

ISOTROPIC POSITION AND INERTIA ELLIPSOIDS AND ZONOIDS OF THE UNIT BALL OF A NORMED n -DIMENSIONAL SPACE

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Introduction

We start with some notations which will be used throughout the paper. We consider the euclidean space \mathbb{R}^n with the canonical inner product denoted $\langle \cdot, \cdot \rangle$ and the norm $|x|^2 = |\langle x, x \rangle| = \sum_1^n |x_i|^2$, $x = (x_i)_{i=1}^n \in \mathbb{R}^n$. Denote $D_n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ be the unit euclidean ball, vol_n be the Lebesgue n -dimensional volume form normalized such that

$$\text{vol}_n D_n = \pi^{n/2} / \Gamma(1 + n/2).$$

Note also that we usually omit a subindex n if the dimension is clear from the text and write D instead of D_n and vol instead of vol_n . We denote $S^{n-1} = \partial D$ to be the unit euclidean sphere and $\sigma_{n-1} (= \sigma)$ to be probability rotation invariant measure on S^{n-1} .

We study in this paper convex symmetric bodies in \mathbb{R}^n . We standardly denote such a body by K . Then $\|\cdot\|_K$ is a norm with $K = \{x \in \mathbb{R}^n \mid \|x\|_K \leq 1\}$ be the unit ball of this norm.

One of the important problems in Local Theory of normed spaces is introducing the "right" euclidean structure in a normed space $X = (\mathbb{R}^n, \|\cdot\|_K)$ or, in other words, to construct a special ellipsoid corresponding to the body K . There exist many such ellipsoids which we use for different purposes (see, e.g., [M.Sch] for a few of the most famous ones). We think that a more "mechanical" view of K may be helpful in this respect and we devote this paper to the study of different inertia ellipsoids. Much of the material covered is old and classical. However, from our own experience we know that experts in Geometric Functional Analysis are not familiar with most of these classic facts. Also revision of results of 50 to 80 years ago seems to be useful especially as new and

unexpected applications have been found for them. Naturally, we improve, on the way, some known inequalities and find interesting connections between classical convexity theory (in the spirit of Blaschke, F. John, Busemann and others) and the modern asymptotic theory of normed spaces. Indeed, at the start of convexity theory, a low dimensional case was the center of attention. At the present time, the high dimensional effects and asymptotic properties of a convex body (when dimension grows to infinity) are the most studied subjects. So, among other purposes, our revision aims to update the old approaches and questions to the new spirit of convexity theory.

The paper contains 5 sections. We introduce Legendre and Binet inertia ellipsoids in the first section and study them from different points of view. We also connect them with the *centroid body* of K which, in this paper, we call the *zonoid of inertia* of K . We also introduce the isotropic position of body K which will be studied in the next sections. The second section is devoted to different inequalities of a Khinchine type for convex bodies and to some of their applications. In sections 3 and 4 we study volume of sections of convex bodies and connect this study with many other remarkable problems of convexity theory in the spirit of Blaschke and Busemann. The major open problem of this theory is formulated in section 5, although it is, indeed, discussed throughout this paper. In section 5, we also connect this problem with many other known open problems of finite dimensional convexity theory.

Before starting, let us introduce one more standard notation: $A \lesssim B$ will mean that for some numerical constant c we have $A \leq c \cdot B$; similarly $A \sim B$ (or $A \simeq B$) means that for some numerical constants $c_1 > 0$ and C_2 we have $c_1 A \leq B \leq C_2 A$.

1. Inertia Ellipsoids and Zonoids

All through this section, we consider a fixed convex symmetric compact body K in \mathbb{R}^n .

1.1. Legendre's ellipsoid of inertia: Analytic description.

A classical fact, having its roots in mechanics, is stated thus: there exists a unique ellipsoid, say $\mathcal{L} = \mathcal{L}(K)$, which has the same moments of inertia as K , with respect to every axis. This ellipsoid is called the Legendre ellipsoid of K . It is therefore defined by the equalities:

$$(1.1) \quad \int_K |\langle x, \theta \rangle|^2 dx = \int_{\mathcal{L}} |\langle x, \theta \rangle|^2 dx \quad \text{for every } \theta \text{ in } \mathbb{R}^n.$$

Legendre's ellipsoid may also be introduced from a *probabilistic point of view*. Let X_1, \dots, X_m be independent random vectors uniformly distributed on K . From the central limit theorem, the

density of $(X_1 + X_2 + \dots + X_m)/\sqrt{m}$ converges as m tends to infinity towards the density of a gaussian vector in \mathbb{R}^n , say G , such that

$$\mathbb{E}|\langle G, \theta \rangle|^2 = \frac{1}{\text{vol}(K)} \int_K |\langle x, \theta \rangle|^2 dx$$

for every θ in \mathbb{R}^n .

The covariance matrix Γ of G defines an euclidean norm with the unit ball C by $\|x\|_C^2 = \langle x, \Gamma^{-1}x \rangle/2$, for every x in \mathbb{R}^n . We have

$$\mathbb{E}|\langle G, \theta \rangle|^2 = \frac{1}{(2\pi)^{n/2}(\det \Gamma)^{1/2}} \int_{\mathbb{R}^n} e^{-\|x\|_C^2} |\langle x, \theta \rangle|^2 dx$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-\|x\|_C^2} |\langle x, \theta \rangle|^2 dx &= \int_{\mathbb{R}^n} |\langle x, \theta \rangle|^2 \left(\int_{\|x\|_C^2}^{+\infty} e^{-t} dt \right) dx \\ &= \int_0^{+\infty} e^{-t} \left(\int_{\|x\|_C^2 \leq t} |\langle x, \theta \rangle|^2 dx \right) dt \\ &= \int_0^{+\infty} e^{-t} t^{(n+2)/2} dt \int_C |\langle x, \theta \rangle|^2 dx. \end{aligned}$$

Therefore we get

$$\mathbb{E}|\langle G, \theta \rangle|^2 = \frac{\Gamma(n+2)}{(2\pi)^{n/2}(\det \Gamma)^{1/2}} \int_C |\langle x, \theta \rangle|^2 dx$$

and we conclude that there exists a positive scalar λ such that the ellipsoid λC satisfies (1.1).

1.2 Geometric approach to the Legendre ellipsoid; the centroid body.

Let K_1 and K_2 be subsets of \mathbb{R}^n and $\lambda_1 \geq 0$, $\lambda_2 \geq 0$. The Minkowski sum $\lambda_1 K_1 + \lambda_2 K_2$ is defined by

$$\lambda_1 K_1 + \lambda_2 K_2 = \{ \lambda_1 x_1 + \lambda_2 x_2 : x_1 \in K_1, x_2 \in K_2 \}.$$

If μ is a discrete measure on \mathbb{R}^n , say $\mu = \sum_{i=1}^m \alpha_i \delta_{x_i}$, $\alpha_i \geq 0$, then we define

$$\int [0, x] d\mu(x) = \sum_{i=1}^m \alpha_i [0, x_i]$$

where $[0, x] = \{ \lambda x : 0 \leq \lambda \leq 1 \}$, $x \in \mathbb{R}^n$.

Now if μ is compactly supported, it is possible, by passing to a limit, to give a sense to the integral: $\int [0, x] d\mu(x)$.

Definition. The *zonoid of inertia* of K (= the *centroid body* of K in the terminology of Petty [Pet]), denoted $Z(K)$ is the convex body defined by

$$(1.2) \quad Z(K) = \frac{1}{\text{vol}(K)} \int_K [0, x] dx .$$

Since K is symmetric, the inertia zonoid $Z(K)$ is also symmetric. Now the supporting functional of the Minkowski sum of convex bodies is the sum of the supporting functional of these bodies. Hence from the symmetry of $Z(K)$, the supporting functional of $Z(K)$, that is the norm associated with the polar body $Z(K)^\circ$ is:

$$(1.3) \quad \|\theta\|_{Z(K)^\circ} = \frac{1}{2} \cdot \frac{1}{\text{vol}(K)} \int_K |\langle x, \theta \rangle| dx .$$

By definition, the centroid body (= inertia zonoid) $Z(K)$ is obtained from a limit of Minkowski sum of segments. Therefore $Z(K)$ is a zonoid. Recall that a zonotope is a sum of segments and a zonoid is a limit of zonotopes, in the Hausdorff sense. For more information on the centroid body see [Pet] or [BLM].

We will see in section 1.5 that $Z(K)$ is uniformly equivalent (in the Hausdorff sense) up to a homothety to the Legendre ellipsoid and this is our reason for calling $Z(K)$ "the inertia zonoid".

1.3 Binet Ellipsoid of inertia.

The body K also defines naturally another ellipsoid when considering the quadratic form $q(\theta) = \int_K |\langle x, \theta \rangle|^2 dx$. This ellipsoid, say β , is called the Binet fundamental ellipsoid of K . It is defined by

$$\|\theta\|_\beta^2 = \int_K |\langle x, \theta \rangle|^2 dx , \quad \theta \in \mathbb{R}^n .$$

The connection between Legendre and Binet ellipsoids is well known in mechanics (see [J]). They are up to a constant dual to each other. More precisely

$$(1.4) \quad \beta = \sqrt{\frac{n+2}{\text{vol}(\mathcal{L})}} \mathcal{L}^\circ .$$

Indeed, let \mathcal{L} in \mathbb{R}^n be an ellipsoid. Then for every θ in \mathbb{R}^n , we have

$$(1.5) \quad \frac{1}{\text{vol}(\mathcal{L})} \int_{\mathcal{L}} |\langle x, \theta \rangle|^2 dx = \frac{1}{n+2} \|\theta\|_{\mathcal{L}^\circ}^2 .$$

To prove (1.5), write $\mathcal{E} = T(D)$ so that $\|\theta\|_{\mathcal{E}^0} = |T^*\theta|$ for every θ in \mathbb{R}^n . Then

$$\frac{1}{\text{vol}(\mathcal{E})} \int_{\mathcal{E}} |\langle x, \theta \rangle|^2 dx = \frac{1}{\text{vol}(D)} \int_D |\langle x, T^*\theta \rangle|^2 dx.$$

Now

$$\begin{aligned} \frac{1}{\text{vol}(D)} \int_{\infty} |\langle x, \theta \rangle|^2 dx &= n \int_{S^{n-1}} |\langle x, \theta \rangle|^2 \left(\int_0^1 r^{n+1} dr \right) d\sigma(x) \\ &= \frac{n}{n+2} \int_{S^{n-1}} |\langle x, \theta \rangle|^2 d\sigma(x) \\ &= \frac{1}{n+2} |\theta|^2. \end{aligned}$$

1.4 Khinchine type inequalities (inverse Hölder inequalities).

When K is a convex compact subset of \mathbb{R}^n , the measure μ defined by $\mu(A) = \text{vol}(A \cap K) / \text{vol}(K)$ is log-concave, that is satisfies $\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$ for all compact sets A and B in \mathbb{R}^n and all $0 < \lambda < 1$. Indeed, this follows from the Brunn-Minkowski theorem (see, e.g., [B.Z]). For such a probability measure μ we have the following result (see [M.Sch], Appendix III).

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a seminorm. Then the L_p norms $\|f\|_p = (\int |f|^p d\mu)^{1/p}$, are equivalent. In other terms, if $p > q > 0$

$$\|f\|_q \leq \|f\|_p \leq c_{p,q} \|f\|_q$$

where $c_{p,q}$ depends only on p and q .

The first inequality is the Hölder inequality whereas the second follows from the concentration of volume in convex bodies as expressed by Borell's lemma [Bo.3] (see [M.Sch], Appendix III).

In particular we may state the following generalization of the Khinchine inequality (see [Gr.M]).

Lemma. Let K be a convex compact body in \mathbb{R}^n and $p > q > 0$. Then for every θ in \mathbb{R}^n ,

$$(1.6) \quad \left(\frac{1}{\text{vol}(K)} \int_K |\langle x, \theta \rangle|^p dx \right)^{1/p} \leq c_{p,q} \left(\frac{1}{\text{vol} K} \int_K |\langle x, \theta \rangle|^q dx \right)^{1/q}$$

where $c_{p,q}$ depends only on p and q .

(If K is the unit cube, (1.6) becomes the standard Khinchine inequality, see also 2.8 for constants $c_{p,q}$.)

1.5 Relation between the zonoid of inertia (centroid body) and Legendre's ellipsoid.

The following proposition states that the zonoid of inertia $Z(K)$ is "almost" a homothety of Legendre's ellipsoid.

