Nonlinear reinforcement problems
with right-hand side in $L^1$

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Abstract

We study the asymptotic behaviour, as $\varepsilon$ goes to zero, of the
entropy solution $u^\varepsilon$ to a class of nonlinear elliptic problems whose
prototype is

$$
-\text{div} \left( (\delta^\varepsilon)^{p-1} |\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon \right) = f^\varepsilon
$$

in $\Omega^\varepsilon$, $u^\varepsilon = 0$ on $\partial \Omega^\varepsilon$, where $1 < p < N$, $\Omega^\varepsilon = \Omega \cup \Sigma^\varepsilon$, $\Omega \subset \mathbb{R}^N$
is a bounded domain of class $C^{1,1}$, surrounded along its boundary
by a layer $\Sigma^\varepsilon$ of small thickness $\varepsilon$, $\delta^\varepsilon = 1$ in $\Omega$, $\delta^\varepsilon = \varepsilon$ in $\Sigma^\varepsilon$, and $f^\varepsilon$
belongs to $L^1(\Omega^\varepsilon)$. The solutions $u^\varepsilon$ also satisfies natural transmission
conditions on $\partial \Omega$.

1 Introduction

The prototype of so-called “reinforcement problems” considered in this
paper is the following one. Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain of
class $C^{1,1}$, surrounded along its boundary by a layer $\Sigma^\varepsilon$ of small thickness $\varepsilon$.
We denote by $\Omega^\varepsilon$ the domain defined by $\Omega^\varepsilon = \Omega \cup \Sigma^\varepsilon$ and we consider the
entropy solution of the nonlinear elliptic equation

$$(1.1) \quad \text{div} \left( |\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon \right) = f^\varepsilon \text{ in } \Omega, \quad -\varepsilon^{p-1} \text{div} \left( |\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon \right) = f^\varepsilon \text{ in } \Sigma^\varepsilon,$$

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with Dirichlet condition, \( u^\varepsilon = 0 \) on \( \partial \Omega^\varepsilon \), and with natural transmission conditions on \( \partial \Omega \). Here we assume that \( 1 < p < N \) and that \( f^\varepsilon \) is a function belonging to \( L^1(\Omega^\varepsilon) \). The aim of this paper is to investigate the asymptotic behaviour of the entropy solution \( u^\varepsilon \) of (1.1), as \( \varepsilon \) goes to zero.

The equation (1.1) presents two difficulties: the first one is due to the low summability of the term \( f^\varepsilon \) belonging to \( L^1(\Omega^\varepsilon) \), which leads us to consider entropy solutions ([3]; see also [15], [16]), and the second one is due to the degenerancy of the operator on \( \Sigma^\varepsilon \), as \( \varepsilon \) goes to zero.

Asymptotic properties of solutions to reinforcement problems have been investigated by several authors in different frameworks (see, for instance, [11], [1], [10], [12], [7], [8]).

In the present paper we study the behaviour of the entropy solution \( u^\varepsilon \) of reinforcement problem (1.1), as \( \varepsilon \) goes to zero, and we prove that the sequence \( \{u^\varepsilon\}_\varepsilon \) converges almost everywhere to the entropy solution \( u \) of the “limit problem” which can be formally written as

\[
\left\{ \begin{array}{l}
-\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = f & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} + |u|^{p-2} u = 0 & \text{in } \partial \Omega.
\end{array} \right.
\]

Here \( n \) denotes the outer unit normal to \( \Omega \) and \( f \) is the weak limit of \( f^\varepsilon \) in \( L^1(\Omega) \). In Section 2 we give the definition of entropy solution for this boundary value problem. Such a definition is based on a notion of trace for measurable functions which are not necessarily in Sobolev spaces. Indeed we define the trace on \( \partial \Omega \) for a measurable function \( v : \Omega \to \mathbb{R} \), finite almost everywhere, whose truncations \( T_k(v), k > 0 \), belong to \( W^{1,p}(\Omega) \) (see Section 3). This trace is the \( H_{N-1} \)-measurable function \( \tau : \partial \Omega \to \overline{\mathbb{R}} \) denoted by \( \gamma(v) \) and defined by

\[
\tau(\sigma) = \lim_{k \to \infty} \gamma(T_k(v))(\sigma) \quad \mathcal{H}_{N-1} - \text{a.e. } \sigma \in \partial \Omega.
\]

Here \( \gamma(T_k(v)) \) denotes the classical trace of \( T_k(v) \), which is well-defined since the truncations of \( v \) belong to \( W^{1,p}(\Omega) \). We will prove that the limit in (1.3) exists and we will study its properties.

Obviously our notion of trace coincides with the classical notion of trace if \( v \) is a function belonging to \( W^{1,p}_0(\Omega) \). Moreover it is a generalization of the notion of trace given in [2], where a more restrictive class of measurable function is considered (see Section 3).
2 Definitions and main results

In the whole paper we denote by $\Omega$ is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with $C^{1,1}$-boundary $\partial \Omega$. Then the outer normal vector $n(\sigma)$, at the point $\sigma \in \partial \Omega$, is $C^{0,1}(\partial \Omega)$. Moreover for $\varepsilon$ sufficiently small, one can consider the boundary layer $\Sigma^\varepsilon$ defined by

$$\Sigma^\varepsilon = \{\sigma + tn(\sigma) : \sigma \in \partial \Omega, \ 0 < t < h^\varepsilon(\sigma)\},$$

where $h^\varepsilon : \partial \Omega \to \mathbb{R}$ is a given function of class $C^1(\partial \Omega)$, satisfying the condition

$$0 < h^\varepsilon(\sigma) \leq \varepsilon, \ \forall \sigma \in \partial \Omega.$$  

Besides the function, denoted $\sigma(x)$, which transforms $x \in \Sigma^\varepsilon$ into its projection $\sigma(x)$ on $\partial \Omega$, is $C^{0,1}(\Sigma^\varepsilon)$. Observe that $\Sigma^\varepsilon$ surrounds $\Omega$, that its thickness varies along $\partial \Omega$ and is given by the function $h^\varepsilon$. We denote by $\Omega^\varepsilon$ the enlarged domain

$$\Omega^\varepsilon = \overline{\Omega} \cup \Sigma^\varepsilon, \ \varepsilon > 0.$$  

Now we consider the nonlinear elliptic problem which can be formally written as

$$\begin{cases} 
-\text{div} (\beta^\varepsilon a(\delta^\varepsilon \nabla u^\varepsilon)) = \mu^\varepsilon & \text{in } \Omega^\varepsilon, \\
u^\varepsilon = 0 & \text{on } \partial \Omega^\varepsilon.
\end{cases}$$

Here $a : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function satisfying

$$a(\xi) \xi \geq \alpha_1 |\xi|^p, \ \alpha_1 > 0, \ 1 < p < N,$$

$$|a(\xi) - a(\eta)| \leq \alpha_2 \left(1 + |\xi|^{\frac{p-1}{r}} + |\eta|^{\frac{p-1}{r'}}\right)|\xi - \eta|^{\frac{p-1}{r}},$$

$$\alpha_2 > 0, \quad 1 \leq r < +\infty, \quad \frac{1}{r} + \frac{1}{r'} = 1,$$

for every $\xi, \eta \in \mathbb{R}^N$. We assume that $a(\xi)$ is colinear to $\xi$, if $\xi \neq 0$, that is

$$a(\xi) = \frac{a(\xi) \xi}{|\xi|^2} \xi, \quad \text{if } \xi \neq 0.$$
Coming back to (2.4), let $b^\varepsilon : \partial \Omega \rightarrow \mathbb{R}$ be a positive function, $\beta^\varepsilon$ and $\delta^\varepsilon : \Omega^\varepsilon \rightarrow \mathbb{R}$ are defined by

\begin{align}
\beta^\varepsilon(x) &= \begin{cases} 1, & x \in \Omega, \\ b^\varepsilon(\sigma(x)), & x \in \Sigma^\varepsilon, \end{cases} \\
\delta^\varepsilon(x) &= \begin{cases} 1, & x \in \Omega, \\ h^\varepsilon(\sigma(x)), & x \in \Sigma^\varepsilon. \end{cases}
\end{align}

As for the source term in (2.4), we assume that $\mu^\varepsilon$ belongs to $L^1(\Omega^\varepsilon) + W^{-1,p'}(\Omega^\varepsilon)$, i.e.

\begin{align}
\mu^\varepsilon &= f^\varepsilon - \text{div} g^\varepsilon, \\
f^\varepsilon &\in L^1(\Omega^\varepsilon), \quad g^\varepsilon \in \left(L^{p'}(\Omega^\varepsilon)\right)^N.
\end{align}

We assume also that

\begin{align}
f^\varepsilon &\rightharpoonup f \text{ weakly in } L^1(\Omega), \text{ for some } f \in L^1(\Omega), \\
\int_{\Sigma^\varepsilon} |f^\varepsilon| \, dx &\rightarrow 0, \\
g^\varepsilon &\rightarrow g \text{ strongly in } \left(L^{p'}(\Omega)\right)^N, \text{ for some } g \in \left(L^{p'}(\Omega)\right)^N, \\
\int_{\Sigma^\varepsilon} \frac{|g^\varepsilon|^{p'}}{h^\varepsilon \circ \sigma} \, dx &\rightarrow 0, \\
\nabla h^\varepsilon &\rightarrow 0 \text{ strongly in } \left(L^p(\partial \Omega)\right)^{N-1},
\end{align}

as $\varepsilon$ tends to zero.

