mathrm{Vol}(wqT(B_2^k))/\mathrm{Vol}(B_2^k))^{1/k}$$
$$\leqslant 2\varepsilon(\mathrm{Vol}(w(B_Z))/\mathrm{Vol}(B_2^k))^{1/k}$$
$$\leqslant 2\varepsilon\, vr_k(v).$$

The latter inequality follows from the obvious formula $\mathrm{Vol}(w(B_Z)) = \mathrm{Vol}(wq(B_{X^*})) = \mathrm{Vol}(Pv(B_{X^*}))$. Since the estimate is valid for arbitrary $\varepsilon > e_k(qT)$, we get

$$|\det(wqT)|^{1/k} \leqslant 2e_k(qT)\, cr_k(v).$$

Observe now that Z^* is a k-dimensional subspace of X and that the norm of $(qT)^*$, as an operator to l_∞^k, satisfies $\|(qT)^*: Z^* \to l_\infty^n\| = \max\|x_j\| = 1$. Thus $\pi_2((qT)^*) \leqslant \sqrt{k}$ and, by Proposition 2.1(iii), $e_k(qT) \leqslant C\pi_2((qT)^*)/\sqrt{k} \leqslant C$. Combining these estimates with (2.3) we get

$$(\det[(y_i^*, vx_j)])^{1/k} \leqslant 2Cvr_k(v), \tag{2.4}$$

where $C \leqslant a(wC_2(X))^2 \log(1 + wC_2(X))$ and $a \geqslant 1$ is a universal constant. Combining with (2.2) and taking $\eta \to 0$ shows (ii).

(ii) $\Rightarrow$ (iii(a)) and (ii) $\Rightarrow$ (iii(b)). For $u: l_2 \to X$ we have $a_k(u) = c_k(u) = d_k(u^*) = a_k(u^*)$ (cf. [Pie, Chap. 11]), hence the implications follow from Theorem 1.2 and the equality $e_k(u) = e_k(u^{**})$.

(iii(a)) $\Rightarrow$ (iv) and (iii(b)) $\Rightarrow$ (iv) follows directly from (0.1).

(iv) $\Rightarrow$ (i). Let $u: l_2^n \to X$. Then $d_k(u) = c_k(u^*)$ and so the conclusion is an immediate consequence of (0.7). ∎

In the statement of the next corollary we use the following convention. Given Banach space X, functions f and g on $L(l_2, X)$. whose values are positive numbers or $+\infty$, and $c \geqslant 1$, we write $f(u) \underset{c}{\sim} g(u)$ if $c^{-1} g(u) \leqslant f(u) \leqslant c g(u)$, for every $u \in L(l_2, X)$.

COROLLARY 2.4. *Let X be a Banach space of weak cotype 2. Let $u: l_2 \to X$ be an operator. Then for every $\alpha > 0$,*

$$\sup_n n^\alpha a_n(u) \underset{c_\alpha}{\sim} \sup_n n^\alpha vr_n(u^*) \underset{c_\alpha}{\sim} \sup_n n^\alpha e_n(u)$$

$$\underset{c_\alpha}{\sim} \sup_n n^\alpha e_n(u^*) \underset{c_\alpha}{\sim} \sup_n n^\alpha d_n(u), \tag{2.5}$$

where $c_\alpha \geqslant 1$ depends only on α.

Moreover, if every operator $u: l_2 \to X$ satisfies $\sup_n \sqrt{n}\, a_n(u) \underset{c}{\sim} \sup_n \sqrt{n}\, s_n(u)$, where $s_n(u)$ is $vr_n(u^*)$ or $e_n(u)$ or $e_n(u^*)$ or $d_n(u)$, then X is of weak cotype 2.

Proof. It is well known that for arbitrary Banach space X and an operator $u: l_2 \to X$, $\sup_n n^\alpha a_n(u)$ dominates any of other four functions (cf. [Pie, 11.2.3]). The opposite inequalities follow directly from Theorem 2.3. Conversely, each of the conditions implies (iv) of Theorem 2.3. ∎

Spaces of weak cotype 2 admit the following simple characterization in terms of operator ideal norms.

