Gelfand numbers and metric entropy of convex hulls in Hilbert spaces

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Abstract

We establish optimal estimates of Gelfand numbers or Gelfand widths of absolutely convex hulls $\mathrm{cov}(K)$ of precompact subsets $K \subset H$ of a Hilbert space H by the metric entropy of the set K where the covering numbers $N(K, \varepsilon)$ of K by ε-balls of H satisfy the Lorentz condition

$$\int_0^\infty \big(\log_2 N(K,\varepsilon)\big)^{r/s}\, d\varepsilon^s < \infty$$

for some fixed $0 < r, s \leq \infty$ with the usual modifications in the cases $r = \infty, 0 < s < \infty$ and $0 < r < \infty, s = \infty$. Moreover, we obtain optimal estimates of Gelfand numbers of absolutely convex hulls if the metric entropy satisfies the entropy condition

$$\sup_{\varepsilon>0} \varepsilon \big(\log_2 N(K,\varepsilon)\big)^{1/r} \big(\log_2(2 + \log_2 N(K,\varepsilon))\big)^{\beta} < \infty$$

fore some fixed $0 < r < \infty, -\infty < \beta < \infty$. Using inequalities between Gelfand and entropy numbers we also get optimal estimates of the metric entropy of the absolutely convex hull $\mathrm{cov}(K)$.

As an interesting feature of the estimates, a sudden jump of the asymptotic behavior of Gelfand numbers as well as of the metric entropy of absolutely convex hulls occurs for fixed s if the parameter r crosses the point $r = 2$ and, if $r = 2$ is fixed, if the parameter β crosses the point $\beta = 1$.

The results established in Hilbert spaces extend and recover corresponding results of several authors. The proofs are based on two inequalities already discovered in [CKP99].

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1 Introduction and results

Let (M,d) be a metric space and let $B(x_o,\varepsilon) := \{x \in M \,:\, d(x_o,x) \le \varepsilon\}$ be the closed ball with radius ε and center x_o. For a bounded set $K \subset M$, let

$$N(K,\varepsilon) := \min\left\{N \in \mathbb{N} \,:\, \exists x_1,\dots,x_N \in M \text{ with } K \subset \bigcup_{k=1}^{N} B(x_k,\varepsilon)\right\}$$

be the covering number of K by ε-balls. We denote the metric entropy of K by $\log_2 N(K,\varepsilon)$. Moreover, for $n \in \mathbb{N}$,

$$\varepsilon_n(K) := \inf\{\varepsilon \ge 0 \,:\, N(K,\varepsilon) \le n\}$$

are the entropy numbers of K and

$$e_n(K) := \varepsilon_{2^{n-1}}(K)$$

are the dyadic entropy numbers of K. The closed unit ball of a Banach space X is denoted by $B(X)$. The Banach space of all summable families $(\xi_t)_{t\in K}$ of real numbers over the index set K with the norm

$$\|(\xi_t)\| = \sum_{t\in K} |\xi_t|$$

is denoted by $\ell_1(K)$.

The entropy numbers $e_n(\mathrm{cov}(K))$ of the absolutely convex hull $\mathrm{cov}(K)$ of a bounded subset $K \subset X$ of a Banach space X can be expressed in terms of the entropy numbers of operators. For a compact operator $S : X \to Y$ from a Banach space X into a Banach space Y, the entropy numbers $e_n(S)$ are defined by

$$e_n(S) := e_n(S(B(X))).$$

Then

$$e_n(\mathrm{cov}(K)) = e_n(T) = e_n(T(B(\ell_1(K)))$$

where $T : \ell_1(K) \to X$ is the operator defined on the canonical basis $(e_t)_{t\in K}$ of $\ell_1(K)$ by $Te_t := t$.

The study of convex hulls of precompact sets K in Hilbert and Banach spaces is universal insofar as the entropy numbers $e_n(S)$ of a compact operator $S : X \to Y$ between Banach spaces X and Y are always shared by the entropy numbers of a compact operator $T : \ell_1(K) \to Y$ defined on a $\ell_1(K)$ space in the sense that

$$e_n(S) = e_n(T).$$

This fact indicates why we study the asymptotic behavior of entropy numbers of absolutely convex hulls of precompact subsets.

We show in this paper how the rate of decay of dyadic entropy numbers $e_n(K)$ of a precompact set $K \subset H$ of a Hilbert space H expressed in terms of Lorentz spaces reflects the rate of decay of the dyadic entropy numbers $e_n(\mathrm{cov}(K))$ of the absolutely convex hull $\mathrm{cov}(K)$ of K. It will be convenient to express the arguments in terms of entropy numbers. We also give reformulations in terms of integral conditions of the metric entropy $\log N(\mathrm{cov}(K), \varepsilon)$ of $\mathrm{cov}(K)$.

For our purpose we need the notion of Lorentz sequence spaces. A null sequence $x = (\xi_k)$ is said to belong to the Lorentz sequence space $\ell_{r,s}$ if the non-increasing rearrangement $(a_k(x))$ of its absolute values $|\xi_k|$ satisfies

$$\sum_{k=1}^{\infty} \left(k^{1/r-1/s} a_k(x)\right)^s < \infty$$

if $0 < r \leq \infty$ and $0 < s < \infty$ and

$$\sup_{1 \leq k < \infty} k^{1/r} a_k(x) < \infty$$

if $0 < r < \infty$ and $s = \infty$.

The problem of investigating the metric entropy of convex hulls in Hilbert and Banach spaces is nowadays of particular interest in different branches of analysis and probability. We give sharp estimates for the metric entropy of precompact subsets of Hilbert spaces. In a forthcoming paper we will treat diverse consequences of the statements in the present paper yielding widespread applications. For example, we get there precise estimates for the metric entropy of diverse integral operators and new insight into the metric entropy of weakly singular integral operators from a Hilbert space into the space of continuous functions. The entropy of convex hulls has useful applications in analysis, approximation theory, geometry, as well as in probability (cf. the treatises [CS90, ET96, LT91, LL99, P89]).

Related with eigenvalue distributions in Banach spaces, the metric entropy of convex hulls in the so-called diagonal case was already studied in [C82]. The main result given there can be reformulated as follows. If X is a Banach space of type p, $1 < p \leq 2$ (cf. [P89] for the definition) and if $K = \{x_1, x_2, \ldots\} \subset X$ is a sequence such that $(\|x_n\|) \in \ell_{r,s}$ for some $0 < r < \infty, 0 < s < \infty$ then the sequence of dyadic entropy numbers $(e_n(\mathrm{cov}(K))$ of the absolutely convex hull of K belongs to the Lorentz sequence space $\ell_{q,s}$ where q is given by $\frac{1}{q} = \frac{1}{r} + 1 - \frac{1}{p}$. The result is optimal and can be reformulated in terms of the metric entropy by the integral condition

$$\int_0^\infty \big(\log_2 N(\mathrm{cov}(K),\varepsilon)\big)^{s/q}\, d\varepsilon^s < \infty.$$

More generally, it has turned out that in the non-diagonal case the result remains true for precompact subsets $K \subset X$ of a type p space X where the assumption now is that the entropy numbers $\varepsilon_n(K)$ of K belong to the Lorentz sequence space $\ell_{r,s}$ which is equivalent to the integral condition

$$\int_0^{e_1(K)} N(K,\varepsilon)^{s/r}\, d\varepsilon^s < \infty.$$

This was proved in the case $s = \infty$ in [CKP99] and, for arbitrary $0 < s < \infty$ in [St04] using a refined decomposition method for the precompact subset K.

The paper [CKP99] has been the starting point of a comprehensive and systematic study of the behavior of the metric entropy of convex hulls of precompact subsets in Hilbert and Banach spaces. However, it should be noted that this problem was already raised by Dudley [Du87] in the context of empirical processes where he was able to prove a first result. He showed that, for a precompact subset $K \subset H$ of a Hilbert space H, the property

$$\sup_{\varepsilon>0} \varepsilon\, N(K,\varepsilon)^{1/r} < \infty$$

implies that

$$\sup_{\varepsilon>0} \varepsilon \big(\log N(\mathrm{cov}(K),\varepsilon)\big)^{1/t} < \infty$$

for any t with $\frac{1}{t} < \frac{1}{r} + \frac{1}{2}$. It was proved in [C97] that the implication also remains true for $\frac{1}{t} = \frac{1}{r} + \frac{1}{2}$ which is sharp.

As already noticed in [CKP99] the situation changes completely if the metric entropy of the precompact subset K satisfies the condition

$$\sup_{\varepsilon>0} \varepsilon \big(\log N(K,\varepsilon)\big)^{1/r} < \infty$$

which is equivalent to the condition

$$\sup_n n^{1/r} e_n(K) < \infty$$

for the dyadic entropy numbers.

Several results were given by Ball and Pajor [BP90], Talagrand [T87] and Li and Linde [LL00] (cf. also [CE03]) in the diagonal case where the set $K = \{x_1, x_2, \ldots\} \subset H$ of a Hilbert space H satisfies the condition

$$\sup_n n^{1/r} \|x_n\| < \infty.$$

They showed that this condition implies

$$\sup_n n^{1/r} e_n(\operatorname{cov}(K)) < \infty \quad \text{if } 2 \le r < \infty$$

and

$$\sup_n n^{1/2} \big(\log(n+1) \big)^{1/r-1/2} e_n(\operatorname{cov}(K)) < \infty \quad \text{if } 0 < r \le 2.$$

In the non-diagonal case where the entropy condition on the subset K is

$$\sup_n n^{1/r} e_n(K) < \infty$$

the above implication remains true for $r \neq 2$. However, if r crosses the point $r = 2$ there is a sudden jump which was discovered by Gao in [Ga01]. His ingenious example which was extended by Creutzig and Steinwart in [CS02] shows that the estimate

$$\sup_n n^{1/2} \big(\log(n+1) \big)^{-1} e_n(\operatorname{cov}(K)) < \infty$$

is optimal. For Gelfand numbers, and consequently also for entropy numbers, this was already implicitly contained in [CKP99], cf. [CE03].

