Convex Bodies with Few Faces

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Abstract. It is proved that if $u_1, \ldots, u_n$ are vectors in $\mathbf{R}^k, k \le n, 1 \le p < \infty$ and

$$r = \left(\frac{1}{k} \sum_1^n |u_i|^p \right)^{\frac{1}{p}}$$

then the volume of the symmetric convex body whose boundary functionals are $\pm u_1, \ldots, \pm u_n$, is bounded from below as

$$|\{x \in \mathbf{R}^k \colon\ |\langle x, u_i \rangle| \le 1 \text{ for every } i\}|^{\frac{1}{k}} \ge \frac{1}{\sqrt{\rho} r}.$$

An application to number theory is stated.

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§0. Introduction.

In [V], Vaaler proved that if $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ is the central unit cube in $\mathbf{R}^n$ and U is a subspace of $\mathbf{R}^n$ then the volume $|U \cap Q_n|$, of the section of Q_n by U is at least 1. This result may be reformulated as follows: if $u_1, \ldots, u_n$ are vectors in $\mathbf{R}^k, 1 \le k \le n$ whose Euclidean lengths satisfy $\sum_1^n |u_i|^2 \le k$ then

$$|\{x \in \mathbf{R}^k\colon\ |\langle x, u_i\rangle| \le 1 \text{ for every } i\}|^{\frac{1}{k}} \ge 2.$$

A related theorem, (Theorem 1, below) in which the condition $\sum |u_i|^2 \le k$ is replaced by $\max_i |u_i| \le 1$ was proved by Carl and Pajor [C-P] and Gluskin [G]. Gluskin's methods enable him to obtain sharp results in limiting cases which in turn have applications in harmonic analysis. Results closely related to Theorem 1 were also obtained by Bárány and Füredi [B-F] and Bourgain, Lindenstrauss and Milman [B-L-M].

Theorem 1. There is a constant $\delta > 0$ so that if $u_1, \ldots, u_n \in \mathbf{R}^k, 1 \le k \le n$ are vectors of length at most 1 then

$$|\{x \in \mathbf{R}^k\colon\ |\langle x, u_i\rangle| \le 1 \text{ for every } i\}|^{\frac{1}{k}} \ge \frac{\delta}{\sqrt{1 + \log \frac{n}{k}}}.$$

The estimate is best possible if n is at most exponential in k, apart from the value of the constant δ. This is demonstrated by an example which had appeared some time earlier in a paper of Figiel and Johnson, [F-J]. Theorem 1 gives a lower bound on the volume ratios of the unit balls of k-dimensional subspaces of ℓ_∞^n and hence on the distance of these subspaces from Euclidean space.

Regarding Theorem 1 as a "$p = \infty$" version of Vaaler's "$p = 2$" result, Kashin asked whether a similar result holds for $2 < p < \infty$. This question is answered in the affirmative by the following theorem.

Theorem 2. Suppose $u_1, \ldots, u_n \in \mathbf{R}^k$ with $k \le n, 1 \le p < \infty$ and let $r = (\frac{1}{k}\sum_1^n |u_i|^p)^{\frac{1}{p}}$. Then

$$|\{x \in \mathbf{R}^k\colon\ |\langle x, u_i\rangle| \le 1 \text{ for every } i\}|^{\frac{1}{k}} \ge \begin{cases} \frac{2\sqrt{2}}{\sqrt{p}r} & \text{if} \quad p \ge 2 \\ \frac{1}{r} & \text{if} \quad 1 \le p \le 2. \end{cases}$$

The lower bound is best possible (up to a constant) provided $e^p k \le n \le e^k$.

Remark. The slightly stronger result for $p \ge 2$ is isolated since for $p = 2$ it gives back exactly Vaaler's result.

Theorem 1 follows immediately from Theorem 2 by a standard optimisation argument. If $(u_i)_1^n$ in $\mathbf{R}^k$ all have norm at most 1 then for any $p \in [1, \infty)$,

$$\left(\frac{1}{k}\sum_1^n |u_i|^p\right)^{\frac{1}{p}} \le \left(\frac{n}{k}\right)^{\frac{1}{p}}$$

so that

$$|\{x\colon\ |\langle x, u_i\rangle| \le 1 \text{ for every } i\}|^{\frac{1}{k}} \ge \frac{2\sqrt{2}}{\sqrt{p}(\frac{n}{k})^{\frac{1}{p}}}$$

(for $p \ge 2$) and the latter is at least $\frac{2}{\sqrt{e}\sqrt{1+\log\frac{n}{k}}}$ when $p = 2(1 + \log\frac{n}{k})$.