PROPOSITION 2.5. *Let X be a Banach space. Then X is of weak cotype 2 if and only if $\exists C\ \forall E \subset X$, $\dim E = n < \infty$ $\forall u: l_2^n \to E$ one-to-one*

$$n \leqslant C l(u)\, \pi_2(u^{-1}). \tag{2.6}$$

Proof. Fix a subspace $E \subset X$ with $\dim E = n < \infty$ and let $u: l_2^n \to E$ be a one-to-one operator. Set $k = [(n+1)/2]$ and let $H \subset l_2^n$ be a subspace with codim $H = k - 1$ such that $\|u|_H\| = a_k(u)$. Set $w = u|_H$, so that $w^{-1} = (u^{-1})|_{u(H)}$. Since $\pi_2(w) \leqslant \sqrt{\dim H}\, \|w\|$, we get, by (2.1),

$$\begin{aligned} (n+1)/2 &\leqslant \operatorname{trace} w^{-1} w \leqslant \pi_2(w^{-1})\, \pi_2(w) \\ &\leqslant \pi_2(w^{-1}) \sqrt{k}\, \|w\| \\ &\leqslant \pi_2(u^{-1} \sqrt{k} a_k(u) \\ &\leqslant C l(u)\, \pi_2(u^{-1}). \end{aligned}$$

This shows (2.6) with $C = 2wC_2(X)$.

Conversely, we shall show that (2.6) implies the existence of a constant $0<\delta_0<1$ such that every finite-dimensional subspace $E \subset X$ contains $F \subset E$ with $k=\dim F \geqslant \delta_0 \dim E$ and $d(F, l_2^n) \leqslant 2$. This condition, in turn, was shown in [M-P1] to imply that X is of weak cotype 2.

Fix $E \subset X$ with $\dim E = n$. Let $u: l_2^n \to E$ be the isomorphism associated to the John's ellipsoid of maximal volume contained in B_E. In particular, $\pi_2(u^{-1}) = \sqrt{n}$ and $\|u\| = 1$ (cf., e.g., [M-P1, T2, Chap. III]). The former equality and (2.6) yield $l(u) \geqslant C^{-1}\sqrt{n}$ which, in turn, combined with the latter equality, can be used for finding an Euclidean subspace F of E. We will use the result of [F-L-M, Theorem 2.6] (cf. also [M-Sch]) with $|||x||| = \|u^{-1}x\|_2$, for $x \in E$. Then $b=1$ and $M_r = l(u)/\sqrt{n} \geqslant C^{-1}$. It follows that there exists $F \subset E$ with $k = \dim F \geqslant \delta_0 n$ and $d(F, l_2^k) \leqslant 2$. Here δ_0 depends only on C, so this completes the proof. ∎

Theorem 2.3 and its proof suggest the new parameter which might be of an independent interest. For an operator $u: X \to Y$ between Banach spaces X and Y, and for a positive integer n, set

$$D_n(u) = \sup\{\det[(y_i^*, ux_j)]^{1/n} \mid \{x_j\}_1^n \subset B_X,\ \{y_i^*\}_1^n \subset B_{Y^*}\}. \tag{2.7}$$

For operators from l_2 into a space of weak cotype 2, $D_n(u)$ admits a number of interesting upper estimates. Most of them actually characterize spaces of weak cotype 2.

THEOREM 2.6. *Let X be a Banach space. The following conditions are equivalent.*

(i) *X is of weak cotype* 2,

(ii) $\exists C\ \forall u: l_2 \to X\ \forall n \geqslant 1$

$$D_n(u) \leqslant C v r_n(u^*),$$

(iii) $\exists C\ \forall u: l_2 \to X\ \forall n \geqslant 1$

(a) $D_n(u) \leqslant C e_n(u)$,

(b) $D_n(u) \leqslant C e_n(u^*)$,

(iv) $\exists C\ \forall n \geqslant 1\ \forall u: l_2^n \to X$

$$\sup_k \sqrt{k}\, D_k(u) \leqslant C l(u).$$

Moreover, condition (i) *implies*

(v) $\exists C\ \forall u: l_2 \to X\ \forall n \geqslant 1$

$$D_{2n}(u) \leqslant C \left(\prod_{i=1}^n c_i(u) \right)^{1/n}.$$

Proof. As $D_n(u) = D_n(u^*)$, the implication (i) $\Rightarrow$ (ii) follows from (2.4). The implications (ii) $\Rightarrow$ (iii(a)) $\Rightarrow$ (iv) $\Rightarrow$ (iii(b)) $\Rightarrow$ (iv) have the same proof as similar implications in Theorem 2.3. (iv) $\Rightarrow$ (i) is an immediate consequence of Lemma 2.2.