\textbf{Remark 2.1} It is easy to check that assumptions (2.5) and (2.6) imply that $a(0) = 0$ and that $a$ satisfies the growth condition

\begin{align}
|a(\xi)| &\leq \alpha_2(1 + 2|\xi|^{p-1}).
\end{align}

Moreover conditions (2.5) to (2.8) are satisfied for $a(\xi) = |\xi|^{p-2}\xi$. In particular, since

$$
||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leq \begin{cases} C (|\xi|^{p-2} + |\eta|^{p-2}) |\xi - \eta|, & \text{if } p \geq 2, \\ C|\xi - \eta|^{p-1}, & \text{if } 1 < p < 2,
\end{cases}
$$
for some $C > 0$, the continuity condition (2.6) holds true with $r = p - 1$ if $p \geq 2$ and with $r = 1$ if $1 < p < 2$.

Let us recall the definition of entropy solution of (2.4). For $k > 0$, we denote by $T_k : \mathbb{R} \rightarrow \mathbb{R}$ the usual truncation at level $k$, that is $T_k(s) = \max(-k, \min(k, s))$ for every $s \in \mathbb{R}$.

**Definition 2.2** We say that $u^\varepsilon$ is an entropy solution of (2.4) if it satisfies the following conditions

\[
\begin{aligned}
&u^\varepsilon \text{ is a measurable function on } \Omega^\varepsilon, \text{ almost everywhere finite,} \\
&\text{such that } T_k(u^\varepsilon) \in W^{1,p}_0(\Omega^\varepsilon), \quad \forall k > 0,
\end{aligned}
\]

and

\[
\begin{align}
\int_{\Omega^\varepsilon} \beta^\varepsilon a(\delta^\varepsilon \nabla u^\varepsilon) \nabla T_k(u^\varepsilon - \varphi) \, dx \\
\leq \int_{\Omega^\varepsilon} f^\varepsilon T_k(u^\varepsilon - \varphi) \, dx + \int_{\Omega^\varepsilon} g^\varepsilon \nabla T_k(u^\varepsilon - \varphi) \, dx,
\end{align}
\]

for every $k > 0$ and every $\varphi \in W^{1,p}_0(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$.

Thanks to [6] or [3], we know that such a solution of (2.4) exists and is unique. Let us recall that the gradient of $u^\varepsilon$, is defined according to Lemma 2.1 in [3]; it is the unique measurable function $v^\varepsilon : \Omega^\varepsilon \rightarrow \mathbb{R}^N$, such that

\[
\nabla T_k(u^\varepsilon) = v^\varepsilon \chi_{(|u^\varepsilon| \leq k)} \quad \text{almost everywhere in } \Omega^\varepsilon, \quad \forall k > 0.
\]

We define the gradient $\nabla u^\varepsilon$ of $u^\varepsilon$ as this function $v^\varepsilon$, and write $\nabla u^\varepsilon = v^\varepsilon$.

Our first and main result states that, if $b^\varepsilon$ is uniformly bounded in $\varepsilon$ and $\sigma$ and if $b^\varepsilon$ tends to a positive function $b$, weakly* in $L^\infty(\partial \Omega)$, then $u^\varepsilon$ converges almost everywhere to the entropy solution $u$ of the “limit problem” formally written as

\[
\begin{aligned}
&-\text{div}(a(\nabla u)) = f - \text{div} g \quad \text{in } \Omega, \\
&a(\nabla u) n - ba(-u) n = g n \quad \text{on } \partial \Omega.
\end{aligned}
\]

The notion of entropy solution for such a problem is given in Definition 2.3 below; it is based on the notion of trace $\gamma(u)$ of a measurable function $u$ finite almost everywhere and such that $T_k(u)$ belongs to $W^{1,p}(\Omega)$, for all $k > 0$.

We give such a definition of trace with related properties in Section 3 (see also (1.3)).
**Definition 2.3** We say that \( u \) is an entropy solution of (2.19) if it satisfies the following conditions

\[
(2.20) \quad \begin{cases} 
    u \text{ is a measurable function on } \Omega, \text{ almost everywhere finite,} \\
    \text{such that } T_k(u) \in W^{1,p}(\Omega), \quad \forall k > 0, \\
    \text{and such that } |\gamma(u)|^{p-1} \in L^1(\partial\Omega),
\end{cases}
\]

where \( \gamma(u) \) denotes the trace of \( u \) on \( \partial\Omega \), as defined in Definition 3.3 of Section 3 (see also (1.3)), and

\[
\int_\Omega a(\nabla u) \nabla T_k(u - \varphi) \, dx \\
- \int_{\partial\Omega} b a(-\gamma(u)n) n \gamma(T_k(u - \varphi)) \, d\mathcal{H}_{N-1}(\sigma)
\leq \int_\Omega f T_k(u - \varphi) \, dx + \int_\Omega g \nabla T_k(u - \varphi) \, dx,
\]

for every \( k > 0 \) and for every \( \varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \).

**Remark 2.4** Observe that the integral on \( \partial\Omega \) is well-defined. Indeed, since \( T_k(u - \varphi) \) belongs to \( W^{1,p}(\Omega) \), the function \( \gamma(T_k(u - \varphi)) \) is its trace in the usual sense and therefore it is bounded. Since \( |\gamma(u)|^{p-1} \) belongs to \( L^1(\partial\Omega) \), then by the growth condition (2.15), we have that \( a(-\gamma(u)n)n \in L^1(\partial\Omega) \). Therefore, if \( b \) belongs to \( L^\infty(\partial\Omega) \), the function \( b a(-\gamma(u)n)n \gamma(T_k(u - \varphi)) \) belongs to \( L^1(\partial\Omega) \).

In Section 4 we will prove the following result

**Theorem 2.5** Suppose that the assumptions (2.5) to (2.11) and (2.12) to (2.14) hold true. Assume also that, for some \( \beta_1, \beta_2 \),

\[
0 < \beta_1 \leq b^\varepsilon(\sigma) \leq \beta_2, \quad \mathcal{H}_{N-1} - \text{a.e. } \sigma \in \partial\Omega,
\]

so that, up to a subsequence,

\[
(2.23) \quad b^\varepsilon \rightharpoonup b \text{ weakly* in } L^\infty(\partial\Omega), \quad \text{for some } b \in L^\infty(\partial\Omega), \, b > 0.
\]

Then the entropy solution \( u^\varepsilon \) of problem (2.4) converges almost everywhere to the unique entropy solution \( u \) of problem (2.19).
Under suitable assumptions on the convergence of $b^\varepsilon$, “limit problem” (2.19) is the Dirichlet problem

\[
\begin{aligned}
-\text{div}(a(\nabla u)) &= f - \text{div} g \quad \text{in} \quad \Omega, \\
    u &= 0 \quad \text{on} \quad \partial\Omega.
\end{aligned}
\]

More precisely, our second result, whose proof is a simple variant of the proof of Theorem 2.5 (see Section 5), is the following one

**Theorem 2.6** Suppose that the assumptions (2.5) to (2.11) and (2.12) to (2.14) hold true. Assume also that, for some $\beta_1,$

\[
b^\varepsilon(\sigma) \geq \beta_1 > 0, \ H_{N-1} - \text{a.e.} \ \sigma \in \partial\Omega,
\]

and

\[
\left(\frac{1}{b^\varepsilon}\right)^{p'-1} \to 0 \quad \text{strongly in} \ L^1(\partial\Omega).
\]

Then the entropy solution $u^\varepsilon$ of problem (2.4) converges almost everywhere to the unique entropy solution $u$ of problem (2.24).