With the careful use of well-known methods for estimating the entropy of convex bodies it is possible to obtain more general (but less precise) estimates than that provided by Theorem 2; (see [B-P]). The purpose of this paper is to provide a very short proof of Theorem 2 and, a fortiori, Theorem 1.

Vaaler originally proved his theorem because of its applications to the geometry of numbers. The last section of this paper includes a statement of the generalisation of Siegel's lemma which follows from Theorem 2.

§1. The lower bound.

The proof of Theorem 2 makes use of the following result from [Me-P] which was designed to extend Vaaler's theorem in a different direction: it estimates the volumes of sections of the unit balls of the spaces $\ell_p^n, 1 \le p \le \infty$. For $1 \le p \le \infty, n \in \mathbf{N}$ let

$$B_p^n = \left\{ x \in \mathbf{R}^n \colon \sum_1^n |x_i|^p \le 1 \right\}$$

be the unit ball of ℓ_p^n.

Theorem 3. Let U be a k-dimensional subspace of $\mathbf{R}^n$; if $1 \le p \le q \le \infty$ then

$$\frac{|B_p^n \cap U|}{|B_p^k|} \le \frac{|B_q^n \cap U|}{|B_q^k|}. \qquad \square$$

Remark. The case $p = 2, q = \infty$ is Vaaler's theorem since then, the left side is 1 and the inequality states that

$$|B_\infty^n \cap U| \ge |B_\infty^k| \ge 2^k.$$

For notational convenience, the proof of Theorem 2 is divided into several short lemmas. The first is no more than a convenient form of Hölder's inequality. For $k \in \mathbf{N}, S^{k-1}$ will denote the Euclidean sphere in $\mathbf{R}^k$ and $\sigma = \sigma_{k-1}$, the rotationally invariant probability measure on S^{k-1}. Also let v_k be the volume of the Euclidean unit ball in $\mathbf{R}^k$.

Lemma 4. Let C and B be symmetric convex bodies in $\mathbf{R}^k$ with Minkowski gauges $\|\cdot\|_C$ and $\|\cdot\|_B$ respectively. Then for $p > 0$

$$\left(\frac{|C|}{|B|}\right)^{\frac{1}{k}} \ge \left(\frac{k+p}{k|B|}\int_B \|x\|_C^p dx\right)^{-\frac{1}{p}}.$$

Proof.

$$\begin{aligned}\left(\frac{|C|}{|B|}\right)^{\frac{1}{k}} &= \left(\frac{v_k}{|B|}\int_{S^{k-1}} \|\theta\|_C^{-k} d\sigma(\theta)\right)^{\frac{1}{k}} \\ &= \left(\frac{kv_k}{|B|}\int_{S^{k-1}} \left(\frac{\|\theta\|_B}{\|\theta\|_C}\right)^k \int_0^{\|\theta\|_B^{-1}} r^{k-1} dr d\sigma(\theta)\right)^{\frac{1}{k}}\end{aligned}$$

$$= \left(\frac{1}{|B|}\int_B \left(\frac{\|x\|_B}{\|x\|_C}\right)^k dx\right)^{\frac{1}{k}}$$

$$\geq \left(\frac{1}{|B|}\int_B \left(\frac{\|x\|_B}{\|x\|_C}\right)^{-p} dx\right)^{-\frac{1}{p}}$$

$$= \left(\frac{k+p}{k|B|}\int_B \|x\|_C^p dx\right)^{-\frac{1}{p}}. \qquad \square$$

(Lemma 4 appears in [Mi-P] as Corollary 2.2.)