(i) $\Rightarrow$ (v). Let $u: l_2 \to X$ and $n \geqslant 1$. Let $\{x_j\}_1^{2n} \subset B_{l_2}$ and $\{x_i^*\}_1^{2n} \subset B_{X^*}$. Let $H = \text{span}(x_1, ..., x_{2n}) \subset l_2$ and $Z = u(H) \subset X$ and set $u_0 = u|_H: H \to Z$. Define $R: Z \to l_2^{2n}$ by $Rx = \sum (x_i^*, x)\, e_i$ for $x \in Z$, where $\{e_i\}$ is the unit vector basis in l_2^{2n}. Define $U: H \to l_2^{2n}$ by $U = Ru_0$.

Recall that $c_{2k-1}(ST) \leqslant c_k(S)\, c_k(T)$, for arbitrary operators S and T [Pie, 11.9.2]. Thus

$$|\det[(x_i^*, ux_j)]|^{1/2n} = |\det U|^{1/2n} = \left(\prod_{i=1}^{2n} c_i(U)\right)^{1/2n}$$
$$\leqslant \left(\prod_{i=1}^{n} c_i(u_0) \prod_{j=1}^{n} c_j(R)\right)^{1/n}.$$

Similarly as in Theorem 2.3 we have $\pi_2(R) \leqslant \sqrt{n}$. So, by Proposition 2.1,

$$c_j(R) \leqslant a'(n/j)^{3/2} C, \qquad \text{for } j = 1, ..., 2n,$$

where $C \leqslant a'(wC_2(X))^2 \log(1 + wC_2(X))$ and $a' \geqslant 1$ is a universal constant. Thus $(\prod_{j=1}^n c_j(R))^{1/n} \leqslant a''C$ and this clearly concludes the proof of (v). ∎

Combining condition (v) of the last theorem with Lemma 2.2 we see that if X is of weak cotype 2 then for $u: l_2 \to X$, $D_n(u)$ admits both upper and lower estimates in terms of products of $c_i(u)$. More precisely,

$$\left(\prod_{j=1}^{2n} c_j(u)\right)^{1/2n} \leqslant D_{2n}(u) \leqslant C \left(\prod_{i=1}^{n} c_i(u)\right)^{1/n}, \tag{2.8}$$

where $C \leqslant a(wC_2(X))^2 \log(1 + wC_2(X))$ and $a \geqslant 1$ is a universal constant.

Remark. Some related results can be found in [Pa, P2]. In particular, in the latter paper it is shown that for a Banach space X the condition $\sup_n D_n(id_X) < \infty$ is equivalent to the condition X and X^* are of weak cotype 2 and X is K-convex.

3. More on Entropy Numbers versus Gelfand Numbers

In this section we study s-numbers of operators from X into Y, where X^* and Y are of weak cotype 2 and either X or Y is K-convex. Due to properties of X and Y strong inequalities between individual entropy numbers and Gelfand and Kolmogorov numbers can be proved. This shows that some of the results of Section 2 remain valid in a more general context, when a Hilbert space is replaced by an arbitrary Banach space whose dual is of weak cotype 2.

On the other hand, the theorem which follows is a generalization of a result from [G-K-S], where a similar result was proved if X and Y^* were of type 2.

THEOREM 3.1. *Let X and Y be Banach spaces. Let X^* and Y be of weak cotype 2 and let X or Y be K-convex. Let $u: X \to Y$ be an operator. Then, for $n = 1, 2, \ldots$,*

$$\max(d_n(u), c_n(u)) \leqslant A \min(e_n(u), e_n(u^*)), \tag{3.1}$$

where A depends only on the constants $wC_2(X^*)$, $wC_2(Y)$, and $\min(K(X), K(Y))$.