Proposition. *Let K be a convex compact body in \mathbb{R}^n . Then*

$$c_1 \frac{1}{\sqrt{n}} \left(\frac{\text{vol } \mathcal{L}}{\text{vol } K} \right)^{1/2} \mathcal{L} \subset Z(K) \subset c_2 \frac{1}{\sqrt{n}} \left(\frac{\text{vol } \mathcal{L}}{\text{vol } K} \right)^{1/2} \mathcal{L}$$

where c_2 and $c_1 > 0$ are universal constants.

Proof: From (1.3) and Lemma 1.4 we get

$$\|\theta\|_{Z(K)^\circ} = \frac{1}{2} \cdot \frac{1}{\text{vol}(K)} \int_K |\langle x, \theta \rangle| dx \geq \frac{1}{2} c_{2,1} \left(\frac{1}{\text{vol } K} \int_K |\langle x, \theta \rangle|^2 dx \right)^{1/2}$$

for every θ in \mathbb{R}^n . Now using (1.5) we have

$$\|\theta\|_{Z(K)^\circ} \geq \frac{1}{2} c_{2,1} \left(\frac{\text{vol } \mathcal{L}}{\text{vol } K} \right)^{1/2} \frac{1}{\sqrt{n+2}} \|\theta\|_{\mathcal{L}^\circ}.$$

This inequality shows that

$$Z(K)^\circ \subseteq \left(\frac{1}{2} c_{2,1} \left(\frac{\text{vol } \mathcal{L}}{\text{vol } K} \right)^{1/2} \frac{1}{\sqrt{n+2}} \right)^{-1} \mathcal{L}^\circ$$

and gives by polarity the left hand side inclusion of the proposition. The right hand side inclusion is obtained in the same way, using now a Hölder inequality.

1.6 Isotropic position.

Let us first note that if T is a linear isomorphism, then

$$\mathcal{L}(T(K)) = T(\mathcal{L}(K)) \quad \text{as well as} \quad Z(T(K)) = T(Z(K)).$$

Therefore there exist a position of K , that is a linear isomorphism T such that $\mathcal{L}(T(K))$ is homothetic to the canonical euclidean ball D , and $\text{vol}(T(K)) = 1$. We say that a compact body K is *isotropic* with *constant of isotropy* $L_K = L$ if $\text{vol } K = 1$ and

$$\int_K |\langle x, \theta \rangle|^2 dx = L^2 \quad \text{for every } \theta \text{ in } S^{n-1} = \partial D.$$

Every body K has an isotropic position, say $T(K)$ which is unique up to an orthogonal rotation.

We will note

$$L_K \stackrel{\text{def}}{=} L_{T(K)}.$$

Important remark. *The isotropic position corresponds to the situation where Legendre and Binet ellipsoids are homothetic and the normalization $\text{vol } K = 1$.*

Lemma. *The isotropic position minimalizes the integral $\int_{T(K)} |x|^2 dx$ among positions such that $\text{vol}(T(K)) = 1$.*

Proof: Let $M = (m_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$ be the matrix of inertia of K with entries $m_{ij} = \int_K x_i x_j dx$. Then for every θ in \mathbb{R}^n

$$\int_K |(x, \theta)|^2 dx = \langle M\theta, \theta \rangle \quad \text{and} \quad \int_K |x|^2 dx = \text{trace}(M).$$

Since the matrix of inertia of $T(K)$ is TMT^* , one has

$$\int_{T(K)} |x|^2 dx = \text{trace}(TMT^*).$$

The minimum of $\text{trace}(TMT^*)$ with a fixed determinant of T is then clearly obtained (from a comparison of arithmetic and geometric means) when TMT^* is a multiple of the identity.

Equivalently we may reformulate this property as follows.

Let

$$(1.7) \quad \lambda^2 = \min_{\mathcal{E}} \int_K \|x\|_{\mathcal{E}}^2 dx$$

where the minimum runs over all ellipsoids with the same volume as D . Then the minimum is attained for a unique ellipsoid which is homothetic to Legendre's ellipsoid and

$$(1.8) \quad \lambda^2 = nL_K^2(\text{vol } K)^{1+2/n}.$$

Important Remark. Note some kind of duality which appears in the definitions of a very useful in Local Theory of normed spaces \mathcal{L} -ellipsoid and Legendre's ellipsoid \mathcal{L} as reflected in (1.7). First we describe an \mathcal{L} -ellipsoid of a symmetric convex body K (for a canonical definition see, e.g., [M.Sch] and references in this book).

Let $v_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transform such that

$$\min \left\{ \int_D \|x\|_{v_0 K}^2 dx \mid \det v = 1 \right\} = \int_D \|x\|_{v_0 K}^2 dx.$$

Then $v_0^{-1}D = \mathcal{E}$ is the ℓ -ellipsoid of the body K (up to homothetic factor).

As it follows from (1.7), if $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transform such that

$$\min \left\{ \int_K \|x\|_{u_0 D}^2 dx \mid \det u = 1 \right\} = \int_K \|x\|_{u_0 D}^2 dx .$$

then $u_0 D = \lambda \mathcal{L}$ is homothetic to Legendre's ellipsoid.

Note also the relation between L_K and the matrix of inertia M of K which follows from the fact that M is a diagonal matrix in the isotropic position:

$$(1.9) \quad L_K = (\det M)^{1/2n} (\text{vol } K)^{-(n+2)/2n} .$$

Remark. Let K be an isotropic body and let X be the normed space \mathbb{R}^n equipped with the norm associated with K , that is with the unit ball K . Let $I : X^* \rightarrow \mathcal{L}_2^n$ be the identity operator. The isotropic position gives an information on the π_2 -summing norm of I , precisely $\pi_2(I) \leq 1/L_K$. This follows immediately from the definition of $\pi_2(I)$ (see [Pie]) since K represents the unit ball of the dual of X^* .

1.7 The volume of Legendre's ellipsoid.

Let M be the matrix of inertia of K . We have

$$\|\theta\|_B^2 = \langle M\theta, \theta \rangle \quad \text{for every } \theta \in \mathbb{R}^n$$

and therefore

$$\text{vol } B = (\det M)^{-1/2} \text{vol } D .$$

Using (1.4) and the fact that $\text{vol } \mathcal{E} \cdot \text{vol } \mathcal{E}^0 = (\text{vol } D)^2$ for any ellipsoid \mathcal{E} , we deduce

$$(1.10) \quad \text{vol } \mathcal{L} = (\det M)^{1/(n+2)} (\text{vol } D)^{2/(n+2)} (n+2)^{n/(n+2)} .$$

We note that $c_1 \leq (\text{vol } D)^{2/(n+2)} (n+2)^{n/(n+2)} \leq c_2$ where c_2 and $c_1 > 0$ are universal constants.

There is a well known formula (which goes back to the 19th century, see [Bla.1]) to express $\det M$ (and therefore $\text{vol}(\mathcal{L})$) as follows:

$$(1.11) \quad \det M = \frac{1}{n!} \int_K \dots \int_K |\det(x_1, x_2, \dots, x_n)|^2 dx_1 \dots dx_n .$$

(The proof is direct by expanding the determinant.)

We summarize in the following.

Proposition. Let K be a compact body and M be its matrix of inertia. Then

$$\det M = \frac{1}{n!} \int_K \dots \int_K |\det(x_1, x_2, \dots, x_n)|^2 dx_1 \dots dx_n$$

and

$$c_1(\det M)^{1/(n+2)} \leq \text{vol}(\mathcal{L}) \leq c_2(\det M)^{1/(n+2)}$$

where $c_1 > 0$ and c_2 are universal constants.

Using (1.9) we obtain the following

Corollary. Let K be a compact body and $\text{vol} K = 1$. Then

$$\text{vol} \mathcal{L} \sim L_K^2.$$

1.8 Stability of the isotropic position.

Assume again that $\text{vol} K = 1$. Let a and b be numbers such that for every $\theta \in S^{n-1}$

$$\frac{1}{a}L \leq \left(\int_K |\langle x, \theta \rangle|^2 dx \right)^{1/2} \leq b \cdot L.$$

Then

$$c_1 \frac{1}{a}L \leq L_K \leq c_2 bL$$

for some universal constants c_1 and c_2 .

Indeed, by Lemma 1.4 we may state

$$\frac{1}{a}L \lesssim \int_K |\langle x, \theta \rangle| dx \lesssim bL$$

(as usually, $A \lesssim B$ means that for a numerical constant c we have $A \leq c \cdot B$). Because $\int_K |\langle x, \theta \rangle| dx$ defines the dual norm to the norm defined by the body $Z(K)$, we have

$$\frac{c_2}{a}LD \subset Z(K) \subset c_3 bLD$$

for some universal constants $c_2 > 0$ and c_3 . To derive our statement recall that by 1.5 and Corollary of 1.7 we also have that

$$(\text{vol} Z(K))^{1/n} \sim L_K / \sqrt{n} \quad \square$$

1.9 Volume of the zonoid of inertia (the centroid body).

As for Legendre's ellipsoid, there is a simple integral formula to express the volume of the inertia zonoid (see [Pet]).

Proposition. *Let K be a compact body in \mathbb{R}^n . Then*

$$\text{vol}(Z(K)) = \frac{1}{(\text{vol } K)^n} \int_K \cdots \int_K (|\det(x_1, \dots, x_n)|/n!) dx_1 \dots dx_n$$

Actually the same type of formula is valid for a general zonoid. We check the proof for completeness.

Proof: Since the problem may be reduced to a discrete sum, it is sufficient to prove that for $m \geq n$ and x_1, \dots, x_m in \mathbb{R}^n we have

$$(1.12) \quad \text{vol} \left(\sum_{i=1}^m [0, x_i] \right) = \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq m} |\det(x_{i_1}, \dots, x_{i_n})|.$$

Indeed if $\mu = \sum_{i=1}^m \alpha_i \delta_{x_i}$, $\alpha_1, \dots, \alpha_m$ positive, the last formula may be written

$$\text{vol} \left(\int [0, x] d\mu(x) \right) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \frac{|\det(y_1, \dots, y_n)|}{n!} d\mu(y_1) \cdots d\mu(y_n).$$

We prove (1.12) by induction on m and n , $m \geq n$, of course if $m = n$, $\text{vol} \left(\sum_{i=1}^n [0, x_i] \right) = |\det(x_1, \dots, x_n)|$, $x_i \in \mathbb{R}^n$. Now let A be a convex body in \mathbb{R}^n . Then one can check easily that for any x in \mathbb{R}^n :

$$\text{vol}(A + [0, x]) = \text{vol}(A) + |x| \text{vol}(P(A))$$

where P denotes the orthogonal projection onto the hyperplane orthogonal to x . Using induction hypothesis and the fact that

$$|x| |\det(Px_1, \dots, Px_{n-1})| = |\det(x_1, \dots, x_{n-1}, x)|; x_i \in \mathbb{R}^n$$

we see that

$$\begin{aligned} \text{vol} \left(\sum_{i=1}^n [0, x_i] + [0, x] \right) &= |\det(x_1, \dots, x_n)| + |x| \text{vol} \left(\sum_{i=1}^n [0, Px_i] \right) \\ &= |\det(x_1, \dots, x_n)| + \sum_{1 \leq i_1 < \dots < i_{n-1} \leq n} |\det(x_{i_1}, \dots, x_{i_{n-1}}, x)|. \end{aligned}$$

This allows us to conclude (1.12).

Remark. The formula (1.12) is a particular case of a more general result. If K_1, \dots, K_m are convex bodies in \mathbb{R}^n , $m \geq n$, then $\text{vol} \left(\sum_{i=1}^m K_i \right) = \sum_I \text{vol} \left(\sum_{i \in I} K_i \right)$ where I runs over all subsets of $\{1, 2, \dots, m\}$ with cardinality n (see [BF]).