Lemma 5. Suppose $u_1, \ldots, u_n \in \mathbf{R}^k$ with $k \leq n$ and $1 \leq p < \infty$. Then

$$|\{x \in \mathbf{R}^k\colon\ |\langle x_i, u_i\rangle| \leq 1 \text{ for every } i\}|^{\frac{1}{k}}$$
$$\geq 2\left(\frac{k+p}{k}\sum_{i=1}^n \frac{1}{|B_p^k|}\int_{B_p^k} |\langle x, u_i\rangle|^p dx\right)^{-\frac{1}{p}}.$$

Proof. Define $T\colon\ \mathbf{R}^k \to \mathbf{R}^n$ by $(Tx)_i = \langle x, u_i\rangle, 1 \leq i \leq n$ and let $U = T(\mathbf{R}^k)$. The problem is to estimate from below

$$|T^{-1}(B_\infty^n)|^{\frac{1}{k}} = |T^{-1}(U \cap B_\infty^n)|^{\frac{1}{k}}.$$

By Theorem 3,

$$|U \cap B_\infty^n|^{\frac{1}{k}} \geq 2\left(\frac{|U \cap B_p^n|}{|B_p^k|}\right)^{\frac{1}{k}}$$

and so

$$|T^{-1}(B_\infty^n)|^{\frac{1}{k}} \geq 2\left(\frac{|T^{-1}(B_p^n)|}{|B_p^k|}\right)^{\frac{1}{k}}.$$

Regard T as an operator: $\ell_p^k \to \ell_p^n$. Then by Lemma 4,

$$2\left(\frac{|T^{-1}(B_p^n)|}{|B_p^k|}\right)^{\frac{1}{k}} \geq 2\left(\frac{k+p}{k|B_p^k|}\int_{B_p^k} \|Tx\|^p dx\right)^{-\frac{1}{p}}$$
$$= 2\left(\frac{k+p}{k|B_p^k|}\int_{B_p^k} \sum_1^n |\langle x, u_i\rangle|^p dx\right)^{-\frac{1}{p}}. \qquad \square$$

Proof of Theorem 2. Let $(u_i)_1^n$ and p be as above. For each i let v_i be the unit vector in the direction of u_i. By Lemma 5,

$$
\begin{aligned}
&|\{x \in \mathbf{R}^k\colon\ |\langle x, u_i\rangle| \le 1 \text{ for every } i\}|^{\frac{1}{k}} \\
&\ge 2\bigg(\frac{k+p}{k}\sum_{i=1}^{n}\frac{1}{|B_p^k|}\int_{B_p^k}|\langle x, u_i\rangle|^p dx\bigg)^{-\frac{1}{p}} \\
&= 2\bigg(\frac{k+p}{k}\sum_{1}^{n}|u_i|^p\cdot\frac{1}{|B_p^k|}\int_{B_p^k}|\langle x, v_i\rangle|^p dx\bigg)^{-\frac{1}{p}} \\
&\ge 2\bigg(\frac{1}{k}\sum_{i=1}^{n}|u_i|^p\bigg)^{-\frac{1}{p}}\ \min\bigg(\frac{k+p}{|B_p^k|}\int_{B_p^k}|\langle x, v\rangle|^p dx\bigg)^{-\frac{1}{p}}
\end{aligned}
$$

where the minimum is taken over all vectors v of Euclidean length 1. So to complete the proof it suffices to show that for such a vector v,

$$
\bigg(\frac{k+p}{|B_p^k|}\int_{B_p^k}|\langle x, v\rangle|^p dx\bigg)^{\frac{1}{p}} \le \begin{cases} \sqrt{\frac{p}{2}} & \text{if} \quad p \ge 2 \\ 2 & \text{if} \quad 1 \le p < 2. \end{cases}
$$

Let $(x^{(j)})_1^k$ and $(v^{(j)})_1^k$ be the coordinates of the vectors x and v in $\mathbf{R}^k$. For $p \ge 2$, observe that the functions $(x^{(j)}v^{(j)})$ on B_p^k form a conditionally symmetric sequence, so by Khintchine's inequality and Hölder's inequality (for $\sum_1^n v^{(j)2} = 1$),

$$
\begin{aligned}
&\bigg(\frac{k+p}{|B_p^k|}\int_{B_p^k}\bigg|\sum_{1}^{k}x^{(j)}v^{(j)}\bigg|^p dx\bigg)^{\frac{1}{p}} \\
&\qquad \le \sqrt{\frac{p}{2}}\bigg(\frac{k+p}{|B_p^k|}\int_{B_p^k}\bigg(\sum_{1}^{k}x^{(j)2}v^{(j)2}\bigg)^{\frac{p}{2}} dx\bigg)^{\frac{1}{p}} \\
&\qquad \le \sqrt{\frac{p}{2}}\bigg(\frac{k+p}{|B_p^k|}\int_{B_p^k}\sum_{1}^{n}|x^{(j)}|^p\cdot v^{(j)2}dx\bigg)^{\frac{1}{p}} \\
&\qquad = \sqrt{\frac{p}{2}}\bigg(\frac{k+p}{|B_p^k|}\int_{B_p^k}|x^{(1)}|^p dx\bigg)^{\frac{1}{p}} \\
&\qquad = \sqrt{\frac{p}{2}}.
\end{aligned}
$$

