

On the Limiting Empirical Measure of the sum of rank one matrices with log-concave distribution

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Abstract

We consider $n \times n$ real symmetric and hermitian random matrices $H_{n,m}$ equals the sum of a non-random matrix $H_n^{(0)}$ matrix and the sum of m rank-one matrices determined by m i.i.d. isotropic random vectors with log-concave probability law and i.i.d. random amplitudes $\{\tau_\alpha\}_{\alpha=1}^m$. This is a generalization of the case of vectors uniformly distributed over the unit sphere, studied in [17]. We prove as in [17] that if $n \rightarrow \infty$, $m \rightarrow \infty$, $m/n \rightarrow c \in [0, \infty)$ and that the empirical eigenvalue measure of $H_n^{(0)}$ converges weakly, then the empirical eigenvalue measure of $H_{n,m}$ converges in probability to a non-random limit, found in [17].

1 Introduction: Problem and Main Result

Let $\{Y_\alpha\}_{\alpha=1}^m$ be i.i.d. random vectors of \mathbb{R}^n (or \mathbb{C}^n), and $\{\tau_\alpha\}_{\alpha=1}^m$ be i.i.d. random variables with common probability law σ . Set

$$M_{n,m} = \sum_{\alpha=1}^m \tau_\alpha L_{Y_\alpha}, \quad (1.1)$$

where $L_Y = Y \otimes Y$ is the rank-one matrix, corresponding to $Y \in \mathbb{R}^n$ (or \mathbb{C}^n) and defined as

$$L_Y X = (X, Y)Y, \quad \forall X \in \mathbb{R}^n(\mathbb{C}^n), \quad (1.2)$$

with (\cdot, \cdot) denoting the standard euclidian (or hermitian) scalar product in \mathbb{R}^n (or \mathbb{C}^n).

Let also $H_n^{(0)}$ be a real symmetric (or hermitian) $n \times n$ matrix. We then consider the real symmetric (or hermitian) $n \times n$ random

$$H_{n,m} = H_n^{(0)} + M_{n,m}. \quad (1.3)$$

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Denote

$$-\infty < \lambda_1 \leq \dots \leq \lambda_n < \infty \quad (1.4)$$

the eigenvalues of $H_{n,m}$ and introduce their Normalized Counting (or empirical) Measure $N_{n,m}$, setting for any interval $\Delta \subset \mathbb{R}^n$

$$N_{n,m}(\Delta) = \text{Card}\{l \in [1, n] : \lambda_l \in \Delta\}/n. \quad (1.5)$$

Likewise, we define the Normalized Counting Measure $N_n^{(0)}$ of eigenvalues $\{\lambda_l^{(0)}\}_{l=1}^n$ of $H_n^{(0)}$

$$N_n^{(0)}(\Delta) = \text{Card}\{l \in [1, n] : \lambda_l^{(0)} \in \Delta\}/n \quad (1.6)$$

and we assume that the sequence $\{N_n^{(0)}\}$ converges weakly to a probability measure $N^{(0)}$:

$$\lim_{n \rightarrow \infty} N_n^{(0)} = N^{(0)}. \quad (1.7)$$

It was shown in [17] that if $\{Y_\alpha\}_{\alpha=1}^m$ are uniformly distributed over the unit sphere of $\mathbb{R}^n(\mathbb{C}^n)$ and

$$n \rightarrow \infty, m \rightarrow \infty, m/n \rightarrow c \in [0, \infty), \quad (1.8)$$

then there exists a non-random probability measure N ($N(\mathbb{R}) = 1$) such that for any interval $\Delta \subset \mathbb{R}$ we have the convergence in probability

$$\lim_{n \rightarrow \infty, m \rightarrow \infty, m/n \rightarrow c} N_{n,m}(\Delta) = N(\Delta). \quad (1.9)$$

The limiting non-random measure N can be found as follows. Introduce the Stieltjes transform

$$f^{(0)}(z) = \int_{\mathbb{R}} \frac{N^{(0)}(d\lambda)}{\lambda - z}, \quad \Im z \neq 0 \quad (1.10)$$

of the measure $N^{(0)}$ of (1.7) and the Stieltjes transform

$$f(z) = \int_{\mathbb{R}} \frac{N(d\lambda)}{\lambda - z}, \quad \Im z \neq 0 \quad (1.11)$$

of the measure N of (1.9). Then f is uniquely determined by the functional equation

$$f(z) = f^{(0)} \left(z - c \int_{\mathbb{R}} \frac{\tau \sigma(d\tau)}{1 + \tau f(z)} \right), \quad (1.12)$$

considered in the class of functions analytic in $\mathbb{C} \setminus \mathbb{R}$ and such that $\Im f(z) \Im z \geq 0$, $\Im z \neq 0$. Since the Stieltjes transform determines N uniquely by the formula

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_{\mathbb{R}} \varphi(\lambda) \Im f(\lambda + i\varepsilon) d\lambda = \int_{\mathbb{R}} \varphi(\lambda) N(d\lambda), \quad (1.13)$$

valid for any continuous function of compact support, (1.12) determines N uniquely.

The same result is valid if the components $\{Y_{\alpha j}\}_{j=1}^n$ of Y_α , $\alpha = 1, \dots, m$ are i.i.d. random variables of zero mean and of unit variance. A particular case of this for $H_n^{(0)} = 0$, $\tau_\alpha = 1$, $\alpha = 1, \dots, m$ and Gaussian $\{Y_{\alpha j}\}_{j=1}^n$ is known since the 30th in statistics as the Wishart matrix (see e.g. [18]). The same random matrix appears also in the local theory of Banach

spaces or so-called asymptotic convex geometry (see e.g. [8, 22]). One particular and important case that enters this framework is the study of some geometric parameters associated to i.i.d. random points Y_α , $\alpha = 1, \dots, m$ uniformly distributed on a convex body in \mathbb{R}^n and the asymptotic geometry of the random convex polytope generated by these points (see e.g. [6, 11, 12, 16, 3]).

In this paper we prove the same result for the case where the common probability law of the i.i.d. vectors Y_α , $\alpha = 1, \dots, m$ is isotropic and log-concave. A preliminary non-published result was obtained in 2004 by the authors when the vectors are random points uniformly distributed in the unit ball $\{\sum_1^n |x_i|^p \leq 1\}$. The latter case was also obtained by a different approach in [2]. Here are the corresponding definitions.