Proof. Assume first that X is K-convex. We shall use a similar argument as in Theorem 2.3. Let $\eta > 0$ and pick, by Lemma 2.2, $x_1, \ldots, x_n$ in B_X and $y_1^*, \ldots, y_n^*$ in B_{Y^*} such that

$$\max(d_n(u), c_n(u)) \leqslant (1+\eta)(\det[(y_i^*, ux_j)])^{1/n}. \tag{3.2}$$

Set $X_0 = \operatorname{span}(x_1, \ldots, x_n)$ and $Y_0 = u(X_0)$. Similarly as in Theorem 2.3 define operators $T: l_2^n \to X_0$ and $S: Y_0 \to l_2^n$, by $Te_j = x_j$ for $j = 1, \ldots, n$ and $Sy = \sum_{j=1}^n (y_j^*, y)\, e_j$, for $y \in Y_0$. Denoting by $u_0: X_0 \to Y_0$ the restriction of u we get

$$\begin{aligned}(\det[(y_i^*, ux_j)])^{1/n} &= \det(Su_0T) = \operatorname{Vol}(Su_0T(B_2^n))/\operatorname{Vol}(B_2^n)\\ &\leqslant 2^3 e_{3n}(Su_0T) \leqslant 2^3 e_n(S)\, e_n(u_0)\, e_n(T).\end{aligned} \tag{3.3}$$

By Proposition 2.1(ii) we have $e_n(S) \leqslant a(wC_2(Y))^2 \log(1 + wC_2(Y))$ and by Proposition 2.1(iii), $e_n(T) \leqslant a(wC_2(X_0^*))^2 \log(1 + wC_2(X_0^*))$, where a is a universal constant. It can be shown that $wC_2(X_0^*) \leqslant wT_2(X_0) \leqslant wT_2(X) \leqslant K(X)\, wC_2(X^*)$ (here $wT_2(\cdot)$ denotes the weak type 2 constant (cf. [M-P1, P2])). Combining (3.2) and (3.3) we get $\max(d_n(u), c_n(u)) \leqslant Ae_n(u)$. The estimate in terms of $e_n(u^*)$ can be proved analogously, concluding the proof of (3.1).

The case when Y is K-convex follows from (3.1) by duality, using the principle of local reflexivity and the fact that $c_k(u) = d_k(u^*)$. ∎

Remark. It is interesting to observe that for (3.1) to be valid, some strong assumptions on X and Y are in fact needed. For instance, it can be easily checked from (1.15) and (1.16) that for $1 < p < 2$, the identity operator $i_{p2}: l_p^n \to l_2^n$ satisfies, for $k \sim n^\alpha$, with $0 < \alpha < 1$,

$$d_k(i_{p2})/e_k(i_{p2}) \geqslant c_k(i_{p2})/e_k(i_{p2}) \to \infty,$$

as $n \to \infty$. Also, from Section 2 it follows that if Y is a Hilbert space and (3.1) is valid for all operators $u: X \to Y$ then X^* is of weak cotype 2. A similar situation occurs when X is a Hilbert space.

Now we would like to address briefly the so-called problem of duality of entropy numbers. This problem consists of, roughly speaking, comparison of the asymptotic behaviour of sequences $\{e_n(u)\}$ and $\{e_n(u^*)\}$, for a compact operator u. For some recent partial solutions and related results, the reader can consult [T1] and references therein.

The following theorem is due to König and Milman [K-M] (see also [P3, Chap. 8]).

THEOREM 3.2. *Let X and Y be Banach spaces. Let $u: X \to Y$ be an operator of rank n, then*

$$e_{[an]}(u) \leqslant 2e_n(u^*) \qquad \text{and} \qquad e_{[an]}(u^*) \leqslant 2e_n(u), \tag{3.4}$$

where $a > 1$ is a universal constant.

Combining the general estimate (3.4) with Theorem 3.1 we get

THEOREM 3.3. *Let X and Y be Banach spaces. Let X^* and Y be of weak cotype 2 and let X or Y be K-convex. Let $u: X \to Y$ be an operator. Then, for every n, one has*

$$B^{-1} e_{[an]}(u^*) \leqslant e_n(u) \leqslant B e_{[a^{-1}n]}(u^*),$$

where $a \geqslant 1$ is a universal constant and B depends only on the constants $wC_2(X^*)$, $wC_2(Y)$, and $\min(K(X), K(Y))$.