### 3 Definition of generalized trace

In this section we will assume that $1 \leq p < +\infty$ and we give the definition of trace for a measurable function, in general not belonging to a Sobolev space. Such a definition is based on the following result

**Proposition 3.1** For every $u: \Omega \to \mathbb{R}$ measurable, finite almost everywhere in $\Omega$ and such that $T^k(u) \in W^{1,p}(\Omega)$ for every $k > 0,$ there exists a unique $H_{N-1}$-measurable function $\tau: \partial\Omega \to \mathbb{R}$ such that

\[
\tau(\sigma) = \lim_{k \to \infty} \gamma(T^k(u))(\sigma) \quad H_{N-1} - \text{a.e.} \ \sigma \in \partial\Omega.
\]
Proof of Proposition 3.1 For every $k > 0$, we set
\[
(\partial \Omega)_k = \{ \sigma \in \partial \Omega : |\gamma(T_k(u))(\sigma)| < k \},
\]
\[
(\partial \Omega)_+ = \bigcap_{k>0} \{ \sigma \in \partial \Omega : \gamma(T_k(u))(\sigma) = k \},
\]
\[
(\partial \Omega)_- = \bigcap_{k>0} \{ \sigma \in \partial \Omega : \gamma(T_k(u))(\sigma) = -k \}.
\]
Let $k < k'$, hence $T_k(T_{k'}(u)) = T_k(u)$. Moreover, since $T_k \in C^{0,1}(\mathbb{R})$, we have $T_k \circ \gamma = \gamma \circ T_k$, so
\[
(3.2) \quad T_k(\gamma(T_{k'}(u))) = \gamma(T_k(T_{k'}(u))) = \gamma(T_k(u)), \text{ a.e. in } \partial \Omega.
\]
In particular
\[
\{ \sigma \in \partial \Omega : |\gamma(T_k(u))(\sigma)| < k \} = \{ \sigma \in \partial \Omega : |T_k(\gamma(T_{k'}(u)))(\sigma)| < k \},
\]
therefore $T_k(\gamma(T_{k'}(u))) = \gamma(T_{k'}(u))$ a.e. in $(\partial \Omega)_k$ and, using (3.2), we get
\[
(3.3) \quad \gamma(T_k(u)) = \gamma(T_{k'}(u)), \text{ a.e. in } (\partial \Omega)_k, \text{ if } k < k',
\]
which clearly implies that
\[
(3.4) \quad (\partial \Omega)_k \subset (\partial \Omega)_{k'}, \text{ if } k < k'.
\]
It follows from (3.3) that, for each $\sigma \in \bigcup_{k'>0}(\partial \Omega)_{k'}$, $\gamma(T_k(u))(\sigma)$ does not depend on $k$ and is finite, for $k$ large enough, and hence
\[
(3.5) \quad \tau(\sigma) = \lim_{k \to \infty} \gamma(T_k(u))(\sigma)
\]
is well defined in $\mathbb{R}$, for any $\sigma \in \bigcup_{k'>0}(\partial \Omega)_{k'}$. For $\sigma \in (\partial \Omega)_+$, we set $\tau(\sigma) = +\infty$ and for $\sigma \in (\partial \Omega)_-$, $\tau(\sigma) = -\infty$, so that (3.5) is still valid in $(\partial \Omega)_+ \cup (\partial \Omega)_-$. In order to have (3.1), it remains to prove that, up to a negligible set,
\[
(3.6) \quad \partial \Omega = \bigcup_{k>0} (\partial \Omega)_k \cup (\partial \Omega)_+ \cup (\partial \Omega)_-.
\]
Actually, the three above sets are disjoint and if $\sigma$ is a point in $\partial \Omega$ which is in none of them, then for any $k > 0$, $|\gamma(T_k(u))(\sigma)| = k$, and for some $s, t > 0$, $\gamma(T_s(u))(\sigma) = s$ and $\gamma(T_t(u))(\sigma) = -t$. If, for example, $s < t$, then $T_s(\gamma(T_t(u)))(\sigma) = \gamma(T_s(T_t(u)))(\sigma) = \gamma(T_s(u))(\sigma) = s$, hence $s \leq \gamma(T_t(u))(\sigma) = -t < 0$, in contradiction with $s > 0$. 8
Remark 3.2 We explicitely observe that the assumption that $T_k(u) \in W^{1,p}(\Omega)$ assures the existence of the traces, in the classical sense, $\gamma(T_k(u))$. Moreover according to Proposition 3.1, the trace of the function $u$ is a function which takes values in $\overline{\mathbb{R}}$, but it is finite at any point $\sigma$ such that $|\gamma(T_k(u))(\sigma)| < k$, for some $k > 0$.

Definition 3.3 We say that the function $\tau : \partial \Omega \rightarrow \overline{\mathbb{R}}$ defined in (3.1) is the trace of $u$ and we write $\gamma(u) = \tau$.

Remark 3.4 Observe that $\gamma(u)$ coincides with the classical notion of trace if $u \in W^{1,1}(\Omega)$. In contrast there are examples of functions $u \notin W^{1,1}(\Omega)$ (and thus such that the trace in the classical sense is not defined) for which the trace $\gamma(u)$ in the sense of Definition 3.3 is defined. An example is given by the function

$$u(x) = \frac{1}{x} \text{ in } ]0,1[,$$

which does not belong to $L^1([0,1])$ and therefore the usual trace $\gamma(u)$ is not defined. Nevertheless the trace of $u$ in the sense of Definition 3.3 is well-defined: $\gamma(u)(1) = 1$, $\gamma(u)(0) = +\infty$.

Remark 3.5 Our definition is a slight extension of the notion of trace given in [2], where the authors define the trace of a function $u$ which is finite almost everywhere in $\Omega$, such that $T_k(u) \in W^{1,p}(\Omega)$ for every $k > 0$ and which satisfies a further condition, namely:

there exists a sequence $u_n \in W^{1,p}(\Omega)$ such that $u_n \rightarrow u$ a.e. in $\Omega$, $\nabla T_k(u_n) \rightarrow \nabla T_k(u)$ strongly in $L^1(\Omega)$ for any $k > 0$ and the sequence $\gamma(u_n)$ converges a.e. on $\partial \Omega$.

Observe that the notion of trace in Definition 3.3 is given for a wider class of function. For example the function $u(x) = \frac{1}{x}$ in $]0,1[$ has no trace in the sense of the definition given in [2], but it has a trace in the sense of Definition 3.3.

We will use the following properties of $\gamma(u)$.

Proposition 3.6 Assume that $u$, $v$ and $w$ are measurable functions, finite almost everywhere in $\Omega$ and such that $T_k(u)$, $T_k(v)$ and $T_k(w)$ belong to $W^{1,p}(\Omega)$ for every $k > 0$. Then we have
1) \{ |\gamma(T_k(u))| \}_k is an increasing sequence of functions converging to |\gamma(u)|, as \( k \) tends to infinity;

2) |\gamma(u)| = \gamma(|u|);

3) If \( u \leq v \), then \( \gamma(u) \leq \gamma(v) \);

4) |\gamma(u) - \gamma(v)| \leq |\gamma(u - v)|, which implies that |\gamma(u + v)| \leq |\gamma(u)| + |\gamma(v)|;

5) Using the convention that \( T_k(\pm \infty) = k \) and \( T_k(-\infty) = -k \), then
\[
\gamma(T_k(u)) = T_k(\gamma(u));
\]

6) If \( v = w \) in \{ |u| < h \}, then \( \gamma(v) = \gamma(w) \) in \{ |\gamma(u)| < h \}.

Proof The proof is a simply consequence of Proposition 3.1. For example, let us prove the last property. We have \{ |u| < h \} = \{ |T_h(u)| < h \} and hence, if \( v = w \) in \{ |u| < h \}, then for any \( k \), \( T_k(v) = T_k(w) \) in \{ |T_h(u)| < h \}, which gives, for the classical trace, \( \gamma(T_k(v)) = \gamma(T_k(w)) \) in \{ |\gamma(T_h(u))| < h \}. Let \( \sigma \in \partial \Omega \) be such that \( |\gamma(u)(\sigma)| < h \). Then, by the first property, \( |\gamma(T_h(u))(\sigma)| \leq |\gamma(u)(\sigma)| < h \). It follows that \( \gamma(T_k(v))(\sigma) = \gamma(T_k(w))(\sigma) \) and, by letting \( k \) tend to infinity, \( \gamma(v)(\sigma) = \gamma(w)(\sigma) \).

4 Proof of Theorem 2.5

4.1 A priori estimates of \( |\gamma(u^\varepsilon)|^{p-1}, |u^\varepsilon|^{p-1} \) and \( |\nabla u^\varepsilon|^{p-1} \)

Let \( u^\varepsilon \) be the entropy solution of (2.4). We are going to prove some a priori estimates for \( u^\varepsilon \), for its gradients and its traces in Marcinkiewicz space \( \mathcal{M}^r(\Omega) \). This is done by adapting to the present case a Lemma proved in [3] (see also [4]).