Definition 1.1 [Isotropic vectors] A random real vector $Y \in \mathbb{R}^n$ is called isotropic if

$$\mathbf{E}\{(Y, X)\} = 0 \quad \text{and} \quad \mathbf{E}\{|(Y, X)|^2\} = n^{-1}|X|^2, \quad \forall X \in \mathbb{R}^n, \quad (1.14)$$

where $|X|$ denotes the euclidian norm of X .

A random complex vector $Y \in \mathbb{C}^n$ is called isotropic if $(\Re Y, \Im Y) \in \mathbb{R}^{2n}$ is isotropic and $\Re Y$ and $\Im Y$ are independent.

The definition implies that an isotropic complex random vector satisfies that $\mathbf{E}\{(Y, X)\} = 0$ and $\mathbf{E}\{|(Y, X)|^2\} = n^{-1}|X|^2$ for any $X \in \mathbb{C}^n$, but the reverse is not true.

Recall also that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called log-concave if for any $\theta \in [0, 1]$ and any $X_1, X_2 \in \mathbb{R}^n$, then $f(\theta X_1 + (1 - \theta)X_2) \geq f(X_1)^\theta f(X_2)^{1-\theta}$.

Definition 1.2 [Log-concave measure] A measure μ on \mathbb{R}^n (or \mathbb{C}^n) is log-concave if for any measurable subsets A, B of \mathbb{R}^n (or \mathbb{C}^n) and any $\theta \in [0, 1]$,

$$\mu(\theta A + (1 - \theta)B) \geq \mu(A)^\theta \mu(B)^{(1-\theta)}$$

whenever $\theta A + (1 - \theta)B = \{\theta X_1 + (1 - \theta)X_2 : X_1 \in A, X_2 \in B\}$ is measurable.

Remark 1.3 If Y is a random vector with a log-concave distribution, then any affine image of Y has a log-concave distribution. If Y_1 and Y_2 are independent random vector with log-concave distributions, then the pair (Y_1, Y_2) has a log-concave distribution and $Y_1 + Y_2$ has a log-concave distribution as well (see [7, 4, 21]). The Brunn-Minkowski inequality provides examples of log-concave measures, that are the uniform Lebesgue measure on compact convex subsets of \mathbb{R}^n as well as their marginals. More generally Borell's theorem [5] characterizes the log-concave measures that are not supported by any hyperplane as the absolutely continuous measures (with respect to Lebesgue measure) with a log-concave density. Note that the distribution of an isotropic vector is not supported by any hyperplane.

The paper is organized as follows. In Section 2 we present necessary spectral and probabilistic facts, including recent ones on the isotropic random vectors. Section 3 contains the proof of our main result (Theorem 3.3). Our proof is different from that of [17], and follows essentially the scheme outlined in [20].

2 Necessary Spectral and Probabilistic Facts

We will begin by recalling several facts on the resolvent of real symmetric (hermitian) matrices. Here and below $|\dots|$ denotes the euclidian (or hermitian) norm of vectors and matrices.

Lemma 2.1 *Let A be a real symmetric (hermitian) matrix and*

$$G_A(z) = (A - z)^{-1}, \quad \Im z \neq 0, \quad (2.1)$$

be its resolvent. We have:

$$(i) \quad |G_A(z)| \leq |\Im z|^{-1}, \quad (2.2)$$

(ii) if for $Y \in \mathbb{R}^n(\mathbb{C}^n)$ L_Y is the corresponding rank-one matrix defined in (1.2), and $\tau \in \mathbb{R}$, then

$$G_{A+\tau L_Y}(z) = G_A(z) - \tau G_A(z) L_Y G_A(z) (1 + \tau(G_A(z)Y, Y))^{-1}, \quad \Im z \neq 0. \quad (2.3)$$

The proof of Lemma 2.1 is elementary. Next is a version of the martingal-difference bounds for the variance of a function of independent random variables (see [9]). The technique of martingal differences was used in studies of random matrices by Girko (see e.g. [13]) for results and references).

Lemma 2.2 *Let $\{Y_\alpha\}_{\alpha=1}^m$ be a collection of i.i.d. random vectors of $\mathbb{R}^n(\mathbb{C}^n)$ with a common probability law F , and $\Phi : \mathbb{R}^{nm}(\mathbb{C}^{nm}) \rightarrow \mathbb{C}$ be a bounded borelian function, satisfying the inequalities:*

$$\sup_{X_1, \dots, X_m \in \mathbb{R}^n(\mathbb{C}^n)} |\Phi - \Phi|_{X_\alpha=0} \leq C < \infty, \quad \alpha = 1, \dots, m. \quad (2.4)$$

Then

$$\mathbf{Var}\{\Phi(Y_1, \dots, Y_m)\} \leq 4C^2 m \quad (2.5)$$

Proof. Denote for $\alpha = 1, \dots, m-1$

$$\bar{\Phi}_\alpha := \mathbf{E}\{\Phi|Y_1, \dots, Y_\alpha\} = \int_{\mathbb{R}^{\alpha n}} \Phi(Y_1, \dots, Y_\alpha, Y_{\alpha+1}, \dots, Y_m) F(dY_{\alpha+1}) \dots F(dY_m),$$

and $\Phi_0 = \mathbf{E}\{\Phi\}$, $\Phi_m = \Phi$, and $\Delta_\alpha = \bar{\Phi}_\alpha - \bar{\Phi}_{\alpha-1}$, $\alpha = 1, \dots, m$. Then

$$\Phi - \mathbf{E}\{\Phi\} = \sum_{\alpha=1}^m \Delta_\alpha,$$

hence

$$\begin{aligned} \mathbf{Var}\{\Phi\} & : = \mathbf{E}\{|\Phi - \mathbf{E}\{\Phi\}|^2\} = \sum_{\alpha=1}^m \mathbf{E}\{|\Delta_\alpha|^2\} \\ & + \sum_{1 \leq \alpha < \beta \leq m} \mathbf{E}\{\Delta_\alpha \bar{\Delta}_\beta + \Delta_\beta \bar{\Delta}_\alpha\}. \end{aligned}$$

It follows then from the definition of Δ_α that all the term of the second sum on the r.h.s. are zero and the terms of the first sum are bounded by $4C^2$ in view of condition (2.4). Hence, we obtain (2.5). ■

We will also use two recent fundamental results about log-concave measures. The first is a large deviation inequality due to G. Paouris [19]:

Theorem 2.3 *There exists a positive constant $C > 0$ such that for any integer $n \geq 1$ and any isotropic random vector $Y \in \mathbb{R}^n$ with a log-concave distribution we have*

$$\mathbf{P}\{|Y| \geq Ct\} \leq \exp(-t\sqrt{n}). \quad (2.6)$$

for every $t \geq 1$.

