On the Euclidean metric entropy of convex bodies

A. E. Litvak V. D. Milman A. Pajor
N. Tomczak-Jaegermann *

1 Introduction

In this note we study the Euclidean metric entropy of convex bodies and its relation to classical geometric parameters such as diameters of sections or mean widths. We provide an exact analysis of a classical Sudakov's inequality relating Euclidean covering numbers of a body to its mean width, and we obtain some new upper and lower bounds for these covering numbers.

We will explain the subject in a little more detail while briefly describing the organization of the paper. In order to be more precise, let B_2^n denote the unit Euclidean ball in $\mathbb{R}^n$. For a symmetric convex body $K \subset \mathbb{R}^n$ let $N(K, \varepsilon B_2^n)$ be the smallest number of Euclidean balls of radius ε needed to cover K, and finally let $M^*(K)$ be a half of the mean width of K (see (2.1) and (2.2) below).

Section 2 collects the notation and preliminary results used throughout the paper. Sudakov's inequality gives an upper bound for $N(K, tB_2^n)$ in terms of $M^*(K)$, and we show (in Section 3) that if this upper bound is essentially sharp, then diameters of all k-codimensional sections of K are large, for an appropriate choice of k. On the other hand, in Section 4 we discuss conditions that ensure that the covering can be significantly decreased by cutting the body K by a Euclidean ball of a certain radius, in which case "most" of the entropy of K lies outside of this Euclidean ball. In Section 5 this leads to further consequences of sharpness in Sudakov's inequality which turn out to

*This author holds the Canada Research Chair in Geometric Analysis.

be close to a well-known concept of M-position. Finally, in Section 6 we obtain lower estimates for covering numbers $N(K, B_2^n)$ in terms of diameters of sections of a body. It is worthwhile to point out here that the most satisfactory results involve a smaller body T intimately related to K, its skeleton. In Sections 5 and 6 we will use notions of random projections and sections, however we shall not try to specify any probability estimates, as they will be not needed.

2 Notation and preliminaries

We denote by $|\cdot|$ the canonical Euclidean norm on $\mathbb{R}^n$, by $\langle\cdot,\cdot\rangle$ the canonical inner product and by B_2^n the Euclidean unit ball.

By a convex body we always mean a closed convex set with non-empty interior. By a symmetric convex body we mean centrally symmetric (with respect to the origin) convex body. Let K be a convex body in $\mathbb{R}^n$ with the origin in its interior. The gauge of K is denoted by $\|\cdot\|_K$. The space $\mathbb{R}^n$ endowed with such a gauge is denoted by $(\mathbb{R}^n, \|\cdot\|_K)$ or just by $(\mathbb{R}^n, K)$. The radius of K is the smallest number R such that $K \subset RB_2^n$, and is denoted by $R(K)$. Note that if K is centrally symmetric then $2R(K)$ is the diameter of K.

Let K be a symmetric convex body in $\mathbb{R}^n$ and let $k \geq 1$. By $c_k(K)$ we denote the infimum of $R(K \cap E)$ taken over all $(k-1)$-codimensional subspaces $E \subset \mathbb{R}^n$. Clearly, $2c_1(K) = 2R(K)$ is the diameter of K and $2c_{k+1}(K)$ is the smallest possible diameter of k-codimensional section of K. Below we call $c_{k+1}(K)$ the k-diameter of K.

Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior. We denote by $|K|$ the volume of K, and by K^0 the polar of K, i.e.

$$K^0 = \{x \mid \langle x, y\rangle \leq 1 \ \text{ for every } \ y \in K\}\,.$$

Given $\rho > 0$, we denote

$$K_\rho = K \cap \rho B_2^n \quad \text{and} \quad K_\rho^0 = (K_\rho)^0\,.$$

Let X be a linear space and K, L be subsets of X. We recall that covering number $N(K, L)$ is defined as the minimal number N such that there exist vectors x_1, ..., x_N in X satisfying

$$K \subset \bigcup_{i=1}^{N} (x_i + L). \tag{2.1}$$

We also will use the notions of ε-net and ε-separated set. Let K, A be sets in $\mathbb{R}^n$ and $\varepsilon > 0$. The set A is called an ε-net for K if $K \subset A + \varepsilon B_2^n$; it is called an ε-separated set for K if $A \subset K$ and for any two different points x and y in A one has $|x - y| > \varepsilon$. It is well known (and easy to check) that any maximal (in sense of inclusion) ε-separated set is an ε-net and that any ε-net has cardinality not smaller than the cardinality of any (2ε)-separated set.