1.10 The distance of K to its Legendre's ellipsoid.

Proposition. Let K be a symmetric convex body in \mathbb{R}^n and \mathcal{L} its Legendre's ellipsoid. Then

$$\sqrt{\frac{3}{n+2}} \left(\frac{\text{vol } \mathcal{L}}{\text{vol } K} \right)^{1/2} \mathcal{L} \subset K \subset \sqrt{\frac{n+1}{2}} \left(\frac{\text{vol } \mathcal{L}}{\text{vol } K} \right)^{1/2} \mathcal{L}.$$

Proof: Let $\theta \in S^{n-1}$ and set

$$\varphi_\theta(t) = \text{vol} (K \cap \{x \in \mathbb{R}^n ; \langle x, \theta \rangle = t\}).$$

The function φ_θ has its support on $[-\|\theta\|_{K^\circ}, \|\theta\|_{K^\circ}]$. Set

$$\psi(u) = (2\varphi(u\|\theta\|_{K^\circ})\|\theta\|_{K^\circ} / \text{vol } K)^{1/(n-1)}$$

so that we have $\int_0^1 \psi^{n-1}(u) du = 1$. Moreover, from Brunn's theorem ψ is a decreasing concave function on $[0, 1]$. Now

$$\begin{aligned} \|\theta\|_B^2 &= \int_K |\langle x, \theta \rangle|^2 dx = 2 \int_0^\infty t^2 \varphi_\theta(t) dt \\ &= 2\|\theta\|_{K^\circ}^2 \int_0^1 u^2 \|\theta\|_{K^\circ} \varphi(u\|\theta\|_{K^\circ}) du \\ &= \|\theta\|_{K^\circ}^2 (\text{vol } K) \int_0^1 u^2 \psi(u)^{n-1} du. \end{aligned}$$

Since ψ is decreasing, we easily have

$$\int_0^1 u^2 \psi(u)^{n-1} du \leq \int_0^1 u^2 du = 1/3$$

and from an inequality on concave function (see [Bo.1]) we get

$$\int_0^1 u^2 \psi(u)^{n-1} du \geq \int_0^1 u^2 n(1-u)^{n-1} du = \frac{2}{(n+1)(n+2)}.$$

Therefore

$$\frac{2 \text{vol } K}{(n+1)(n+2)} \|\theta\|_{K^\circ}^2 \leq \|\theta\|_B^2 \leq \frac{\text{vol } K}{3} \|\theta\|_{K^\circ}^2.$$

This yields

$$(1.13) \quad \left(\frac{2 \text{vol } K}{(n+1)(n+2)} \right)^{1/2} B \subset K^\circ \subset \left(\frac{\text{vol } K}{3} \right)^{1/2} B$$

we conclude by using (1.4) and polarity.

Remark. Separately, each of the inclusions of the proposition is sharp. The left inclusion is sharp for the unit ball of \mathcal{L}_∞^n that is the cube, the right inclusion is sharp for the unit ball of \mathcal{L}_1^n . The distance of K to its Legendre ellipsoid, that is the smallest ratio a/b such that $a\mathcal{L} \subset K \subset b\mathcal{L}$ is therefore bounded by cn for some absolute constant c . This is the right order, for instance by mixing the unit ball of \mathcal{L}_1^n and \mathcal{L}_∞^n , that is with K in \mathbb{R}^{2n} defined by $|x_i| \leq 1$ for $i = 1, 2, \dots, n$ and $\sum_{i=n+1}^{2n} |x_i| \leq 1, x = (x_i)_{i=1, \dots, 2n} \in \mathbb{R}^{2n}$.

2. More on Khinchine Type Inequalities and Applications to Geometric Inequalities

Khinchine type inequalities (= inverse Hölder inequalities) which we started to discuss in section 1.4 play such an important role in the study of convex bodies that we are forced to investigate them in more detail.

2.1. Let K be a symmetric convex body in \mathbb{R}^n and $\|\cdot\|_K$ its associated norm on \mathbb{R}^n . Then

$$\begin{aligned} \int_K \|x\|_K^p dx &= \int_0^1 pt^{p-1} \text{vol}\{x \in \mathbb{R}^n ; \|x\|_K \geq t\} dt \\ &= \text{vol}(K) \int_0^1 pt^{p-1}(1-t^n) dt . \end{aligned}$$

Therefore

$$\int_K \|x\|_K^p dx = \frac{n}{n+p} \text{vol}(K) .$$

Lemma. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a measurable function such that $\|f\|_\infty = 1$ and let K be a symmetric convex body in \mathbb{R}^n . Then the function

$$F(p) = \left(\frac{\int_{\mathbb{R}^n} \|x\|_K^p f(x) dx}{\int_K \|x\|_K^p dx} \right)^{1/(n+p)}$$

is an increasing function of p on $] -n, +\infty[$.

(The inequality $F(2) \geq F(0)$ when K is the euclidean ball was proved by Hensley [H].)

Remark. The equality $F(p) = F(q)$ for $p \neq q$ in the lemma occurs if and only if f is the indicator of K .

Proof of the lemma. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$, $\|f\|_\infty = 1$ and $p > q > -n$.

$$\begin{aligned} \int_{\mathbb{R}^n} \|x\|_K^p f(x) dx &\geq t^{p-q} \int_{\mathbb{R}^n} \|x\|_K^q f(x) dx - \int_{\|x\|_K \leq t} (t^{p-q} \|x\|_K^q - \|x\|_K^p) f(x) dx \\ &\geq t^{p-q} \int_{\mathbb{R}^n} \|x\|_K^q f(x) dx - \int_K t^{p+n} (\|x\|_K^q - \|x\|_K^p) dx. \end{aligned}$$

We now optimize the last inequality by choosing $t = \left((n+q) \int_{\mathbb{R}^n} \|x\|_K^q f(x) dx \right)^{1/(n+q)}$. An elementary computation gives $F(p) \geq F(q)$.

2.2. Applying the lemma 2.1 to $f = 1_{K_1}$ and writing $F(p) \geq F(0)$ leads to

a. Corollary. Let K_1 and K_2 be two symmetric convex bodies in \mathbb{R}^n and let $p > 0$. Then

$$\left(\frac{1}{\text{vol } K_1} \int_{K_1} \|x\|_{K_2}^p dx \right)^{1/p} \geq \left(\frac{n}{n+p} \right)^{1/p} \left(\frac{\text{vol } K_1}{\text{vol } K_2} \right)^{1/n}.$$

Sending $p \rightarrow 0$ we have

b. Corollary.

$$\exp \left[\frac{1}{\text{vol } K_1} \int_{K_1} \log \|x\|_{K_2} dx \right] \geq e^{-1} \left(\frac{\text{vol } K_1}{\text{vol } K_2} \right)^{1/n}.$$

Remark. Equality in the corollary only occurs when $K_1 = \lambda K_2$.

2.3. The inequality in Corollary 2.2 a and b may be applied in different contexts. We show first that it generalizes Urysohn's inequality. For simplicity we rewrite the inequality for $p = 1$

$$(2.1) \quad \left(\frac{\text{vol } K_1}{\text{vol } K_2} \right)^{1/n} \leq \frac{n+1}{n} \cdot \frac{1}{\text{vol } K_1} \int_{K_1} \|x\|_{K_2} dx.$$

Use now Santaló's inequality (see [Bou.M]) which says that

$$\text{vol } K \cdot \text{vol } K^\circ \leq (\text{vol } D)^2$$

where, as usual, K° denotes the polar body of K .

Then we rewrite

$$(2.2) \quad \left(\frac{\text{vol } K_2^\circ \cdot \text{vol } K_1}{(\text{vol } D)^2} \right)^{1/n} \leq \frac{n+1}{n} \frac{1}{\text{vol } K_1} \int_{K_1} \|x\|_{K_2} dx.$$

In the case $K_1 = D$ the last inequality (2.2) becomes the Urysohn inequality because

$$\frac{n+1}{n} \frac{1}{\text{vol } D} \int_D \|x\|_{K_2} dx = \int_{S^{n-1}=\partial D} \|x\|_{K_2} d\sigma(x)$$

where $\sigma(x)$ is the probability rotation invariant measure on S^{n-1} . In particular this means that using Corollary 2.2a for $p < 1$ or Corollary 2.2b we derive stronger versions of Urysohn's inequality. For example, the following is true.

Corollary. For any convex symmetric body $K \subset \mathbb{R}^n$

$$\left(\frac{\text{vol } K}{\text{vol } D} \right)^{1/n} \leq e \cdot \exp \left\{ \frac{1}{\text{vol } D} \int_D \log \|x\|_{K^\circ} dx \right\}.$$

Returning to (2.1) in a general case we use besides the Santaló inequality also the inverse Santaló inequality [Bou.M] which says that there exists a universal constant $c > 0$ such that

$$c^n \leq \frac{\text{vol } K \cdot \text{vol } K^\circ}{(\text{vol } D)^2}.$$

Substitute the last inequality in (2.1) and exchange the places of K_i and K_i° . Then

$$(2.3) \quad \left(\frac{\text{vol } K_2}{\text{vol } K_1} \right)^{1/n} \leq \frac{1}{c} \cdot \frac{1}{\text{vol } K_1^\circ} \int_{K_1^\circ} \|x\|_{K_2} dx.$$

This inequality may be used in entropy estimation in the same way as Urysohn's inequality was used (see, e.g., [Pa]). An importance of the use of polars instead of bodies by itself comes from the fact that $\|x\|_{(K_2 + \lambda K_1)^\circ} = \|x\|_{K_2^\circ} + \lambda \|x\|_{K_1^\circ}$.

2.4 Isotropic position and inverse Brunn-Minkowski inequality.

To describe the next connection of inequalities from 2.2 with known geometric inequalities, we will need the next definition.

Let K be a convex symmetric body and \mathcal{E} an ellipsoid such that $\text{vol } K = \text{vol } \mathcal{E}$. Fix (universal) constants $c_1 > 0$ and C_2 . We call \mathcal{E} an M -ellipsoid (for constants c_1 and C_2) if

$$\text{vol}(K \cap \mathcal{E}) \geq c_1^n \quad \text{and} \quad \text{vol}(K + \mathcal{E}) \leq C_2^n.$$

It was proved in [Mi2] (see also much simplified versions in [Mi3] and [Pi]) that there exist universal constants $0 < c_1$ and C_2 such that for every $n \in \mathbb{N}$ every body $K \subset \mathbb{R}^n$ has an M -ellipsoid (for these constants c_1 and C_2). In the coming discussion we will usually not refer to specific universal

constants. As it was shown in [Pi] (or [Mi3]) an ellipsoid \mathcal{E} , such that $\text{vol } \mathcal{E} = \text{vol } K$ and the covering number

$$(2.4) \quad N(K, \mathcal{E}) = \min \left\{ N \mid \exists x_1, \dots, x_N \text{ and } K \subset \bigcup_{i=1}^N (x_i + \mathcal{E}) \right\} \leq \exp(C_3 n)$$

(for some universal constant $C_3 > 0$), is an M -ellipsoid (for some universal $c_1 > 0$ and C_2 depending only on C_3). The importance of M -ellipsoid is in the fact that it gives an inverse form of the Brunn-Minkowski inequality: if \mathcal{E} is M -ellipsoid for two convex bodies K and T ($\text{vol } K = \text{vol } T$) then for any $\lambda > 0$

$$(2.5) \quad \text{vol}(K + \lambda T)^{1/n} \leq C((\text{vol } K)^{1/n} + \lambda(\text{vol } T)^{1/n})$$

where C is a universal constant (depending only on a constant C_3 in (2.4)).

After recalling these facts we return to (2.3) and use it for an ellipsoid \mathcal{E} and body T . Then, if \mathcal{E} is homothetic to the Legendre ellipsoid of T^0 and $\text{vol } T^0 = 1 = \text{vol } \mathcal{E}$ then

$$\left(\frac{\text{vol } \mathcal{E}^0}{\text{vol } T} \right)^{1/n} \leq \frac{1}{c} \cdot \frac{1}{\text{vol } T^0} \int_{T^0} \|x\|_{\mathcal{E}} dx \simeq \frac{1}{c} L_{T^0} .$$

Also

$$\left(\frac{\text{vol}(\mathcal{E}^0 + \lambda T)}{\text{vol } T} \right)^{1/n} \leq \frac{1}{c} (L_{T^0} + \lambda)$$

and therefore by the standard argument (using Santalo's inequality and its inverse) we have

$$N(T, \mathcal{E}) \lesssim \exp(n \log L_{T^0}/c) .$$

Therefore, *conditional statement*: if for some universal constant C_4 , $L_{T^0} \leq C_4$ (this is the main problem discussed in section 5) then the Binet ellipsoid \mathcal{B} for T^0 (which is the dual one up to homothety to the Legendre ellipsoid to T^0) is (up to homothety) an M -ellipsoid for T (for constants c_1 and C_2 depending on C_4 and a constant c in the inverse Santalo inequality).

Note, that K. Ball [Ba.1] observed that the main problem of a universal boundness of L_K will imply the inverse Brunn-Minkowski inequality (2.5) however, as it seems for us, not in such direct way.