The second is an inequality on the concentration of the euclidian norm. It is a reformulation of the main result from [14]:

Theorem 2.4 *There exist positive constants C, α with $\alpha < 1$, such that for any integer $n \geq 1$ and any isotropic random vector $Y \in \mathbb{R}^n$ with a log-concave distribution we have*

$$\mathbf{Var}\{|Y|^2\} \leq C/n^\alpha. \quad (2.7)$$

Weaker forms of (2.7) with a power of $\ln n$ instead of n^α were first proved in [15] and later in [10].

We can now state one main technical tool about log-concave measures that we will need.

Lemma 2.5 *Let A be a complex matrix, with operator norm $\|A\| \leq 1$ and let $Y \in \mathbb{R}^n$ (or \mathbb{C}^n) be an isotropic random vector with a log-concave distribution. Then*

$$\mathbf{Var}\{(AY, Y)\} \leq 4C/n^\alpha \quad (2.8)$$

where $C < \infty$ and $\alpha \in (0, 1)$ are the constants from (2.7).

Proof. Let us first consider the case $Y \in \mathbb{R}^n$. Writing $A = R + iI$, where R and I are hermitian and denoting $\xi = \xi_1 + i\xi_2 = (RY, Y) + i(IY, Y)$, $\overset{\circ}{\xi}_{1,2} = \xi_{1,2} - \mathbf{E}\{\xi_{1,2}\}$, we have

$$\mathbf{Var}\{(AY, Y)\} := \mathbf{E}\{|\overset{\circ}{\xi}|^2\} = \mathbf{E}\{(\overset{\circ}{\xi}_1)^2\} + \mathbf{E}\{(\overset{\circ}{\xi}_2)^2\} = \mathbf{Var}\{\xi_1\} + \mathbf{Var}\{\xi_2\}.$$

Hence the proof reduces to the case where A is hermitian, for which we have $(AY, Y) = (BY, Y)$ where $B = \Re A$ is a real symmetric matrix with $\|B\| \leq 1$. Thus we may assume without loss of generality that A is a real symmetric matrix. If V is an isometry then VY is also an isotropic random vector with a log-concave distribution. Hence, we can assume that A is diagonal, i.e., $A = \text{diag}\{a_i\}_{i=1}^n$ with $|a_i| \leq 1$ for all $1 \leq i \leq n$ and set

$$\varphi(a_1, \dots, a_n) = \mathbf{Var}\{(AY, Y)\} = \mathbf{E}\left\{\left|\sum a_i y_i^2\right|^2\right\} - \left|n^{-1} \sum a_i\right|^2,$$

where $Y = \{y_i\}_{i=1}^n$. Since φ is a positive quadratic form, its maximum on the cube $[-1, 1]^n$ is attained on one of its vertices $\{-1, 1\}^n$. In order to estimate φ on a vertex, let I be a subset of $\{1, \dots, n\}$. Then

$$\mathbf{Var}\left\{\sum_{i \in I} y_i^2 - \sum_{i \notin I} y_i^2\right\} \leq \left(\mathbf{Var}^{1/2}\left\{\sum_{i \in I} y_i^2\right\} + \mathbf{Var}^{1/2}\left\{\sum_{i \notin I} y_i^2\right\}\right)^2,$$

and we get

$$\mathbf{Var}\{(AY, Y)\} \leq 4 \max_I \mathbf{Var} \left\{ \sum_{i \in I} y_i^2 \right\},$$

where the maximum is taken over all subset $I \subset \{1, \dots, n\}$.

Now observe that the projection $Y_I = (y_i)_{i \in I}$ of Y onto \mathbb{R}^I is clearly an isotropic random vector and that by [7] it has a log-concave distribution. Thus Klartag's result [14] (see also [10]) can be applied and gives that there exist $\alpha \in (0, 1)$ such that

$$\mathbf{Var}\{|Y_I|^2\} = \mathbf{E} \left\{ \sum_I y_i^2 \right\} - n^{-2}|I|^2 \leq cn^{-2}|I|^{2-\alpha} \leq c/n^\alpha,$$

where c, α are universal constant (do not depend on $Y, n, |I|$) and $|I|$ denotes $\text{Card}I$. We conclude that

$$\mathbf{Var}\{(AY, Y)\} \leq 4c/n^\alpha.$$

This complete the proof when Y is real; the complex case is similar. ■

3 Proof of the Main Result

We will prove first auxiliary facts that can be of independent interest.

Proposition 3.1 *Let $N_{n,m}$ be the Normalized Counting Measure of eigenvalues of (1.3), in which $\{Y_\alpha\}_{\alpha=1}^m$ are i.i.d. random vectors (not necessarily isotropic and/or with log-concave distribution) and $\{\tau_\alpha\}_{\alpha=1}^m$ are i.i.d. random variables, independent of $\{Y_\alpha\}_{\alpha=1}^m$. Denote*

$$g_{n,m}(z) = \int_{\mathbb{R}} \frac{N_{n,m}(d\lambda)}{\lambda - z}, \quad \Im z \neq 0, \quad (3.1)$$

the Stieltjes transform of $N_{n,m}$. Then we have:

$$\mathbf{Var}\{N_{n,m}(\Delta)\} \leq 4m/n^2 \quad (3.2)$$

for any interval $\Delta \subset \mathbb{R}$, and

$$\mathbf{Var}\{g_{n,m}(z)\} \leq 4m/n^2 |\Im z|^2 \quad (3.3)$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. To prove (3.2) we use Lemma 2.2 with $\Phi = nN_{n,m}(\Delta)$, the number of eigenvalues of H_{nm} in Δ . Since

$$H_{n,m} - H_{n,m}|_{Y_\alpha=0} = \tau_\alpha L_{Y_\alpha}$$

is a rank 1 matrix, we have by the mini-max principle

$$|nN_{n,m} - nN_{n,m}|_{Y_\alpha=0}| \leq 1,$$

i.e. the constant C in (2.4) is 1 in this case. This and (2.5) lead to (3.2).