Following [MSz] we say that A is an ε-skeleton of K if $A \subset K$ and A is an ε-net for K. If, in addition, A is convex we say that A is a convex ε-skeleton.

We say that A is an ε-separated skeleton of K if A is a maximal ε-separated set for K. Note that every ε-separated skeleton of K is also an ε-skeleton of K. We say that A is a convex ε-separated skeleton of K if A is the convex hull of an ε-separated skeleton of K. We say that A is an absolute ε-separated skeleton of K if A is the absolute convex hull of an ε-separated skeleton of K.

Given a convex body $K \subset \mathbb{R}^n$ with the origin in its interior, we let

$$M_K = M(K) = \int_{S^{n-1}} \|x\|_K \, d\nu \quad \text{and} \quad \ell(K) = \mathbb{E} \left\| \sum_{i=1}^n g_i e_i \right\|_K ,$$

where $d\nu$ is the probability Lebesgue measure on S^{n-1}, and g_i's are $N(0,1)$ Gaussian random variables.

It is well known and easy to check that there exists a positive constant ω_n such that for every convex K one has $\ell(K) = \omega_n \sqrt{n}\, M(K)$. In fact

$$1 - \frac{1}{4n} < \omega_n = \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n g_i^2 \right)^{1/2} < 1.$$

We also set

$$M_K^* = M^*(K) = M(K^0). \tag{2.2}$$

It is also well known that for any subspace E of $\mathbb{R}^n$ we have $(PK)^0 = K^0 \cap E$ where P is the orthogonal projection on E and the polar of PK is taken in E. In particular we have

$$\ell \left((PK)^0 \right) \le \ell \left(K^0 \right)$$

hence,

$$M^* \left(PK \right) \le \frac{\omega_n}{\omega_m} \sqrt{\frac{n}{m}} \; M^* \left(K \right),$$

where $m = \dim E$.

Recall Urysohn's inequality (see e.g. [P])

$$\left(\frac{|K|}{|B_2^n|}\right)^{1/n} \leq M^*(K). \tag{2.3}$$

An upper estimate for the k-diameters of K in terms of $M^*(K)$ originated in [M1]. Below we will use the result from [PT], with the best known constant proved in [Go2]. The estimate is also know as "Lower M^*-estimate".

Theorem 2.1 *Let $K \subset \mathbb{R}^n$ be a symmetric convex body and $\rho > 0$. Let $1 \leq k \leq n$ satisfy*

$$k > \left(\frac{\ell(K_\rho^0)}{\omega_k \, \rho}\right)^2 .$$

Then for a "random" k-codimensional subspace $E \subset \mathbb{R}^n$ one has

$$K \cap E \subset \rho B_2^n.$$

Let us also recall the following form of the Dvoretzky Theorem ([MSch], [P], [Go1]). The "moreover" part of it is one-sided estimate which follows from Milman's proof of the Dvoretzky Theorem. The dependence on ε in both parts of the theorem follows from Gordon's work.

Theorem 2.2 *Let $\varepsilon > 0$. Let $K \subset \mathbb{R}^n$ be a convex body with origin in its interior and let $R := R(K)$. Let $m \leq \varepsilon^2 \left(M^*(K)/R\right)^2 n$. Then for "random" projection P of rank m one has*

$$\frac{1-\varepsilon}{1+\varepsilon} \, M^*(K) \, PB_2^n \subset PK \subset \frac{1+\varepsilon}{1-\varepsilon} \, M^*(K) \, PB_2^n.$$

Moreover, if $M^*(K) < A < R$ and $m \leq \varepsilon^2 \left(A/R\right)^2 n$ then for "random" projection P of rank m one has

$$PK \subset \frac{1+\varepsilon}{1-\varepsilon} \, A \, PB_2^n.$$

We also will use Sudakov's inequality ([P], [Lif]).

Theorem 2.3 *Let $K \subset \mathbb{R}^n$ be a convex body with origin in its interior. Then for every $t > 0$ one has*

$$N(K, tB_2^n) \leq \exp\left(\kappa \left(\ell(K^0)/t\right)^2\right),$$

where $1 \leq \kappa \leq 4.8$ (in fact $\kappa \to 1$ very fast as $N(K, tB_2^n)$ grows).

Below we keep the notation κ for the constant from this theorem.