In the present paper we give a complete answer to the behavior of Gelfand and entropy numbers as well as of the metric entropy of absolutely convex hulls $\operatorname{cov}(K)$ of precompact subsets $K \subset H$ of a Hilbert space H under the assumption that the metric entropy of K satisfies the integral Lorentz condition

$$\int_0^\infty \big(\log_2 N(K, \varepsilon) \big)^{s/r} \, d\varepsilon^s < \infty$$

for some $0 < r, s \le \infty$, which is equivalent to the condition

$$\big(e_n(K) \big) \in \ell_{r,s}$$

for the dyadic entropy numbers. Moreover, we also study the interesting case where the entropy condition is

$$\sup_n n^{1/r} \big(\log(n+1) \big)^{\beta} e_n(K) < \infty$$

for some $0 < r < \infty$ and $\beta \in \mathbb{R}$.

We prove new results and recover known results on the metric entropy of convex hulls of precompact subsets in Hilbert spaces. For more than 30 years this question has been extensively treated in different settings (cf. e.g. [C81, C82, C97, CE01, CE03, CHK88, CKP99, CS90, CS02, CT80, Du67, Du73, Du87, ET96, Ga01, GD93, LL00, St00, St04, T93, VW96]). In the dual case we get new insight into the metric entropy of integral and especially weakly singular integral operators from a Hilbert space into the space $C(K)$ of continuous functions on a compact metric space K.

Now we turn to the main results of the paper. For this purpose we introduce the Gelfand numbers $c_n(S)$ of a bounded linear operator $S : X \to Y$ from a Banach space X into a Banach space Y. They are defined by

$$c_n(S) := \inf \big\{ \|S|_M\| \,:\, M \subset X, \operatorname{codim}(M) < n \big\},$$

where $S|_M$ is the restriction of S to the finite codimensional subspace M. The Gelfand numbers of the absolutely convex hull $\operatorname{cov}(K)$ of a precompact subset $K \subset X$ of a Banach space X is defined by

$$c_n(\operatorname{cov}(K)) := c_n(T),$$

where $T : \ell_1(K) \to X$ is the operator already defined at the beginning of this section.

The starting point are the following two inequalities of [CKP99] in the version of [CE03].

Theorem A. *Let $K \subset H$ be a precompact subset of a Hilbert space H. Then the following inequalities hold:*

(i)

$$n^{1/2} c_n(\operatorname{cov}(K)) \le c_K \Big(1 + \sum_{k=1}^{n} k^{-1/2} e_k(K)\Big)$$

for $n \in \mathbb{N}$, where $c_K \le c\big(1 + \sup_{t \in K} \|t\|\big)$ and $c > 0$ is an absolute constant.

(ii)

$$k^{1/2} c_{k+n}(\operatorname{cov}(K)) \le c\Big(\big(\log(n+1)\big)^{1/2} \varepsilon_n(K) + \sum_{j=n+1}^{\infty} \frac{\varepsilon_j(K)}{j(\log(j+1))^{1/2}}\Big)$$

for $k, n \in \mathbb{N}$, where $c > 0$ is an absolute constant.

In particular, we have

$$2^{n/2} c_{2^n}(\mathrm{cov}(K)) \le \tilde{c}\Big(n^{1/2} e_n(K) + \sum_{j=n}^{\infty} j^{-1/2} e_j(K)\Big)$$

for $n \in \mathbb{N}$, where $\tilde{c} > 0$ is an absolute constant. Indeed, this can be derived from *(ii)* as the special case $k = n = 2^{m-1}$

$$\begin{aligned}
&2^{(m-1)/2} c_{2^m}(\mathrm{cov}(K)) \\
&\quad\le\ c\left(\big(\log(2^{m-1}+1)\big)^{1/2} e_m(K) + \sum_{j=2^{m-1}+1}^{\infty} \frac{\varepsilon_j(K)}{j(\log(j+1))^{1/2}}\right) \\
&\quad\le\ c\left(m^{1/2} e_m(K) + \sum_{j=m}^{\infty} \sum_{k=2^{j-1}+1}^{2^j} \frac{\varepsilon_k(K)}{k(\log(k+1))^{1/2}}\right) \\
&\quad\le\ c_o\left(m^{1/2} e_m(K) + \sum_{j=m}^{\infty} j^{-1/2} e_j(K)\right).
\end{aligned}$$

In order to get the corresponding results for the dyadic entropy numbers $e_n(\mathrm{cov}(K))$ of absolutely convex hulls we need the following inequalities between dyadic entropy and Gelfand numbers of absolutely convex hulls of bounded subsets of Banach spaces. They are variants of an inequality in [C81], see also [CKP99, Theorem 1.3].

Theorem B. *(i) Let $0 < r \le \infty$ and $0 < s \le \infty$. Then there exists a constant $c(r,s) \ge 1$ such that*

$$\sum_{k=1}^{n} k^{s/r-1}\big(e_k(\mathrm{cov}(K))\big)^s \le c(r,s) \sum_{k=1}^{n} k^{s/r-1}\big(c_k(\mathrm{cov}(K))\big)^s$$

for all Banach spaces X, all bounded subsets $K \subset X$ and all $n \in \mathbb{N}$.

(ii) Let (b_n) be an increasing sequence of positive real numbers with the property that there exists a constant $\gamma \ge 1$ such that $b_{2n} \le \gamma b_n$ for all $n \in \mathbb{N}$. Then there exists a constant $c(\gamma) \ge 1$ such that

$$\sup_{1\le k\le n} b_k\, e_k(\mathrm{cov}(K)) \le c(\gamma) \sup_{1\le k\le n} b_k\, c_k(\mathrm{cov}(K))$$

for all Banach spaces X, all bounded subsets $K \subset X$ and all $n \in \mathbb{N}$.

To formulate precise estimates, we also need the following generalization of Lorentz sequence spaces. For $0 < r, s \leq \infty$ and $\alpha \in \mathbb{R}$, excluding the case $r = s = \infty$, we say that a null sequence $x = (\xi_k)$ belongs to $\ell_{r,s,\alpha}$ if the non-increasing rearrangement $(a_k(x))$ of its absolute values $(|\xi_k|)$ satisfies

$$\sum_{n=1}^{\infty} \big(\log_2(n+1)\big)^{\alpha} \, n^{s/r-1} \, \big(a_n(x)\big)^s < \infty$$

if $0 < r, s < \infty$,

$$\sum_{n=1}^{\infty} \big(\log_2(n+1)\big)^{\alpha} \, n^{-1} \, \big(a_n(x)\big)^s < \infty$$

if $r = \infty, 0 < s < \infty$ and

$$\sup_{1 \leq n < \infty} \big(\log_2(n+1)\big)^{\alpha} \, n^{1/r} \, a_n(x) < \infty$$

if $0 < r < \infty, s = \infty$. The case $\alpha = 0$ corresponds to the classical Lorentz sequence spaces, $\ell_{r,s} = \ell_{r,s,0}$.

To compare sequences (a_n) and (b_n) of positive numbers, we use the notation $a_n \preceq b_n$ if there exists a constant $c > 0$ such that $a_n \leq c b_n$ for all $n \in \mathbb{N}$. If $a_n \preceq b_n$ and $b_n \preceq a_n$, we write $a_n \asymp b_n$.

Now we are able to formulate estimates of Gelfand and dyadic entropy numbers of absolutely convex hulls $\mathrm{cov}(K)$ provided that the dyadic entropy numbers of K belong to the Lorentz sequence space $\ell_{r,s}$.

Theorem 1.1. *Let (s_n) stand either for the Gelfand numbers (c_n) or for the dyadic entropy numbers (e_n). Let $0 < r, s \leq \infty$ excluding the case $r = s = \infty$ and let $K \subset H$ be a precompact subset of a Hilbert space H. Then there exists a constant $a = a(r, s, K) > 0$ such that the following inequality holds in the respective cases:*

(i) Let $0 < r < 2$. Then

$$\left(\sum_{n=1}^{\infty} \big(\log(n+1)\big)^{s(\frac{1}{r}-\frac{1}{2})-1} \, n^{\frac{s}{2}-1} \big(s_n(\mathrm{cov}(K))\big)^s \right)^{1/s} \leq a \left(1 + \sum_{n=1}^{\infty} n^{\frac{s}{r}-1} \big(e_n(K)\big)^s \right)^{1/s}$$

for $0 < s < \infty$ and

$$\sup_{1 \leq n < \infty} \big(\log(n+1)\big)^{s(\frac{1}{r}-\frac{1}{2})} \, n^{\frac{1}{2}} s_n(\mathrm{cov}(K)) \leq a \left(1 + \sup_{1 \leq n < \infty} n^{\frac{1}{r}} e_n(K) \right)$$

for $s=\infty$. These inequalities mean that

$$\big(e_n(K)\big) \in \ell_{r,s} \quad \Longrightarrow \quad \big(s_n(\mathrm{cov}(K))\big) \in \ell_{2,s,\alpha}$$

with $\alpha = s(\frac{1}{r}-\frac{1}{2})-1$ for $0<s<\infty$ and $\alpha=\frac{1}{r}-\frac{1}{2}$ for $s=\infty$, respectively. The results are optimal in the following sense: If $0<s\le\infty$ and $\beta>\alpha$, then there exists a precompact subset $K\subset\ell_2$ such that $(e_n(K))\in\ell_{r,s}$ and $(s_n(\mathrm{cov}(K)))\notin\ell_{2,s,\beta}$.

(ii) *Let $r=2$. Then*

$$\sup_{1\le k\le n} \big(\log(k+1)\big)^{\min\{0,-1+1/s\}} k^{1/2}\, s_k(\mathrm{cov}(K)) \le a(2,s,K)\left(1+\sum_{k=1}^{n} k^{-1+s/2}\big(e_k(K)\big)^s\right)^{1/s}$$

for $0<s<\infty$ and $n\in\mathbb{N}$ and

$$\sup_{1\le k\le n} \big(\log(k+1)\big)^{-1} k^{1/2}\, s_k(\mathrm{cov}(K)) \le a(2,\infty,K)\left(1+\sup_{1\le k\le n} k^{1/2}\, e_k(K)\right)$$

for $s=\infty$ and $n\in\mathbb{N}$. These inequalities mean that

$$\big(e_n(K)\big) \in \ell_{2,s} \quad \Longrightarrow \quad \big(s_n(\mathrm{cov}(K))\big) \in \ell_{2,\infty,\alpha}$$

with $\alpha=\min\{0,-1+\frac{1}{s}\}$ for $0<s\le\infty$. The results are optimal in the following sense: If $1<s\le\infty$ and $\beta>\frac{1}{s}-1$, then there exists a precompact subset $K\subset\ell_2$ such that $(e_n(K))\in\ell_{2,s}$ and $(s_n(\mathrm{cov}(K)))\notin\ell_{2,\infty,\beta}$. If $0<s\le1$ and $0<t<\infty$ then there exists a precompact subset $K\subset\ell_2$ such that $(e_n(K))\in\ell_{2,s}$ and $(s_n(\mathrm{cov}(K)))\notin\ell_{2,t}$.