2.5. Let K be a symmetric convex body in \mathbb{R}^n and let $\theta \in S^{n-1}$. For every real number t , we denote

$$\varphi_\theta(t) = \text{vol}(K \cap (t\theta + H))$$

where $H = \{y \in \mathbb{R}^n ; \langle y, \theta \rangle = 0\}$ is the hyperplane orthogonal to θ .

The following corollary was proved by Hensley [H.].

Corollary. Let K be a symmetric convex body in \mathbb{R}^n , $p > 0$ and let H be an hyperplane defined by the equation $\langle x, \theta \rangle = 0$ with $\theta \in S^{n-1}$. Then

$$\left(\frac{1}{\text{vol}(K)} \int_K |\langle x, \theta \rangle|^p dx \right)^{1/p} \geq \frac{\text{vol}(K)}{\text{vol}(K \cap H)} \cdot \frac{1}{2(p+1)^{1/p}}.$$

Proof: From Brunn's theorem, φ_θ is maximal for the central section, that is $\|\varphi_\theta\|_\infty = \varphi_\theta(0)$.

Now

$$\int_K |\langle x, \theta \rangle|^p dx = \int_{\mathbb{R}} |t|^p \varphi_\theta(t) dt.$$

The result follows readily from Lemma 2.1 for $n = 1$ and with $f = \varphi_\theta/\varphi_\theta(0)$ (compare p and 0 values).

Remark. Corollary 2.5 is sharp. Equality occurs for a cylinder in the direction of θ (φ_θ is then constant on its support).

2.6. We are now interested in inverse Hölder type inequalities. The inequalities we will use stem from the monotonicity of ratios of means. The following lemma due to Marshall, Olkin and Proschan [M.O.P] is the continuous version of a result of Schur and Ostrowski (see [M.O.P] for more information, see also [Bo.2] for related inequalities.)

Lemma. Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a decreasing function and let $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\Phi(0) = 0$ and such that Φ and $\Phi(x)/x$ are increasing. Then

$$G(p) = \left(\frac{\int_0^\infty h(\Phi(x)) x^p dx}{\int_0^\infty h(x) x^p dx} \right)^{1/(p+1)}$$

is a decreasing function of p on $] -1, +\infty[$.

Proof: Let $\alpha = 1/G(p)$ so that we have

$$\int_0^\infty h(\alpha x) x^p dx = \int_0^\infty h(\Phi(x)) x^p dx.$$

Now set $G(t) = \int_0^{\infty} (h(\alpha x) - h(\Phi(x))) x^p dx$. By definition of α , we have $G(0) = G(\infty) = 0$. Now $\Phi(x)/x$ is increasing and h decreases, so that we can analyse the sign of $h(\alpha x) - h(x\Phi(x)/x)$ to deduce that G is first increasing and then decreasing. Therefore G is positive. This means that for every $t > 0$, we have

$$\int_t^{\infty} h(\alpha x) x^p dx \geq \int_t^{\infty} h(\Phi(x)) x^p dx .$$

The lemma now follows by integration. Indeed let $q > p > -1$,

$$\begin{aligned} \int_0^{\infty} h(\Phi(x)) x^q dx &= (q-p) \int_0^{\infty} t^{q-p-1} \left(\int_t^{\infty} h(\Phi(x)) x^p dx \right) dt \\ &\leq (q-p) \int_0^{\infty} t^{q-p-1} \left(\int_t^{\infty} h(\alpha x) x^p dx \right) dt \\ &\leq \int_0^{\infty} h(\alpha x) x^q dx = \frac{1}{\alpha^{q+1}} \int_0^{\infty} h(x) x^q dx . \end{aligned}$$

This proves $G(q) \leq G(p)$.

A class of functions Φ for which the hypothesis of Lemma 2.6 are satisfied is given by the convex functions $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\Phi(0) = 0$. One special and important case of application of Lemma 2.6 consists of choosing $h(t) = e^{-t}$ and Φ convex, hence the lemma may be applied to log-concave function $f = h \circ \Phi$ satisfying $f(0) = 1$. In this context the inequality $G(2) \geq G(0)$ was also proved by Hensley [H.] and the simple proof of Lemma 2.6 given above is just a copy of Hensley's proof. Still in the context of the log-concave function Lemma 2.6 was proved independently by Ball in [Ba.1], [Ba.2] where the author given many applications to the study of sections of convex bodies, some of them we shall use and state later.

Closely related to Lemma 2.6 is the following inequality of Berwald [Ber] for a concave function φ defined on a convex body K in \mathbb{R}^n

$$\left[\binom{n+p}{n} (\text{vol } K)^{-1} \int_K |\varphi(x)|^p dx \right]^{1/p} \geq \left[\binom{n+q}{n} (\text{vol } K)^{-1} \int_K |\varphi(x)|^q dx \right]^{1/q} , \quad 0 < p \leq q .$$

The relation is clear once we write

$$\frac{1}{\text{vol } K} \int_K \varphi^{p+1}(x) dx = (p+1) \int_0^{+\infty} t^p \mu[x \in K \mid \varphi(x) \geq t] dt$$

where μ is the measure defined as in 1.4 by $\mu(A) = \text{vol}(A \cap K) / \text{vol} K$. Now, from Brunn-Minkowski theorem and the concavity of φ we see that $\Phi(t) = 1 - \mu[x \in \mathbb{R}^n \mid \varphi(x) \geq t]^{1/n}$ is a convex function satisfying $\Phi(0) = 0$. Therefore by setting $h(t) = (1-t)^n$ for $0 \leq t \leq 1$, Berwald inequality follows from Lemma 2.6. This in particular gives a simple proof of this type of inequality (see also [Bo.2]).

2.7 Corollary. Let K be a symmetric convex body in \mathbb{R}^n , $p > -1$ and let H be an hyperplane defined by the equation $\langle x, \theta \rangle = 0$ with $\theta \in S^{n-1}$. Then

$$\left(\frac{1}{\text{vol} K} \int_K |\langle x, \theta \rangle|^p dx \right)^{1/p} \leq \frac{\text{vol} K}{\text{vol}(K \cap H)} \cdot \left(\frac{n}{2} \right) \left(\frac{n!}{(p+1)(p+2)\cdots(p+n)} \right)^{1/p}.$$

Proof: As above let $\varphi_\theta(t) = \text{vol}(K \cap (t\theta + H))$, $t \in \mathbb{R}$. Let $h(x) = (1-x)^{n-1}$ for $0 \leq x \leq 1$, $h(x) = 0$ for $x > 1$ and set $\Phi(x) = 1 - (\varphi_\theta(x)/\varphi_\theta(0))^{1/(n-1)}$. From Brunn's theorem $\Phi(x)$ is convex and all hypotheses of Lemma 2.6 are satisfied so that we derive that

$$G(p) = \left(\frac{\int_0^\infty x^p \varphi_\theta(x) / \varphi_\theta(0) dx}{\int_0^\infty t^p (1-t)^{n-1} dt} \right)^{1/p+1} = \left(\frac{1}{2} \int_K |\langle x, \theta \rangle|^p dx / \varphi_\theta(0) \beta(n, p+1) \right)^{1/p+1}$$

is a decreasing function of p ($p > -1$). Corollary follows from the comparison $G(p) \leq G(0)$, $p \geq 0$.

Corollary 2.7 for $p = 2$ is due to ~~Milman~~^{Ball} [H]. We note that the inequality in Corollary 2.7 is sharp. There is equality for a cone based on H with θ as summit.

2.8. As we saw in Proposition 2.2, sending $p \rightarrow 0$ we derive much better estimates of some integrals important for the understanding of convex bodies. To be able to proceed this in some other cases we have to establish the exact dependence on p (when $p \rightarrow 0$) in some of the above inequalities

We start by revising Lemma 1.4. By Corollaries 2.5 and 2.7, we obtain for $\text{vol} K = 1$

$$\left(\int_K |\langle x, \theta \rangle|^2 dx \right)^{1/2} \leq c(p+1)^{1/p} \left(\int_K |\langle x, \theta \rangle|^p dx \right)^{1/p}.$$

Therefore, we get

Lemma. * There exists a universal constant c such that for every $\theta \in \mathbb{R}^n$ and $0 < p < 2$

$$\left(\frac{1}{\text{vol} K} \int_K |\langle x, \theta \rangle|^2 dx \right)^{1/2} \leq c \exp \left[\frac{1}{\text{vol} K} \int_K \ln |\langle x, \theta \rangle| dx \right] \leq c \left(\frac{1}{\text{vol} K} \int_K |\langle x, \theta \rangle|^p dx \right)^{1/p}.$$

* We learned recently that D. Ullrich [U] has obtained similar inequality for independent Rademacher variables which corresponds to the case $K = [-1, 1]^n$.

3. Sections of a Convex Body

In the previous section we have already derived some inequalities for a volume of hypersections of body K . In this section we delve more deeply into this subject.

3.1. Noticing that $n^{(p+1)/p} \beta(n, p+1)^{1/p}$ is of the order of $\Gamma(p+1)^{1/p}$ we summarize Corollaries 2.5 and 2.7 in the following

Corollary. Let K be a symmetric convex body in \mathbb{R}^n , $p > 0$ and let H be an hyperplane defined by the equation $\langle x, \theta \rangle = 0$ with $\theta \in S^{n-1}$. Then

$$c_1(p) \frac{\text{vol} K}{\text{vol}(K \cap H)} \leq \left(\frac{1}{\text{vol} K} \int_K |\langle x, \theta \rangle|^p dx \right)^{1/p} \leq c_2(p) \frac{\text{vol} K}{\text{vol}(K \cap H)}$$

where $c_1(p) > 0$ and $c_2(p)$ depends only on p .

Remark. Corollary 3.1 also follows directly from the concentration of volume in convex bodies (see [Gr.M]) as already used in (1.4).

3.2 Corollary. Let K be an isotropic body in \mathbb{R}^n . Then

$$c_1 / \text{vol}(K \cap H) \leq L_K \leq c_2 / \text{vol}(K \cap H)$$

for every hyperplane H , where $c_1 > 0$ and c_2 are universal constants.

3.3 Remark. As a consequence of Corollary 3.2, if K is an isotropic body then $c_1 \leq \text{vol}(K \cap H_1) / \text{vol}(K \cap H_2) \leq c_2$ for every hyperplane H_1 and H_2 where $c_1 > 0$ and c_2 are universal constants. This interesting fact was first observed by Hensley [H]. A non-trivial use of this fact may be found in [Bou.1].

3.4 Proposition. Let K be a symmetric convex body in \mathbb{R}^n and let H be an hyperplane defined by the equation $\langle x, \theta \rangle = 0$ with $\theta \in S^{n-1}$. Then

$$\frac{e^{-1}}{2} \cdot \frac{\text{vol}(K)}{\text{vol}(K \cap H)} \leq \exp \left(\frac{1}{\text{vol}(K)} \int_K \ln |\langle x, \theta \rangle| dx \right) \leq \frac{\text{vol}(K)}{\text{vol}(K \cap H)} \cdot \frac{e^{-\gamma}}{2}$$

where γ is the Euler constant.

Proof: The left hand side inequality is obtained from Corollary 2.5 by letting p go to zero. Now from Corollary 2.7 we have also

$$\exp \left(\frac{1}{\text{vol}(K)} \int_K \ln |\langle x, \theta \rangle| dx \right) \leq \frac{\text{vol}(K)}{\text{vol}(K \cap H)} \cdot \frac{1}{2c}$$

where

$$\begin{aligned} c &= \lim_{p \rightarrow 0} \frac{1}{n} \left(\frac{(p+1) \cdots (p+n)}{n!} \right)^{1/p} \\ &= \frac{1}{n} \lim_{p \rightarrow 0} \prod_{k=1}^n \left(1 + \frac{p}{k} \right)^{1/p} \\ &= \exp \left(\sum_{k=1}^n \frac{1}{k} - \ln(n) \right). \end{aligned}$$

Now the sequence $\sum_{k=1}^n \frac{1}{k} - \ln(n)$ decreases towards the Euler constant γ .

Remark. The constants in the previous inequalities are sharp as n tends to infinity (see the remarks after Corollary 2.5 and Corollary 2.7).

3.5. As we have seen in 1.6 (see (1.8));

$$L_K \leq \left(\int_K \frac{|x|^2}{n} dx \right)^{1/2}, \quad (\text{vol } K = 1).$$

We shall now prove a stronger inequality.