3 On the sharpness of Sudakov's inequality

Our starting point is a recent result from [LPT] valid for arbitrary symmetric bodies K and L.

Theorem 3.1 *Let $R > a > 0$ and $1 \leq k \leq n$. Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Let $K \subset RL$ and $K \cap E \subset aL$ for some k-codimensional subspace of $\mathbb{R}^n$. Then for every $r > a$ one has*

$$N(K, 2rL) \leq 2^k \left(\frac{R+r}{r-a}\right)^k.$$

Remark. The factor 2^k above can be replaced by a better function of k (see [LPT]).

Theorem 3.1 was used in [LPT] as an upper bound for the covering numbers, here we would like to interpret it as a lower bound for k-diameters. In this form it will provide an additional insight into Sudakov's inequality.

Theorem 3.2 *Let $R > 1$, $\eta > 0$. Let K and L be symmetric convex bodies in $\mathbb{R}^n$ such that $K \subset RL$. Assume that*

$$N(K, L) \geq \exp(\eta n).$$

Then for every k-codimensional subspace E of $\mathbb{R}^n$ with

$$k = \left[\frac{\eta\, n}{\ln(12R)}\right]$$

one has

$$K \cap E \not\subset \tfrac{1}{4}\, L.$$

Proof: Let $k = \left[\frac{\eta\, n}{\ln(12R)}\right]$. Denote by a the smallest real number $r \geq 0$ such that there exists a k-codimensional subspace E of $\mathbb{R}^n$ satisfying

$$K \cap E \subset r\, L.$$

Assume that $a < 1/2$ (otherwise the proof is finished). Then by Theorem 3.1 we have

$$\exp(\eta n) \leq N(K, L) \leq \left(\frac{2R+1}{1/2 - a}\right)^k.$$

Since $R > 1$ we obtain

$$a \geq \frac{1}{2} - \frac{2R+1}{\exp(\eta n/k)} > \frac{1}{2} - \frac{3R}{\exp(\eta n/k)} \geq 1/4.$$

□

From now on, we restrict ourselves to the case $L = B_2^n$, which is our main interest in this paper. By Sudakov's inequality (Theorem 2.3) we then have $N(K, B_2^n) \leq \exp(\kappa(M_K^*)^2 n)$, and let us now assume that this inequality is almost sharp, i.e.,

$$N(K, B_2^n) \geq \exp\left(\varepsilon (M_K^*)^2 n\right), \tag{3.1}$$

for some $\varepsilon > 0$.

Applying Theorem 3.2 directly, for $k = k_0 := \left[\frac{\varepsilon (M^*)^2 n}{\ln(12R(K))}\right]$, we get that every k-codimensional section of K has diameter at least $1/4$.

This insight into geometry of K can be strengthen even further by considering truncations of the body K. Namely, assuming again that K satisfies (3.1), for some $\varepsilon > 0$, and letting $\beta > 1$, we have two distinct possibilities:

I. Either the covering number $N(K, B_2^n)$ can be significantly decreased by cutting K on the level β, i.e., $N(K \cap \beta B_2^n, B_2^n)$ is essentially smaller (which in turn means that "most" of the entropy of K comes from parts far from B_2^n);

II. or every k'-codimensional section of K has large diameter, for an appropriate choice of $k' > k_0$ depending on β.

In the next section we study sufficient conditions for Case I to hold, while in Section 5 we return to consequences of essential sharpness of Sudakov's estimate.

4 Improving Sudakov's inequality

In connection with Case I in Section 3, we discuss the behavior of the covering numbers of K_β $(= K \cap \beta B_2^n)$, when β varies. Sudakov's inequality relates these numbers to $M^*(K_\beta)$, while our point below is to replace $M^*(K_\beta)$ by smaller $M^*(K_\rho)$, for some parameter $\rho < \beta$.

To prepare the discussion we consider a general statement, which combines Theorems 3.1 and 2.1.

Theorem 4.1 *Let $K \subset \mathbb{R}^n$ be a symmetric convex body. Let $\rho > 0$ and $\beta > 0$. Then for every $\gamma > \rho$ one has*

$$N(K_\beta, 2\gamma B_2^n) \le \left(2\,\frac{\beta+\gamma}{\gamma-\rho}\right)^{2(\ell(K_\rho^0)/\rho)^2}. \tag{4.1}$$

Remark. The proof below gives, actually, the exponent $k = \lceil \left(\ell(K_\rho^0)/\rho\right)^2 + 1/2\rceil$ in (4.1).