Proof. Let $a \geqslant 1$ be a constant from Theorem 3.2. Fix n and set $k = [a^{-1}n]$. Let $i: Y \to l_\infty$ be an isometric embedding and put $v = iu: X \to l_\infty$. Pick $E \subset X$ with codim $E < k$ such that $\|u|_E\| \leqslant 2c_k(u)$. Let $w: X \to l_\infty$ be an extension of $iu|_E$ with $\|w\| = \|iu|_E\| \leqslant 2c_k(u)$.

By subadditivity of entropy numbers, we have

$$\begin{aligned} e_n(u) &\leqslant 2e_n(v) \leqslant 2(\|w\| + e_n(v-w)) \\ &\leqslant 2c_k(u) + 2e_n(v-w). \end{aligned}$$

By Theorem 3.1, $c_k(u) \leqslant Ae_k(u^*)$. Since $\operatorname{rank}(v-w) < k$, by Theorem 3.2,

$$\begin{aligned} e_n(v-w) &\leqslant 2e_k((v-w)^*) \leqslant 2(e_k(v^*) + \|w^*\|) \\ &\leqslant 2(e_k(v^*) + 2c_k(u)) \\ &\leqslant 2(e_k(u^*) + 2Ae_k(u^*)). \end{aligned}$$

Thus

$$e_n(u) \leqslant (4+10A)\, e_k(u^*) = (4+10A)\, e_{[a^{-1}n]}(u^*).$$

The lower estimate is obtained in the same way, and we omit the details. ▮

Remarks. 1. Theorem 3.3 is a strengthening of the duality result from [G-K-S, C2] and the main idea of the proof is the same.

2. In comparison with, e.g., [T1], it should be noted that Theorem 3.3 does not require any assumptions on a "regular" behaviour of sequences $\{e_n(u)\}$ and $\{e_n(u^*)\}$.

References

[B-M] J. Bourgain and V. D. Milman, New volume ratio properties for symmetric convex bodies in $\mathbb{R}^n$, *Invent. Math.* **88** (1987), 319–340 (cf. also Sectiones euclidiennes et volume des corps convexes symétriques dans $\mathbb{R}^n$, *C.R. Acad. Sci. Paris* **300** (1985), 435–438).

[B-Z] Y. D. Burago and V. A. Zalgaler, "Geometric Inequalities," Nauka, Leningrad, 1980 (Russian) and Springer-Verlag, New York/Berlin, 1987.

[C1] B. Carl, Entropy numbers, s-numbers and eigenvalue problems, *J. Funct. Anal.* **41** (1981), 290–306.

[C2] B. Carl, Inequalities of Bernstein–Jackson type and the degree of compactness of operators in Banach spaces, *Ann. Inst. Fourier* **35**, No. 3 (1985), 79–118.

[D] R. M. Dudley, The size of compact subsets of Hilbert space and continuity of Gaussian processes, *J. Funct. Anal.* **1** (1967), 290–330.

[F-L-M] T. Figiel, J. Lindenstrauss, and V. D. Milman, The dimension of almost spherical sections of convex bodies, *Acta Math.* **139** (1977), 53–94.

[F-T] T. Figiel and N. Tomczak-Jaegermann, Projections onto Hilbertian subspaces of Banach spaces, *Israel J. Math.* **33** (1979), 155–171.

[G-G] A. Yu. Garnaev and E. D. Gluskin, On widths of the euclidean ball, *Soviet Math. Dokl.* **30** (1984), 200–204.

[G1] E. D. Gluskin, On some finite-dimensional problems in width theory, *Vestnik Leningrad. Univ.* **13** (1981), 5–10.

[G2] E. D. Gluskin, Norms of random matrices and widths of finite-dimensional sets, *Math. USSR-Sb.* **48** (1984), 173–182.

[G-K-S] Y. Gordon, H. König, and C. Schütt, Geometric and probabilistic estimates for entropy and approximation numbers of operators, *J. Approx. Theory.*

[K-M] H. König and V. D. Milman, Covering numbers for a body and its polar, *in* "Notes of GAFA Seminar, Tel-Aviv, 1985/86," Lecture Notes in Math., Vol. 1267, pp. 82–96, Springer-Verlag, New York/Berlin, 1987.