(iii) *Let $2<r\le\infty$. Then the expressions*

$$1+\sum_{k=1}^{n} k^{s/r-1}\big(e_k(K)\big)^s \quad \textit{and} \quad 1+\sum_{k=1}^{n} k^{s/r-1}\big(s_k(\mathrm{cov}K)\big)^s \quad \textit{for } 0<s<\infty$$

and

$$1+\sup_{1\le k\le n} k^{1/r}\, e_k(K) \quad \textit{and} \quad 1+\sup_{1\le k\le n} k^{1/r}\, s_k(\mathrm{cov}K) \quad \textit{for } s=\infty$$

are asymptotically equivalent, respectively. These equivalences mean that

$$\big(e_n(K)\big) \in \ell_{r,s} \quad \Longleftrightarrow \quad \big(s_n(\mathrm{cov}(K))\big) \in \ell_{r,s}$$

for $0<s\le\infty$.

Results in probability often use the metric entropy instead of entropy numbers. For the convenience of the reader we reformulate Theorem 1.1 in terms of metric entropy.

Theorem 1.2. *Let $K \subset H$ be a precompact subset of a Hilbert space such that the metric entropy $H(K,\varepsilon) := \log_2 N(K,\varepsilon)$ satisfies the integral Lorentz condition*

$$\int_0^\infty \big(H(K,\varepsilon)\big)^{s/r}\, d\varepsilon^s < \infty$$

for some r, s with $0 < r < \infty, 0 < s < \infty$,

$$\sup_{\varepsilon>0} \varepsilon \big(H(K,\varepsilon)\big)^{1/r} < \infty$$

for $0 < r < \infty$, $s = \infty$ or

$$\int_0^\infty \log\big(1 + H(K,\varepsilon)\big)\, d\varepsilon^s < \infty$$

for $r = \infty$, $0 < s < \infty$, respectively. Then the following integral conditions for the metric entropy $H(\mathrm{cov}(K),\varepsilon)$ of the absolutely convex hull of K hold:

(i)

$$\int_0^\infty \big(2 + H(\mathrm{cov}(K),\varepsilon)\big)^{s(\frac{1}{r}-\frac{1}{2})-1} \big(H(\mathrm{cov}(K),\varepsilon)\big)^{s/2} d\varepsilon^s < \infty$$

for $0 < r < 2$ and $0 < s < \infty$, and

$$\sup_{\varepsilon>0} \varepsilon \log\big(2 + H(\mathrm{cov}(K),\varepsilon)\big)^{s(\frac{1}{r}-\frac{1}{2})} \big(H(\mathrm{cov}(K),\varepsilon)\big)^{1/2} < \infty$$

for $0 < r < 2$ and $s = \infty$.

(ii)

$$\sup_{\varepsilon>0} \varepsilon \log\big(2 + H(\mathrm{cov}(K),\varepsilon)\big)^{\min\{0,-1+1/s\}} \big(H(\mathrm{cov}(K),\varepsilon)\big)^{1/2} < \infty$$

for $r = 2$ and $0 < s < \infty$.

(iii)

$$\int_0^\infty \big(H(\mathrm{cov}(K),\varepsilon)\big)^{s/r} d\varepsilon^s < \infty$$

for $2 < r < \infty$ and $0 < s < \infty$,

$$\sup_{\varepsilon>0} \varepsilon \big(H(\mathrm{cov}(K),\varepsilon)\big)^{1/r} < \infty$$

for $2 < r < \infty$ *and* $s = \infty$*, and*

$$\int_0^\infty \log\big(1 + H(\mathrm{cov}(K), \varepsilon)\big)\, d\varepsilon^s < \infty$$

for $r = \infty$*,* $0 < s < \infty$*. In this case, the conditions for the metric entropy of the subset* K *and its convex hull* $\mathrm{cov}(K)$ *are equivalent.*

The following corollary is a conclusion from Theorem 1.1 and is already contained in [St00] and implicitly in [CKP99].

Corollary 1.3. *Let* $2 < r < \infty$ *and let* (s_n) *stand for either one of the Gelfand numbers* (c_n) *or the dyadic entropy numbers* (e_n)*. If* (a_n) *is a sequence of positive numbers such that* $(n^{1/r} a_n)$ *is increasing for* $n \geq n_0$*, then*

$$e_n(K) \preceq a_n \quad \Longleftrightarrow \quad s_n(\mathrm{cov}(K)) \preceq a_n$$

and

$$e_n(K) \asymp a_n \quad \Longleftrightarrow \quad s_n(\mathrm{cov}(K)) \asymp a_n$$

for every compact subset $K \subset H$ *of a Hilbert space* H*.*

Proof. For the entropy numbers the statement immediately follows from Theorem Part (*iii*) of 1.1. For the Gelfand numbers it can be obtained by a trick given in [C85] (cf. also [CP88] and [St00, Corollary 3]). □

Remarks. Theorem 1.1 contains several special cases that recover corresponding earlier results. This applies, for the case of entropy numbers of convex hulls, to the case $0 < r < 2, 0 < s < \infty, \frac{1}{r} = \frac{1}{2} + \frac{1}{s}$, which is contained in [St04], and, for entropy as well as Gelfand numbers, to the case $2 < r < \infty, 0 < s \leq \infty$, which is contained in [CKP99, St00]. In the case $r = 2, s = \infty$, we get the famous result of Gao [Ga01] (cf. also [CS02]) even for Gelfand numbers (cf. also [CE03]).

Moreover, let us say that a precompact subset $K \subset H$ of a Hilbert space H satisfies Dudley's entropy condition, if the integral Lorentz condition

$$\int_0^\infty \big(H(K, \varepsilon)\big)^{1/2}\, d\varepsilon < \infty$$

holds. From Theorem 1.2 (cf. also [St04]) we obtain the following conclusion for the metric entropy of the convex hull of K:

$$\int_0^\infty \big(H(K, \varepsilon)\big)^{3/2}\, d\varepsilon < \infty \quad \Longrightarrow \quad \int_0^\infty \big(H(\mathrm{cov}(K), \varepsilon)\big)^{1/2}\, d\varepsilon < \infty.$$

In terms of Gelfand and dyadic entropy numbers this conclusion states that

$$\big(e_n(K)\big) \in \ell_{2/3,1} \quad \Longrightarrow \quad \big(s_n(\mathrm{cov}(K))\big) \in \ell_{2,1},$$

where again (s_n) stands for either one of the Gelfand numbers (c_n) or the dyadic entropy numbers (e_n). Hence $\mathrm{cov}(K)$ satisfies the Dudley condition provided that $\big(e_n(K)\big) \in \ell_{2/3,1}$ or, equivalently,

$$\int_0^\infty \big(H(K,\varepsilon)\big)^{3/2}\, d\varepsilon < \infty.$$

Finally, from Theorem 1.1 it is interesting to see that for fixed s the dyadic entropy numbers as well as the Gelfand numbers exhibit a sudden jump if r crosses the point $r = 2$. This is why $r = 2$ is often called the critical point.

In the next theorem we assume that the precompact subset $K \subset H$ satisfies the entropy condition

$$\sup_{1\le n<\infty} \big(\log(n+1)\big)^{\beta}\, n^{1/r}\, e_n(K) < \infty$$

for some fixed $\beta \in \mathbb{R}$ and $0 < r < \infty$. In this case we again get surprising results.

Theorem 1.4. *Let (s_n) stand either for the Gelfand numbers (c_n) or for the dyadic entropy numbers (e_n). Let $K \subset H$ be a precompact subset of a Hilbert space H. If $0 < r < \infty$ and $\beta \in \mathbb{R}$ then there exists a constant $a = a(r,\beta,K) > 0$ such that the following inequality holds in the respective cases:*

(i) If $0 < r < 2$ and $\beta \in \mathbb{R}$ then

$$\begin{aligned}\sup_{1\le n<\infty} &\big(\log(1+\log(n+1))\big)^{\beta} \big(\log(n+1)\big)^{1/r-1/2} n^{1/2} s_n(\mathrm{cov}(K)) \\ &\le \; a\Big(1 + \sup_{1\le n<\infty} \big(\log(n+1)\big)^{\beta} n^{1/r} e_n(K)\Big).\end{aligned}$$

The result is asymptotically optimal.

(ii) If $r = 2$, then

$$\begin{aligned}\sup_{1\le k\le n} &\big(\log(k+1)\big)^{\beta-1} k^{1/2} s_k(\mathrm{cov}(K)) \\ &\le a\Big(1 + \sup_{1\le k\le n} \big(\log(k+1)\big)^{\beta} k^{1/2} e_k(K)\Big)\end{aligned}$$

for $-\infty < \beta < 1$ *and* $n \in \mathbb{N}$ *and*

$$\sup_{1\le k\le n} \big(\log(k+1)\big)^{-1} k^{1/2} s_k(\mathrm{cov}(K)) \\ \le a\Big(1 + \sup_{1\le k\le n} \log(k+1) k^{1/2} e_k(K)\Big)$$

for $\beta = 1$ *end* $n \in \mathbb{N}$, *and*

$$\sup_{1\le k<\infty} \big(\log(k+1)\big)^{\beta-1} k^{1/2} s_k(\mathrm{cov}(K)) \\ \le a\Big(1 + \sup_{1\le k\le\infty} \big(\log(k+1)\big)^{\beta} k^{1/2} e_k(K)\Big)$$

for $1 < \beta < \infty$. *In the case* $-\infty < \beta < 1$ *the inequality is asymptotically optimal.*

(iii) *If* $2 < r \le \infty$ *and* $\beta \in \mathbb{R}$, *the expressions*

$$1 + \sup_{1\le k\le n} \big(\log(k+1)\big)^{\beta} k^{1/r} e_k(K)$$

and

$$\sup_{1\le k\le n} \big(\log(k+1)\big)^{\beta} k^{1/r} s_k\big(\mathrm{cov}(K)\big)$$

are asymptotically equivalent.