Proposition 4.1 Let \( \varepsilon > 0 \) fixed and \( u^\varepsilon \) the entropy solution to problem (2.4). Under the assumptions (2.5) - (2.11), then \( |\gamma(u^\varepsilon)|^{p-1} \) belongs to \( \mathcal{M}^{N-1} \), \( |u^\varepsilon|^{p-1} \) belongs to \( \mathcal{M}^{N} \), \( |\nabla u^\varepsilon|^{p-1} \) belongs to \( \mathcal{M}^{N'} \) and

\[
\begin{align*}
\eta^{N-1} & \text{meas } \{ |\gamma(u^\varepsilon)|^{p-1} > \eta \} \leq C \\
\eta^{N} & \text{meas } \{ |u^\varepsilon|^{p-1} > \eta \} \leq C \\
\eta^{N'} & \text{meas } \{ |\nabla u^\varepsilon|^{p-1} > \eta \} \leq C
\end{align*}
\]

where \( C \) is a constant not depending on \( \varepsilon \) or \( \eta \).
Proof of Proposition 4.1 We set
\begin{equation} \lambda^{\varepsilon}(x) = b^{\varepsilon}(\sigma(x)) \left( h^{\varepsilon}(\sigma(x)) \right)^{p-1}, \quad \text{a.e. } x \in \Sigma^{\varepsilon}, \tag{4.4} \end{equation}
and we observe that, by (2.22), it results that
\begin{equation} \lambda^{\varepsilon}(x) \geq \beta_1 \left( h^{\varepsilon}(\sigma(x)) \right)^{p-1} > 0, \quad \text{a.e. } x \in \Sigma^{\varepsilon}. \tag{4.5} \end{equation}

Taking \( \varphi = 0 \) in (2.17), we have
\begin{equation} \int_{\Omega^{\varepsilon}} \beta^{\varepsilon} a(\delta^{\varepsilon} \nabla u^{\varepsilon}) \nabla T_k(u^{\varepsilon}) \, dx \leq \int_{\Omega^{\varepsilon}} f^{\varepsilon} T_k(u^{\varepsilon}) \, dx + \int_{\Omega^{\varepsilon}} g^{\varepsilon} \nabla T_k(u^{\varepsilon}) \, dx. \tag{4.6} \end{equation}

By the ellipticity condition (2.5), the definition (2.9) of \( \beta^{\varepsilon} \) and the definition (2.10) of \( \delta^{\varepsilon} \), the first integral in (4.6) is bounded from below as follows
\begin{align*}
\int_{\Omega^{\varepsilon}} \beta^{\varepsilon} a(\delta^{\varepsilon} \nabla u^{\varepsilon}) \nabla T_k(u^{\varepsilon}) \, dx &= \int_{\Omega} a(\nabla u^{\varepsilon}) \nabla T_k(u^{\varepsilon}) \, dx \\
&\quad + \int_{\Sigma^{\varepsilon}} \frac{b^{\varepsilon} \circ \sigma}{h^{\varepsilon} \circ \sigma} a((h^{\varepsilon} \circ \sigma) \nabla u^{\varepsilon}) \left( (h^{\varepsilon} \circ \sigma) \nabla T_k(u^{\varepsilon}) \right) \, dx \\
&\geq \alpha_1 \int_{\Omega} \left| \nabla T_k(u^{\varepsilon}) \right|^p \, dx + \alpha_1 \int_{\Sigma^{\varepsilon}} \frac{b^{\varepsilon} \circ \sigma}{h^{\varepsilon} \circ \sigma} \left| (h^{\varepsilon} \circ \sigma) \nabla T_k(u^{\varepsilon}) \right|^p \, dx.
\end{align*}

Therefore by definition (4.4) of \( \lambda^{\varepsilon} \), it results that
\begin{align*}
\int_{\Omega^{\varepsilon}} \beta^{\varepsilon} a(\delta^{\varepsilon} \nabla u^{\varepsilon}) \nabla T_k(u^{\varepsilon}) \, dx \geq \alpha_1 \int_{\Omega} \left| \nabla T_k(u^{\varepsilon}) \right|^p \, dx + \alpha_1 \int_{\Sigma^{\varepsilon}} \lambda^{\varepsilon} \left| \nabla T_k(u^{\varepsilon}) \right|^p \, dx. \tag{4.7}
\end{align*}

Now we consider the right-hand side of (4.6). We have
\begin{equation} \left| \int_{\Omega^{\varepsilon}} f^{\varepsilon} T_k(u^{\varepsilon}) \, dx \right| \leq k \int_{\Omega^{\varepsilon}} \left| f^{\varepsilon} \right| \, dx = k \| f^{\varepsilon} \|_{L^1(\Omega^{\varepsilon})}. \tag{4.8} \end{equation}
Moreover, by Young inequality, the definition (4.4) of $\lambda^\varepsilon$ and (4.5), we have

\[
\left| \int_{\Omega^\varepsilon} g^\varepsilon \nabla T_k(u^\varepsilon) \, dx \right|
\leq \frac{1}{p' \alpha_1^\varepsilon} \int_{\Omega^\varepsilon} |g^\varepsilon|' \, dx + \frac{\alpha_1}{p} \int_{\Omega^\varepsilon} |\nabla T_k(u^\varepsilon)|^p \, dx
\leq \frac{1}{p' \alpha_1^\varepsilon} \int_{\Omega^\varepsilon} |g^\varepsilon|' \, dx + \frac{\alpha_1}{p} \int_{\Sigma^\varepsilon} \lambda^\varepsilon |\nabla T_k(u^\varepsilon)|^p \, dx
\]

(4.9)

Combining (4.5) to (4.9), we obtain

\[
\int_{\Omega^\varepsilon} |\nabla T_k(u^\varepsilon)|^p \, dx + \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} |\nabla T_k(u^\varepsilon)|^p \, dx
\leq k \frac{p'}{\alpha_1} \|f^\varepsilon\|_{L^1(\Omega^\varepsilon)} + \frac{1}{\alpha_1^\varepsilon} \|g^\varepsilon\|_{L^{p'}(\Omega^\varepsilon)} + \frac{1}{\alpha_1^\varepsilon \beta_1^{p'}} \left\| \frac{|g^\varepsilon|}{(h^\varepsilon \circ \sigma)^{\frac{p'}{2}}} \right\|_{L^{p'}(\Sigma^\varepsilon)}.
\]

(4.10)

Let us define

\[
M = \frac{p'}{\alpha_1} \sup \varepsilon \|f^\varepsilon\|_{L^1(\Omega^\varepsilon)},
\]

\[
L = \frac{1}{\min(1, \beta_1^\varepsilon)} \frac{1}{p' \alpha_1^\varepsilon} \sup \varepsilon \left( \|g^\varepsilon\|_{L^{p'}(\Omega^\varepsilon)} + \left\| \frac{|g^\varepsilon|}{(h^\varepsilon \circ \sigma)^{\frac{p'}{2}}} \right\|_{L^{p'}(\Sigma^\varepsilon)} \right).
\]

Observe that $M$ and $L$ are finite, due to (2.12) and (2.13). Therefore the inequality (4.10) gives

\[
\int_{\Omega^\varepsilon} |\nabla T_k(u^\varepsilon)|^p \, dx + \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} |\nabla T_k(u^\varepsilon)|^p \, dx \leq Mk + L,
\]

(4.11)
for every $k > 0$. Now, we prove that the following inequality holds true

\begin{equation}
\int_{\partial \Omega} |\gamma(T_k(u^\varepsilon))|^p \, d\mathcal{H}_{N-1}(\sigma) \leq C \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} |\nabla T_k(u^\varepsilon)|^p \, dx.
\end{equation}

We give below a sketch of its proof, but all details can be found in [8]. We have

\begin{equation}
\gamma(T_k(u^\varepsilon))(\sigma) = -\int_0^{h^\varepsilon(\sigma)} \nabla T_k(u^\varepsilon)(\sigma + tn(\sigma))n(\sigma) \, dt, \quad \text{a.e. } \sigma \in \partial \Omega.
\end{equation}

By Hölder inequality, we deduce that

\begin{equation}
|\gamma(T_k(u^\varepsilon))(\sigma)|^p \leq (h^\varepsilon(\sigma))^{p-1} \int_0^{h^\varepsilon(\sigma)} |\nabla T_k(u^\varepsilon)(\sigma + tn(\sigma))|^p dt.
\end{equation}

Therefore by integrating on the boundary of $\Omega$ and by a change of variables, we get (4.12).

Combining (4.12) with (4.11) and with the following Poincaré-type inequality

\begin{equation}
\|T_k(u^\varepsilon)\|_{W^{1,p}(\Omega)} \leq C \left\{\|\nabla T_k(u^\varepsilon)\|_{L^p(\Omega)} + \|\gamma(T_k(u^\varepsilon))\|_{L^p(\partial \Omega)}\right\},
\end{equation}

we get

\begin{equation}
\|T_k(u^\varepsilon)\|_{W^{1,p}(\Omega)} \leq C (Mk + L)^{\frac{1}{p}}, \quad \forall k > 0.
\end{equation}

Now by this estimate of the truncations we deduce (4.1), (4.2) and (4.3) as in [3] or in [4] (Appendix). We just point out that, in order to prove (4.1), we use continuity of trace mapping from $W^{1,p}(\Omega)$ to $W^{1-\frac{1}{p},p}(\partial \Omega)$ and Sobolev imbedding of $W^{1-\frac{1}{p},p}(\partial \Omega)$ in $L^{\tilde{p}}(\partial \Omega)$ for $\tilde{p} = p(N - 1)/(N - p)$.

### 4.2 Passing to the limit by monotonicity

The a priori estimates proved in the previous section allow to pass to the limit in problem (2.4). The main difficulty consists in proving that the term in the variational formulation (2.17) on the layer $\Sigma^\varepsilon$ tends to a term on the boundary of $\Omega$ which involves the trace of the limit function.