Proof: Let $k = \lceil (\ell(K_\rho^0)/\rho)^2 + 1/2\rceil$. Then, since $\omega_k^2 > 1 - 1/(2k)$, we have

$$\frac{\ell(K_\rho^0)}{\rho} \le \sqrt{k-1/2} < \omega_k\,\sqrt{k}.$$

By Theorem 2.1 there exists a k-codimensional subspace $E \subset \mathbb{R}^n$ such that

$$K \cap E \subset \rho B_2^n.$$

Applying Theorem 3.1 to the bodies K_β and B_2^n with $R = \beta$, $r = \gamma$, and $a = \rho$ we obtain the estimate announced in the Remark. Now if $\rho \ge R(K)$ then, clearly, $N(K_\beta, 2\gamma B_2^n) = 1$. If $\rho < R(K)$ then

$$\ell(K_\rho^0)/\rho = \ell\left((B_2^n \cap \tfrac{1}{\rho}K)^0\right) \ge 1,$$

which means that $k \le 2(\ell(K_\rho^0)/\rho)^2$. It proves the theorem. □

The next corollary is a partial case of Theorem 4.1, where we fix some of the parameters, in order to compare the results with Sudakov's inequality.

Corollary 4.2 *Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior. Let $\rho > 0$ and $\beta \ge \rho/3$. Then*

$$N(K_\beta, 4\,\rho\, B_2^n) \le \exp\left(2\left(\frac{\ell\left(K_\rho^0\right)}{\rho}\right)^2 \ln\frac{3\beta}{\rho}\right).$$

Proof: If $\beta \le 4\rho$ then the estimate is trivial. Otherwise let $\gamma = 2\rho$. Then, since $\beta \ge 2\gamma$,

$$2\,\frac{\beta+\gamma}{\gamma-\rho} \le \frac{3\beta}{\rho}.$$

Theorem 4.1 implies the desired result. □

First note that by Sudakov's inequality we have

$$N(K_\beta, 4\rho B_2^n) \leq \exp\left(\kappa\left(\frac{\ell\left(K_\beta^0\right)}{4\rho}\right)^2\right). \tag{4.2}$$

Therefore, if

$$4\sqrt{\tfrac{2}{\kappa}}\sqrt{\ln\tfrac{3\beta}{\rho}}\;\ell\left(K_\rho^0\right) \leq \ell\left(K_\beta^0\right),$$

then Corollary 4.2 improves Sudakov's inequality.

Now we shall consider coverings of the whole body, without additional truncations. Let K be a symmetric convex body. Given $\rho > 0$ define the function $F = F_K$ by

$$F(\rho) = \frac{\ell(K^0)}{\ell\left(K_\rho^0\right)}.$$

This function can be used to measure a possible gain in Sudakov's estimates. Rewriting Theorem 2.3 we get

$$N(K, 8\,\rho\,B_2^n) \leq \exp\left(\kappa\left(\frac{\ell(K_\rho^0)}{8\rho}\right)^2 F(\rho)^2\right),$$

which should be compared with the following:

Theorem 4.3 *Let K be a symmetric convex body and $\rho > 0$. Then*

$$N(K, 8\,\rho\,B_2^n) \leq \exp\left(2\left(\frac{\ell\left(K_\rho^0\right)}{\rho}\right)^2 \ln\left(6\,F(\rho)\right)\right).$$

Proof: It is known (and easy to check) that for every $t > 0$ one has $N(K, tB_2^n) = N(K, (2K) \cap tB_2^n)$. Therefore, for $\beta > 0$ and $\rho > 0$ we have

$$\begin{aligned} N(K, 8\,\rho\,B_2^n) &\leq N(K, (2K) \cap 2\beta B_2^n)\, N((2K) \cap 2\beta B_2^n, 8\,\rho\,B_2^n) \\ &= N(K, 2\beta B_2^n)\, N(K_\beta, 4\,\rho\,B_2^n). \end{aligned}$$

Now we apply Sudakov's inequality to estimate the first factor and Corollary 4.2 to estimate the second one. We obtain

$$N(K, 8\,\rho\, B_2^n) \le \exp\left(\kappa \left(\frac{\ell(K^0)}{2\beta}\right)^2 + 2\left(\frac{\ell\left(K_\rho^0\right)}{\rho}\right)^2 \ln\frac{3\beta}{\rho}\right).$$

Notice that $F(\rho) \ge 1$ and choose

$$\beta = \tfrac{1}{2}\sqrt{\kappa}\, F(\rho)\rho \ge \tfrac{1}{2}\,\rho.$$

Then

$$N(K, 8\,\rho\, B_2^n) \le \exp\left(2\left(\frac{\ell\left(K_\rho^0\right)}{\rho}\right)^2 \ln\left(1.5\sqrt{e\,\kappa}\, F(\rho)\right)\right),$$

which proves the theorem, since $\kappa < 5$. $\square$

5 Quasi M-position

We are now prepared to obtain a further consequence of sharpness of Sudakov's inequality.