Lemma. Let K be a symmetric convex body in \mathbb{R}^n , with $\text{vol}(K) = 1$. Then

$$L_K \leq c \exp \left(\int_K \log \left(\prod_{i=1}^n |x_i| \right)^{1/n} dx \right),$$

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$, where c is a universal constant.

Proof: Let $0 < p < 2$ and $T \in SL_n$ such that $T(K)$ is an isotropic body. Therefore $\det T = \text{vol}(K) = 1$ and, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$\begin{aligned} \int_K \sum_{i=1}^n |x_i|^p dx &= \int_{T(K)} \|T^{-1}x\|_p^p dx = \\ &= \int_{T(K)} \sum_{i=1}^n |\langle x, T^{-1*} e_i \rangle|^p dx. \end{aligned}$$

From the preceding lemma we get

$$\begin{aligned} \int_K \sum_{i=1}^n |x_i|^p dx &\geq \frac{1}{c^p} \sum_{i=1}^n \left(\int_{T(K)} |\langle x, T^{-1*} e_i \rangle|^2 dx \right)^{p/2} \\ &\geq \frac{1}{c^p} L_K^p \sum_{i=1}^n |T^{-1*} e_i|^p. \end{aligned}$$

Hence

$$\left(\frac{1}{n} \int_K \sum_{i=1}^n |x_i|^p dx \right)^{1/p} \geq \frac{1}{c} L_K \left(\sum_{i=1}^n \frac{|T^{-1} e_i|^p}{n} \right)^{1/p}.$$

From the comparison between the arithmetic and geometric means and from Hadamard inequality we arrive at

$$\begin{aligned} \left(\frac{1}{n} \int_K \sum_{i=1}^n |x_i|^p dx \right)^{1/p} &\geq \frac{L_K}{c} \left(\prod_{i=1}^n |T^{-1} e_i| \right)^{1/n} \\ &\geq \frac{L_K}{c} \det(T^{-1})^{1/n} = \frac{L_K}{c}. \end{aligned}$$

The lemma is now proved by letting p go down to 0.

3.6. In particular we get from the last lemma, that

$$L_K \leq c \int_K \left(\prod_{i=1}^n |x_i| \right)^{1/n} dx_1, \dots, dx_n \leq c \int_K \frac{\sum_{i=1}^n |x_i|}{n} dx, \quad \text{vol } K = 1.$$

This fact has the following immediate consequence.

a. Proposition. Let K be a symmetric convex body in \mathbb{R}^n and let $C = \text{conv}(\pm x_1, \pm x_2, \dots, \pm x_n)$, where x_1, \dots, x_n are points in \mathbb{R}^n . Suppose $K \subset C$, then

$$L_K \leq c \left(\frac{\text{vol } C}{\text{vol } K} \right)^{1/n}$$

where c is a universal constant.

Proof: Since the constants L_K are affine invariant, we may suppose that $\text{vol}(K) = 1$ and that $c = \lambda B_1^n$, where B_1^n is the unit ball of ℓ_1^n . Then from above we deduce that (we denote $|x|_1 = \sum_{i=1}^n |x_i|$)

$$L_K \leq c \int_K \frac{|x|_1}{n} dx \leq \frac{c\lambda}{n}$$

since $K \subset \lambda B_1^n$. Now $(\text{vol } C)^{1/n} = \lambda (\text{vol } B_1^n)^{1/n} \geq \frac{c_1}{n} \lambda$ for some absolute constant $c_1 > 0$.

Therefore

$$L_K \leq \frac{c}{c_1} (\text{vol } C)^{1/n}.$$

In particular if B_X is the unit ball of a Banach space with 1-unconditional basis, then it is known from Lozanovski result [Lo] that B_X admits a representation such that $B_X = K \subset \mathbb{R}^n$

$$K \subset B_1^n \quad \text{and} \quad \left(\frac{\text{vol } B_1^n}{\text{vol } K} \right)^{1/n} \leq c$$

where c is a universal constant.

From this remark and the proposition above we may extend the proposition to

b. Proposition. Let K be a symmetric convex body in \mathbb{R}^n and C be the unit ball of a 1-unconditional norm on \mathbb{R}^n . Let $K \subset C$, then

$$L_K \leq c \left(\frac{\text{vol} C}{\text{vol} K} \right)^{1/n}$$

where c is a universal constant.

In particular if K is the unit ball of a 1-unconditional norm on \mathbb{R}^n then

$$L_K \leq c.$$

Now if K is isomorphic to the unit ball C of a 1-unconditional norm, that is

$$aC \subset K \subset bC.$$

Then $L_K \leq c \frac{b}{a}$.

Remark. Propositions a and b of this section were received earlier by J. Bourgain (by a different approach).

3.7. We bring one more result of the same nature.

Proposition. Let K be a symmetric convex body in \mathbb{R}^n with $\text{vol} K = 1$. Then for $0 < p < 2$

$$L_K \leq \frac{c}{\sqrt{n}} \left(\int_{K^n} |\det(x_1, \dots, x_n)|^p dx_1 \dots dx_n \right)^{1/pn}$$

for some universal constant c .

Proof: We may suppose without restriction that K is an isotropic body. Let $I_k(x_1, x_2, \dots, x_k) = \int_{K^{n-k}} |\det(x_1, \dots, x_n)|^p dx_{k+1} \dots dx_n$.

$$\begin{aligned} I_{n-1}(x_1, \dots, x_{n-1}) &= \int_K |\det(x_1, \dots, x_n)|^p dx_n \\ &= \int_K \left| \sum_{\sigma \in \Pi_n} \varepsilon_\sigma x_1^{\sigma(1)}, \dots, x_n^{\sigma(n)} \right|^p dx_n \end{aligned}$$

where $(x_i^j)_{j=1, \dots, n}$ are the coordinates of x_i , $i = 1, 2, \dots, n$, Π_n the permutation group and ε_σ is the signature of σ . Therefore from Lemma 2.8 we have

$$\begin{aligned} I_{n-1}(x_1, \dots, x_{n-1}) &\geq \frac{1}{c^p} \left(\int_K \left| \sum_{\sigma \in \Pi_n} \varepsilon_\sigma x_1^{\sigma(1)} \dots x_n^{\sigma(n)} \right|^2 dx_n \right)^{p/2} = \\ &= \frac{1}{c^p} \left(\int_K \sum_{\sigma, \sigma'} \varepsilon_\sigma \varepsilon_{\sigma'} x_1^{\sigma(1)} x_1^{\sigma'(1)} \dots x_n^{\sigma(n)} x_n^{\sigma'(n)} dx_n \right)^{p/2} = \end{aligned}$$

by definition of L_k and isotropic position of K

$$\begin{aligned}
&= \frac{L_K^p}{c^p} \left(\sum_{\sigma(n)=\sigma'(n)} \varepsilon_\sigma \varepsilon_{\sigma'} x_1^{\sigma(1)} x_1^{\sigma'(n)} \dots x_{n-1}^{\sigma(n-1)} x_{n-1}^{\sigma'(n-1)} \right)^{p/2} = \\
&= \frac{L_K^p}{c^p} \left(\sum_{i=1}^n \left| \sum_{\sigma(n)=i} \varepsilon_\sigma x_1^{\sigma(1)} \dots x_{n-1}^{\sigma(n-1)} \right|^2 \right)^{p/2} \\
&\geq \frac{L_K^p}{c^p} n^{\frac{p}{2}-1} \sum_{i=1}^n \left| \sum_{\sigma(n)=i} \varepsilon_\sigma x_1^{\sigma(1)} \dots x_{n-1}^{\sigma(n-1)} \right|^p.
\end{aligned}$$

Integrate the above inequality by $x_{n-1} \in K$ and repeat the above argument

$$\begin{aligned}
I_{n-2}(x_1, \dots, x_{n-2}) &\geq \frac{L_K^p}{c^{2p}} n^{\frac{p}{2}-1} \sum_{i=1}^n \left(\int \left| \sum_{\sigma(n)=i} \varepsilon_\sigma x_1^{\sigma(1)} \dots x_{n-1}^{\sigma(n-1)} \right|^2 dx_{n-1} \right)^{p/2} \\
&\geq \frac{L_K^{2p}}{c^{2p}} n^{\frac{p}{2}-1} \sum_{i=1}^n \left| \sum_{\substack{\sigma \in \Pi_n \\ \sigma(n)=i=\sigma'(n) \\ \sigma(n-1)=\sigma'(n-1)}} \varepsilon_\sigma \varepsilon_{\sigma'} x_1^{\sigma(1)} x_1^{\sigma'(1)} \dots x_{n-2}^{\sigma(n-2)} x_{n-2}^{\sigma'(n-2)} \right|^{p/2} \\
I_{n-2}(x_1, \dots, x_{n-2}) &\geq \frac{L_K^{2p}}{c^{2p}} n^{\frac{p}{2}-1} \sum_{i_n=1}^n \left| \sum_{\substack{i_{n-1}=1 \\ i_{n-1} \neq i_n}}^n \left(\sum_{\substack{\sigma \in \Pi_n \\ \sigma(n)=i_n \\ \sigma(n-1)=i_{n-1}}} \varepsilon_\sigma x_1^{\sigma(1)} \dots x_{n-2}^{\sigma(n-2)} \right)^2 \right|^{p/2} \\
&\geq \frac{L_K^{2p}}{c^{2p}} n^{\frac{p}{2}-1} (n-1)^{\frac{p}{2}-1} \sum_{i_n=1}^n \sum_{\substack{i_{n-1}=1 \\ i_{n-1} \neq i_n}}^n \left| \sum_{\substack{\sigma \in \Pi_n \\ \sigma(n)=i_n \\ \sigma(n-1)=i_{n-1}}} \varepsilon_\sigma x_1^{\sigma(1)} \dots x_{n-2}^{\sigma(n-2)} \right|^p.
\end{aligned}$$

Finally, we get

$$I_1(x_1) \geq \left(\frac{L_K}{c} \right)^{(n-1)p} n^{\frac{p}{2}-1} (n-1)^{\frac{p}{2}-1} \dots 2^{\frac{p}{2}-1} \sum_{i_n \neq i_{n-1} \neq \dots \neq i_2} \left\| \varepsilon_\sigma x_1^{\sigma(1)} \right\|^p$$

where σ is defined by $\sigma(n) = i_n \dots \sigma(2) = i_2$.

Therefore

$$\begin{aligned}
I_0 &= \int_{K^n} |\det(x_1, \dots, x_n)|^p dx_1 \dots dx_n = \int_K I_1(x_1) dx_1 \\
&\geq \left(\frac{L_K}{c} \right)^{np} (n!)^{\frac{p}{2}-1} n! = \left(\frac{L_K}{c} \right)^{np} (n!)^{p/2}.
\end{aligned}$$

Hence

$$\left(\int_{K^n} |\det(x_1, \dots, x_n)|^p dx_1 \dots dx_n \right)^{1/p} \geq \frac{L_K^n (n!)^{1/2}}{c^n}.$$

and we deduce that for $0 < p < 2$

$$L_K \leq \frac{c}{\sqrt{n}} \left(\int_{K^n} |\det(x_1, \dots, x_n)|^p dx_1 \dots dx_n \right)^{1/np}$$

for some new universal constant c .

Two partial cases have a special interest. The case $p = 1/n$ and the case $p \rightarrow 0$ which give

Corollary.

- a. $L_K \leq \frac{c}{\sqrt{n}} \int_{K^n} |\det(x_1, \dots, x_n)|^{1/n} dx_1 \dots dx_n .$
- b. $L_K \leq \frac{c}{\sqrt{n}} \exp \left(\int_{K^n} \log |\det(x_1, \dots, x_n)|^{1/n} dx_1 \dots dx_n \right) .$

3.8. We now extend Corollary 3.2 to a higher codimension.

Theorem. *There exists a function f on \mathbb{N} so that for every integer $n, n > k$, every n -dimensional isotropic body K and for every k -codimensional subspaces E_1 and E_2 of \mathbb{R}^n we have:*

$$(\text{vol}(K \cap E_1) / \text{vol}(K \cap E_2))^{1/k} \leq f(k) .$$

This result is obtained by Hensley [H] with an estimate of $f(k)$ of the order of $k!$. The latter estimate was improved by Ball [Ba.1], [Ba.2] who get a function $f(k)$ of the order of \sqrt{k} . This estimate may also be improved and the question whether $f(k)$ can be bounded above by a universal constant is open and would follow from the main problem of section 4 as is clear from Proposition 3.11 below.

We shall give below a more precise result but before this we need some statements of independent interest.