In the case of $g_{n,m}$ we choose $\Phi = ng_{n,m}(z)$. Then we have according to (1.5), (3.1), and the spectral theorem for real symmetric (hermitian) matrices

$$ng_{n,m}(z) = \text{Tr}(H_{n,m} - z)^{-1} := \text{Tr} G_{n,m}(z).$$

Then (2.3) implies

$$|ng_{n,m}(z) - ng_{n,m}(z)|_{Y_\alpha=0} \leq \frac{|\tau_\alpha|(G_\alpha^2 Y_\alpha, Y_\alpha)}{|1 + \tau_\alpha(G_\alpha Y_\alpha, Y_\alpha)|}. \quad (3.4)$$

By spectral theorem for real symmetric (hermitian) matrices there exists non-negative measure m_α such that for any integer p

$$(G_\alpha^p Y_\alpha, Y_\alpha) = \int_{\mathbb{R}} \frac{m_\alpha(d\lambda)}{(\lambda - z)^p}.$$

Thus

$$|\tau_\alpha| |(\pi G_\alpha^2 Y_\alpha, Y_\alpha)| \leq |\tau_\alpha| \int_{\mathbb{R}} \frac{m_\alpha(d\lambda)}{|\lambda - z|^2},$$

and

$$|1 + \tau_\alpha(G_\alpha Y_\alpha, Y_\alpha)| \geq |\tau_\alpha| |\Im(G_\alpha Y_\alpha, Y_\alpha)| = |\tau_\alpha| |\Im z| \int_{\mathbb{R}} \frac{m_\alpha(d\lambda)}{|\lambda - z|^2}.$$

This implies

$$|ng_{n,m}(z) - ng_{n,m}(z)|_{Y_\alpha=0} \leq |\Im z|^{-1}. \quad (3.5)$$

Thus we can choose $|\Im z|^{-1}$ as C in (2.5) and obtain (3.3) from (2.5). ■

Proposition 3.2 *Let $N^{(0)}$ and σ be probability measures on \mathbb{R} and $f^{(0)}$ is the Stieltjes transform (1.10) of $N^{(0)}$. Consider a non-negative measure N on \mathbb{R} and assume that its Stieltjes transform (1.11) satisfies (1.12). We have*

(i) *if σ in (1.12) is such that*

$$\bar{\tau} = \int_{\mathbb{R}} |\tau| \sigma(d\tau) \leq \infty, \quad (3.6)$$

then N is a probability measure and is determined uniquely by (1.12);

(ii) *if N is a probability measure (i.e., $N(\mathbb{R}) = 1$), then it is uniquely determined by (1.12) with any σ (i.e. not necessarily satisfying (3.6)).*

Proof. (i). Note that $N \neq 0$, otherwise $f = 0$ and (1.12) has the form

$$0 = f^{(0)}(z - c\bar{\tau}), \quad \Im z \neq 0 \quad (3.7)$$

which is impossible since $N^{(0)} \neq 0$ (see e.g. (1.13)). Let us show that N is a probability measure, i.e., that $N(\mathbb{R}) = 1$. It suffices to show that

$$\lim_{y \rightarrow \infty} y |f(iy)| = 1. \quad (3.8)$$

(see [1], Section 59). Note that

$$y \Im f(iy) = \int_{\mathbb{R}} \frac{y^2}{\lambda^2 + y^2} N(d\lambda) \rightarrow N(\mathbb{R}) > 0, \quad y \rightarrow \infty. \quad (3.9)$$

Denote

$$\zeta(z) = -c \int_{\mathbb{R}} \frac{\tau \sigma(d\tau)}{1 + \tau f(z)}. \quad (3.10)$$

and prove that

$$\lim_{y \rightarrow \infty} y^{-1} \zeta(iy) = 0. \quad (3.11)$$

Indeed, write

$$\zeta(iy) = -c \int_{|\tau| < \sqrt{y}} \frac{\tau \sigma(d\tau)}{1 + \tau f(iy)} - c \int_{|\tau| \geq \sqrt{y}} \frac{\tau \sigma(d\tau)}{1 + \tau f(iy)} := I_1 + I_2.$$

We have for $y \rightarrow \infty$ in view of the bound $|f(iy)| \leq 1/y$:

$$|I_1| \leq \frac{c\sqrt{y}}{1 - \sqrt{y}/y} = O(\sqrt{y}),$$

thus $\lim_{y \rightarrow \infty} y^{-1} I_1 = 0$. We have also

$$|I_2| \leq \frac{c}{\Im f(iy)} \int_{|\tau| \geq \sqrt{y}} \sigma(d\tau),$$

and in view of (3.9) and the obvious relation

$$\lim_{y \rightarrow \infty} \int_{|\tau| \geq \sqrt{y}} \sigma(d\tau) = 0$$

we conclude that $\lim_{y \rightarrow \infty} y^{-1} I_2 = 0$. Thus, (3.11) is proved. This implies that $iy + \zeta(iy) = iy(1 + o(1))$, $y \rightarrow \infty$ and since $N^{(0)}$ is a probability measure, we have

$$\lim_{y \rightarrow \infty} y |f^{(0)}(iy)| = 1.$$

It follows then from (1.12) that (3.8) is true, i.e., N is a probability measure.