Theorem 5.1 *Let $\varepsilon \in (0,1)$. Let K be a symmetric convex body normalized in such a way that $M^*(K) = 1$. Assume that*

$$N(K, 8\, B_2^n) \ge \exp(\varepsilon n)\,.$$

Then there exist a constant $0 < c_\varepsilon < 1$ depending only on ε such that for a "random" projection P of rank $m = [c_\varepsilon^2 n]$ one has

$$c_\varepsilon P B_2^n \subset PK \quad \text{and} \quad \left(\frac{|PK|}{|PB_2^n|}\right)^{1/m} \le 1/c_\varepsilon. \tag{5.1}$$

We refer to property (5.1) by saying, informally, that PK has a finite volume ratio.

Proof: Let

$$\gamma := F(1) = \frac{M^*(K)}{M^*(K \cap B_2^n)} = \frac{\ell(K^0)}{\ell\left((K \cap B_2^n)^0\right)} = \frac{\omega_n \sqrt{n}}{\ell\left((K \cap B_2^n)^0\right)}.$$

Applying Theorem 4.3 with $\rho = 1$ and using the assumption of our Theorem we conclude that

$$2 \ln (6\,\gamma) \geq \frac{\varepsilon\, n}{\left(\ell\left((K \cap B_2^n)^0\right)\right)^2} = \frac{\varepsilon\, \gamma^2}{\omega_n^2}.$$

Therefore there exists an absolute positive constant C such that

$$\gamma \leq C'_\varepsilon := C \sqrt{\frac{1}{\varepsilon} \ln\left(\frac{2}{\varepsilon}\right)}.$$

Therefore $M^*(K \cap B_2^n) \geq 1/C'_\varepsilon$ and, applying Theorem 2.2 to the body $K \cap B_2^n$, we obtain that for "random" projection P of rank $m = [n/(2C'_\varepsilon)^2]$ one has

$$\frac{1}{3\, C'_\varepsilon} P B_2^n \subset PK \cap B_2^n \subset PK.$$

On the other hand, by Urysohn's inequality (2.3) we obtain

$$\left(\frac{|PK|}{|PB_2^n|}\right)^{1/m} \leq M^*(PK) \leq \frac{\omega_n \sqrt{n}}{\omega_m \sqrt{m}} M^*(K) \leq 4 C'_\varepsilon.$$

That completes the proof. □

Remark The proof shows that c_ε can be taken as

$$c_\varepsilon = c_0 \sqrt{\varepsilon \,/\, \ln\left(\tfrac{2}{\varepsilon}\right)},$$

where c_0 is an absolute positive constant.

The property (5.1) exhibited above appeared in the theory already long time ago, in the context of M-positions of convex bodies. The existence of M-position was first proved in [M2], and we refer the interested reader to [P] and references therein for the definition and properties of M-position. Here let us just recall that an arbitrary convex body K in M-position has this property (5.1), moreover, "random proportional projection" of K has "finite volume ratio" for any proportion $0 < \lambda < 1$ of the dimension n. This nowadays appears to be the main property of bodies in M-position used in applications. Theorem 5.1 shows that such a property for some proportion of n (with some dependence of parameters), is a consequence of some tightness of covering estimates. We feel that this property may be important for understanding the geometry of convex bodies, especially when we investigate covering numbers by ellipsoids. With this in mind we introduce a new (slightly informal) definition:

Definition. Let K be a convex body in $\mathbb{R}^n$. We say that K is in a *quasi M-position* (for a proportion $0 < \lambda < 1$) if "random" proportional projection of K onto λn-dimensional subspace has finite volume ratio.

The next corollary gives another example of bodies in quasi M-position. This is a variant of Theorem 5.1 in which the hypothesis about $M^*(K)$ is replaced by a weaker condition of an upper estimate for entropies.