We begin by observing that the same arguments used in [13](Section 5) allow to prove that $\{u_\varepsilon\}$ is a Cauchy sequence in measure. This implies that
there exists a further subsequence, still denoted by \( u^\varepsilon \), and a measurable function \( u \), which is finite almost everywhere, such that \( u^\varepsilon \) converges to \( u \) almost everywhere in \( \Omega \). Therefore \( T_k(u^\varepsilon) \) converges to \( T_k(u) \) almost everywhere. Moreover, since \( T_k(u^\varepsilon) \) is bounded in \( W^{1,p}(\Omega) \), \( T_k(u^\varepsilon) \) converges to \( T_k(u) \) weakly in \( W^{1,p}(\Omega) \), for every \( k > 0 \), and, up to a subsequence,

\[
T_k(u^\varepsilon - \varphi) \text{ converges to } T_k(u - \varphi) \text{ weakly in } W^{1,p}(\Omega),
\]

for every \( k > 0 \) and for every \( \varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \), and, since the trace mapping from \( W^{1,p}(\Omega) \) in \( L^p(\partial \Omega) \) is compact, we get

\[
\gamma(T_k(u^\varepsilon)) \to \gamma(T_k(u)) \text{ strongly in } L^p(\partial \Omega) \text{ and almost everywhere,}
\]

up to a subsequence. On the other hand, since \((4.1)\) holds true, \( \mathcal{M}^{\frac{N-1}{N-p}}(\partial \Omega) \subset L^q(\partial \Omega) \) for \( q < \frac{N-1}{N-p} \), and the properties 2) and 3) of Proposition 3.6, we have

\[
\int_{\partial \Omega} |\gamma(T_k(u^\varepsilon))|^{(p-1)q} d\mathcal{H}_{N-1}(\sigma) \leq \int_{\partial \Omega} |\gamma(u^\varepsilon)|^{(p-1)q} d\mathcal{H}_{N-1}(\sigma) \leq C,
\]

for every \( q < \frac{N-1}{N-p} \). Letting \( \varepsilon \) go to zero, by applying Lebesgue theorem, we get

\[
\int_{\partial \Omega} |\gamma(T_k(u))|^{(p-1)q} d\mathcal{H}_{N-1}(\sigma) \leq C.
\]

Now, letting \( k \) tend to infinity and applying the property 1) of Proposition 3.6 and Beppo-Levi theorem gives

\[
\int_{\partial \Omega} |\gamma(u)|^{(p-1)q} d\mathcal{H}_{N-1}(\sigma) \leq C.
\]

In particular, this implies that \( |\gamma(u)|^{p-1} \) belongs to \( L^1(\partial \Omega) \).

Using the monotonicity of \( a \) (2.7) and \( \delta^\varepsilon > 0 \), one get

\[
a(\delta^\varepsilon \nabla \varphi) \nabla T_k(u^\varepsilon - \varphi) \leq a(\delta^\varepsilon \nabla u^\varepsilon) \nabla T_k(u^\varepsilon - \varphi) \quad \text{a.e.}
\]

Hence, for \( \varepsilon > 0 \) fixed, since \( u^\varepsilon \) is an entropy solution of (2.4) (see (2.17)), the following inequality holds true

\[
\int_{\Omega^\varepsilon} \beta^\varepsilon a(\delta^\varepsilon \nabla \varphi) \nabla T_k(u^\varepsilon - \varphi) dx \\
\leq \int_{\Omega^\varepsilon} f^\varepsilon T_k(u^\varepsilon - \varphi) dx + \int_{\Omega^\varepsilon} g^\varepsilon \nabla T_k(u^\varepsilon - \varphi) dx,
\]

(4.19)
for every $k \geq 0$ and every $\varphi \in W_0^{1,p}(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$.

Let $\psi$ be a function in $C^1(\overline{\Omega})$ and let us define $\psi^\varepsilon : \Omega \to \mathbb{R}$ by

$$\psi^\varepsilon(x) = \begin{cases} \psi(x), & \text{if } x \in \Omega, \\ \psi(\sigma(x)) \left(1 - \frac{d(x, \partial \Omega)}{h^\varepsilon(\sigma(x))}\right), & \text{if } x \in \Sigma^\varepsilon, \end{cases}$$

where $d(x, \partial \Omega)$ denotes the distance from $x \in \Sigma^\varepsilon$ to the boundary of $\Omega$. As $\psi \circ \sigma$ and the inverse of $h^\varepsilon \circ \sigma$ are Lipschitz in $\Sigma^\varepsilon$ and since $d$ has regularity $C^1$ (notice that $\nabla d = n \circ \sigma$), it is clear that $\psi^\varepsilon$ is Lipschitz in $\Sigma^\varepsilon$. Moreover $\psi^\varepsilon$ is continuous in $\Omega^\varepsilon$ and vanishes on $\partial \Omega^\varepsilon$. Hence $\psi^\varepsilon \in W_0^{1,p}(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$, we can choose $\varphi = \psi^\varepsilon$ in (4.19) and get

$$\int_{\Omega} a(\nabla \psi) \nabla T_k(u^\varepsilon - \psi) \; dx + \int_{\Sigma^\varepsilon} \beta^\varepsilon a(\delta^\varepsilon \nabla \psi^\varepsilon) \nabla T_k(u^\varepsilon - \psi^\varepsilon) \; dx$$

$$\leq \int_{\Omega^\varepsilon} f^\varepsilon T_k(u^\varepsilon - \psi^\varepsilon) \; dx + \int_{\Omega^\varepsilon} g^\varepsilon \nabla T_k(u^\varepsilon - \psi^\varepsilon) \; dx. \quad (4.20)$$

Now we want to pass to the limit as $\varepsilon$ goes to zero in each term of (4.20). As pointed out at the beginning of this section, the main difficulty concerns the second term in (4.20); we will prove that it converges to a term on the boundary of $\Omega$, involving the trace of $u$. Precisely we prove the following equalities

$$\lim_{\varepsilon \to 0} \int_{\Omega} a(\nabla \psi) \nabla T_k(u^\varepsilon - \psi) \; dx = \int_{\Omega} a(\nabla \psi) \nabla T_k(u - \psi) \; dx, \quad (4.21)$$

$$\lim_{\varepsilon \to 0} \int_{\Sigma^\varepsilon} \beta^\varepsilon a(\delta^\varepsilon \nabla \psi^\varepsilon) \nabla T_k(u^\varepsilon - \psi^\varepsilon) \; dx$$

$$= - \int_{\partial \Omega} b a(\psi n) n \gamma(T_k(u - \psi)) \; d\mathcal{H}^{N-1}(\sigma), \quad (4.22)$$

$$\lim_{\varepsilon \to 0} \int_{\Omega^\varepsilon} f^\varepsilon T_k(u^\varepsilon - \psi^\varepsilon) \; dx = \int_{\Omega} f T_k(u - \psi) \; dx, \quad (4.23)$$

$$\lim_{\varepsilon \to 0} \int_{\Omega^\varepsilon} g^\varepsilon \nabla T_k(u^\varepsilon - \psi^\varepsilon) \; dx = \int_{\Omega} g \nabla T_k(u - \psi) \; dx. \quad (4.24)$$

**Proof of (4.21)** The equality (4.21) is an immediate consequence of (4.16).

**Proof of (4.22)** First let us observe that

$$\nabla \psi^\varepsilon(x) = \Phi_1^\varepsilon(x) + \Phi_2^\varepsilon(x), \quad \text{a. e. in } \Sigma^\varepsilon, \quad (4.25)$$
\(\Phi_\varepsilon(x) = -\frac{\psi(\sigma(x))}{h^\varepsilon(\sigma(x))} d(x, \partial \Omega) = -\frac{\psi(\sigma(x))}{h^\varepsilon(\sigma(x))} n(\sigma(x)), \)

where

\[
(4.26)
\]

and

\[
(4.27)
\]

Since \(\psi \in C^1(\Omega)\) and \(d(x, \partial \Omega) \leq h^\varepsilon(\sigma(x))\), it follows that

\[
|\nabla \psi^\varepsilon| \leq C\frac{h^\varepsilon}{h^\varepsilon \circ \sigma} + C\frac{|\nabla (h^\varepsilon \circ \sigma)|}{h^\varepsilon \circ \sigma} \text{ a.e. in } \Sigma^\varepsilon.
\]

Using decomposition (4.25), we have

\[
\int_{\Sigma^\varepsilon} \beta^\varepsilon a(\delta^\varepsilon \nabla \psi^\varepsilon) \nabla T_k(u^\varepsilon - \psi^\varepsilon) \, dx \\
= \int_{\Sigma^\varepsilon} b^\varepsilon \circ \sigma a(h^\varepsilon \circ \sigma \nabla \psi^\varepsilon) \nabla T_k(u^\varepsilon - \psi^\varepsilon) \, dx = I_1^\varepsilon + I_2^\varepsilon,
\]

where

\[
I_1^\varepsilon = \int_{\Sigma^\varepsilon} (b^\varepsilon \circ \sigma) a(h^\varepsilon \circ \sigma \Phi_1^\varepsilon) \nabla T_k(u^\varepsilon - \psi^\varepsilon) \, dx,
\]

\[
I_2^\varepsilon = \int_{\Sigma^\varepsilon} (b^\varepsilon \circ \sigma) \left[ a((h^\varepsilon \circ \sigma)(\Phi_1^\varepsilon + \Phi_2^\varepsilon)) - a((h^\varepsilon \circ \sigma) \Phi_1^\varepsilon) \right] \nabla T_k(u^\varepsilon - \psi^\varepsilon) \, dx.
\]

Now we prove that

\[
\lim_{\varepsilon \to 0} I_1^\varepsilon = -\int_{\partial \Omega} b a(-\psi n) \gamma (T_k(u - \psi)) \, d\mathcal{H}_{N-1}(\sigma),
\]

and

\[
\lim_{\varepsilon \to 0} I_2^\varepsilon = 0.
\]

This will imply (4.22).