Again we reformulate the preceding theorem in terms of metric entropy.

Theorem 1.5. *Let* $K \subset H$ *be a precompact subset of a Hilbert space such that the metric entropy* $H(K,\varepsilon) := \log_2 N(K,\varepsilon)$ *satisfies the condition*

$$\sup_{\varepsilon>0} \varepsilon \,\log\big(1 + H(K,\varepsilon)\big)^{\beta} \big(H(K,\varepsilon)\big)^{1/r} < \infty$$

for some $0 < r < \infty$ *and* $\beta \in \mathbb{R}$. *Then the following integral conditions for the metric entropy* $H(\mathrm{cov}(K),\varepsilon)$ *of the absolutely convex hull of* K *hold:*

(i) *If* $0 < r < 2$ *and* $\beta \in \mathbb{R}$ *then*

$$\sup_{\varepsilon>0} \varepsilon \big(\log(2 + \log(H(\mathrm{cov}(K),\varepsilon)))\big)^{\beta} \big(1 + \log(H(\mathrm{cov}(K),\varepsilon))\big)^{1/r-1/2} \times \\ \times \big(H(\mathrm{cov}(K),\varepsilon)\big)^{1/2} < \infty.$$

(ii) *If* $r = 2$ *then*

$$\sup_{\varepsilon>0} \varepsilon \left(1 + \log(H(\mathrm{cov}(K), \varepsilon)\right)^{\beta-1} \left(H(\mathrm{cov}(K), \varepsilon)\right)^{1/2} < \infty$$

for $-\infty < \beta < 1$,

$$\sup_{\varepsilon>0} \varepsilon \left(\log(2 + \log(H(\mathrm{cov}(K), \varepsilon))\right)^{-1} \left(H(\mathrm{cov}(K), \varepsilon)\right)^{1/2} < \infty$$

for $\beta = 1$ *and*

$$\sup_{\varepsilon>0} \varepsilon \left(\log(1 + H(\mathrm{cov}(K), \varepsilon)\right)^{\beta-1} \left(H(\mathrm{cov}(K), \varepsilon)\right)^{1/2} < \infty$$

for $1 < \beta < \infty$.

(iii) *If* $2 < r < \infty$ *and* $\beta \in \mathbb{R}$ *then*

$$\sup_{\varepsilon>0} \varepsilon \log \left(1 + H(\mathrm{cov}(K), \varepsilon)\right)^{\beta} \left(H(\mathrm{cov}(K), \varepsilon)\right)^{1/r} < \infty.$$

In this case, the conditions for the metric entropy of the subset K *and its convex hull* $\mathrm{cov}(K)$ *are equivalent.*

Remarks. We should note that as in Theorem 1.1 we again have, for fixed β, a sudden jump of the Gelfand and dyadic entropy numbers of the convex hull if r crosses the point $r = 2$. Moreover, in the case $r = 2$, we get another sudden jump at least in our upper estimates if β crosses the point $\beta = 1$. However, we do not know whether the upper estimates are sharp for $r = 2$ and $\beta \geq 1$.

2 Basic tools

In this section we provide basic tools for the proofs of our results. The first lemma relates the metric entropy $H(K, \varepsilon) = \log_2 N(K, \varepsilon)$ of a precompact subset $K \subset X$ of a metric space X with its dyadic entropy numbers.

Lemma 2.1. *Let* $f, F : [0, \infty) \to [0, \infty)$ *be non-negative continuous and increasing functions with* $F(0) = 0$. *Moreover, let* $K \subset X$ *be a precompact subset of a metric space* X *such that* $e_1 := e_1(K) > 0$ *and such that for any* $0 < a \leq e_1$ *the Stieltjes integral*

$$\int_a^{e_1} f(H(K, \varepsilon))\, dF(\varepsilon)$$

exists. Let also $e_k := e_k(K)$ *for* $k \in \mathbb{N}$. *Then* *(i) If*

$$\sum_{k=1}^{\infty} \big(f(k) - f(k-1)\big) F(e_k)$$

is finite then the improper Stieltjes integral

$$\int_0^{e_1} f(H(K,\varepsilon))\, dF(\varepsilon)$$

is finite.

(ii) If the improper Stieltjes integral

$$\int_0^{e_1} f(H(K,\varepsilon))\, dF(\varepsilon)$$

is finite then

$$\sum_{k=1}^{\infty} \big(f(k) - f(k-1)\big) F(e_{k+1})$$

is finite.

Proof. We initially assume that

$$e_1(K) > e_2(K) > \ldots > e_n(K) > \ldots > 0. \tag{1}$$

Note that

$$n-2 \le H(K,\varepsilon) \le n-1 \quad \text{for} \quad e_n(K) < \varepsilon < e_{n-1}(K).$$

Let us first assume that

$$S := \sum_{k=1}^{\infty} \big(f(k) - f(k-1)\big) F(e_k)$$

is finite. Then integration by parts yields

$$\begin{aligned}
&\int_{e_n}^{e_1} f(H(K,\varepsilon))\, dF(\varepsilon) \\
&= f(H(K,e_1))F(e_1) - f(H(K,e_n))F(e_n) - \int_{e_n}^{e_1} F(\varepsilon)\, df(H(K,\varepsilon)) \\
&= f(0)F(e_1) - f(n-1)F(e_n) - \sum_{k=2}^{n} \int_{e_k}^{e_{k-1}} F(\varepsilon)\, df(H(K,\varepsilon)) \\
&\le f(0)F(e_1) - f(n-1)F(e_n) + \sum_{k=2}^{n} F(e_{k-1})\big(f(k-1) - f(k)\big) \\
&\le f(0)F(e_1) + \sum_{k=1}^{n} \big(f(k) - f(k-1)\big) F(e_k) \le f(0)F(e_1) + S.
\end{aligned}$$

Since $\lim_{n\to\infty} e_n = 0$, this proves that the integral

$$\int_0^{e_1} f(H(K,\varepsilon))\, dF(\varepsilon)$$

is finite and assertion (i) is shown.

Now we assume that

$$I := \int_0^{e_1} f(H(K,\varepsilon))\, dF(\varepsilon)$$

is finite. Then, similar as above,

$$\begin{aligned}
&\int_{e_n}^{e_1} f(H(K,\varepsilon))\, dF(\varepsilon) \\
&= f(0)F(e_1) - f(n-1)F(e_n) - \sum_{k=2}^{n} \int_{e_k}^{e_{k-1}} F(\varepsilon)\, df(H(K,\varepsilon)) \\
&\geq f(0)F(e_1) - f(n-1)F(e_n) - \sum_{k=2}^{n} F(e_k)\big(f(k-2) - f(k-1)\big) \\
&= f(0)F(e_1) - f(n-1)F(e_n) + \sum_{k=2}^{n} \big(f(k-1) - f(k-2)\big)F(e_k).
\end{aligned}$$

Because of $F(0) = 0$ we have

$$\int_0^{e_n} f(H(K,\varepsilon))\, dF(\varepsilon) \geq f(n-1)\big(F(e_n) - F(0)\big) = f(n-1)F(e_n).$$

Hence

$$I \geq f(0)F(e_1) + \sum_{k=2}^{n} \big(f(k-1) - f(k-2)\big)F(e_k).$$

Since this holds for any $n = 2, 3, \ldots$, we conclude that

$$\sum_{k=1}^{\infty} \big(f(k) - f(k-1)\big)F(e_{k+1})$$

is finite and assertion (ii) is proved.

If (1) does not hold, then either there exits some n such that $e_n = 0$ or for any n there is a k such that $e_n = e_{n+1} = \ldots = e_{n+k-1} > e_k$. In both cases, the above inequalities remain true with the obvious modifications in the proofs. □

The next lemma relates certain integral conditions of the metric entropy with dyadic entropy numbers. For this purpose we need the function

$$\Gamma(\alpha,\beta,x) := \int_1^x \big(\log_2(t+1)\big)^\alpha\, t^\beta\, dt \quad \text{for } \alpha,\beta\in\mathbb{R} \text{ and } x\geq 1.$$

It turns out that

$$\Gamma(\alpha,\beta,x) \asymp \big(\log_2(x+1)\big)^\alpha\, x^{\beta+1}$$

for fixed $\beta > -1$ and $\alpha\in\mathbb{R}$.

Lemma 2.2. *Let $K\subset X$ be a precompact subset of a metric space X. Then the following characterization of integral conditions of the metric entropy by the dyadic entropy numbers $e_n(K)$ hold.*

(i) Let $0<r,s<\infty$ and $\alpha\in\mathbb{R}$. Then

$$\int_0^{e_1} \Big(\log_2\big(2+H(k,\varepsilon)\big)\Big)^\alpha \big(H(K,\varepsilon)\big)^{s/r}\, d\varepsilon^s < \infty$$

if and only if $\big(e_n(K)\big)\in\ell_{r,s,\alpha}$.