3.9 Theorem. *Let K be a symmetric convex body in \mathbb{R}^n . Let $1 \leq k \leq n$ and let E be a k -codimensional subspace of \mathbb{R}^n . Let F be the orthogonal subspace of E and for every non-zero vector θ in F denote by $E(\theta)$ the $(k-1)$ -codimensional subspace generated by E and θ . Then the function*

$$\frac{|\theta|}{\text{vol}(K \cap E(\theta))} , \quad \theta \in F$$

is a norm on F .

Proof: Let θ_1 and θ_2 be non colinear vectors in F and set $\theta_3 = \theta_1 + \theta_2$. Let $f_i(x) = \text{vol}[K \cap (x \frac{\theta_i}{|\theta_i|} + E)]$, $x > 0$, $i = 1, 2, 3$ and set

$$F_i = \int_0^{+\infty} f_i(u) du = \frac{1}{2} \text{vol}(K \cap E(\theta_i)) , \quad i = 1, 2, 3 .$$

Since the function $|\theta| / \text{vol}(K \cap E(\theta))$ is positively homogeneous, the theorem will be proved by showing that

$$\frac{F_3}{|\theta_3|} \geq \left(\frac{|\theta_1|}{F_1} + \frac{|\theta_2|}{F_2} \right)^{-1} .$$

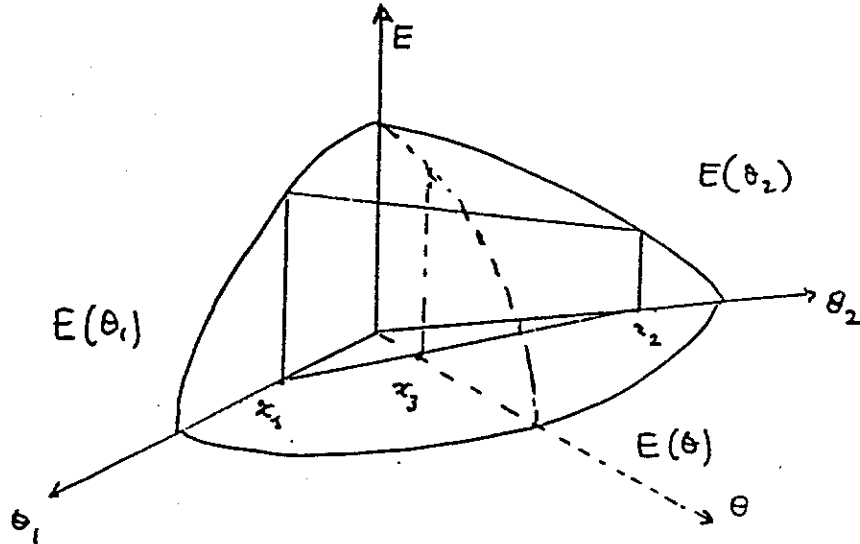
Let x_1, x_2 be positive numbers and let P_1, P_2 be the points defined by $OP_i = x_i \theta_i / |\theta_i|$, $i = 1, 2$. The segment $P_1 P_2$ intersects the straight line in the direction of θ_3 at a point P_3 such that $OP_3 = x_3 \theta_3 / |\theta_3|$ where x_3 satisfies

$$(3.1) \quad \frac{|\theta_3|}{x_3} = \frac{|\theta_1|}{x_1} + \frac{|\theta_2|}{x_2}.$$

On the other hand we look at P_3 as a barycenter of P_1 and P_2 by writing $P_2 P_3 = \alpha P_2 P_1$ with $0 \leq \alpha \leq 1$ given by the relation $(1 - \alpha)x_2 / |\theta_2| = \alpha x_1 / |\theta_1|$, so that

$$(3.2) \quad \alpha = \frac{x_2 / |\theta_2|}{x_1 / |\theta_1| + x_2 / |\theta_2|}$$

Note that $\frac{\alpha x_1}{|\theta_1|} \theta_1 + \frac{(1-\alpha)x_2}{|\theta_2|} \theta_2 = \frac{x_3}{|\theta_3|} \theta_3$.



Now we map the intervals where $f_i(u) \neq 0$, $i = 1, 2$ onto $[0, 1]$ through the relations

$$t = \frac{1}{F_1} \int_0^{x_1} f_1(u) du = \frac{1}{F_2} \int_0^{x_2} f_2(u) du, \quad 0 \leq t \leq 1.$$

Hence, we get $\frac{dx_1}{dt} = \frac{F_1}{f_1(x_1)}$, $\frac{dx_2}{dt} = \frac{F_2}{f_2(x_2)}$ and taking the derivative of (3.1) we obtain $\frac{|\theta_3|}{x_3^2} \frac{dx_3}{dt} = \frac{|\theta_1|}{x_1^2} \frac{F_1}{f_1(x_1)} + \frac{|\theta_2|}{x_2^2} \frac{F_2}{f_2(x_2)}$. From Brunn's theorem we have

$$f_3(x_3) \geq f_1(x_1)^\alpha f_2(x_2)^{1-\alpha}.$$

Therefore

$$(3.3) \quad \frac{F_3}{|\theta_3|} \geq \int_0^1 \frac{1}{|\theta_3|} f_3(x_3(t)) \frac{dx_3}{dt} dt$$

and

$$\frac{1}{|\theta_3|} f_3(x_3) \frac{dx_3}{dt} \geq \frac{x_3^2}{|\theta_3|^2} \left(\frac{|\theta_1|}{x_1^2} \cdot \frac{F_1}{f_1(x_1)} + \frac{|\theta_2|}{x_2^2} \cdot \frac{F_2}{f_2(x_2)} \right) f_1(x_1)^\alpha f_2(x_2)^{1-\alpha}.$$

Setting $a = |\theta_1|/x_1$ and $b = |\theta_2|/x_2$ and using the relations (3.1) and (3.2), we get that the integrand in (3.3) is at least

$$\frac{1}{(a+b)^2} \left(a^2 \frac{F_1}{|\theta_1| f_1(x_1)} + b^2 \frac{F_2}{|\theta_2| f_2(x_2)} \right) f_1(x_1)^{a/(a+b)} f_2(x_2)^{b/(a+b)}.$$

Comparing the arithmetic and geometric mean we get

$$a \left(a \frac{F_1}{|\theta_1| f_1(x_1)} \right) + b \left(b \frac{F_2}{|\theta_2| f_2(x_2)} \right) \geq (a+b) \left(\frac{aF_1}{|\theta_1| f_1(x_1)} \right)^{a/(a+b)} \left(\frac{bF_2}{|\theta_2| f_2(x_2)} \right)^{b/(a+b)}$$

and using this comparison one more time we have

$$\begin{aligned} \left(\frac{aF_1}{|\theta_1|} \right)^{a/(a+b)} \left(\frac{bF_2}{|\theta_2|} \right)^{b/(a+b)} &\geq \left(\frac{a}{a+b} \cdot \frac{|\theta_1|}{aF_1} + \frac{b}{a+b} \cdot \frac{|\theta_2|}{bF_2} \right)^{-1} \\ &\geq (a+b) \left(\frac{|\theta_1|}{F_1} + \frac{|\theta_2|}{F_2} \right)^{-1}. \end{aligned}$$

These inequalities show that the integrand in (2.3) is at least $\left(\frac{|\theta_1|}{F_1} + \frac{|\theta_2|}{F_2} \right)^{-1}$ and this accomplishes the proof. \square

Theorem 3.9 was proved by Busemann [Bus.1] for $k = 1$. The proof given above for any k follows the same lines. When $k = 1$, Theorem 3.9 may be stated differently. For θ in \mathbb{R}^n , let $\{\theta^\perp\}$ be the hyperplane orthogonal to θ . Then $|\theta|/\text{vol}(K \cap \{\theta^\perp\})$ is a norm in \mathbb{R}^n . This precisely the statement in [Bus.1]. This precise statement actually gives no real contribution to our problem because we already know from Corollary 3.1 that this function $|\theta|/\text{vol}(K \cap \{\theta^\perp\})$ is "equivalent" to an euclidean norm, the norm associated to the Binet ellipsoid of K . We shall use a more general result which follows from some inequalities of Ball [Ba.1].

3.10 Lemma. *Under the same assumptions and notations as in the preceding theorem, the function*

$$|\theta|^{1+\frac{p}{p+1}} / \left(\int_{K \cap E(\theta)} |\langle x, \theta \rangle|^p dx \right)^{1/p+1}, \quad \theta \in F, \quad p \geq 0$$

is a norm on F .

(Proof is the same as was demonstrated for $p = 0$).

3.11 Proposition. Let K be an isotropic body in \mathbb{R}^n . Let $1 \leq k \leq n$ and let E be a k -codimensional subspace. If C is the unit ball defined in Lemma 3.10 for $p = k + 1$ on the subspace F orthogonal to E , then

$$c_1 L_C / L_K \leq \text{vol}(K \cap E)^{1/k} \leq c_2 L_C / L_K$$

where $c_1 > 0$ and c_2 are universal constants.

This proposition implies immediately the Theorem 3.8 in which the estimate of the function f depends on an estimate on the constants L_C which we shall study in section 5.

Proof: Let y be in F . Put $x = x' + \rho\theta$ where $\theta \in S(F)$ (we use a cylindrical coordinate). Note that $(x'; \rho)$ is a point of $E(\theta)$. Then

$$\int_K |\langle x, y \rangle|^2 dx = k \text{vol } D_k \int_{\theta \in S(F)} |\langle \theta, y \rangle|^2 \left(\int_{\substack{x = \rho\theta + x' \in K \cap E(\theta) \\ x' \in E}} \rho^{k+1} d\rho dx' \right) d\theta =$$

(note that $\rho = |\langle x, \theta \rangle|$)

$$= \frac{k}{2} \text{vol } D_k \int_{\theta \in S^{k-1}} |\langle \theta, y \rangle|^2 \left(\int_{K \cap E(\theta)} |\langle x, \theta \rangle|^{k+1} dx \right) d\sigma(\theta).$$

Therefore

$$\begin{aligned} L_K^2 &= \frac{k}{2} (\text{vol } D_k) \int_{S^{k-1}} |\langle \theta, y \rangle|^2 \frac{1}{\|\theta\|_C^{k+2}} d\sigma(\theta) \\ &= \frac{k+2}{2} \int_C |\langle x, y \rangle|^2 dx. \end{aligned}$$

This shows that $\frac{C}{(\text{vol } C)^{1/k}}$ is isotropic, therefore

$$L_K^2 = \frac{k+2}{2} L_C^2 (\text{vol } C)^{1+2/k}.$$

And

$$\text{vol } C = (\text{vol } D_k) \int_{S^{k-1}} \left(\int_{K \cap E(\theta)} |\langle x, \theta \rangle|^{k+1} dx \right)^{k/k+2} d\sigma(\theta).$$

Using Lemma 2.6 we get that

$$\left(\frac{1}{2} \int_{K \cap E(\theta)} |\langle x, \theta \rangle|^{k+1} dx / \text{vol}(K \cap E)(k+1)! \right)^{1/k+2} \cdot \left(\frac{1}{2} \int_{K \cap E(\theta)} |\langle x, \theta \rangle|^{k-1} / \text{vol}(K \cap E)(k-1)! \right)^{1/k}.$$

Therefore

$$\begin{aligned} \text{vol } C &\leq (\text{vol } D_k) \int_{S^{k-1}} \left(\int_{K \cap E(\theta)} | \langle x, \theta \rangle |^{k-1} dx \right) d\sigma(\theta) \cdot \frac{k(k+1)((k+1)!)^{-2/k+2}}{(2 \text{vol}(K \cap E))^{2/k+2}} \\ &\leq (\text{vol } K) \frac{(k+1)((k+1)!)^{-2/(k+2)}}{(2 \text{vol}(K \cap E))^{2/k+2}}. \end{aligned}$$

Hence

$$L_K^2 \leq L_C^2 \cdot \frac{1}{\text{vol}(K \cap E)^{2/k}} \cdot \frac{(k+1)(k+2)}{((k+1)!)^{2/k}} 2^{-1-2/k}$$

and

$$L_K^2 \leq c_2 L_C^2 / \text{vol}(K \cap E)^{2/k},$$

for some absolute constant c_2 .

The inverse inequality is obtained through the same line now using Lemma 2.1 instead of Lemma 2.6.

Remark. Proposition 3.11 can be found implicit in K. Ball [Ba1].