Let us prove now that (1.12) determines N uniquely. Assume that there exists two probability measures N' and N'' , whose Stieltjes transforms f' and f'' satisfy (1.12). Since f' and f'' are analytic in the upper halfplane, there exists a sequence $\{z_j\}_{j \geq 1}$ such that $z_j \rightarrow \infty$ as $j \rightarrow \infty$ and $(f' - f'')(z_j) \neq 0$. Subtracting (1.12) for f' from that for f'' we find

$$1 = c \int_{\mathbb{R}} \frac{N^{(0)}(d\lambda)}{(\lambda - z - \zeta'(z_j))(\lambda - z - \zeta''(z_j))} \int_{\mathbb{R}} \frac{\tau^2 \sigma(d\tau)}{(1 + \tau f'(z_j))(1 + \tau f''(z_j))},$$

where ζ' and ζ'' are given by (3.10) for $f = f', f''$. By using a bit more sophisticated version of the proof of (3.8) it can be proved that the limit of the r.h.s. of this relation is zero as $j \rightarrow \infty$. Thus we have that $f' = f''$. Since the Stieltjes transform of a non-negative measure determines the measure uniquely (see e.g. (1.13)) we conclude that $N' = N''$.

(ii). If N is a probability measure, then we have $N(\mathbb{R}) = 1$ in the r.h.s. of (3.9). Now it is easy to check that the argument proving assertion (i) is applicable to the case, where (3.6) is not valid and we obtain the proof of assertion (ii) of the proposition. ■

Now we are ready to prove

Theorem 3.3 *Let $\{Y_\alpha\}_{\alpha=1}^m$ be i.i.d. isotropic random vectors of \mathbb{R}^n (or \mathbb{C}^n) with a log-concave distribution and $\{\tau_\alpha\}_{\alpha=1}^m$ be i.i.d. random variables with a common probability law*

σ . Consider random matrices $H_{n,m}$ of (1.1) – (1.3) and assume (1.7). Then there exist a probability measure N such that for any interval $\Delta \subset \mathbb{R}$ we have in probability

$$\lim_{n \rightarrow \infty, m \rightarrow \infty, m/n \rightarrow c \in [0, \infty)} N_{n,m}(\Delta) = N(\Delta) \quad (3.12)$$

for the Normalized Counting Measure $N_{n,m}$ of (1.5) of eigenvalues of $H_{n,m}$

The limiting non-random measure N is uniquely determined by equation (1.12) for its Stieltjes transform (1.11).

Proof. In view of (3.2) it suffices to prove that the expectations

$$\overline{N}_{n,m} = \mathbf{E}\{N_{n,m}\} \quad (3.13)$$

of the Normalized Counting Measure (1.5) of eigenvalues of H_{nm} will converge weakly to the measure, whose Stieltjes transform solves of (1.12). Recall now that weak convergence of probability measures to a probability measure N is equivalent to the convergence of their Stieltjes transforms to the Stieltjes transform f of (1.11) N on a compact set of $\mathbb{C} \setminus \mathbb{R}$ and to the relation

$$\lim_{y \rightarrow \infty} y|f(iy)| = 1. \quad (3.14)$$

(see e.g. [1], Section 59). We are going to prove below these two facts.

We derive first equation (1.12). Assume first that the random variables τ_α are bounded:

$$|\tau_\alpha| \leq T < \infty. \quad (3.15)$$

Write the resolvent identity for the resolvents

$$G = (H_{n,m} - z)^{-1}, \quad \mathcal{G} = (H_n^{(0)} - z)^{-1} \quad (3.16)$$

(we will omit below the sub-indices n and m and the argument z in the resolvents). We have in view of (1.3):

$$G = \mathcal{G} - \sum_{\alpha=1}^m \tau_\alpha G L_{Y_\alpha} \mathcal{G},$$

and if

$$\overline{G} = \mathbf{E}\{G\},$$

then

$$\overline{G} = \mathcal{G} - \sum_{\alpha=1}^m \mathbf{E}\{\tau_\alpha G L_{Y_\alpha}\} \mathcal{G}. \quad (3.17)$$

Choose $t \geq 1$ and write (3.17) as

$$\overline{G} = \mathcal{G} - \sum_{\alpha=1}^m \mathbf{E}\{\tau_\alpha G L_{Y_\alpha} \mathbf{1}_{|Y_\alpha| < t}\} \mathcal{G} - R_1, \quad (3.18)$$

where

$$R_1 = \sum_{\alpha=1}^m \mathbf{E}\{\tau_\alpha G L_{Y_\alpha} \mathbf{1}_{|Y_\alpha| \geq t}\} \mathcal{G}. \quad (3.19)$$

Denote

$$G_\alpha = G|_{Y_\alpha=0}. \quad (3.20)$$

Then we have from (2.3)

$$GL_{Y_\alpha} = G_\alpha L_{Y_\alpha} (1 + \tau_\alpha(G_\alpha Y_\alpha, Y_\alpha))^{-1}.$$

This allows us to write (3.18) as

$$\bar{G} = \mathcal{G} - \sum_{\alpha=1}^m \mathbf{E} \left\{ \frac{\tau_\alpha}{1 + \tau_\alpha(G_\alpha Y_\alpha, Y_\alpha)} G_\alpha L_{Y_\alpha} \mathbf{1}_{|Y_\alpha|<t} \right\} \mathcal{G} + R_1$$

and then, denoting

$$f_{n,m}(z) := \int_{\mathbb{R}} \frac{\bar{N}_{n,m}(d\lambda)}{\lambda - z} = \mathbf{E}\{n^{-1} \text{Tr}(H_{n,m} - z)^{-1}\}, \quad \Im z \neq 0, \quad (3.21)$$

as

$$\bar{G} = \mathcal{G} - \frac{m}{n} \int_{\mathbb{R}} \frac{\tau_\alpha \sigma(d\tau)}{1 + \tau f_{n,m}(z)} \bar{G} \mathcal{G} - \sum_{q=1}^6 R_q, \quad (3.22)$$

where R_1 is given by (3.19) and

$$R_2 = \sum_{\alpha=1}^m \mathbf{E} \left\{ \left(\frac{\tau_\alpha}{1 + \tau_\alpha(G_\alpha Y_\alpha, Y_\alpha)} - \frac{\tau_\alpha}{1 + \tau_\alpha f_{n,m}} \right) G_\alpha L_{Y_\alpha} \mathbf{1}_{|Y_\alpha|<t} \right\} \mathcal{G}, \quad (3.23)$$

$$R_3 = \sum_{\alpha=1}^m \mathbf{E} \left\{ \frac{\tau_\alpha}{1 + \tau_\alpha f_{n,m}} (G_\alpha L_{Y_\alpha} - n^{-1} G_\alpha) \mathbf{1}_{|Y_\alpha|<t} \right\} \mathcal{G}, \quad (3.24)$$