(ii) Let $0<r<\infty, s=\infty$, and $\alpha\in\mathbb{R}$. Then

$$\sup_{\varepsilon>0} \varepsilon \Big(\log_2\big(2+H(k,\varepsilon)\big)\Big)^\alpha \big(H(K,\varepsilon)\big)^{1/r} < \infty$$

if and only if $\big(e_n(K)\big)\in\ell_{r,\infty,\alpha}$. In particular, for $\alpha=0$ we have

$$\sup_{\varepsilon>0} \varepsilon \big(H(K,\varepsilon)\big)^{1/r} < \infty \quad \textit{iff} \quad \big(e_n(K)\big)\in\ell_{r,\infty}.$$

(iii) Let $0<s<\infty, r=\infty$, and $\alpha\neq -1$. Then

$$\int_0^{e_1} \Big(\log_2\big(2+H(K,\varepsilon)\big)\Big)^{\alpha+1} d\varepsilon^s < \infty$$

if and only if $\big(e_n(K)\big)\in\ell_{\infty,s,\alpha}$. In particular, for $\alpha=0$ we have

$$\int_0^{e_1} \log_2\big(1+H(K,\varepsilon)\big)\, d\varepsilon^s < \infty \quad \textit{iff} \quad \big(e_n(K)\big)\in\ell_{\infty,s}.$$

Proof. (i): Using the functions $f(x) = \Gamma(\alpha, \frac{s}{r} - 1, x)$ and $F(x) = x^s$ we check with the mean value theorem that, for $n \geq 2$, there is a $0 < \theta < 1$ with

$$f(n) - f(n-1) = f'(n-1+\theta) = \big(\log_2(n+\theta)\big)^\alpha\,(n-1+\theta)^{s/r-1}.$$

Furthermore, there are constants $a_i := a_i(\alpha, r, s)$ for $i = 1, 2$ so that

$$a_0\,\big(\log_2(n+2)\big)^\alpha\,(n+1)^{s/r-1} \leq f(n)-f(n-1) \leq a_1\,\big(\log_2(n+2)\big)^\alpha\,(n+1)^{s/r-1}.$$

Applying Lemma 2.1 we conclude the assertion from

$$\sum_{n=2}^{\infty}\big(f(n)-f(n-1)\big)F(e_n) \leq a_1\sum_{n=2}^{\infty}\big(\log_2(n+1)\big)^\alpha\,n^{s/r-1}\big(e_n(K)\big)^s$$

and

$$\sum_{n=2}^{\infty}\big(f(n)-f(n-1)\big)F(e_n) \geq a_0\sum_{n=2}^{\infty}\big(\log_2(n+2)\big)^\alpha\,(n+1)^{s/r-1}\big(e_{n+1}(K)\big)^s.$$

(ii): First, let

$$\sup_n\big(\log_2(n+1)\big)^\alpha\,n^{1/r}\,e_n(K) < \infty.$$

In the case where $e_n(K) = 0$ for some $n \in \mathbb{N}$ the assertion is trivial. So let us assume $e_n(K) > 0$ for all $n \in \mathbb{N}$. Since $H(K,\varepsilon) = 0$ for $\varepsilon > e_1(K)$ it is enough to consider the interval $0 < \varepsilon \leq e_1(K)$. Because of $e_n(K) \to 0$ we can find $n \in \mathbb{N}$ such that

$$e_{n+1}(K) < \varepsilon \leq e_n(K).$$

Then $n - 1 \leq H(K,\varepsilon) \leq n$ and

$$\begin{aligned}
&\varepsilon\Big(\log_2\big(2+H(k,\varepsilon)\big)\Big)^\alpha\,\big(H(K,\varepsilon)\big)^{1/r}\\
&\quad\leq\ \max\{1,2^\alpha\}\,\big(\log_2(n+1)\big)^\alpha\,n^{1/r}\,e_n(K)\\
&\quad\leq\ \max\{1,2^\alpha\}\,\sup_n\big(\log_2(n+1)\big)^\alpha\,n^{1/r}\,e_n(K) < \infty.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sup_{\varepsilon>0}\varepsilon\Big(\log_2\big(2+H(k,\varepsilon)\big)\Big)^\alpha\,\big(H(K,\varepsilon)\big)^{1/r}\\
&\quad\leq\ \max\{1,2^\alpha\}\,\sup_n\big(\log_2(n+1)\big)^\alpha\,n^{1/r}\,e_n(K) < \infty.
\end{aligned}$$

Now let

$$\sup_{\varepsilon>0} \varepsilon \Big(\log_2 \big(2 + H(k,\varepsilon)\big)\Big)^{\alpha} \big(H(K,\varepsilon)\big)^{1/r} < \infty.$$

Without loss of generality we assume $e_{n-1}(K) > e_n(K)$. We choose ε such that $e_n(K) < \varepsilon < e_{n-1}(K)$. Then $n-2 \leq H(K,\varepsilon) \leq n-1$. For $n \geq 3$ we have

$$\begin{aligned} &\big(\log_2(n+1)\big)^{\alpha} n^{1/r} e_n(K) \\ \leq\ & \max\{1, 2^{\alpha}\} 3^{1/r} \varepsilon \Big(\log_2 \big(2 + H(k,\varepsilon)\big)\Big)^{\alpha} \big(H(K,\varepsilon)\big)^{1/r}. \end{aligned}$$

This implies

$$\sup_{n\in\mathbb{N}} \big(\log_2(n+1)\big)^{\alpha} n^{1/r} e_n(K) \leq a \sup_{\varepsilon>0} \varepsilon \Big(\log_2 \big(2+H(k,\varepsilon)\big)\Big)^{\alpha} \big(H(K,\varepsilon)\big)^{1/r}.$$

(iii): Put $f(x) = \big(\log_2(x+1)\big)^{\alpha+1}$ and $F(x) = x^s$ for $x \geq 1$. Similarly as in the proof of *(i)* we conclude the assertion. □

To compare Gelfand numbers with entropy numbers we use the following generalization of one of Hardy's inequalities.

Lemma 2.3. *(i) Let $0 < r < \infty$, $0 < s \leq \infty$, $\alpha \in \mathbb{R}$ and $0 < p < r$. If $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ is a non-increasing sequence of non-negative numbers then*

$$\begin{aligned} &\sum_{n=1}^{N} \big(\log(n+1)\big)^{\alpha} n^{-1+s/r} \left(\frac{1}{n}\sum_{k=1}^{n} \sigma_k^p\right)^{s/p} \\ \leq\ & a \sum_{n=1}^{N} \big(\log(n+1)\big)^{\alpha} n^{-1+s/r} \sigma_n^s \end{aligned}$$

for $N \in \mathbb{N}$, where $a = a(\alpha, p, r, s)$ is a constant depending only on α, p, r and s.

(ii) Let $0 < q < 1$ and $0 < s \leq \infty$. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ be a non-increasing sequence of non-negative numbers satisfying

$$\sum_{n=1}^{\infty} n^{-1+s/q} \sigma_n^s < \infty \quad \textit{if } 0 < s < \infty$$

and

$$\sup_n n^{1/q}\,\sigma_n < \infty \quad \textit{if } s = \infty.$$

Then, in the case $0 < s < \infty$, *there exists a constant* $a = a(q,s) \geq 1$ *such that*

$$\sum_{n=1}^{\infty} n^{-1-s+s/q} \left(\sum_{k=n}^{\infty} \sigma_k \right)^s \leq a \sum_{n=1}^{\infty} n^{-1+s/q}\, \sigma_n^s,$$

and, in the case $s = \infty$, *we have*

$$\sum_{n=1}^{\infty} n^{-1+1/q} \sum_{k=n}^{\infty} \sigma_k \leq \frac{1}{1-q} \sup_n n^{1/q}\, \sigma_n.$$

In the context of Lorentz sequence spaces these inequalities mean that

$$(\sigma_n) \in \ell_{q,s} \quad \Longrightarrow \quad \left(\sum_{k=n}^{\infty} \sigma_k \right) \in \ell_{t,s}$$

with $\frac{1}{t} = \frac{1}{q} - 1$.

Proof. The inequalities in (ii) are well-known and often used in the literature. The proof is based on a geometric progression, the monotonicity of ℓ_q-norms in the case $0 < s \leq 1$ and Hölder's inequality in the case $1 < s < \infty$. The crucial point is to find an appropriate factorization of the item σ_{2^n} in the sum $\sum_{n=k}^{\infty} \sigma_{2^n}$.

We roughly sketch the proof of (i) by following the crucial steps as in [P87, 2.1.7.]. We define $\frac{1}{q} := \frac{1}{p} + \frac{1}{s}$ and choose any number ϱ with $\frac{1}{r} < \varrho < \frac{1}{p}$. We check from

$$k\sigma_k^p \leq \sum_{j=1}^{n} \sigma_j^p \quad \text{for } k = 1, \ldots, n$$

and Hölder's inequality that

$$
\begin{aligned}
&\left(\sum_{k=1}^{n}\sigma_k^p\right)^{1/q}\\
&= \left(\sum_{k=1}^{n}\Big[\big(\log(k+1)\big)^{-\alpha/s}\,k^{-\varrho}\,\big(\big(\log(k+1)\big)^{\alpha/s}k^{\varrho-1/s}\sigma_k\big)\,\big(k\sigma_k^p\big)^{1/s}\Big]^q\right)^{1/q}\\
&\le \left(\sum_{k=1}^{n}\big(\log(k+1)\big)^{-\alpha p/s}\,k^{-\varrho p}\right)^{1/p}\times\\
&\quad\times\left(\sum_{k=1}^{n}\Big(\big(\log(k+1)\big)^{\alpha/s}\,k^{\varrho-1/s}\,\sigma_k\Big)^s\,k\sigma_k^p\right)^{1/s}\\
&\le \left(\sum_{k=1}^{n}\big(\log(k+1)\big)^{-\alpha p/s}\,k^{-\varrho p}\right)^{1/p}\times\\
&\quad\times\left(\sum_{k=1}^{n}\Big(\big(\log(k+1)\big)^{\alpha/s}\,k^{\varrho-1/s}\,\sigma_k\Big)^s\right)^{1/s}\left(\sum_{j=1}^{n}\sigma_j^p\right)^{1/s}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left(\sum_{k=1}^{n}\sigma_k^p\right)^{1/p} &\le \left(\sum_{k=1}^{n}\big(\log(k+1)\big)^{-\alpha p/s}\,k^{-\varrho p}\right)^{1/p}\times\\
&\times\left(\sum_{k=1}^{n}\Big(\big(\log(k+1)\big)^{\alpha/s}\,k^{\varrho-1/s}\,\sigma_k\Big)^s\right)^{1/s}.
\end{aligned}
$$

Because of $\varrho p<1$ we have

$$
\left(\sum_{k=1}^{n}\big(\log(k+1)\big)^{-\alpha p/s}\,k^{-\varrho p}\right)^{1/p} \le a_0(\alpha,\varrho,p)\,\big(\log(n+1)\big)^{-\alpha/s}\,n^{-\varrho+1/p}
$$

yielding

$$
\begin{aligned}
&\sum_{n=1}^{N}\big(\log(n+1)\big)^{\alpha}\,n^{-1+s/r}\,\left(\frac{1}{n}\sum_{k=1}^{n}\sigma_k^p\right)^{s/p}\\
&\quad\le\ a_0(\alpha,\varrho,p)^s\sum_{n=1}^{N}n^{s/r-\varrho s-1}\sum_{k=1}^{n}\big(\log(k+1)\big)^{\alpha}\,\big(k^{\varrho-1/s}\sigma_k\big)^s.
\end{aligned}
$$

Changing the order of summation on the right-hand side and using that $s/r - \varrho s < 0$ we arrive at the desired inequality

$$
\begin{aligned}
&\sum_{n=1}^{N} \big(\log(n+1)\big)^{\alpha} n^{-1+s/r} \left(\frac{1}{n}\sum_{k=1}^{n} \sigma_k^p\right)^{s/p} \\
&\quad \leq \; a_0(\alpha,\varrho,p)^s \sum_{k=1}^{N} \big(\log(k+1)\big)^{\alpha} k^{\varrho s-1}\sigma_k^s \sum_{k=n}^{\infty} n^{s/r-\varrho s-1} \\
&\quad \leq \; a_1(\alpha,\varrho,p)^s \sum_{k=1}^{N} \big(\log(k+1)\big)^{\alpha} k^{s/r-1}\sigma_k^s .
\end{aligned}
$$

□

Finally, in order to prove the optimality of our results we need the following lemma. The first part goes back to [CKP99], the second part to [Ga01] with extensions due to [CS02].