3.12. We end this section with one application which is a corollary to subsection 3.5. The concentration property of a function

$$\varphi(t) = \frac{\text{vol}(K \setminus tD)}{\text{vol } K}$$

was discussed in [Mi.1]. We were interested there in the behavior of $\varphi(t)$ around its maximum.

Clearly, if $\text{vol } K = \text{vol } D$, then $\varphi(t) < t^n$ (for $t < 1$). It is also known by Borell's lemma (see [M.Sch], Appendix 3) that if $\varphi(t_0) < 1/2$ then, for $t > t_0$, $\varphi(x) < c_1 e^{-c_2(t-t_0)}$ for universal constants c_1 and c_2 . So, we would have the concentration of $\varphi(t)$ around its maximum if t_0 is close enough to 1.

We will show this below for one important example. Let $K_n = \sqrt{n} \text{Conv}\{\pm e_1, \dots, \pm e_n\}$ where $\{e_i\}_1^n$ is the canonical orthonormal basis. Then K is homothetic to the unit ball of \mathcal{L}_1^n , $D \subset K_n$ and $(\text{vol } K_n / \text{vol } D)^{1/n} \leq C$ for some universal constant C (easily computable). By 3.6, L_{K_n} is uniformly bounded and therefore for a universal constant C_1

$$\frac{1}{\text{vol } K_n} \int_{K_n} |x|^2 dx \leq C_1.$$

This immediately implies that there exists a universal t_0 such that $\text{vol}(K_n \setminus t_0 D) / \text{vol } K_n < 1/2$.

Therefore,

Let $\varphi_n(t) = \text{vol}(K_n \setminus tD) / \text{vol } K_n$. Then $\varphi(t) < t^n$ for $t < 1$ and $\varphi(t) < c_1 \exp(-c_2 t)$ for $t > 1$ where c_1 and c_2 are some universal constants.

Remark. Professor S. Smale asked one of the authors this question a few years ago. He was more interested in a non-symmetric version, i.e., when $K_n = \text{conv}\{e_1, \dots, e_n\}$. This case may be obtained from the symmetric case.

4. Busemann Formula and Projections of a Convex Body

Let K be a convex body in \mathbb{R}^n . Bounds for L_K , $\text{vol } \mathcal{L}$ and $\text{vol } Z(K)$ (as functions of volumes of K) have been investigated and we begin by recalling some relevant facts.

4.1 Lemma. Let K be an n -dimensional body in \mathbb{R}^n . Then $L_K \geq L_{D_n} \geq c$ where $c > 0$ is a universal constant

Proof: We may suppose that K is isotropic. Then if rD is the euclidean ball of volume one, we have

$$L_K^2 n = \int_K |x|^2 dx \geq \int_{rD} |x|^2 dx = L_D^2 n$$

since the euclidean norm of any vectors in $K \setminus rD$ is always larger than for any vectors in $rD \setminus K$. Therefore $L_K \geq L_D$ with equality if and only if $K = D$. Moreover,

$$L_{D_n} = (n+2)^{-1} (\text{vol } D_n)^{-2/n} \geq c > 0$$

for some absolute constant c .

As an immediate consequence of the relation between L_K and Legendre ellipsoid and its volume (see 1.7) we get that $\text{vol } \mathcal{L}(K) \geq \text{vol } \mathcal{L}(rD)$ for $\text{vol}(rD) = \text{vol } K$. Therefore

4.2 Corollary. Let K be an n -dimensional body in \mathbb{R}^n . Then

$$\text{vol}(\mathcal{L}(K)) \geq \text{vol } K$$

with equality if and only if K is an ellipsoid.

Remark. Lemma 4.1 and Corollary 4.2 are due to Blaschke [Bla.1], [Bla.2] (see also [J]).

4.3 Proposition. Let K be a symmetric convex body in \mathbb{R}^n . Then

$$\text{vol } Z(K) \geq \text{vol } Z(\lambda D_n)$$

where homothetical normalization λ is chosen such that

$$\text{vol}(\lambda D_n) = \text{vol } K .$$

In particular

$$\begin{aligned} (\text{vol } Z(K))^{1/n} &\geq \left(\frac{\text{vol}(Z(D_n))}{\text{vol } D_n} \right)^{1/n} (\text{vol } K)^{1/n} \\ &\geq \frac{c}{\sqrt{n}} (\text{vol } K)^{1/n} \end{aligned}$$

where $c > 0$ is a universal constant.

Proposition 4.3 is due to Busemann [Bus.2] and uses a symmetrization method of Blaschke developed in particular in the treatment of the so-called Sylvester's problem. In dimension n , Sylvester's problem is the following question.

4.4. Find the probability $p(K)$ that $(n+2)$ points chosen at random inside a convex set K form a convex polytope. Considering complementary probability leads to

$$(4.1) \quad 1 - p(K) = (n+1) \frac{1}{(\text{vol } K)^{n+1}} \int_K \dots \int_K T(x_1, x_2, \dots, x_{n+1}) dx_1 \dots dx_{n+1}$$

where $T(x_1, x_2, \dots, x_{n+1})$ denotes the volume of the simplex with vertices x_1, \dots, x_{n+1} .

The number $p(K)$ is an affine invariant. Blaschke showed [Bla.2] that this probability is the greatest for ellipsoid. The proof uses a Steiner symmetrization method (see [B.M.M.P]), and involves proving that the average of the determinant $T(x_1, \dots, x_{n+1})$ decreases through such a symmetrization. The problem of lower bound for $p(K)$ is open and is in some sense equivalent to the main problem in section 5. Blaschke showed [Bla.2 and 3] that in dimension 2, $p(K)$ has its smallest value for a triangle. (See [S] and for historical remarks see [Kl].)

4.5 Proposition. Let K be a convex body in \mathbb{R}^n . Then

$$(\text{vol } K)^{n-1} = n! \text{vol } D_n \int_{G_{n,n-1}} (\text{vol } Z(K \cap H)) (\text{vol}(K \cap H))^{n-1} dH$$

where dH denotes the invariant probability measure on $G_{n,n-1}$.

Proposition 4.5 is a particular case of Busemann's formula, stating that if K_1, \dots, K_{n-1} are convex bodies in \mathbb{R}^n , then

$$\prod_{i=1}^{n-1} \text{vol } K_i = n! \text{vol } D_n \int_{G_{n,n-1}} T(K_1 \cap H, \dots, K_{n-1} \cap H) dH$$

where $T(K_1 \cap H, \dots, K_{n-1} \cap H) = \int_{K_1 \cap H} \dots \int_{K_{n-1} \cap H} T(0, x_1, \dots, x_{n-1}) dx_1 \dots dx_{n-1}$ and $T(0, x, \dots, x_{n-1})$ denotes as before the volume of the simplex with vertices $0, x_1, \dots, x_{n-1}$, thus $T(0, x, \dots, x_{n-1}) = \frac{1}{(n-1)!} |\det(x_1, \dots, x_{n-1})|$, x_1, \dots, x_{n-1} in H . Proposition 4.5 follows thus from the last formula and from (1.8).

Remark. Actually there is a more general formula due to Blaschke and Petkantschin (see [S]). Denote by $dx_i(E)$ the volume element in $E \in G_{n,k}$ at point x_i and dE the volume measure on $G_{n,k}$, then

$$\prod_{i=1}^k dx_i = (k! T(0, x_1, \dots, x_k))^{n-k} \left(\prod_{i=1}^k dx_i(E) \right) dE.$$

For a short proof of this result and related formula see [S], page 200. (Note that for $k = 1$ this is just integration in polar coordinates.)

4.6 Corollary. Let K be a symmetric convex body in \mathbb{R}^n . Then

$$(\text{vol } K)^{\frac{n-1}{n}} \geq c \left(\int_{G_{n,n-1}} (\text{vol}(K \cap H))^n dH \right)^{1/n}$$

where $c > 0$ is a universal constant.

Proof: From Proposition 4.5 and 4.3 we have

$$(\text{vol } K)^{n-1} \geq n! \text{vol } D_n \left(\frac{\text{vol } Z(D_n)}{\text{vol } D_{n-1}} \right) \int_{G_{n,n-1}} \text{vol}(K \cap H)^n dH.$$

Taking the n -th root in the latter, gives the corollary.

Remark. A consequence of Corollary 4.5 is the following result of Busemann [Bus.2]: if K is a convex body and

$$\text{vol}(K \cap H) \geq \text{vol}(D_n \cap H) = \text{vol } D_{n-1}$$

then

$$\text{vol } K \geq \text{vol } D_n.$$

The question whether D_n may be replaced by any convex body in the last statement is discussed in section 5.

4.7. If we combine formula in 4.5 with Corollary 3.2, we see that if K is an isotropic body in \mathbb{R}^n then

$$(4.1) \quad L_K \sim \sqrt{n} \left(\int_{G_{n,n-1}} \text{vol} Z(K \cap H) dH \right)^{1/n-1}$$

Another relation which we note here is the following.

Let K be in the isotropic position. By Proposition 3.11, which we will use for every subspace $E \hookrightarrow H$, $\text{codim} E = 2$, and $\text{codim} H = 1$,

$$\text{vol}(K \cap E) \sim (1/L_K)^2 \quad \text{and} \quad \text{vol}(K \cap H) \sim 1/L_K.$$

By 3.1 it means that

$$\frac{1}{\text{vol}(K \cap H)} \int_{K \cap H} |\langle x, \theta \rangle| dx \sim \frac{1}{L_K}.$$

Stability property 1.8 shows now that

$$(4.2) \quad L_{K \cap H} \sim L_K.$$

5. The Main Problem; Equivalent Formulations

As is already clear from the previous section, it is very important to estimate L_K of an arbitrary body K . We saw in section 4 that $0 < c \leq L_K$ for a universal constant c independent of dimension n or body $K \subset \mathbb{R}^n$.

It is also probably a generally excepted hypothesis that L_K is also bounded from above.

Problem. Is it true that for a universal constant C

$$L_K \leq C?$$

We saw above that it is true in many interesting cases (see, e.g., Proposition 3.6).

Define $d(K, D) = \inf\{a \cdot b \mid K \subset aD \subset abK\}$. Also define the Banach-Mazur distance d_K of K to D as

$$d_K = \inf\{d(K, \mathcal{E}) \mid \mathcal{E} \text{ runs through all ellipsoids in } \mathbb{R}^n\}.$$

Note that it is easy to see from Lemma 1.6 that

$$L_K \lesssim d_K.$$

This means that in general $L_K \lesssim \sqrt{n}$. Recently J. Bourgain [Bou.2] has shown that $L_K \lesssim n^{1/4}$.

From the results of section 3 (see 3.6) it easily follows that for large classes of bodies, the problem has a positive solution. We add that the class of zonoids satisfies the condition of Proposition 3.6b (i.e., L_K is uniformly bounded for all zonoids).

In this section we will show some equivalent formulations to this problem which show that the problem is closely related to many old classical problems.