$$R_4 = \frac{1}{n} \sum_{\alpha=1}^m \mathbf{E} \left\{ \frac{\tau_\alpha}{1 + \tau_\alpha f_{n,m}} (G_\alpha - G) \mathbf{1}_{|Y_\alpha|<t} \right\} \mathcal{G}, \quad (3.25)$$

$$R_5 = \frac{m}{n} \mathbf{E} \left\{ \frac{1}{m} \sum_{\alpha=1}^m \left(\frac{\tau_\alpha}{1 + \tau_\alpha f_{n,m}} - \int_{\mathbb{R}} \frac{\tau \sigma(d\tau)}{1 + \tau f_{n,m}(z)} \right) G \mathbf{1}_{|Y_\alpha|<t} \right\} \mathcal{G}, \quad (3.26)$$

$$R_6 = \frac{m}{n} \int_{\mathbb{R}} \frac{\tau \sigma(d\tau)}{1 + \tau f_{n,m}(z)} \mathbf{E} \{ G \mathbf{1}_{|Y_\alpha| \geq t} \}. \quad (3.27)$$

It follows from (3.22) that if

$$\tilde{\mathcal{G}} = \mathcal{G}(\tilde{z}_{n,m}(z)), \quad \tilde{z}_{n,m}(z) = z - c \int_{\mathbb{R}} \frac{\tau \sigma(d\tau)}{1 + \tau f_{n,m}(z)}, \quad (3.28)$$

then

$$\bar{G} = \tilde{\mathcal{G}} - \sum_{q=1}^6 \tilde{R}_q, \quad (3.29)$$

where \tilde{R}_q , $q = 1, \dots, 6$ is obtained from R_q , $q = 1, \dots, 6$ by replacing \mathcal{G} by $\tilde{\mathcal{G}}$ in (3.19), (3.23) – (3.27). Note that

$$\Im \tilde{z}_{n,m}(z) = \Im z + c \Im f_{n,m}(z) \int_{\mathbb{R}} \frac{\tau^2 \sigma(d\tau)}{|1 + \tau f_{n,m}(z)|},$$

and since $\Im f_{n,m}(z)\Im z > 0$, $\Im z \neq 0$ by (3.21) we have

$$|\Im \tilde{z}_{n,m}(z)| \geq |\Im z|$$

and $\tilde{\mathcal{G}}(z)$ is well defined for $\Im z \neq 0$.

Applying to (3.22) the operation $n^{-1}\text{Tr}$ and recalling (1.6), (3.16), (3.21), and the spectral theorem, we obtain

$$f_{n,m}(z) = f_n^{(0)}(\tilde{z}_{n,m}(z)) - \sum_{q=1}^6 \tilde{r}_q, \quad (3.30)$$

where

$$f_n^{(0)}(z) = n^{-1}\text{Tr}\mathcal{G}(z) = \int_{\mathbb{R}} \frac{N_n^{(0)}(d\lambda)}{\lambda - z},$$

and

$$\tilde{r}_q(z) = n^{-1}\text{Tr}\tilde{R}_q.$$

We will prove now that if

$$|\Im z| \geq 2t^2T, \quad (3.31)$$

where $t \geq 1$ and T is defined in (3.15), then there exist $C_{t,T} < \infty$ and $b > 0$ such that

$$|\tilde{r}_q(z)| = o(1), \quad q = 1, \dots, 6, \quad (3.32)$$

and we write here and below $o(1)$ for a quantity that tends to zero under condition (1.8) and (3.31).

We begin from R_1 . Note first that in view of (3.21) $\Im f_{n,m}(z)\Im z \geq 0$, $\Im z \neq 0$, hence

$$|\tilde{z}_{n,m}(z)| \geq |\Im z| \geq 2t^2T. \quad (3.33)$$

This, (3.19), (3.31), and (2.6) imply

$$\begin{aligned} |\tilde{r}_1(z)| &= \left| \frac{1}{n} \sum_{\alpha=1}^m \mathbf{E} \{ \tau_{\alpha}(\tilde{\mathcal{G}}GY_{\alpha}, Y_{\alpha}) \mathbf{1}_{|Y_{\alpha}| \geq t} \} \right| \\ &\leq \frac{m}{4nt^4T^2} \mathbf{E} \{ |Y_{\alpha}|^2 \mathbf{1}_{|Y_{\alpha}| \geq t} \} = o(1). \end{aligned}$$

Next, we have from (3.23)

$$\tilde{r}_2(z) = \frac{1}{n} \sum_{\alpha=1}^m \mathbf{E} \left\{ \frac{(\tau_{\alpha})^2((G_{\alpha}Y_{\alpha}, Y_{\alpha}) - f_{n,m})}{(1 + \tau_{\alpha}(G_{\alpha}Y_{\alpha}, Y_{\alpha}))(1 + \tau_{\alpha}f_{n,m})} (\tilde{\mathcal{G}}GY_{\alpha}, Y_{\alpha}) \mathbf{1}_{|Y_{\alpha}| < t} \right\}.$$

According to (2.2) and (3.21) we have $|(G_{\alpha}Y_{\alpha}, Y_{\alpha})| \leq |\Im z|^{-1}|Y_{\alpha}|^2$, $|f_{n,m}| \leq |\Im z|^{-1}$. Thus (3.31) and (3.33) yield for $|Y_{\alpha}| < t$:

$$\begin{aligned} |1 + \tau_{\alpha}(G_{\alpha}Y_{\alpha}, Y_{\alpha})| &\geq 1 - Tt^2/|\Im z| \geq 1/2, \\ |1 + \tau_{\alpha}f_{n,m}| &\geq 1 - T/|\Im z| \geq 1/2. \end{aligned}$$

This, (2.2), and (3.33) lead to the bound

$$|\tilde{r}_2(z)| \leq \frac{4mt^2T^2}{n|\Im z|^2} \mathbf{E} \{ \mathbf{E}_{\alpha} \{ |(G_{\alpha}Y_{\alpha}, Y_{\alpha}) - f_{n,m}| \} \}, \quad (3.34)$$

where $\mathbf{E}_\alpha\{\dots\}$ denotes the expectation only with respect to Y_α . Since Y_α is isotropic and G_α does not depend on Y_α , we have (see (3.20))

$$\mathbf{E}_\alpha\{(G_\alpha Y_\alpha, Y_\alpha)\} = n^{-1}\text{Tr}G_\alpha := g_\alpha.$$