Corollary 5.2 *Let $0 < \delta < \varepsilon < 1 < A$. Let K be a symmetric convex body. Assume that*

$$N(K, B_2^n) \geq \exp(\varepsilon n) \quad \textit{and} \quad N(K, A\, B_2^n) \leq \exp(\delta n)$$

Then there exist positive constants c, $\bar{c}$, C depending only on ε, δ, A such that for "random" projection P of rank $m = [cn]$ one has

$$\bar{c}\, P B_2^n \subset PK \quad \textit{and} \quad \left(\frac{|PK|}{|PB_2^n|}\right)^{1/m} \leq C,$$

i.e. K is in a quasi M-position for a proportion c.

Proof: First note that the estimate for volumes follows immediately from covering estimates.

To show the existence of the desired projection note that we have

$$\begin{aligned} e^{\varepsilon n} &\leq N(K, B_2^n) \leq N(K, (2K) \cap A\, B_2^n)\, N((2K) \cap A\, B_2^n, B_2^n) \\ &\leq e^{\delta n}\, N(K \cap \tfrac{A}{2}\, B_2^n, \tfrac{1}{2} B_2^n). \end{aligned}$$

To estimate $(\varepsilon - \delta)n$ we can use either of two following ways:

[i] Sudakov's inequality implies

$$(\varepsilon-\delta)n \le 4\kappa\left(\ell\left(\left(K\cap \tfrac{A}{2}\,B_2^n\right)^0\right)\right)^2 .$$

[ii] Corollary 4.2 implies

$$(\varepsilon-\delta)n \le 2\left(8\ell\left(\left(K\cap \tfrac{1}{8}\,B_2^n\right)^0\right)\right)^2 \ln(12A).$$

Now the result follows from Theorem 2.2 in the same way as in the proof of Theorem 5.1. □

Remark. The proof above shows that Corollary 5.2 holds with (at least) two choices of constants c, $\bar{c}$, C:

[i]

$$c=\frac{c_0\,(\varepsilon-\delta)}{A^2},\qquad \bar{c}=c_1\sqrt{\varepsilon-\delta},\qquad C=A\exp(2\delta/c),$$

[ii]

$$c=\frac{c_0\,(\varepsilon-\delta)}{\ln(12A)},\qquad \bar{c}=\frac{c_1\sqrt{\varepsilon-\delta}}{\sqrt{\ln(12A)}},\qquad C=A\exp(2\delta/c),$$

where c_0, c_1 are absolute positive constant.

6 Comparing k-diameters and covering numbers

Here we discuss lower estimates for Euclidean covering numbers of a body in terms of k-diameters of its skeleton. More precisely, we get inequalities between k-diameters of a body (or its skeleton) and a covering number of K_β $(=K\cap\beta B_2^n)$ for some β, by small balls. We have already seen (Theorem 3.1) that a small k-diameter of K implies an upper bound for covering of K_β. On the other hand we show here that if such a covering is small then, for some m, m-diameter of any absolute skeleton is small as well.

Theorem 6.1 *Let $1 \le k \le n$. Let K be a symmetric convex body in $\mathbb{R}^n$. Let $\beta > a := c_{k+1}(K)$. Let $\rho > 0$, $0 < \delta < 1$ be such that*

$$M^*(K_{\rho/2}) \le \tfrac{1}{8}\delta M^*(K_\beta). \tag{6.1}$$

Then

$$N\left(K_\beta, \rho\, B_2\right) \ge \left(\tfrac{1}{\delta}\right)^m,$$

where

$$m = \left[\tfrac{1}{9}\left(\tfrac{M^*(K_\beta)}{\beta}\right)^2 n\right] > \tfrac{1}{18}\left(\tfrac{a}{\beta}\right)^2 k - 1.$$

Remark. 1. In fact we show that for every $0 < \varepsilon < 1$ and every $\rho > 0$, $0 < \delta < 1$ such that

$$2M^*(K_{\rho/2}) \le \delta \left(\tfrac{1-\varepsilon}{1+\varepsilon}\right)^2 M^*(K_\beta). \tag{6.2}$$

one has the lower bound from the Theorem with

$$m = \left[\varepsilon^2\left(\tfrac{M^*(K_\beta)}{\beta}\right)^2 n\right] > \left(\tfrac{\varepsilon\, a}{\beta}\right)^2 \left(k - \tfrac{1}{2}\right) - 1.$$

2. Condition (6.1) can be viewed in two different ways. Firstly, if we fix ρ and let δ be the smallest satisfying (6.1), we obtain a lower bound for covering numbers in terms of the ratio of M^*'s. Secondly, if we fix δ and chose the best ρ satisfying (6.1), we get a lower estimate for the entropy number (see [P] for the precise definition).