Lemma 2.4. *Let (s_n) stand either for the Gelfand numbers or for the dyadic entropy numbers.*

(i) If $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ is a non-increasing sequence of non-negative numbers and

$$K = \{\sigma_n u_n \,:\, n \in \mathbb{N}\} \subset \ell_2$$

is a subset of the Hilbert space ℓ_2, where (u_n) denotes the unit vector basis of ℓ_2, then

$$e_n(K) \leq \sigma_{2^{n-1}} \;\; \text{and} \;\; s_n(\mathrm{cov}(K)) \geq a \max\left\{n^{-1/2}\big(\log(n+1)\big)^{1/2}\sigma_{n^2}, \sigma_{2^n}\right\}$$

for $n \in \mathbb{N}$, where $a > 0$ is an absolute constant.

(ii) If $-\infty < \beta < 1$, then there exists a precompact subset $K \subset H$ of an infinite dimensional Hilbert space and constants $a, b > 0$ such that for all $n \in \mathbb{N}$ we have

$$e_n(K) \leq a n^{-1/2}\big(\log(n+1)\big)^{-\beta} \;\; \text{and} \;\; s_n(\mathrm{cov}(K)) \geq b n^{-1/2}\big(\log(n+1)\big)^{1-\beta}.$$

Proof. (i): The estimate $e_n(K) \leq \sigma_{2^{n-1}}$ is obvious by definition of the dyadic entropy numbers. For the absolutely convex sections

$$\Delta_{n,m} := \mathrm{cov}\big(\{\sigma_k u_k \;:\; n \leq k \leq m\}\big)$$

for $m,n = 1,2,\ldots$ with $m > n$ we have with a positive constant a_0

$$s_n\big(\mathrm{cov}(K)\big) \geq s_n(\Delta_{n,m}) \geq a_0\, \sigma_m\, s_n\big(id : \ell_1^{m-n} \to \ell_2^{m-n}\big),$$

where $id : \ell_1^{m-n} \to \ell_2^{m-n}$ stands for the identity operator. By a result of Schütt [S84] and Garnaev/Gluskin [GG84] (cf. also [CP88] for a generalization) we have

$$s_n\big(id : \ell_1^{m-n} \to \ell_2^{m-n}\big) \geq b_0 \min\left\{1, \left(\frac{\log(m/n)}{n}\right)^{1/2}\right\},$$

where $b_0 > 0$ is an absolute constant. Choosing $m = n^2$ and $m = 2^n$, respectively, we check the desired estimate

$$s_n(\mathrm{cov}(K)) \geq a \max\left\{n^{-1/2}\big(\log(n+1)\big)^{1/2}\sigma_{n^2}, \sigma_{2^n}\right\}$$

for $n \in \mathbb{N}$, where $a > 0$ is an absolute constant.

(i): In the case of entropy numbers the result was proved by Gao [Ga01] for $\beta = 0$ and extended by Creutzig and Steinwart [CS02] for $-\infty < \beta < 1$ even for Banach spaces of type p with $1 < p \leq 2$. By a trick given in [C85] it can also be carried over to Gelfand numbers (cf. [CE03]). $\square$

3 Proof of the results

The constants $a_0, a_1, \ldots$ appearing in the proofs may depend on parameters $r, s, \ldots$ and on the size of the precompact subset $K \subset H$ but not on $n \in \mathbb{N}$.

Proof of Theorem 1.1. (i): The case $0 < r < 2$ and $s = \infty$ was already treated in [CKP99]. So we only need to consider the case $0 < r < 2$ and $0 < s < \infty$. From (ii) in Theorem 1 we have

$$2^{n/2} c_{2^n}(\mathrm{cov}(K)) \leq a_0\Big[n^{1/2}\, e_n(K) + \sum_{j=n}^{\infty} j^{-1/2}\, e_j(K)\Big]$$

which implies

$$2^{sn/2}\big(c_{2^n}(\mathrm{cov}(K))\big)^s \leq a_1\left[n^{s/2}\,\big(e_n(K)\big)^s + \Big(\sum_{j=n}^{\infty} j^{-1/2}\, e_j(K)\Big)^s\right].$$

Hence

$$\sum_{n=1}^{\infty} n^{s(\frac{1}{r}-\frac{1}{2})-1} 2^{sn/2} \big(c_{2^n}(\mathrm{cov}(K))\big)^s$$
$$\leq \; a_1 \Big[\sum_{n=1}^{\infty} n^{-1+s/r} \big(e_n(K)\big)^s + \sum_{n=1}^{\infty} n^{s(\frac{1}{r}-\frac{1}{2})-1} \Big(\sum_{j=n}^{\infty} j^{-1/2}\, e_j(K) \Big)^s \Big]$$

Now we estimate the second summand on the right-hand side by (ii) in Lemma 2.3 with $\frac{1}{q} = \frac{1}{r} + \frac{1}{2}$ and obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{s(\frac{1}{r}-\frac{1}{2})-1} \Big(\sum_{j=n}^{\infty} j^{-1/2}\, e_j(K) \Big)^s &= \sum_{n=1}^{\infty} n^{-1-s+s/q} \Big(\sum_{j=n}^{\infty} j^{-1/2}\, e_j(K) \Big)^s \\
\leq \; a_2 \sum_{n=1}^{\infty} n^{-1+s/q} \big(n^{-1/2}\, e_n(K)\big)^s &= a_2 \sum_{n=1}^{\infty} n^{s(\frac{1}{q}-\frac{1}{2})-1} \big(e_n(K)\big)^s \\
= \; a_2 \sum_{n=1}^{\infty} n^{-1+s/r} \big(e_n(K)\big)^s &
\end{aligned}$$

Combining the previous estimates we obtain

$$\sum_{n=1}^{\infty} n^{s(\frac{1}{r}-\frac{1}{2})-1} 2^{sn/2} \big(c_{2^n}(\mathrm{cov}(K))\big)^s \leq a_3 \sum_{n=1}^{\infty} n^{-1+s/r} \big(e_n(K)\big)^s.$$

From

$$\begin{aligned}
&\sum_{k=2}^{\infty} \big(\log k\big)^{s(\frac{1}{r}-\frac{1}{2})-1}\, k^{\frac{s}{2}-1} \big(c_k(\mathrm{cov}(K))\big)^s \\
&= \; \sum_{n=1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} \big(\log k\big)^{s(\frac{1}{r}-\frac{1}{2})-1}\, k^{\frac{s}{2}-1} \big(c_k(\mathrm{cov}(K))\big)^s \\
&\leq \; \sum_{n=1}^{\infty} 2^n \big(\log(2^{n+1})\big)^{s(\frac{1}{r}-\frac{1}{2})} \big(\log 2^n\big)^{-1}\, 2^{\frac{s}{2}(n+1)}\, 2^{-n} \big(c_{2^n}(\mathrm{cov}(K))\big)^s \\
&\leq \; 2 \sum_{n=1}^{\infty} (n+1)^{s(\frac{1}{r}-\frac{1}{2})-1}\, 2^{\frac{s}{2}(n+1)} \big(c_{2^n}(\mathrm{cov}(K))\big)^s \\
&\leq \; a_4 \sum_{n=1}^{\infty} n^{s(\frac{1}{r}-\frac{1}{2})-1}\, 2^{\frac{s}{2}n} \big(c_{2^n}(\mathrm{cov}(K))\big)^s
\end{aligned}$$

we now conclude

$$\Big(\sum_{n=1}^{\infty} \big(\log(n+1)\big)^{s(\frac{1}{r}-\frac{1}{2})-1}\, n^{\frac{s}{2}-1}\big(c_n(\mathrm{cov}(K))\big)^s\Big)^{1/s} \le a_5 \Big(1+\sum_{n=1}^{\infty} n^{\frac{s}{r}-1}\big(e_n(K)\big)^s\Big)^{1/s}$$

which is the inequality we had to prove in this case if $s_n = c_n$.

Let us note that if K consists of more than one point then we have $e_1(K) > 0$ and the right-hand expression of the proved inequality can be simplified with the estimate

$$1+\sum_{n=1}^{\infty} n^{\frac{s}{r}-1}\big(e_n(K)\big)^s \le \frac{2}{e_1(K)} \sum_{n=1}^{\infty} n^{\frac{s}{r}-1}\big(e_n(K)\big)^s.$$

The corresponding inequality for the dyadic entropy numbers $e_n(\mathrm{cov}(K))$ of the convex hulls follows from the inequality

$$\sum_{k=1}^{n} \big(e_k(\mathrm{cov}(K))\big)^p \le a_5 \sum_{k=1}^{n} \big(c_k(\mathrm{cov}(K))\big)^p$$

which is a special case of (i) in Theorem B and from inequality (i) of Lemma 2.3.