We begin by evaluating \(I_1^\varepsilon\). By the definition (4.26) of \(\Phi_1^\varepsilon\) and the assumption (2.8), we have

\[
I_1^\varepsilon = \int_{\Sigma^\varepsilon} (b^\varepsilon \circ \sigma) \left[ a((-\psi \circ \sigma) \circ \sigma) \circ \sigma \right] \nabla T_k(u^\varepsilon - \psi^\varepsilon) \, dx.
\]
Moreover by a change of variable, we get
\[
I_1^\varepsilon = \int_{\partial\Omega} \int_0^{h^\varepsilon(\sigma)} b^\varepsilon(\sigma)[a(-\psi(\sigma)n(\sigma)) n(\sigma)] \\
\times [n(\sigma) \nabla T_k(u^\varepsilon - \psi^\varepsilon)(\sigma + tn(\sigma))] (1 + q(\sigma, t) t) \, dt \, dH_{N-1}(\sigma),
\]
where \(q(\sigma, t)\) is a bounded function. Let us denote by \(\omega(\varepsilon)\) a function which tends to zero with \(\varepsilon\). Since \(t \leq h^\varepsilon(\sigma) \leq \varepsilon\), it results that
\[
I_1^\varepsilon = \int_{\partial\Omega} b^\varepsilon(\sigma)[a(-\psi(\sigma)n(\sigma)) n(\sigma)] \\
\times \, n(\sigma) \left[ \int_0^{h^\varepsilon(\sigma)} \nabla T_k(u^\varepsilon - \psi^\varepsilon)(\sigma + tn(\sigma)) \, dt \right] \, dH_{N-1}(\sigma) + \omega(\varepsilon) \\
= - \int_{\partial\Omega} b^\varepsilon(\sigma) a(-\psi(n(\sigma)) n(\sigma) \gamma(T_k(u^\varepsilon - \psi^\varepsilon))(\sigma) dH_{N-1}(\sigma) + \omega(\varepsilon).
\]
On the other hand, since \(T_k\) is Lipschitz,
\[
|\gamma(T_k(u^\varepsilon - \psi^\varepsilon)) - \gamma(T_k(u^\varepsilon - \psi))| = |\gamma(T_k(u^\varepsilon - \psi^\varepsilon) - T_k(u^\varepsilon - \psi))| = \\
\gamma(|T_k(u^\varepsilon - \psi^\varepsilon) - T_k(u^\varepsilon - \psi)|) \leq \gamma(|\psi^\varepsilon - \psi|) = |\gamma(\psi^\varepsilon - \psi)| = 0
\]
and hence \(\gamma(T_k(u^\varepsilon - \psi^\varepsilon)) = \gamma(T_k(u^\varepsilon - \psi))\). Therefore we get
\[
I_1^\varepsilon = - \int_{\partial\Omega} b^\varepsilon a(-\psi n(\sigma)) n(\sigma) \gamma(T_k(u^\varepsilon - \psi)) dH_{N-1}(\sigma) + \omega(\varepsilon).
\]
Moreover (4.16) implies that \(\gamma(T_k(u^\varepsilon - \psi))\) converges to \(\gamma(T_k(u - \psi))\) in \(L^p(\partial\Omega)\) strongly; therefore (4.29) follows from (4.31) and the assumption (2.23).

Now we evaluate \(I_2^\varepsilon\). By Hölder inequality and assumption (2.22), we have
\[
|I_2^\varepsilon| = \left| \int_{\Sigma^\varepsilon} b^\varepsilon \circ \sigma \left[ a(h^\varepsilon \circ \sigma (\Phi_1^\varepsilon + \Phi_2^\varepsilon)) - a(h^\varepsilon \circ \sigma \Phi_1^\varepsilon) \right] \nabla T_k(u^\varepsilon - \psi^\varepsilon) \, dx \right| \\
\leq \beta_2 \int_{\Sigma^\varepsilon} \left| a(h^\varepsilon \circ \sigma (\Phi_1^\varepsilon + \Phi_2^\varepsilon)) - a(h^\varepsilon \circ \sigma \Phi_1^\varepsilon) \right| (h^\varepsilon \circ \sigma)^{\frac{p-1}{p}} \times (h^\varepsilon \circ \sigma)^{\frac{p-1}{p}} \left| \nabla T_k(u^\varepsilon - \psi^\varepsilon) \right| \, dx \\
\leq \beta_2 \left( \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} \left| \nabla T_k(u^\varepsilon - \psi^\varepsilon) \right|^p \, dx \right)^{\frac{1}{p}} \\
\times \left( \int_{\Sigma^\varepsilon} \frac{|a(h^\varepsilon \circ \sigma (\Phi_1^\varepsilon + \Phi_2^\varepsilon)) - a(h^\varepsilon \circ \sigma \Phi_1^\varepsilon)|}{h^\varepsilon \circ \sigma} \left| h^\varepsilon \circ \sigma \right|^{\frac{p}{p'}} \, dx \right)^{\frac{1}{p'}}.
\]
Observe that the first integral in the right-hand side of (4.32) is bounded. Indeed, from (4.11), since \( \|\psi\|_{L^\infty(\Omega)} \leq \|\psi\|_{L^\infty(\Omega)} \), we have

\[
\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} \left| \nabla T_k (u^\varepsilon - \psi^\varepsilon) \right|^p \, dx \\
\leq \int_{\Sigma^\varepsilon \cap \{|u^\varepsilon| \leq \|\psi\|_{L^\infty(\Omega)} + k\}} (h^\varepsilon \circ \sigma)^{p-1} \left| \nabla (u^\varepsilon - \psi^\varepsilon) \right|^p \, dx \\
\leq C \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} \left| \nabla T_k \|\psi\|_{L^\infty(\Omega)} + k \right|^p \, dx + C \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} \left| \nabla \psi^\varepsilon \right|^p \, dx \\
\leq C \left[ M(\|\psi\|_{L^\infty(\Omega)} + k) + L \right] + C \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} \left| \nabla \psi^\varepsilon \right|^p \, dx.
\]

On the other hand, from (4.28), assumption (2.14) and the fact that \( h^\varepsilon \leq \varepsilon \), we get

\[
\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} \left| \nabla \psi^\varepsilon \right|^p \, dx \\
\leq C \int_{\Sigma^\varepsilon} \frac{1}{h^\varepsilon \circ \sigma} \, dx + C \varepsilon^p + C \int_{\Sigma^\varepsilon} \frac{|\nabla (h^\varepsilon \circ \sigma)|^p}{h^\varepsilon \circ \sigma} \, dx \\
\leq C \int_{\partial \Omega} \int_0^{h^\varepsilon(\sigma)} \frac{1}{h^\varepsilon(\sigma)} dt \, dH_{N-1}(\sigma) + C \varepsilon^p \\
+ C \int_{\partial \Omega} \int_0^{h^\varepsilon(\sigma)} \frac{|\nabla h^\varepsilon(\sigma)|^p}{h^\varepsilon(\sigma)} dt \, dH_{N-1}(\sigma) \\
\leq C + C \varepsilon^p + C \int_{\partial \Omega} |\nabla h^\varepsilon|^p \, dH_{N-1}(\sigma) \leq C.
\]

Therefore it results that

\[
(4.33) \quad \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} \left| \nabla T_k (u^\varepsilon - \psi^\varepsilon) \right|^p \, dx \leq C.
\]

Now we evaluate the second integral in the right-hand side of (4.32). Observe that, by assumption (2.6), we have

\[
|a (h^\varepsilon \circ \sigma(\Phi^\varepsilon_1 + \Phi^\varepsilon_2)) - a(h^\varepsilon \circ \sigma \Phi^\varepsilon_1)|^p' \leq C \left( 1 + |h^\varepsilon \circ \sigma(\Phi^\varepsilon_1 + \Phi^\varepsilon_2)|^p + |h^\varepsilon \circ \sigma \Phi^\varepsilon_1|^p \right) |h^\varepsilon \circ \sigma \Phi^\varepsilon_2|^p.
\]

