

Essential uniqueness of M-ellipsoid of a given convex body *

V.D. Milman A. Pajor

Abstract

We show that different M-ellipsoids associated to a convex body, that are introduced in Asymptotic convexity, are “essentially” unique, in the sense that they admit equivalent large dimensional sections.

1 Introduction

This is a short note, just an observation, but curious and important enough to be broadly known. So we decided to write it down.

We use the following standard notation. The space \mathbb{R}^n is equipped with the canonical Euclidean scalar product (\cdot, \cdot) . Its unit ball is denoted by D .

Let K be a convex body in \mathbb{R}^n such that 0 is in its interior. Its polar K° is defined as usual by

$$K^\circ = \{x \in \mathbb{R}^n : (x, y) \leq 1 \text{ for every } y \in K\}.$$

Let A and B be two subsets of \mathbb{R}^n . The covering number $N(A, B)$ is defined as usual as

$$N(A, B) = \min \{\sharp \Lambda : \Lambda \subset \mathbb{R}^n, A \subset \Lambda + B\}$$

where $\Lambda + B = \{x + y : x \in \Lambda, y \in B\}$.

The volume of a measurable subset A of \mathbb{R}^n is denoted by $|A|$. Let $\sigma > 0$ and let K be a convex compact subset of \mathbb{R}^n with 0 in its interior. We say

*The first author was partially supported by the Israel Science Foundation founded by the Academy of Sciences and Humanities.

that an ellipsoid \mathcal{E} of \mathbb{R}^n is an M-ellipsoid of K with constant σ , or shortly an M-ellipsoid of K , if, setting $\lambda = (|K|/|\mathcal{E}|)^{1/n}$ in order that $|K| = |\lambda \mathcal{E}|$, we have

$$N(K, \lambda \mathcal{E}) \leq e^{\sigma n}.$$

We assume below in the paper that ellipsoids are centered at 0.

It is proved in [M1] (see also [M2] and [Pi] for simplified proofs) that there exists a universal constant such that for every n , every n -dimensional symmetric convex body has an M-ellipsoid with respect to this constant. Many interesting properties of centrally symmetric convex bodies and corresponding normed spaces were revealed using M-ellipsoids. The existence of an M-ellipsoid was established even for non centrally-symmetric bodies in [M-P1]. We refer to the survey in [G-M].

It is proved in [M-P1] that if \mathcal{E} is an M-ellipsoid of $K \subset \mathbb{R}^n$ with constant σ , and if 0 is the barycenter of K , then there exist $\sigma_1 \geq \sigma$ depending only on σ such that

$$N(\lambda \mathcal{E}, K) \leq e^{\sigma_1 n}, N(K^\circ, (\lambda \mathcal{E})^\circ) \leq e^{\sigma_1 n} \text{ and } N((\lambda \mathcal{E})^\circ, K^\circ) \leq e^{\sigma_1 n}.$$

Of course, we cannot expect the uniqueness of such an ellipsoid and since the definition is of isomorphic type, it should at least depend on the parameter σ . At the same time, one can construct different M-ellipsoids, having deep and non-trivial additional properties. We say that an ellipsoid \mathcal{E} is a regular M-ellipsoid of order α with constant σ for a convex body $K \subset \mathbb{R}^n$, if $|\mathcal{E}| = |K|$ and for any $t > 0$,

$$N(K, t\mathcal{E}) \leq e^{\sigma n/t^\alpha} \text{ and } N(\mathcal{E}, tK) \leq e^{\sigma n/t^\alpha}. \quad (1)$$

It is proved in [Pi] that for any $0 < \alpha < 2$, there exists $\sigma(\alpha) = O((2-\alpha)^{1/2})$ as $\alpha \rightarrow 2$, such that every centrally symmetric convex body K has an ellipsoid \mathcal{E}_α which is a regular M-ellipsoid of order α with constant $\sigma(\alpha)$ for K . Note that the covering numbers in (1) do not depend on the choice of the center. The previous assumption on the baricenter of K was needed because this statement was using polarity.

These ellipsoids depend on α and the purpose of this note is to show that surprisingly, in some “essential” way these M-ellipsoids are unique. This question arise from consideration of different M-ellipsoids in [M-P2].

2 Essential uniqueness of M-ellipsoids

We begin by a definition of essential uniqueness.