For $1 \le p < 2$ it is easily checked that

$$\left(\frac{k+p}{|B_p^k|}\int_{B_p^k}|\langle x,v\rangle|^p dx\right)^{\frac{1}{p}}$$
$$\leq (k+p)^{\frac{1}{p}}\left(\frac{1}{|B_p^k|}\int_{B_p^k}\langle x,v\rangle^2 dx\right)^{\frac{1}{2}}$$
$$= (k+p)^{\frac{1}{p}}\left(\frac{1}{|B_p^k|}\int_{B_p^k}(x^{(1)})^2 dx\right)^{\frac{1}{2}}$$

and the last expression can be (rather roughly) estimated by 2 using standard inequalities involving logarithmically concave functions. □

Remark. The proof of Theorem 2 can be simplified even further if the integration over B_p^k is replaced by integration over S^{k-1} (and Hölder's inequality applied here). The proof was presented as above because Lemma 5 has some intrinsic interest: for example it may be used to recover Gluskin's precise estimate as follows. Suppose $m \in \mathbf{N}$ and the vectors $(z_i)_1^m \in \mathbf{R}^k$ satisfy

$$|z_i| \leq \left(\log\left(1+\frac{m}{k}\right)\right)^{-\frac{1}{2}}, \quad 1 \leq i \leq m.$$

For $\varepsilon > 0$, let $W(\varepsilon)$ be the set

$$\left\{x \in \mathbf{R}^k\colon \max_j |x^{(j)}| \leq 1, \max_i |\langle x, z_i\rangle| \leq \frac{1}{\varepsilon}\right\}:$$

that is, $W(\varepsilon)$ is the intersection of the cube B_∞^k with m "bands" of width at most $\frac{2}{\varepsilon}\sqrt{\log(1+\frac{m}{k})}$. Then $|W(\varepsilon)|^{\frac{1}{k}} \to 2$ as $\varepsilon \to 0$, uniformly in k and m. To see this, apply Lemma 5 with $n = k+m$, the first k, u_i's being the standard basis vectors of $\mathbf{R}^k$ and the remaining m being the vectors $(\varepsilon z_i)_1^m$. If e_j is a standard basis vector,

$$\frac{k+p}{|B_p^k|}\int_{B_p^k}|\langle x, e_j\rangle|^p dx = 1$$

and so Lemma 5 (and the proof of Theorem 2) show that for each $p \geq 2, |W(\varepsilon)|^{\frac{1}{k}} \geq 2(1+\frac{m}{k}(\frac{p}{2})^{\frac{p}{2}}\varepsilon^p(\log(1+\frac{m}{k}))^{-\frac{p}{2}})^{-\frac{1}{p}}$ and the latter is at least $\frac{2}{1+\sqrt{e}\cdot\varepsilon}$ if $p = \max(2, 2\log(1+\frac{m}{k}))$. □

As was briefly mentioned earlier, more general estimates than that of Theorem 2 are obtained in [B-P] (for entropy numbers instead of volumes). It is worth noting however that even the argument of Theorem 2 can be used to give the following: there is a constant c so that if $u_1, \ldots, u_n \in \mathbf{R}^k, k \leq n$ and $T\colon\ \ell_2^k \to \ell_\infty^n$ is given by $(Tx)_i = \langle u_i, x\rangle,\ 1 \leq i \leq n$, then the k^{th} entropy number of T satisfies

$$e_k(T) \leq \frac{c\sqrt{p}}{\sqrt{k}} \left(\frac{1}{k} \sum_1^n |u_i|^p \right)^{\frac{1}{p}} \left(1 + \log \frac{n}{k} \right)^{\frac{1}{p}}$$

and hence

$$e_k(T) \leq \frac{ec}{\sqrt{k}} \sqrt{1 + \log \frac{n}{k}} \cdot \|T\|$$

(taking $p = 2(1 + \log \frac{n}{k})$). To obtain this one uses Schütt's estimates, [5], for the entropy numbers of the formal identity from ℓ_p^n to ℓ_∞^n in place of the result of Meyer and Pajor, and the dual Sudakov inequality of Pajor and Tomczak, [P-T] in place of the application of Hölder's inequality.