Hence, by using Holder inequality and the fact that \( \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{-1} \, dx \) is
uniformly bounded, we get

\[
\int_{\Sigma^\varepsilon} \frac{|a(h^\varepsilon \circ \sigma(\Phi^\varepsilon_1 + \Phi^\varepsilon_2)) - a(h^\varepsilon \circ \sigma \Phi^\varepsilon_1)|^{p'}}{h^\varepsilon \circ \sigma} \, dx \leq C \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{\frac{p-1}{2}} |\Phi^2_2|^{\frac{p}{2}} \, dx \\
+ C \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} \left(|\Phi^\varepsilon_1 + \Phi^\varepsilon_2|^{\frac{p}{2}} + |\Phi^\varepsilon_1|^{\frac{p}{2}}\right) |\Phi^\varepsilon_2|^{\frac{p}{2}} \, dx \\
\leq C \left( \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} |\Phi^\varepsilon_2|^{p} \, dx \right)^{\frac{1}{2}} \left( \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{-1} \, dx \right)^{\frac{1}{2}} \\
+ \left( \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} |\Phi^\varepsilon_2|^{p} \, dx \right)^{\frac{1}{2}} \\
\times \left( \int_{\Sigma^\varepsilon} (|\Phi^\varepsilon_1 + \Phi^\varepsilon_2|^{\frac{p}{2}} + |\Phi^\varepsilon_1|^{\frac{p}{2}})^{p'} (h^\varepsilon \circ \sigma)^{p-1} \, dx \right)^{\frac{1}{p'}} \\
\leq C \left( \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} |\Phi^\varepsilon_2|^{p} \, dx \right)^{\frac{1}{2}} \\
\times \left[ 1 + \left( \int_{\Sigma^\varepsilon} |\Phi^\varepsilon_1|^{p} (h^\varepsilon \circ \sigma)^{p-1} + |\Phi^\varepsilon_2|^{p} (h^\varepsilon \circ \sigma)^{p-1} \, dx \right) \right]^{\frac{1}{2}}.
\]

(4.34)

Now, by definition (4.27) of \( \Phi^\varepsilon_2 \), we have

\[
\int_{\Sigma^\varepsilon} |\Phi^\varepsilon_2|^{p} (h^\varepsilon \circ \sigma)^{p-1} \, dx \leq C \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} \, dx + C \int_{\Sigma^\varepsilon} \frac{\nabla (h^\varepsilon \circ \sigma)^{p}}{h^\varepsilon \circ \sigma} \, dx \\
\leq C \varepsilon^{p-1} |\Sigma^\varepsilon| + C \int_{\Sigma^\varepsilon} \frac{\nabla (h^\varepsilon \circ \sigma)^{p}}{h^\varepsilon \circ \sigma} \, dx \\
\leq C \varepsilon^{p-1} |\Sigma^\varepsilon| + \int_{\partial \Omega} \frac{\nabla h^\varepsilon}{h^\varepsilon} \, dH_{N-1}(\sigma).
\]

Therefore, by assumption (2.14), we deduce that

(4.35) \[ \lim_{\varepsilon \to 0} \int_{\Sigma^\varepsilon} |\Phi^\varepsilon_2|^{p} (h^\varepsilon \circ \sigma)^{p-1} \, dx = 0. \]

Moreover, since it results from (4.26) that \( |\Phi^\varepsilon_1| \leq C (h^\varepsilon \circ \sigma)^{-1} \), we get

(4.36) \[ \int_{\Sigma^\varepsilon} |\Phi^\varepsilon_1|^{p} (h^\varepsilon \circ \sigma)^{p-1} \, dx \leq C \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{-1} \, dx \leq C. \]

Combining (4.33) to (4.36), it comes

(4.37) \[ \lim_{\varepsilon \to 0} \int_{\Sigma^\varepsilon} \frac{|a(h^\varepsilon \circ \sigma(\Phi^\varepsilon_1 + \Phi^\varepsilon_2)) - a(h^\varepsilon \circ \sigma \Phi^\varepsilon_1)|^{p'}}{h^\varepsilon \circ \sigma} \, dx = 0. \]
Finally, by (4.32), (4.33) and (4.37) we obtain (4.30). This concludes the proof of (4.22).

**Proof of (4.23)** Since \( f^\varepsilon \rightharpoonup f \) in \( L^1(\Omega) \) and since \( T_k(u^\varepsilon - \psi^\varepsilon) \) is a sequence of functions that is bounded in \( L^\infty(\Omega) \) and converges to \( T_k(u - \psi) \) almost everywhere in \( \Omega \), Egoroff and Dunford-Pettis theorems imply (4.23).

**Proof of (4.24)** Obviously it results
\[
\int_{\Omega} g^\varepsilon \nabla T_k(u^\varepsilon - \psi^\varepsilon) \, dx = \int_{\Omega} g^\varepsilon \nabla T_k(u^\varepsilon - \psi) \, dx + \int_{\Sigma^\varepsilon} g^\varepsilon \nabla T_k(u^\varepsilon - \psi^\varepsilon) \, dx.
\]

By (2.13) and (4.16),
\[
(4.38) \quad \lim_{\varepsilon \to 0} \int_{\Omega} g^\varepsilon \nabla T_k(u^\varepsilon - \psi) \, dx = \int_{\Omega} g \nabla T_k(u - \psi) \, dx.
\]

On the other hand, by Hölder inequality, we have
\[
(4.39) \quad \int_{\Sigma^\varepsilon} g^\varepsilon \nabla T_k(u^\varepsilon - \psi^\varepsilon) \, dx \leq \left( \int_{\Sigma^\varepsilon} |g^\varepsilon|^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{p-1} |\nabla T_k(u^\varepsilon - \psi^\varepsilon)|^p \, dx \right)^{\frac{1}{p}}.
\]

By (2.13) the first integral in the right-hand side tends to zero and by (4.33) the second integral in the right-hand side is bounded by a constant which does not depend on \( \varepsilon \). This conclude the proof of (4.24).

Finally combining (4.20) to (4.24), we get
\[
(4.40) \quad \int_{\Omega} a(\nabla \psi) \nabla T_k(u - \psi) \, dx - \int_{\partial \Omega} b a(-\gamma(\psi)n) n \gamma(T_k(u - \psi)) \, dH_{N-1}(\sigma) \leq \int_{\Omega} f T_k(u - \psi) \, dx + \int_{\Omega} g \nabla T_k(u - \psi) \, dx,
\]
for all \( \psi \in C^1(\overline{\Omega}) \). By a density argument we deduce that (4.40) holds true for all \( \psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \).
4.3 The function $u$ is an entropy solution of (2.19)

According to Section 4.2, $T_k(u)$ belongs to $W^{1,p}(\Omega)$, for every $k > 0$ and $|\gamma(u)|^{p-1} \in L^1(\partial \Omega)$. Therefore in order to prove that $u$ is an entropy solution of (2.19), we prove that (4.40) implies (2.21).

We proceed by using the Minty argument ([14]; see also [5]). We take, in (4.40), the test function

$$
\psi = T_h(u) - t T_k(u - \varphi),
$$

where $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $t > 0$ and we obtain

$$
\int_\Omega a(\nabla T_h(u) - t \nabla T_k(u - \varphi)) \nabla T_k(u - T_h(u) + t T_k(u - \varphi))
- \int_{\partial \Omega} b a(-\gamma(T_h(u) - t T_k(u - \varphi))) n 
\times \gamma(T_k(u - T_h(u) + t T_k(u - \varphi)))
\, dH_{N-1}(\sigma)
\leq \int_\Omega f T_k(u - T_h(u) + t T_k(u - \varphi))
+ \int_\Omega g \nabla T_k(u - T_h(u) + t T_k(u - \varphi))
\, dx.
$$

Let us denote by (A) to (D) the four integrals in (4.41) and let us take their limits when $h$ tends to infinity.

Classically (see ([5])), the following equalities hold true

$$
\lim_{h \to +\infty} (A) = t \int_\Omega a(\nabla u - t \nabla T_k(u - \varphi)) \nabla T_k(u - \varphi) \, dx,
$$

$$
\lim_{h \to +\infty} (C) = t \int_\Omega f T_k(u - \varphi) \, dx,
$$

$$
\lim_{h \to +\infty} (D) = t \int_\Omega g \nabla T_k(u - \varphi) \, dx.
$$

Now we evaluate (B). We decompose $\partial \Omega$ in the following way

$$
\partial \Omega = \{|\gamma(u - T_h(u) + t T_k(u - \varphi))| \geq k\}
\cup \{|\gamma(u - T_h(u) + t T_k(u - \varphi))| < k\} \cap \{|\gamma(u)| \geq h\}
\cup \{|\gamma(u - T_h(u) + t T_k(u - \varphi))| < k\} \cap \{|\gamma(u)| < h\}.
$$

Here and in the following we will denote e.g. by $\{|\gamma(v)| > k\}$ the subset of $\partial \Omega$

$\{\sigma \in \partial \Omega : |\gamma(v)(\sigma)| > k\}$ where $v$ is a function defined on $\partial \Omega$.  