Now we prove the optimality claim in the theorem, that means that, for $0 < s < \infty$ and $\beta > \alpha = s(\frac{1}{r} - \frac{1}{2}) - 1$, we have to find a precompact subset $K \subset \ell_2$ such that $\big(e_n(K)\big) \in \ell_{r,s}$ and $\big(s_n(\mathrm{cov}(K))\big) \notin \ell_{2,s,\beta}$. To this end, put

$$K = \{\sigma_n u_n \,:\, n \in \mathbb{N}\} \subset \ell_2$$

in part (i) of Lemma 2.4, where

$$\sigma_n = \big(\log(n+1)\big)^{-1/r} \big(\log\log(n+2)\big)^{-1/s} \big(\log\log\log(n+4)\big)^{-\gamma}$$

with $\gamma \in \mathbb{R}$ satisfying $\gamma\, s > 1$. Part (i) of Lemma 2.4 yields

$$e_n(K) \le \sigma_{2^{n-1}} \le a_6\, n^{-1/r} \big(\log(n+1)\big)^{-1/s}\big(\log\log(n+2)\big)^{-\gamma}.$$

Hence we obtain

$$\sum_{n=1}^{\infty} n^{\frac{s}{r}-1}\big(e_n(K)\big)^s \le a_6 \sum_{n=1}^{\infty} n^{-1} \big(\log(n+1)\big)^{-1}\big(\log\log(n+2)\big)^{-\gamma s} < \infty$$

which shows that $\big(e_n(K)\big) \in \ell_{r,s}$. On the other hand, part (i) of Lemma 2.4 also yields

$$\begin{aligned} s_n(\mathrm{cov}(K)) &\geq a n^{-1/2}\big(\log(n+1)\big)^{1/2}\sigma_{n^2} \\ \geq\ & a_7\, n^{-1/2}\big(\log(n+1)\big)^{1/2-1/r}\big(\log\log(n+2)\big)^{-1/s}\big(\log\log\log(n+4)\big)^{-\gamma} \end{aligned}$$

which implies

$$\begin{aligned} &\sum_{n=1}^{\infty}\big(\log(n+1)\big)^{\beta}\, n^{s/2-1}\,\big(s_n(\mathrm{cov}(K))\big)^{s} \\ \geq\ & a_7\sum_{n=1}^{\infty} n^{-1}\,\big(\log(n+1)\big)^{\beta-\alpha-1}\big(\log\log(n+2)\big)^{-1}\big(\log\log\log(n+4)\big)^{-\gamma s} \end{aligned}$$

Since $\beta > \alpha$, the latter sum is infinite which shows that $(s_n(\mathrm{cov}(K))) \notin \ell_{2,s,\beta}$ and finishes the proof of part (i) of the theorem.

(ii): First we treat the case $r = 2$ and $1 \leq s \leq \infty$. Using Part (i) of Theorem A together with Hölder's inequality we conclude

$$\begin{aligned} n^{1/2}\, c_n(\mathrm{cov}(K)) &\leq a_0\Big(1 + \sum_{k=1}^{n} k^{-1/2}\, e_k(K)\Big) \\ &\leq a_0\Big(1 + \sum_{k=1}^{n} k^{-1}\Big)^{1-1/s}\Big(1 + \sum_{k=1}^{n} k^{s/2-1}\,\big(e_k(K)\big)^{s}\Big)^{1/s} \\ &\leq a_1\big(\log(n+1)\big)^{1-1/s}\Big(1 + \sum_{k=1}^{n} k^{s/2-1}\,\big(e_k(K)\big)^{s}\Big)^{1/s} \end{aligned}$$

which is the desired inequality in this case.

In the case $r = 2$ and $0 < s < 1$ the inequality follows from Part (i) of Theorem A and $\ell_{2,s} \subset \ell_{2,1}$.

Again, by Theorem B we can replace the Gelfand numbers $c_n(\mathrm{cov}(K))$ by the dyadic entropy numbers $e_n(\mathrm{cov}(K))$.

To prove the optimality assertion in the case $1 < s \leq \infty$ let $\beta > \frac{1}{s} - 1$. Choose γ such that $\frac{1}{s} < \gamma < \beta + 1 < 1$. By (ii) of Lemma 2.4 there exists a precompact subset K of a Hilbert space such that

$$e_n(K) \leq a n^{-1/2}\big(\log(n+1)\big)^{-\gamma} \quad \text{and} \quad s_n\big(\mathrm{cov}(K)\big) \geq b n^{-1/2}\big(\log(n+1)\big)^{1-\gamma}.$$

Since $1 < s\gamma$ we conclude that $\big(e_n(K)\big) \in \ell_{2,s}$. Because of $1+\beta-\gamma > 0$ we find

$$\sup_{n\in\mathbb{N}} n^{1/2}\,\big(\log(n+1)\big)^{\beta}\, s_n\big(\mathrm{cov}(K)\big) \geq b \sup_{n\in\mathbb{N}} \big(\log(n+1)\big)^{1+\beta-\gamma} = \infty.$$

Hence $(s_n(\mathrm{cov}(K))) \notin \ell_{2,\infty,\beta}$.

Finally, we deal with optimality in the case $0 < s \leq 1$. Then there exists a precompact subset $K \subset \ell_2$ such that

$$(e_n(K)) \in \ell_{2,s} \;\text{ and }\; \big(s_n(\mathrm{cov}(K))\big) \notin \ell_{2,t} \text{ for } t < \infty$$

Indeed, for the set $K = \{\sigma_n u_n \;:\; n \in \mathbb{N}\}$ of (i) in Lemma 2.4 with

$$\sigma_n = \big(\log(n+1)\big)^{-1/2}\,\big(\log\log(n+2)\big)^{-1/s}\,\big(\log\log\log(n+4)\big)^{-\gamma}$$

for some $\gamma > 1/s$. Then we have

$$e_n(K) \leq \sigma_{2^{n-1}} \;\text{ and }\; s_n\big(\mathrm{cov}(K)\big) \geq a n^{-1/2}\big(\log(n+1)\big)^{1/2}\sigma_{n^2}.$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty} n^{s/2-1}\,\big(e_n(K)\big)^s &\leq \sum_{n=1}^{\infty} n^{s/2-1}\,\sigma^s_{2^{n-1}}\\ &\preceq \sum_{n=1}^{\infty} n^{-1}\,\big(\log(n+1)\big)^{-1}\big(\log\log(n+4)\big)^{-\gamma s} < \infty\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} n^{t/2-1}\,\big(s_n(\mathrm{cov}(K))\big)^t &\geq a\sum_{n=1}^{\infty} n^{t/2-1} n^{-t/2}\big(\log(n+1)\big)^{t/2}\sigma^t_{n^2}\\ &\succeq \sum_{n=1}^{\infty} n^{-1}\,\big(\log\log(n+2)\big)^{-t/s}\,\big(\log\log\log(n+4)\big)^{-\gamma t} = \infty.\end{aligned}$$

(iii): First we treat the case $2 < r < \infty$ and $s = \infty$. By Part (i) of Theorem A we immediately get

$$\begin{aligned}k^{1/2}c_k(\mathrm{cov}(K)) &\leq a_o\Big(1+\sum_{j=1}^{k} j^{-1/2}e_j(K)\Big)\\ &\leq a_o\Big(1+\sum_{j=1}^{k} j^{-1/2-1/r}\sup_{1\leq i\leq k} i^{1/r}e_i(K)\Big)\\ &\leq a_1\,k^{1/2-1/r}\Big(1+\sup_{1\leq i\leq k} i^{1/r}e_i(K)\Big).\end{aligned}$$

Hence

$$k^{1/r}c_k(\mathrm{cov}(K)) \le a_1\Big(1+\sup_{1\le i\le k} i^{1/r}e_i(K)\Big)$$

yields the desired inequality

$$\sup_{1\le k\le n} k^{1/r}c_k(\mathrm{cov}(K)) \le a_1\Big(1+\sup_{1\le k\le n} k^{1/r}e_k(K)\Big).$$

By Theorem B, the inequality remains true for the dyadic entropy numbers $e_n(\mathrm{cov}(K))$ instead of the Gelfand numbers $c_n(\mathrm{cov}(K))$.

Now we turn to the case $2 < r < \infty$ and $0 < s < \infty$. Choose p such that $2 < p < r$. From the just proved inequality we conclude

$$k^{1/p}s_k(\mathrm{cov}(K)) \le a_2\Big(1+\sup_{1\le j\le k} j^{1/p}e_j(K)\Big) \le a_3\Big(1+\sum_{1\le j\le k}\big(e_j(K)\big)^p\Big)^{1/p}.$$

Since $p < r$ we can use Part (i) of Lemma 2.3 to get the inequality

$$\begin{aligned} 1+\sum_{k=1}^{n} k^{s/r-1}\big(s_k(\mathrm{cov}K)\big)^s &\le a_5\sum_{k=1}^{n} k^{s/r-1}\bigg(\frac{1}{k}\Big(1+\sum_{1\le j\le k}\big(e_j(K)\big)^p\Big)\bigg)^{s/p} \\ &\le a_6\Big(1+\sum_{k=1}^{n} k^{s/r-1}\big(e_k(K)\big)^s\Big). \end{aligned}$$

Since $e_n(K) \le e_n(\mathrm{cov}(K)$ we obtain that the expressions

$$1+\sum_{k=1}^{n} k^{s/r-1}\big(e_k(K)\big)^s \quad \text{and} \quad 1+\sum_{k=1}^{n} k^{s/r-1}\big(s_k(\mathrm{cov}K)\big)^s \quad \text{for } 0<s<\infty$$

and

$$1+\sup_{1\le k\le n} k^{1/r}\, e_k(K) \quad \text{and} \quad 1+\sup_{1\le k\le n} k^{1/r}\, s_k(\mathrm{cov}K) \quad \text{for } s=\infty$$

are asymptotically equivalent.