§2. An application to linear forms.

As stated in the introduction, Vaaler's original result has applications to the geometry of numbers. One such, a sharpened form of Siegel's lemma, is given in [B-V]. Using the arguments of Bombieri and Vaaler and Theorem 2, one can obtain the generalisation of their result, contained in Theorem 6, below. Some notation is needed. If A is a $k \times n$ matrix of reals with independent rows ($1 \le k \le n$), denote by $v_j = v_j(A),\ 1 \le j \le k$, the rows of A. Let $(e_i)_1^n$ be the standard basis of $\mathbf{R}^n$ and denote by c_i, the distance (in the Euclidean norm) of e_i from the span of the v_j's in $\mathbf{R}^n$. (So if A_i is the matrix with $k+1$ rows, $v_1, \ldots, v_n, e_i$ then

$$c_i^2 = \frac{\det(A_i A_i^*)}{\det(AA^*)} \quad \text{for} \quad 1 \le i \le n.)$$

Theorem 6. Let A be a $k \times n$ matrix with rank k and integral entries. With the notation above, the system $Ax = 0$ admits $n-k$ linearly independent solutions

$$z^{(r)} = (z_1^{(r)}, \ldots, z_n^{(r)}) \in \mathbf{Z}^n,\ 1 \le r \le n-k$$

so that for every $p \ge 2$,

$$\prod_{1 \le r \le n-k} \max_i |z_i^{(r)}| \le D^{-1} \sqrt{\frac{p}{2}} \left(\frac{1}{n-k} \sum_1^n c_i^p \right)^{\frac{1}{p}} \sqrt{\det AA^*}$$

where D denotes the G-C-D of all $k \times k$ determinants extracted from A. □

Remark. The principal importance of such a generalisation of Bombieri and Vaaler's result is that it takes into account, more strongly, the form of the matrix A. If the c_i's are all about the same size, then for $p > 2$, the expression

$$\left(\frac{1}{n-k} \sum c_i^p \right)^{\frac{1}{p}}$$

is small compared with the corresponding expression in which p is replaced by 2.

References.

[B-F] I. Bárány and Z. Füredi, Computing the volume is difficult, Discrete Comput. Geom. 2 (1987), 319-326.

[B-L-M] J. Bourgain, J. Lindenstrauss and V.D. Milman, Approximation of zonoids by zonotopes, Acta Math. 162 (1989), 73-141.

[B-P] K.M. Ball and A. Pajor, On the entropy of convex bodies with "few" extreme points, in preparation.

[B-V] E. Bombieri and J. Vaaler, On Siegel's lemma, Invent. Math. 73 (1983), 11-32.

[C-P] B. Carl and A. Pajor, Gelfand numbers of operators with values in a Hilbert space, Invent. Math. 94 (1988), 479-504.

[F-J] T. Figiel and W.B. Johnson, Large subspaces of ℓ_∞^n and estimates of the Gordon-Lewis constants, Israel J. Math. 37 (1980), 92-112.

[G] E.D. Gluskin, Extremal properties of rectangular parallelipipeds and their applications to the geometry of Banach spaces, Math. Sbornik, 136 (178) (1988), 85-95.

[Me-P] M. Meyer and A. Pajor, Sections of the unit ball of ℓ_p^n, J. Funct. Anal. 80 (1988), 109-123.

[Mi-P] V.D. Milman and A. Pajor, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. Israel seminar on G.A.F.A. (1987-88), Springer-Verlag, Lecture notes in Math. #1376 (1989).

[P-T] A. Pajor and N. Tomczak-Jaegermann, Subspaces of small codimension of finite-dimensional Banach spaces, Proc. A.M.S. 97 (1986), 637-642.

[S] C. Schütt, Entropy numbers of diagonal operators between symmetric Banach spaces, Journal Approx. Th. 40 (1984), 121-128.

[V] J.D. Vaaler, A geometric inequality with applications to linear forms, Pacific J. Math. 83 (1979), 543-553.