[L-T] J. Lindenstrauss and L. Tzafriri, "Classical Banach Spaces," Springer-Verlag, New York/Berlin, Vol. I, 1977, Vol. II, 1979.

[M1] V. D. Milman, Almost Euclidean quotient spaces of subspaces of finite dimensional normed spaces, *Proc. Amer. Math. Soc.* **94** (1985), 445–449.

[M2] V. D. Milman, Random subspaces of proportional dimension of finite dimensional normed spaces. Approach through the isoperimetric inequality, *in* "Proceeding, Missouri 1984" (N. Kalton and E. Saab, Eds.), pp. 106–115, Springer Lecture Notes, Vol. 1166, Springer-Verlag, New York/Berlin, 1985.

[M-P1] V. D. MILMAN AND G. PISIER, Banach spaces with a weak cotype 2 property, *Israel J. Math.* **54** (1986), 139–158.

[M-P2] V. D. MILMAN AND G. PISIER, Gaussian processes and mixed volumes, *Ann. Probab.* **15** (1987), 292–304.

[M-Sch] V. D. MILMAN AND G. SCHECHTMAN, "Asymptotic theory of finite dimensional normed spaces," Lecture Notes in Math. Vol. 1200, Springer-Verlag, New York/Berlin, 1986.

[Me-Pa] M. MEYER AND A. PAJOR, Sections of the unit ball of l_p^n, *J. Funct. Anal.*, in press.

[Pa] A. PAJOR, Quotient volumique et espace de Banach de type 2 faible, *Israel J. Math.* **57** (1987), 101–106.

[P-T1] A. PAJOR AND N. TOMCZAK-JAEGERMANN, Subspaces of small codimension of finite-dimensional Banach spaces, *Proc. Amer. Math. Soc.* **97** (1986), 637–642.

[P-T2] A. PAJOR AND N. TOMCZAK-JAEGERMANN, Remarques sur les nombres d'entropie d'un operateur et son transposé, *C. R. Acad. Sci. Paris* **301** (1985), 743–746.

[P-T3] A. PAJOR AND N. TOMCZAK-JAEGERMANN, Gelfand numbers and Euclidean sections of large dimensions, *in* "Proceedings, VI Int. Conf. in Probab. in Banach Spaces, Sandbjerg, Denmark, 1986," Lecture Notes in Math., Springer-Verlag, New York/Berlin.

[Pie] A. PIETSCH, "Operator Ideals," VEB Deut. Verlag, Berlin, North-Holland, Amsterdam, 1980.

[P1] G. PISIER, Holomorphic semigroups and geometry of Banach spaces, *Ann. of Math.* **115** (1982), 375–392.

[P2] G. PISIER, Weak Hilbert spaces, *Proc. London Math. Soc.*, in press.

[P3] G. PISIER, "Volume of convex bodies and geometry of Banach spaces," Cambridge Univ. Press.

[S] L. A. SANTALÓ, Un invariante afin para los cuerpos convexos del espacio de n-dimensiones, *Portugal. Math.* **8** (1949), 155–161 (cf. also Integral geometry and geometric probability, "Encyclopedia of Mathematics and its Applications," Addison–Wesley, Reading, MA, 1976).

[Sch] C. SCHÜTT, Entropy numbers of diagonal operators between symmetric spaces, *J. Approx. Theory* **40** (1983), 121–128.

[Su] V. N. SUDAKOV, Gaussian random processes and measures of solid angles in Hilbert space, *Soviet Math. Dokl.* **12** (1971), 412–415.

[S-T] S. J. SZAREK AND N. TOMCZAK-JAEGERMANN, On nearly Euclidean decompositions for some classes of Banach spaces, *Comp. Math.* **40** (1980), 367–385.

[T1] N. TOMCZAK-JAEGERMANN, Dualité des nombres d'entropie pour des opérateurs à valeurs dans un espace de Hilbert, *C. R. Acad. Sci. Paris Sér. I Math.* **305** (1987), 299–301.

[T2] N. TOMCZAK-JAEGERMANN, "Banach–Mazur distances and finite-dimensional operator ideals," Longman Scientific & Technical, Harlow, 1989.

[V] J. D. VAALER, A geometric inequality with applications to linear forms, *Pacific J. Math.* **83** (1979), 543–553.

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