Besides, we have $\mathbf{E}\{g_{n,m}\} = f_{n,m}$ by definitions (1.5), (3.1), and (3.21). Thus

$$\mathbf{E}\{|\mathbf{E}_\alpha\{|(G_\alpha Y_\alpha, Y_\alpha) - f_{n,m}|\}\} \leq \mathbf{E}\left\{\mathbf{Var}_\alpha^{1/2}\{(G_\alpha Y_\alpha, Y_\alpha)\}\right\} + \mathbf{E}\{|g_\alpha - g_{n,m}|\} + \mathbf{Var}_\alpha^{1/2}\{g_{n,m}\}.$$

Now (2.7) and (2.2) yield the bound $2C^{1/2}/|\Im z|n^{a/2}$ for the first term of the r.h.s., (3.5) yields the bound $1/n|\Im z|$ for the second term, and (3.3) yields the bound $2m^{1/2}/n|\Im z|$ for the third term. It follows then from (3.34) that $\tilde{r}_2(z) = o(1)$.

Write now

$$\tilde{r}_3(z) = \frac{1}{n} \sum_{\alpha=1}^m \mathbf{E} \left\{ \frac{\tau_\alpha}{1 + \tau_\alpha f_{n,m}} \left((\tilde{\mathcal{G}}G_\alpha Y_\alpha, Y_\alpha) - n^{-1}\text{Tr}\tilde{\mathcal{G}}G_\alpha \right) \mathbf{1}_{|Y_\alpha| < t} \right\},$$

and obtain similarly to the case of \tilde{r}_2 :

$$|\tilde{r}_3(z)| \leq \frac{2Tm}{n} \mathbf{E} \left\{ \mathbf{Var}_\alpha^{1/2}\{(\tilde{\mathcal{G}}G_\alpha Y_\alpha, Y_\alpha)\} \right\} \leq \frac{2Tmc^{1/2}}{n|\Im z|n^{a/2}} = O(1/n^{a/2}).$$

In the case of

$$\tilde{r}_4(z) = \frac{1}{n} \sum_{\alpha=1}^m \mathbf{E} \left\{ \frac{\tau}{1 + \tau f_{n,m}} n^{-1}\text{Tr}\tilde{\mathcal{G}}(G_\alpha - G) \mathbf{1}_{|Y_\alpha| < t} \right\}$$

we use again (2.3) to write

$$\tilde{r}_4(z) = \frac{1}{n^2} \sum_{\alpha=1}^m \mathbf{E} \left\{ \frac{\tau_\alpha}{1 + \tau_\alpha f_{n,m}} \frac{\tau_\alpha(G_\alpha \tilde{\mathcal{G}}G_\alpha Y_\alpha, Y_\alpha)}{(1 + \tau_\alpha(G_\alpha Y_\alpha, Y_\alpha))} \mathbf{1}_{|Y_\alpha| < t} \right\}.$$

Thus, similarly to the case of $\tilde{r}_2(z)$ we have

$$|\tilde{r}_4(z)| \leq \frac{2Tm}{n^2|\Im z|} = O(1/n).$$

Denote

$$\xi_\alpha = \frac{\tau_\alpha}{1 + \tau_\alpha f_{n,m}} - \int_{\mathbb{R}} \frac{\tau \sigma(d\tau)}{1 + \tau f_{n,m}(z)}, \quad \alpha = 1, \dots, m.$$

Then it follows from (3.26) that

$$\tilde{r}_5(z) = \frac{m}{n^2} \mathbf{E} \left\{ \frac{1}{m} \sum_{\alpha=1}^m \xi_\alpha \text{Tr}\tilde{\mathcal{G}}G \mathbf{1}_{|Y_\alpha| < t} \right\}$$

Since $\{\xi_\alpha\}_{\alpha=1}^m$ is a collection of i.i.d. random variables, and $\mathbf{E}\{\xi_\alpha\} = 0$ we can write in view of (2.2), (3.33), and (3.15)

$$|\tilde{r}_5(z)| \leq \frac{m}{n|\Im z|^2} \mathbf{E}^{1/2} \left\{ \left(\frac{1}{m} \sum_{\alpha=1}^m \xi_\alpha \right)^2 \right\} \leq \frac{m^{1/2}}{n|\Im z|^2} \mathbf{E}^{1/2} \{\xi_1^2\} = O(1/n^{1/2}).$$

Finally we have

$$\tilde{r}_6(z) = \frac{m}{n} \int_{\mathbb{R}} \frac{\tau \sigma(d\tau)}{1 + \tau f_{n,m}(z)} \mathbf{E} \left\{ n^{-1} \text{Tr} \tilde{\mathcal{G}} G \mathbf{1}_{|Y_\alpha| \geq t} \right\},$$

hence

$$|\tilde{r}_6| \leq \frac{2T}{|\Im z|^2} e^{-t\sqrt{n}/C}.$$

Since the collection $\{\bar{N}_{nm}\}$ consists of probability measures, there exists a non-negative measure N which is a limiting point of the collection in the sense (1.8) with respect to the vague convergence. The Stieltjes transforms of the corresponding subsequence of the collection converge to the Stieltjes transform f of N uniformly on a compact $K \subset \mathbb{C} \setminus \mathbb{R}$. Choosing $K \subset \{z \in \mathbb{C} : |\Im z| \geq 2t^2T\}$ and using (1.7) and (3.32) we can make the limit (3.30) along the subsequence for $|\Im z| \geq 2t^2T$. The limiting equation, i.e. (1.12) for $|\Im z| \geq 2t^2T$ implies that $\Im f(z) \neq 0$, $\Im z \neq 0$, otherwise we would have (3.7) for some z_0 , $|\Im z_0| \geq 2t^2T$. Thus the both parts of obtained (1.12) are analytic for $\Im z \neq 0$ and the equation (1.12) is valid everywhere in $\mathbb{C} \setminus \mathbb{R}$. By Proposition 3.2 (i) N is uniquely determined by the equation. This proves the theorem under conditions (3.15).