Proof: We will show the estimate from Remark 1. The Theorem follows by taking $\varepsilon = 1/3$. Without loss of generality we assume that $\beta \le R(K)$.

Denote $N := N(K_\beta, \rho\, B_2^n)$. Then there are $x_i \in \mathbb{R}^n$, $i \le N$, such that

$$K_\beta \subset \bigcup_{i=1}^{N} x_i + \rho\, B_2^n. \tag{6.3}$$

Since we cover K_β by the Euclidean balls, without loss of generality we can assume $x_i \in K_\beta$. Therefore

$$K_\beta \subset \bigcup_{i=1}^{N}(x_i + \rho\, B_2^n) \cap K_\beta \subset \bigcup_{i=1}^{N} x_i + (\rho\, B_2^n) \cap (K_\beta - x_i). \tag{6.4}$$

Since K is centrally symmetric and, by (6.1), $\rho/2 < \beta$, we obtain

$$K_\beta \subset \bigcup_{i=1}^N x_i + (\rho\, B_2^n) \cap (2K_\beta) = \bigcup_{i=1}^N x_i + 2K_{\rho/2}.$$

Denote

$$m := \left[\varepsilon^2 \left(\frac{M^*(K_\beta)}{\beta}\right)^2 n\right] \le \left[\varepsilon^2 \left(\frac{M^*(K_{\rho/2})}{\rho/2}\right)^2 n\right].$$

By Theorem 2.2 we obtain that for a random projection P of rank m one has

$$\frac{1-\varepsilon}{1+\varepsilon}\, M^*(K_\beta)\, PB_2^n \subset PK_\beta$$

and, by “moreover” part of Theorem 2.2, for every $i \le N$

$$P\left(2K_{\rho/2}\right) \subset 2\frac{1+\varepsilon}{1-\varepsilon}\, M^*\left(K_{\rho/2}\right)\, PB_2^n.$$

It implies that

$$\frac{1-\varepsilon}{1+\varepsilon}\, M^*(K_\beta)\, PB_2^n \subset \bigcup_{i=1}^N Px_i + 2\frac{1+\varepsilon}{1-\varepsilon}\, M^*\left(K_{\rho/2}\right)\, PB_2^n.$$

Thus, by comparison of volumes, for every ρ satisfying

$$2M^*(K_{\rho/2}) \le \delta \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 M^*(K_\beta)$$

one has

$$N \ge \delta^{-m}.$$

Finally notice that, by Theorem 2.1,

$$a < \frac{M^*(K_\beta)\sqrt{n}}{\omega_k \sqrt{k}}, \tag{6.5}$$

which implies

$$(\varepsilon\omega_k a/\beta)^2\, k < m+1.$$

Since $\omega_k^2 > 1 - 1/(2k)$, we obtain the desired result. □

The following Theorem shows that the use of skeletons allows to avoid estimating the ratio of M^*'s (and the parameter ρ) in Theorem 6.1, as explained in Remark 2 after that theorem. Thus, it provides another lower estimate for covering numbers.

Theorem 6.2 *Let $\delta \in (0,1)$, and $\beta > 2a > 0$. There exists a constant α, depending only δ such that the following statement holds:*

Let K be a symmetric convex body such that $R(K) \geq \beta$. Let T be an absolute $(2\alpha a)$-separated skeleton of K_β and m be such that $c_{m+1}(T) \geq a$ (i.e. the m-diameter of T is not smaller than a). Then

$$\delta^{-m_0} \leq N\left(K_\beta, \alpha\, \beta\, B_2^n\right),$$

where

$$m_0 = \left[\frac{a^2\, m}{2\, \beta^2}\right].$$

Remarks. 1. In fact we will prove slightly stronger result, namely that for every $\varepsilon \in (0,1)$, $\delta \in (0,1)$, and $\beta > 2a > 0$ there exists a constant α, depending only on ε and δ such that the statement above holds with

$$m_0 = \left[\left(\tfrac{\varepsilon\, a}{\beta}\right)^2 \left(m - \tfrac{1}{2}\right)\right].$$

Moreover, our proof gives that α can be taken to be equal to $C_{\varepsilon\,\delta} \ln^{-3}\left(1/C_{\varepsilon\,\delta}\right)$, where

$$C_{\varepsilon\,\delta} = \min\left\{c,\ \frac{\delta}{\sqrt{\ln(1/\delta)}}\ \frac{(1-\varepsilon)^2}{\varepsilon}\right\},$$

for some absolute constant $c \in (0, 1/4]$.