Definition. Let c be a positive function on $(0, 1)$. Two ellipsoids \mathcal{E}_1 and \mathcal{E}_2 in \mathbb{R}^n are called *essentially c -equivalent* if for every $0 < \varepsilon < 1$, there is a subspace $E \subset \mathbb{R}^n$ of dimension larger than $(1 - \varepsilon)n$, such that

$$\frac{1}{c(\varepsilon)} \mathcal{E}_2 \cap E \subset \mathcal{E}_1 \cap E \subset c(\varepsilon) \mathcal{E}_2 \cap E. \quad (2)$$

As we will actually compare two ellipsoids in a given n -dimensional space, we put “ $c(\varepsilon)$ ” in the definition. However, in Asymptotic Convexity, one compares two family of ellipsoids, say $\mathcal{E}_1(n)$ and $\mathcal{E}_2(n)$, $n \in \mathbb{N}$, and call these families *essentially equivalent* if for every $0 < \varepsilon < 1$, there exists $c(\varepsilon) > 0$ such that for every integer n , there is a subspace $E(\varepsilon) \subset \mathbb{R}^n$ of dimension larger than $(1 - \varepsilon)n$ such that

$$\frac{1}{c(\varepsilon)} \mathcal{E}_2(n) \cap E(\varepsilon) \subset \mathcal{E}_1(n) \cap E(\varepsilon) \subset c(\varepsilon) \mathcal{E}_2(n) \cap E(\varepsilon).$$

We can now state our result on uniqueness of M-ellipsoid.

Theorem. *Let $K \subset \mathbb{R}^n$ be any convex compact body. Let \mathcal{E}_1 and \mathcal{E}_2 be two M-ellipsoids of K , with the same volume $|K| = |\mathcal{E}_1| = |\mathcal{E}_2|$ and the same constant σ . Then there exists a function $c > 0$ on $(0, 1)$, depending only on σ , such that \mathcal{E}_1 and \mathcal{E}_2 are essentially c -equivalent.*

Proof. In the proof, we use notation $\sigma_1, \sigma_2, ..$ to denote positive numbers depending only on the parameter σ associated to the M-ellipsoids \mathcal{E}_1 and \mathcal{E}_2 given above. As noticed previously, there exists $\sigma_1 > 0$ such that

$$N(K, \mathcal{E}_i) \leq e^{\sigma_1 n} \text{ and } N(\mathcal{E}_i, K) \leq e^{\sigma_1 n}, \quad i = 1, 2.$$

Therefore,

$$N(\mathcal{E}_1, \mathcal{E}_2) \leq N(\mathcal{E}_1, K) \cdot N(K, \mathcal{E}_2) \leq e^{\sigma_2 n}$$

for some new constant σ_2 . This means that \mathcal{E}_2 is an M-ellipsoid of \mathcal{E}_1 , with constant σ_2 .

It is a well known property of an M-ellipsoid \mathcal{E} of a symmetric convex body C with constant σ and same volume as C , that

$$|C \cap \mathcal{E}| \geq \sigma_3^n |C|$$

where $\sigma_3 > 0$ depends only on σ (see [M2]). Applying this property to the pair \mathcal{E}_1 and \mathcal{E}_2 , we have

$$|\mathcal{E}_1 \cap \mathcal{E}_2| \geq \sigma_3^n |K|.$$

At the same time, the intersection of two ellipsoids is $\sqrt{2}$ -equivalent to an ellipsoid, which means that there exists an ellipsoid \mathcal{E} such that

$$\mathcal{E} \subset \mathcal{E}_1 \cap \mathcal{E}_2 \subset \sqrt{2} \mathcal{E}.$$

Thus $N(\mathcal{E}_i, \mathcal{E}) \leq e^{\sigma_4 n}$, $i = 1, 2$, for some new constant σ_4 . We conclude that

$$\mathcal{E} \subset \mathcal{E}_i \text{ and } N(\mathcal{E}_i, \mathcal{E}) \leq e^{\sigma_4 n}, i = 1, 2.$$

It is now standard and trivial to show that for any $\varepsilon > 0$, there are subspaces $E_i(\varepsilon)$, $i = 1, 2$, of dimension larger than $(1 - (\varepsilon/2))n$ such that

$$\mathcal{E} \cap E_i(\varepsilon) \subset \mathcal{E}_i \cap E_i(\varepsilon) \subset e^{\sigma_5/\varepsilon} \mathcal{E} \cap E_i(\varepsilon)$$

for some new constant σ_5 . Indeed, let \mathbb{R}^n be equipped with the Euclidean structure associated to \mathcal{E} , then for each $i = 1, 2$, $E_i(\varepsilon)$ is spanned by the $(1 - (\varepsilon/2))n$ axes of \mathcal{E}_i corresponding to the $(1 - (\varepsilon/2))n$ smallest eigenvalues. Now let $E(\varepsilon) = E_1(\varepsilon) \cap E_2(\varepsilon)$. It has dimension larger than $(1 - \varepsilon)n$ and (2) is satisfied with $c(\varepsilon) = e^{\sigma_5/\varepsilon}$, which shows that \mathcal{E}_1 and \mathcal{E}_2 are essentially c -equivalent. \square