Let us first repeat some relations obtained in section 1. For a symmetric convex body $K \subset \mathbb{R}^n$,

$$L_K = (\det M)^{1/2n} / (\text{vol } K)^{n+2/2n} .$$

Therefore, we have

$$\text{vol } \mathcal{L} = L_K^{2n/(n+2)} (\text{vol } D_n)^{2/(n+2)} (n+2)^{n/(n+2)} \text{vol } K$$

and since as we know $0 < c_1 \leq L_K \leq c_2 \sqrt{n}$ for some universal constants c_1 and c_2 , then (see 1.7)

$$5.1 \quad \text{vol } \mathcal{L}(K) \simeq L_K^2 \text{vol } K .$$

Moreover from (1.5) we have

$$(\text{vol } Z(K))^{1/n} \simeq \frac{1}{\sqrt{n}} \frac{(\text{vol } \mathcal{L}(K))^{\frac{1}{n} + \frac{1}{2}}}{(\text{vol } K)^{1/2}} .$$

Therefore

$$5.2 \quad (\text{vol } Z(K))^{1/n} \simeq \frac{L_K}{\sqrt{n}} (\text{vol } K)^{1/n} .$$

Now, from Lemma 2.8, the function

$$\frac{|\theta|}{\text{vol}(K \cap H(\theta))} , \quad \theta \in \mathbb{R}^n , \quad H(\theta) = \{x \in \mathbb{R}^n ; \langle x, \theta \rangle = 0\}$$

defines a norm with C as the unit ball Then

$$\text{vol } C = (\text{vol } D_n) \int_{S^{n-1}} (\text{vol}(K \cap H(\theta)))^n d\sigma_{n-1}(\theta)$$

Note that, from Busemann formula (4.4)

$$\left(\frac{\text{vol } C}{\text{vol } D_n} \right)^{1/n} = \left(\int_{G_{n,n-1}} (\text{vol}(K \cap H))^n dH \right)^{1/n} \leq \frac{1}{c_n} (\text{vol } K)^{\frac{n-1}{n}}$$

where $c_n > 0$ is computed in the equality case, that is for $K = D_n$

$$c_n = \frac{v_n^{\frac{n-1}{n}}}{v_{n-1}} \sim 1 \quad (\text{where } v_N = \text{vol } D_n).$$

On the other hand, from (3.1) we have

$$\frac{|\theta| \text{vol } K}{\text{vol}(K \cap H(\theta))} \sim \left(\frac{1}{\text{vol } K} \int_K |\langle x, \theta \rangle|^2 dx \right)^{1/2}.$$

Therefore

$$\frac{|\theta|}{\text{vol}(K \cap H(\theta))} \sim \frac{1}{(\text{vol } K)^{3/2}} \|\theta\|_{\mathcal{B}(K)}$$

and

$$\begin{aligned} (\text{vol } C)^{1/n} &\sim (\text{vol } K)^{3/2} (\text{vol } \mathcal{B}(K))^{1/n} \\ &\sim (\text{vol } K)^{3/2} (\text{vol } D_n)^{1/n} \frac{(\text{vol } K)^{-\frac{n+2}{2n}}}{L_K}. \end{aligned}$$

Hence

$$5.3 \quad \left(\int_{G_{n,n-1}} (\text{vol}(K \cap H))^n dH \right)^{1/n} \sim \frac{(\text{vol } K)^{\frac{n-1}{n}}}{L_K}$$

We summarize the above discussion in the following proposition, emphasizing connections between different open problems.

5.4 Proposition. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. There exists universal constants c_1, c_2, c_3 such that the following are equivalent*

- (1) $L_K \leq C$
- (2) $\text{vol } \mathcal{L} \leq c_1 C^2 \text{vol } K$
- (3) $(\text{vol } Z(K))^{1/n} \leq c_2 C \frac{(\text{vol } K)^{1/n}}{\sqrt{n}}$
- (4) $(\text{vol } K)^{\frac{n-1}{n}} \leq c_3 C \left(\int_{G(n,n-1)} (\text{vol}(K \cap H))^n dH \right)^{1/n}$

The problems (3) and (4) are going back to Busemann [Bus.2]. More precisely, Busemann asked to estimate expressions on the left sides of (3) and (4) from above, in terms of the expressions from the right side of (3) and (4). However, he did not conjecture that c is a universal constant.

In the next proposition, we have an equivalence of different problems on the class of all symmetric convex bodies.

5.5 Proposition. The following are equivalent:

- (1) There exists a universal constant C such that for any $n \geq 1$ and for any symmetric convex body K of \mathbb{R}^n we have

$$L_K \leq C.$$

- (5) There exist a universal constant c_4 such that for any $n \geq 1$ and any convex body $K \subset \mathbb{R}^n$ we have

$$(\text{vol } K)^{\frac{n-1}{n}} \leq c_4 \max \{ \text{vol}(K \cap H) ; H \in G_{n,n-1} \}$$

(in other words there exists a hyperplane H s.t.

$$(\text{vol } K)^{\frac{n-1}{n}} \leq c_4 \text{vol}(K \cap H) .)$$

- (6) There exists a universal constant c_5 such that whenever, for some convex symmetric bodies K_1 and K_2 and any (central) hyperplane H

$$\text{vol}(K_1 \cap H) \leq \text{vol}(K_2 \cap H)$$

Then

$$\text{vol } K_1 \leq c_5 \text{vol } K_2 .$$

- (7) For every $n \geq 1$ and for any convex body $K \subset \mathbb{R}^n$ of volume 1 there exist a universal constant c_6 and an ellipsoid \mathcal{E} such that

$$(\text{vol } \mathcal{E})^{1/n} \leq c_6 \quad \text{and} \quad \text{vol}(K \cap \mathcal{E}) \geq \frac{1}{2} .$$

Remark. The problem (6) (even with constant $c_5 = 1$) was asked by Busemann (see [Bus.P]). It was shown by Larman and Rogers [La.R] that for some bodies K_1 and K_2 constant c_5 must be strictly more than 1. They have showed this in dimension $n \geq 12$. Recently K. Ball [Ba.3] has shown that indeed it is enough to compare the unit cube $[-1, +1]^n$ and a euclidean ball of the right radius, started dimension $n \geq 10$, to see that $c_5 > 1$.

The problem (5) was considered by Vaaler [Va] who studied sections of a cube. More general cases were considered by [Me.Pa].

Proof:

(1) \Rightarrow (5) immediately follows from statement 4 in the last proposition

(5) \Rightarrow (1) is also immediate from Corollary 2.7, since for an isotropic body K we have

$$L_K \sim \frac{1}{\text{vol}(K \cap H)}$$

for any central hyperplane.

(6) \Rightarrow (5) follows by applying (6) to an euclidean ball $K_2 = rD$.

Indeed let $K_2 = rD_n$ where r is defined by

$$\max_{H \in G_{n,n-1}} \text{vol}(K \cap H) = \text{vol}((rD_n) \cap H) = r^{n-1} \text{vol} D_{n-1}.$$

Then

$$\text{vol} K \leq c_5 \text{vol}(rD_n) = c_5 r^n \text{vol} D_n$$

implies

$$(\text{vol} K)^{\frac{n-1}{n}} \leq c_5^{\frac{n-1}{n}} \frac{(\text{vol} D_n)^{\frac{n-1}{n}}}{\text{vol} D_{n-1}} \max_{H \in G_{n,n-1}} (\text{vol}(K \cap H))$$

and

$$\frac{(\text{vol} D_n)^{n-1/n}}{\text{vol} D_{n-1}} \sim 1.$$

(1) \Rightarrow (6) follows from (4) of Proposition 5.4. Actually the hypothesis

$$\text{vol}(K_1 \cap H) \leq \text{vol}(K_2 \cap H) \quad \forall H \in G_{n,n-1}$$

implies

$$\frac{\text{vol} K_1}{\text{vol} K_2} \leq c_6 \frac{L_{K_1}}{L_{K_2}}.$$

Therefore since L_{K_2} is bounded from below, to compare $\text{vol} K_1$ with $\text{vol} K_2$ it is enough to have control of L_{K_1} from above. The property (6) is therefore satisfied for any couples (K_1, K_2) such that $L_{K_1} \leq C$.

(1) \Rightarrow (7) immediately follows from Borell Lemma as used in 1.4 and as presented in [M.Sch] by applying it in (1.7) and (1.8).

Note, of course, that $\frac{1}{2}$ in (7) is chosen quite arbitrarily and any positive number strictly less than one could be chosen.

We add, to end, another interesting equivalence between the main problem and the old question of Sylvester, which we discussed in section 4.4, adjusted to the n -dimensional asymptotic theory. (Sylvester asked this question in dimension 2 and it was answered by Blaschke.) So, our problem is to estimate (4.1).

Proposition 5.6. *Let $K \subset \mathbb{R}^n$ be a symmetric convex body of $\text{vol} K = 1$ and $p(K)$ is defined in 4.4. Then*

$$(1 - p(K))^{1/n} \sim \frac{L_K}{\sqrt{n}}.$$

Proof: First we rewrite (4.1)

$$1 - p(K) = (n+1) \int_{K^{n+1}} \frac{|\det(x_1 - x, \dots, x_n - x)|}{n!} dx_1 \dots dx_n dx.$$

By expanding the determinant and just by the triangle inequality we have (remember Proposition 1.8)

$$1 - p(K) \leq (n+1)n \operatorname{vol} Z(K).$$

On the other side (note, that $(\dots, \overset{i}{\cdot}, \dots)$ means the i -th place)

$$\frac{1 - p(K)}{n+1} = \frac{1}{n!} \int_{K^{n+1}} \left| \sum_{i=1}^n \det(x_1, \dots, \overset{i}{-x}, \dots, x_n) \right| dx_1 \dots dx_n dx =$$

(by symmetry of K)

$$= \frac{1}{n!} \int_{K^{n+1}} \frac{\left| \sum_{i=1}^n \det(x_1, \dots, \overset{i}{-x}, \dots, x_n) \right| + \left| \sum_{i=1}^n \det(x_1, \dots, \overset{i}{+x}, \dots, x_n) \right|}{2} dx_1 \dots dx_n dx \geq$$

by the triangle inequality

$$\geq \int_{K^n} \frac{|\det(x_1, \dots, x_n)|}{n!} dx_1 \dots dx_n = \operatorname{vol} Z(K).$$

So, we prove

$$(n+1) \operatorname{vol} Z(K) \leq 1 - p(K) \leq (n+1)n \operatorname{vol} Z(K).$$

It remains to apply 5.2.

References

- [Ba.1] K. Ball, Isometric problems in ℓ_p and sections of convex sets. Ph.D. Thesis, Cambridge University 1986.
- [Ba.2] K. Ball, Logarithmic concave functions and sections of convex sets in \mathbb{R}^n . To appear in *Studia Math.*
- [Ba.3] K. Ball, Some remarks on the geometry of convex sets, *GAFSA-Seminar '86-87*, Springer Lecture Notes in Math. 1317 (1988), 224-231.
- [Ber] L. Berwald, Verallgemeinerung eines Mittelwertsatzes von J. Favard, für positive konkave Funktionen. *Acta Math.* 79 (1947), 17-37.
- [B.F] T. Bonnesen and W. Fenchel. *Theorie des Konvexen Körper*. Chelsea Publish. Co. New York, 1948.

- [Bla.1] W. Blaschke, Über affine Geometry XIV: eine minimum Aufgabe für Legendres trägheits Ellipsoid. Ber. verh. sächs. Akad. d. Wiss. 70 (1918), 72-75.
- [Bla.2] W. Blaschke, Über affine Geometry XI: lösing der "Vierpunkproblems" von Sylvester aus der Theorie der geometrischen Wahrsdeinlichkeiten. Leipziger Berichte 69 (1917), 436-453.
- [Bla.3] W. Blaschke, Vorlesung über Differentialgeometrie II. Affine Differentialgeometrie, Springer, Berlin 1923.
- [Bo.1] C. Borell, Inverse Hölder inequalities in one and several dimensions. Jour. of Math. Anal. and Applications 41 (1973), 300-312.
- [Bo.2] C. Borell, Complements of Lyapounov's inequality. Math. Ann. 205 (1973), 323-331.
- [Bo.3] C. Borell, The Brunn-Minkowski inequality in Gauss spaces, Inventiones Math, 30 (1975), 207-216.
- [Bou.1] J. Bourgain, On high dimensional maximal functions associated to convex bodies. Amer. J. Math.
- [Bou.2] J. Bourgain, Private communication.
- [B.L.M] J. Bourgain, J. Lindenstrauss and V. Milman, Approximation of zonoids by zonotopes. Preprint I.H.E.S., September 1987, 62pp., Acta Math.
- [B.M.M.P] J. Bourgain, M. Meyer, V. Milman and A. Pajor, On a geometric inequality. GAFA-Seminar '86-87, Springer Lecture Notes in Math. 1317 (1988), 271-282.
- [Bou.Mi] J. Bourgain and V.D. Milman, New volume ratio properties of convex symmetric bodies in R^n . Invent. Math. 88 (1987), 319-340.
- [Bus.1] H. Busemann, A theorem on convex bodies of Brunn-Minkowski type. Amer. J. Math. 71 (1949), 743-762.
- [Bus.2] H. Busemann, Volume in terms of concurrent cross-section. Pacific J. Math. 3 (1953), 1-12.
- [Bus.P] H. Busemann and C.M. Petty, Problems on convex bodies. Math. Scand. 4 (1956), 88-94.
- [B.Z] Y. Burago and V. Zalgaler, Geometric inequalities. Springer's Series of Comprehensive Studies in Math. 285, 1988 (translated from Russia).
- [Gr.M] M. Gromov and V.D. Milman, Brunn theorem and a concentration of volume of convex bodies. GAFA Seminar Notes, Tel Aviv University, Israel 1983-1984, Exp. V., 12pp.

- [U] D.C. Ullrich, An extension of the Kahane-Khinchine inequality. Bull. of the Amer. Math. Soc. 18 (1988), 52-54.
- [V] J.D. Vaaler, A geometric inequality with applications to linear forms. Pacific J. of Math. 83 (1979), 543-553.