2. Taking $a = 1$, $\delta = 1/4$, $\beta > 2$, and ε close enough to 1 we obtain that if the m-diameter of T is not smaller than 1 then

$$2^{m/\beta^2} \leq N\left(K_\beta, c\, \beta\, B_2^n\right),$$

where c is an absolute positive constant.

In the proof of Theorem 6.2 we will use the following result by Milman-Szarek ([MSz]).

Theorem 6.3 *Let $m \geq 1$. Let $S \subset \mathbb{R}^n$ be a finite set of cardinality m and let T be the convex hull of S. Then for every $0 < r < R(T)$ one has*

$$M^*(T \cap (r\, B_2^n)) \leq Cr \left(\ln \frac{2R(T)}{r}\right)^3 \sqrt{\frac{\ln \max\{m, N\}}{n}},$$

where $N = N(T, (r\, B_2^n))$ and C is an absolute constant.

Proof of Theorem 6.2: Let α be of the form given in Remark 1 for some (small) positive constant c. Denote $\rho := \alpha\beta$ and $N = N(K_\beta, \rho B_2^n)$. We will argue by contradiction. Assume that $N < \delta^{-m_0}$.

Let S be a (2ρ)-separated set for K_β. Then, as is discussed in the first section, the cardinality of S does not exceed N and S is a (2ρ)-net for K. Let T be the absolute convex hull of S. Denote $b = \beta - 2\rho > a$. Since $K \not\subset \beta B_2^n$ and S is a (2ρ)-net for K we obtain $T \not\subset b B_2^n$. Apply Theorem 6.1 to the body T_b with parameter b. We have that whenever ρ satisfies

$$2M^*(T_{\rho/2}) \leq \delta C_\varepsilon M^*(T_b), \tag{6.6}$$

where $C_\varepsilon = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2$, one has

$$N(T_b, \rho B_2) \geq \left(\frac{1}{\delta}\right)^{m_1},$$

where

$$m_1 = \left[\varepsilon^2 \left(\frac{M^*(T_b)}{b}\right)^2 n\right] \geq \left[\left(\frac{\varepsilon\omega_m a}{b}\right)^2 m\right] \geq m_0.$$

This would give a contradiction and thus prove the theorem. Thus to complete the proof it is enough to verify (6.6) for our choice of ρ. First note that by (6.5) we have

$$M^*(T_b) \geq \omega_m\, a\sqrt{m/n}.$$

On the other hand, since $T = \operatorname{conv}\{S, -S\} \subset \beta B_2^n$ and $N(T, 2\rho B_2^n) \leq N$, by Theorem 6.3,

$$2M^*(T_{\rho/2}) \leq 2\,M^*(T_{2\rho}) \leq 4\,C\,\rho\left(\ln\frac{\beta}{\rho}\right)^3 \sqrt{\frac{\ln(2N)}{n}},$$

where C is an absolute positive constant. Therefore to satisfy (6.6) it is enough to have for some absolute positive constant C_1

$$C_1\,\rho\left(\ln\frac{\beta}{\rho}\right)^3 \sqrt{\ln(2N)} \leq \delta\, C_\varepsilon \sqrt{m}\; a,$$

or, using the assumption $N < (1/\delta)^{m_0}$,

$$C_2\,\rho\left(\ln\frac{\beta}{\rho}\right)^3 \varepsilon\sqrt{\ln(1/\delta)} \leq \delta\, C_\varepsilon\, \beta$$

for some absolute positive constant C_2. Clearly there exists a choice of absolute constant c such that our ρ satisfies the last inequality. It proves the result. $\Box$

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A. E. Litvak and N. Tomczak-Jaegermann,
Department of Mathematical and Statistical Sciences,
University of Alberta,
Edmonton, Alberta, Canada T6G 2G1,
alexandr@math.ualberta.ca
nicole.tomczak@ualberta.ca

V. D. Milman
Department of Mathematics,
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv, Israel
milman@math.tau.ac.il

A. Pajor,
Équipe d'Analyse et Mathématiques Appliquëes,
Université de Marne-la-Vallée,
5, boulevard Descartes, Champs sur Marne,
77454 Marne-la-Vallée, Cedex 2, France
Alain.Pajor@univ-mlv.fr