□

Proof of Theorem 1.2. The statements in terms of the metric entropy immediately follow from Theorem 1.1 and Lemma 2.2. □

Proof of Theorem 1.4. (i): Let $0 < r < 2$ and $\beta \in \mathbb{R}$. By Part (ii) of Theorem A we have

$$2^{n/2}c_{2^n}(\mathrm{cov}(K)) \le a_0\Big(n^{1/2}e_n(K)+\sum_{k=n}^{\infty} k^{-1/2}e_k(K)\Big)$$

for $n \in \mathbb{N}$. Furthermore,

$$\sum_{k=n}^{\infty} k^{-1/2} e_k(K) \le \left(\sum_{k=n}^{\infty} k^{-1/2-1/r} \big(\log(k+1)\big)^{-\beta} \right) \sup_{k \ge n} k^{1/r} \big(\log(k{+}1)\big)^{\beta} e_k(K)$$

and, since $0 < r < 2$, we also have

$$\sum_{k=n}^{\infty} k^{-1/2-1/r} \big(\log(k+1)\big)^{-\beta} \le a_1\, n^{1/2-1/r} \, \big(\log(n+1)\big)^{-\beta}.$$

Consequently,

$$\begin{aligned} &2^{n/2} c_{2^n}(\mathrm{cov}(K)) \le \\ &\quad a_2 \Big(n^{1/2} e_n(K) + n^{1/2-1/r} \, \big(\log(n+1)\big)^{-\beta} \sup_{k \ge n} k^{1/r} \big(\log(k+1)\big)^{\beta} e_k(K) \Big) \end{aligned}$$

which further implies

$$\begin{aligned} & n^{1/r-1/2} \, \big(\log(n+1)\big)^{\beta} 2^{n/2} c_{2^n}\big(\mathrm{cov}(K)\big) \\ &\quad \le \; a_2 \Big(n^{1/r} \big(\log(n+1)\big)^{\beta} e_n(K) + \sup_{k \ge n} k^{1/r} \big(\log(k+1)\big)^{\beta} e_k(K) \Big) \\ &\quad \le \; 2a_2 \sup_{k \ge n} k^{1/r} \big(\log(k+1)\big)^{\beta} e_k(K) \\ &\quad \le \; 2a_2 \sup_{1 \le k < \infty} k^{1/r} \big(\log(k+1)\big)^{\beta} e_k(K). \end{aligned}$$

This is the desired inequality for the Gelfand numbers. By Theorem B this inequality remains true with the dyadic entropy numbers $e_n\big(\mathrm{cov}(K)\big)$ instead of the Gelfand numbers $c_n\big(\mathrm{cov}(K)\big)$ of the convex hull.

The inequality is sharp. Indeed, using the precompact subset $K = \{\sigma_n u_n \;:\; n \in \mathbb{N}\} \subset \ell_2$ of (i) in Lemma 2.4 with

$$\sigma_n := \big(\log(n+1)\big)^{-1/r} \, \big(\log\log(n+2)\big)^{-\beta}$$

we have

$$e_n(K) \le \sigma_{2^{n-1}} \preceq n^{-1/r} \, \big(\log(n+1)\big)^{-\beta}$$

and

$$\begin{aligned} s_n\big(\mathrm{cov}(K)\big) \;&\succeq\; n^{-1/2} \, \big(\log(n+1)\big)^{1/2-1/r} \, \sigma_{n^2} \\ &\succeq\; n^{-1/2} \, \big(\log(n+1)\big)^{1/2-1/r} \, \big(\log\log(n+2)\big)^{-\beta}. \end{aligned}$$

This finishes the proof of (i).

(ii): First let $r = 2$ and $-\infty < \beta < 1$. By Part (i) of Theorem A we obtain

$$\begin{aligned}
&n^{1/2} c_n\big(\mathrm{cov}(K)\big) \\
&\le\ a_0\Big(1 + \sum_{k=1}^{n} k^{-1/2} e_k(K)\Big) \\
&\le\ a_0\Big(1 + \sum_{k=1}^{n} k^{-1} \big(\log(k+1)\big)^{-\beta} \sup_{1\le k\le n} k^{1/2} \big(\log(k+1)\big)^{\beta} e_k(K)\Big) \\
&\le\ a_1\Big(1 + \big(\log(n+1)\big)^{-\beta+1} \sup_{1\le k\le n} k^{1/2} \big(\log(k+1)\big)^{\beta} e_k(K)\Big)
\end{aligned}$$

which in turn implies

$$\big(\log(n+1)\big)^{\beta-1} n^{1/2} c_n\big(\mathrm{cov}(K)\big) \le a_2\Big(1 + \sup_{1\le k\le n} k^{1/2} \big(\log(k+1)\big)^{\beta} e_k(K)\Big)$$

for $n = 1, 2, \ldots$. This gives the desired estimate for Gelfand numbers. Again, by Theorem B, this inequality remains true with the dyadic entropy numbers $e_n\big(\mathrm{cov}(K)\big)$ instead of the Gelfand numbers $c_n\big(\mathrm{cov}(K)\big)$ of the convex hull. It follows from (ii) in Lemma 2.4 that the inequality is asymptotically optimal.

The case $r = 2$ and $\beta = 1$ can be handled as the previous case if we use

$$\sum_{k=1}^{n} k^{-1} \big(\log(k+1)\big)^{-1} \le a\, \log\log(n+2)$$

for $n = 1, 2, \ldots$.

Finally, let $r = 2$ and $1 < \beta < \infty$. Now we use the estimate (ii) of Theorem A and get

$$\begin{aligned}
&2^{n/2} c_{2^n}(\mathrm{cov}(K)) \le a_0\Big(n^{1/2} e_n(K) + \sum_{k=n}^{\infty} k^{-1/2} e_k(K)\Big) \\
&\le\ a_0\Big(n^{1/2} e_n(K) + \sum_{k=n}^{\infty} k^{-1}\big(\log(k+1)\big)^{-\beta} \sup_{k\ge n} k^{1/2}\big(\log(k+1)\big)^{\beta} e_k(K)\Big) \\
&\le\ a_1\Big(n^{1/2} e_n(K) + \big(\log\log(n+2)\big)^{-\beta+1} \sup_{k\ge n} k^{1/2}\big(\log(k+1)\big)^{\beta} e_k(K)\Big)
\end{aligned}$$

implying

$$
\begin{aligned}
\big(\log\log(n+2)\big)^{\beta-1}\, 2^{n/2} c_{2^n}(\mathrm{cov}(K)) &\le 2a_1 \sup_{k\ge n} k^{1/2}\big(\log(k+1)\big)^{\beta} e_k(K) \\
&\le 2a_1 \sup_{k\ge 1} k^{1/2}\big(\log(k+1)\big)^{\beta} e_k(K)
\end{aligned}
$$

for $n = 1, 2, \ldots$. This estimate yields the desired estimate for the Gelfand numbers while that for the entropy numbers follows again from Theorem B.

(iii): Let $2 < r < \infty$ and $\beta \in \mathbb{R}$. By Part *(i)* of Theorem A we obtain

$$
\begin{aligned}
n^{1/2} c_n\big(\mathrm{cov}(K)\big) &\le a_0\Big(1 + \sum_{k=1}^{n} k^{-1/2} e_k(K)\Big) \\
&\le a_0\Big(1 + \sum_{k=1}^{n} k^{-1/2-1/r}\big(\log(k+1)\big)^{-\beta} \sup_{1\le k\le n} k^{1/r}\big(\log(k+1)\big)^{\beta} e_k(K)\Big).
\end{aligned}
$$

Because of $2 < r < \infty$ we have

$$
\sum_{k=1}^{n} k^{-1/2-1/r}\big(\log(k+1)\big)^{-\beta} \le a_1\, n^{1/2-1/r}\big(\log(n+1)\big)^{-\beta}.
$$

Combining the previous estimates we arrive at

$$
n^{1/r} c_n\big(\mathrm{cov}(K)\big) \le a_2\Big(1 + \sup_{1\le k\le n} k^{1/r}\big(\log(k+1)\big)^{\beta} e_k(K)\Big).
$$

This yields the desired estimate for the Gelfand numbers and, via Theorem B, also for the entropy numbers.

Moreover, since $e_n(K) \le e_n\big(\mathrm{cov}(K)\big)$, it turns out that the expressions

$$
1 + \sup_{1\le k\le n} \big(\log(k+1)\big)^{\beta} k^{1/r} e_k(K)
$$

and

$$
\sup_{1\le k\le n} \big(\log(k+1)\big)^{\beta} k^{1/r} s_k\big(\mathrm{cov}(K)\big)
$$

are asymptotically equivalent. □

Proof of Theorem 1.5. Again, the statements in terms of the metric entropy immediately follow from Theorem 1.4 and Lemma 2.2. □

4 Metric entropy of integral operators from a Hilbert space into $C(M)$

In this section we illustrate how the inequalities for the entropy numbers of absolutely convex hulls can be used to obtain upper bounds for the entropy numbers and the metric entropy of integral operators from a Hilbert space H into the space of $C(M)$ of continuous functions on a compact metric space M. More applications and, in particular, lower bounds showing the sharpness of our results, are treated in a forthcoming paper.

Let $K : M \to H$ be a continuous function which we call an abstract kernel. The abstract integral operator $T_K : H \to C(M)$ induced by the kernel K is defined by

$$(T_K x)(t) := \langle x, K(t) \rangle$$

for $x \in H$ and $t \in M$. Moreover, let $S : \ell_1(M) \to H$ be the operator defined by $Se_t = K(t)$ and let J_∞ be the canonical embedding of $C(M)$ in $\ell_\infty(M)$. It follows directly from these definitions that

$$J_\infty T_K = S'.$$

In order to use duality, we introduce the symmetrized approximation numbers $(t_n(T))$ for an operator $T : X \to Y$ which are defined as $t_n(T) = c_n(TQ_X)$ where Q_X is the canonical quotient map from $\ell_1(B(X))$ onto X. Then $t_n(T) = t_n(T')$ and $t_n(T) \leq c_n(T)$, see [CS90]. We obtain

$$t_n(T_K) = t_n(J_\infty T_K) = t_n(S') = t_n(S) \leq c_n(S) = c_n\big(\mathrm{cov}(\mathrm{Im}(K))\big).$$

Then Theorems 1.1 and 1.4 can be used to bound the symmetrized approximation numbers of T_K. Furthermore, the entropy numbers of T_K can be bounded by the symmetrized approximation numbers via an analogue of Theorem B, where Gelfand numbers are replaced with symmetrized approximation numbers. Finally, this also provides bounds for the metric entropy of T_K.

Alternatively, one can also use the duality results for operators from a Hilbert to a Banach space obtained in [AMS04] together with Theorems 1.1 and 1.4 for the entropy numbers of T_K and Theorems 1.2 and 1.5 for the metric entropy of T_K.

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