Consider now a general case and introduce the truncated random variables

$$\tau_\alpha^T = \begin{cases} \tau_\alpha, & |\tau_\alpha| < T, \\ 0, & |\tau_\alpha| \geq T, \end{cases} \quad (3.35)$$

for any $T > 0$. Denote

$$H_{n,m}^T = H_n^{(0)} + \sum_{\alpha=1}^m \tau_\alpha^T L_{Y_\alpha}.$$

Then

$$\text{rank}(H_{n,m} - H_{n,m}^T) \leq \text{Card}\{\alpha \in [1, m] : |\tau_\alpha| \geq T\},$$

and if $N_{n,m}^T$ is the Normalized Counting Measure of eigenvalues of $H_{n,m}^T$ and $\bar{N}_{n,m}^T$ is its expectation, then the mini-max principle implies for any interval $\Delta \subset \mathbb{R}$:

$$|\bar{N}_{n,m}(\Delta) - \bar{N}_{n,m}^T(\Delta)| \leq \mathbf{P}\{|\tau_1| \geq T\}.$$

Hence any limiting point N in the sense of (1.8) of the collection $\{\bar{N}_{n,m}\}$ with respect to the vague convergence satisfies the same inequality:

$$|N(\Delta) - N^T(\Delta)| \leq \mathbf{P}\{|\tau_1| \geq T\}, \quad (3.36)$$

where we took into account that according to the first part of the theorem the sequence $\{\bar{N}_{n,m}^T\}$ converges weakly to the probability measure \bar{N}^T for any $0 < T < \infty$. Choosing here $\Delta = \mathbb{R}$, we obtain that N is a probability measure, i.e. N is limiting point of $\{\bar{N}_{n,m}\}$ in the sense (1.8) with respect to the weak convergence.

Besides it follows from (3.36) that N is a weak limiting points of the family $\{N^T\}_{T>0}$.

According to the first part of the theorem and (3.35) the Stieltjes transform f^T of N^T is a unique solution of the functional equation

$$f^T(z) = f_0 \left(z - c \int_{|\tau| < T} \frac{\tau \sigma(d\tau)}{1 + \tau f^T(z)} \right). \quad (3.37)$$

If $\{N^{T_i}\}_{i \geq 1}$ is a subsequence converging weakly to N , then $\{f^{T_i}\}_{i \geq 1}$ converges to the Stieltjes transform f of N uniformly on a compact set $K \subset \mathbb{C} \setminus \mathbb{R}$. Since $N(\mathbb{R}) = 1$, there exists $C > 0$, such that

$$\min_{z \in K} \Im f(z) = C > 0.$$

This and the uniform convergence of $\{f^{T_i}\}_{i \geq 1}$ to f on K implies that we have for all sufficiently big T_i :

$$\min_{z \in K} \Im f^{T_i}(z) = C/2 > 0.$$

Thus $|\tau/(1 + \tau f^{T_i}(z))| \leq (\Im f^{T_i}(z))^{-1} \leq 2/C < \infty$, $z \in K$. This allows us to make the limit $T_i \rightarrow \infty$ in (3.37), i.e. to obtain (1.12) for f . According to Proposition 3.2 (ii) the equation is uniquely soluble for any probability measure σ , hence the limit point f of the sequence $\{f^T\}_{T > 0}$ is unique. Since f determines uniquely N (see (1.13)), the theorem is proved. ■

Remark 3.4 Our result can be used for another random matrix, defined via the vectors $\{Y_\alpha\}_{\alpha=1}^m$ as

$$\mathcal{G}_{n,m} = \{(Y_\alpha, Y_\beta)\}_{\alpha,\beta=1}^m. \quad (3.38)$$

It can be called the Gram matrix of the collection $\{Y_\alpha\}_{\alpha=1}^m$.

Denoting $Y_\alpha = \{y_{\alpha j}\}_{j=1}^n$ we can view $Y = \{y_{\alpha j}\}_{\alpha,j=1}^{m,n}$ as a $m \times n$ random matrix. Then we have for $\mathcal{G}_{n,m}$

$$\mathcal{G}_{n,m} = YY^*,$$

and for $M_{n,m}$ of (1.1) with $\tau_\alpha = 1$, $\alpha = 1, \dots, m$

$$M_{n,m} = Y^*Y.$$

Assume that the law of Y_α is continuous. Then for $n > m$ the vectors $\{Y_\alpha\}_{\alpha=1}^m$ are linearly independent with probability 1, all the eigenvalues of $\mathcal{G}_{n,m}$ are strictly positive and coincide with non-zero eigenvalues of $M_{n,m}$. $M_{n,m}$ has $n - m$ zero eigenvalues, whose eigenvectors form a basis of the complement of the span of $\{Y_\alpha\}_{\alpha=1}^m$ in \mathbb{R}^n (or \mathbb{C}^n). Denote $\tilde{N}_{n,m}$ the Normalized Counting Measure of eigenvalues of $\mathcal{G}_{n,m}$. Then we have

$$\tilde{N}_{n,m} = -\frac{n-m}{m}\delta_0 + \frac{n}{m}N_{n,m}.$$

Hence, if N is the limit of $N_{n,m}$ in the sense of (1.8), then the limit \tilde{N} of $\tilde{N}_{n,m}$ in the same sense also exists and is equal to $\tilde{N} = -(c^{-1} - 1)\delta_0 + c^{-1}N$. Since in this case $N = (1 - c)\delta_0 + N^*$, where $N^*(d\lambda) = \rho^*(\lambda)d\lambda$, where the support of ρ^* is $[a_-, a_+]$, $a_\pm = (1 \pm \sqrt{c})^2$, and

$$\rho^*(\lambda) = \frac{1}{4\pi c\lambda} \sqrt{(a_+ - \lambda)(\lambda - a_-)}, \quad \lambda \in [a_-, a_+], \quad (3.39)$$

(see [17]), we conclude that \tilde{N} is absolutely continuous and has the density $c^{-1}\rho^*$.

Similar argument shows that for $m > n$ we have $\tilde{N} = (1 - c^{-1})\delta_0 + c^{-1}N$, where now N is absolutely continuous and its density is (3.39).

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