Corollary 1. *Let $K \subset \mathbb{R}^n$ be a convex body containing 0. Let \mathcal{E} be an M-ellipsoid of K with constant σ and let $0 < \alpha < 2$. Then for every $\varepsilon > 0$, there exists a subspace $E \subset \mathbb{R}^n$ with dimension larger than $(1 - \varepsilon)n$, such that for every $t > 0$, one has*

$$N(K \cap E, t c(\varepsilon) \mathcal{E} \cap E) \leq e^{\sigma_1 n/t^\alpha}$$

where σ_1 and $c(\varepsilon)$ depend only on α and σ .

Proof. We first assume that K is centrally symmetric. Then as we recalled above, there exists a regular M-ellipsoid \mathcal{E}_α of K of order α and with same volume as \mathcal{E} . Since it is also an M-ellipsoid for K , it follows from the previous Theorem that \mathcal{E} and \mathcal{E}_α are essentially equivalent. Therefore, using (1) and (2) there exists a subspace E with dimension larger than $(1 - \varepsilon)n$, such that for every $t > 0$, we have

$$N(K \cap E, t c(\varepsilon) \mathcal{E} \cap E) \leq N(K \cap E, t \mathcal{E}_\alpha \cap E) \leq e^{\sigma_1 n/t^\alpha}$$

where σ_1 and $c(\varepsilon)$ depend only on α and σ . To get the right-hand side inequality above, observe that because \mathcal{E}_α is centrally symmetric, one has $N(K \cap E, t\mathcal{E}_\alpha \cap E) \leq N(K, (t/2)\mathcal{E}_\alpha)$.

When K is not centrally symmetric the existence of a regular M-ellipsoid of K of order α is still an open problem. Let $K' = K - K$ be the difference body. Since we assume that $0 \in K$, it follows that $K \subset K'$. It is proved in [M-P1], that an M-ellipsoid for K is also an M-ellipsoid for K' and vice-versa, only the constants associated to them differ by a universal factor. Let \mathcal{E} be an M-ellipsoid for K with same volume as K' . Recall that from [R-S], we have $|K| \leq |K'| \leq 4^n|K|$. Thus without loss of generality we can assume that $1 \leq |K|/|\mathcal{E}| \leq 4^n$. We now use the first part of the proof with K' , which is centrally symmetric. Let \mathcal{E}'_α be a regular M-ellipsoid for K' of order α and with same volume as \mathcal{E} . These are M-ellipsoids for K' . From the above Theorem, they are essentially equivalent. Therefore, there exists a subspace E with dimension larger than $(1 - \varepsilon)n$, such that for every $t > 0$, we have

$$N(K' \cap E, tc(\varepsilon)\mathcal{E} \cap E) \leq N(K' \cap E, t\mathcal{E}'_\alpha \cap E) \leq e^{\sigma'_1 n/t^\alpha}$$

where σ'_1 and $c'(\varepsilon)$ depend on α and σ . The result follows from $K \subset K'$. \square

Corollary 2. *Let $K \subset \mathbb{R}^n$ be a convex body with 0 as barycenter. Let \mathcal{E}_1 be an M-ellipsoid of $K \cap (-K)$ and \mathcal{E}_2 be an M-ellipsoid of $K - K$, with the same volume $|K| = |\mathcal{E}_1| = |\mathcal{E}_2|$ and the same constant σ . Then there exists a function $c > 0$ on $(0, 1)$, depending only on σ , such that \mathcal{E}_1 and \mathcal{E}_2 are essentially c -equivalent.*

Proof. It is proved in [M-P1] that when 0 is the barycenter of K , both ellipsoids \mathcal{E}_1 and \mathcal{E}_2 are M-ellipsoids of K . The result follows from the Theorem.

3 References

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V. Milman

Department of Mathematics, Tel Aviv University,
Ramat Aviv, Israel.

e-mail: vitali@math.tau.ac.il

A. Pajor

Equipe d'Analyse et Mathématiques Appliquées,
Université de Marne-la-Vallée

5 boulevard Descartes, Champs sur Marne
77454 Marne-la-Vallee cedex 2, France.

e-mail: pajor@math.univ-mlv.fr