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We denote by \((B_1), (B_2)\) and \((B_3)\) the terms corresponding to such decomposition, that is
\[
(B) = (B_1) + (B_2) + (B_3).
\]

We begin by evaluating the sum \((B_1) + (B_2)\). We are going to prove that
\[
\{\gamma(u - T_h(u) + tT_k(u - \varphi)) \geq k\} \subset \{|\gamma(u)| \geq h\},
\]
(4.45)

or equivalently
\[
\{|\gamma(u)| < h\} \subset \{|\gamma(u - T_h(u) + tT_k(u - \varphi))| < k\}.
\]
(4.46)

Indeed, by the property 6) of Proposition 3.6,
\[
u - T_h(u) + tT_k(u - \varphi) = tT_k(u - \varphi), \quad \text{a.e. in } \{x \in \Omega : |u| < h\}
\]

implies that
\[
\gamma(u - T_h(u) + tT_k(u - \varphi)) = \gamma(tT_k(u - \varphi)), \quad \text{a.e. in } \{|\gamma(u)| < h\}.
\]

Since \(|\gamma(tT_k(u - \varphi))| \leq tk\) a.e. on \(\partial \Omega\), we deduce that
\[
|\gamma(u - T_h(u) + tT_k(u - \varphi))| \leq tk, \quad \text{a.e. in } \{|\gamma(u)| < h\},
\]
for every \(0 < t \leq 1\). This proves (4.46) and therefore (4.45).

Moreover \(|\gamma(T_h(u) - tT_k(u - \varphi))| \leq h + tk\) a.e. on \(\partial \Omega\) and therefore by assumption (2.6) and (4.45), we get
\[
|((B_1) + (B_2))|
\]
\[
\leq \alpha_2 \|b\|_{L^\infty(\partial \Omega)} k \int_{\{|\gamma(u)| \geq h\}} \left(1 + |\gamma(T_h(u) - tT_k(u - \varphi))|^{p-1}\right)
\times |\gamma(T_h(u) - tT_k(u - \varphi))|^{\frac{p-1}{r}} d\mathcal{H}_{N-1}(\sigma)
\leq \alpha_2 k \|b\|_{L^\infty(\partial \Omega)} \int_{\{|\gamma(u)| \geq h\}} \left(1 + (h + tk)^{\frac{p-1}{r}}\right) (h + tk)^{\frac{p-1}{r}} d\mathcal{H}_{N-1}(\sigma)
\leq \alpha_2 k \|b\|_{L^\infty(\partial \Omega)} \int_{\{|\gamma(u)| \geq h\}} \left(1 + (|\gamma(u)| + tk)^{\frac{p-1}{r}}\right)
\times (|\gamma(u)| + tk)^{\frac{p-1}{r}} d\mathcal{H}_{N-1}(\sigma).
\]

We deduce that
\[
\lim_{h \to +\infty} ((B_1) + (B_2)) = 0.
\]
(4.47)
Now we evaluate \((B_3)\). By using the property 6) of Proposition 3.6 as above, we have
\[
\gamma(T_h(u) - tT_k(u - \varphi)) = \gamma(u - tT_k(u - \varphi)), \quad \text{a.e. in } \{|\gamma(u)| < h\}.
\]
Therefore we get
\[
(B_3) = -t \int_{\{|\gamma(u-T_h(u)+tT_k(u-\varphi))|<k\}\cap\{|\gamma(u)|<h\}} ba(-\gamma(u - tT_k(u - \varphi))n) n \\
\times \gamma(T_k(u - \varphi))d\mathcal{H}_{N-1}(\sigma).
\]
By the property 4) of Proposition 3.6, \(|\gamma(u - tT_k(u - \varphi))| \leq |\gamma(u)| + tk\), so that \(|\gamma(u - tT_k(u - \varphi))|^{p-1}\) is integrable on \(\partial\Omega\), as well as \(|\gamma(u)|^{p-1}\) is, the integrand in the above equality is integrable on \(\partial\Omega\) and
\[
\lim_{h \to +\infty} (B_3) = -t \int_{\partial\Omega} ba(-\gamma(u - tT_k(u - \varphi))n) n \\
\times \gamma(T_k(u - \varphi))d\mathcal{H}_{N-1}(\sigma).
\]
Combining (4.41), (4.42), (4.43), (4.44), (4.47) and (4.48), we have
\[
t \int_{\Omega} a(\nabla u - t\nabla T_k(u - \varphi)) \nabla T_k(u - \varphi) \, dx \\
(4.49) \quad - t \int_{\partial\Omega} ba(-\gamma(u - tT_k(u - \varphi))n) n \gamma(T_k(u - \varphi)) \, d\mathcal{H}_{N-1}(\sigma) \\
\leq t \int_{\Omega} fT_k(u - \varphi) \, dx + t \int_{\Omega} g\nabla T_k(u - \varphi) \, dx.
\]
By virtue of the property 4) of Proposition 3.6, \(\gamma(u - tT_k(u - \varphi))\) tends to \(\gamma(u)\) almost everywhere, when \(t\) goes to zero. Therefore we divide (4.49) by \(t\), then we let \(t\) go to zero and, by Fatou lemma, we obtain (2.21).

4.4 Uniqueness of the entropy solution \(u\) of (2.19)

We prove the uniqueness of the entropy solution \(u\) of Problem (2.19) by adapting the arguments used in [3].

Let \(u_1\) and \(u_2\) be two entropy solutions of (2.19).
We choose \( \varphi = T_h(u_2) \) in the inequality (2.21) satisfied by \( u_1 \) and \( \varphi = T_h(u_1) \) in the inequality (2.21) satisfied by \( u_2 \). By adding the resulting inequalities, we get

\[
\int_{\{u_1 - T_h(u_2) < k\}} a(\nabla u_1)(\nabla u_1 - \nabla T_h(u_2)) \, dx \\
+ \int_{\{u_2 - T_h(u_1) < k\}} a(\nabla u_2)(\nabla u_2 - \nabla T_h(u_1)) \, dx \\
- \int_{\partial \Omega} b a(-\gamma(u_1)n) n \gamma(T_k(u_1 - T_h(u_2))) \, d\mathcal{H}_{N-1}(\sigma) \\
- \int_{\partial \Omega} b a(-\gamma(u_2)n) n \gamma(T_k(u_2 - T_h(u_1))) \, d\mathcal{H}_{N-1}(\sigma)
\]

(4.50)

\[
\leq \int_{\Omega} f T_k(u_1 - T_h(u_2)) \, dx + \int_{\Omega} f T_k(u_2 - T_h(u_1)) \, dx \\
+ \int_{\{u_1 - T_h(u_2) < k\}} g(\nabla u_1 - \nabla T_h(u_2)) \, dx \\
+ \int_{\{u_2 - T_h(u_1) < k\}} g(\nabla u_2 - \nabla T_h(u_1)) \, dx.
\]

Let us denote by \((I_1)\) to \((I_8)\) the eight integrals in (4.50). As in [3] (Section 5), we can prove that We prove that

\[
(I_1) + (I_2) \geq \int_{A_0} (a(\nabla u_1) - a(\nabla u_2))(\nabla u_1 - \nabla u_2) + \omega(h),
\]

(4.51)

where

\[
A_0 = \{ x \in \Omega : |u_1 - u_2| < k, |u_1| < h, |u_2| < h \},
\]

and \( \omega(h) \) denotes a function which tends to zero when \( h \) goes to infinity. Moreover, we obtain

\[
(I_5) + (I_6) = \omega(h)
\]

(4.52)

\[
(I_7) + (I_8) = \omega(h).
\]

(4.53)

It remains to evaluate the terms \((I_3)\) and \((I_4)\) on the boundary of \( \Omega \), that is

\[
(I_3) + (I_4) \geq \omega(h).
\]

(4.54)

To this aim let us consider the partition of \( \partial \Omega \) into the sets

\[
B_0 = \{ \sigma \in \partial \Omega : |\gamma(u_1 - T_h(u_2))| < k, |\gamma(u_1)| < h, |\gamma(u_2)| < h \},
\]

(4.55)

\[
B_1 = \{ \sigma \in \partial \Omega : |\gamma(u_1 - T_h(u_2))| \geq k, |\gamma(u_1)| < h, |\gamma(u_2)| < h \},
\]

\[
B_2 = \{ \sigma \in \partial \Omega : |\gamma(u_2)| \geq h \},
\]

(4.56)

\[
B_3 = \{ \sigma \in \partial \Omega : |\gamma(u_1)| \geq h \}.
\]

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We firstly observe that
\[
\left| \int_{B_2} b a(-\gamma(u_1)n) n \gamma(T_k(u_1 - T_h(u_2))) d\sigma \right| \\
\leq k \int_{\{\gamma(u_2) \geq h\}} b |a(-\gamma(u_1)n)| d\sigma = \omega(h).
\]

The same argument proves that also the integral on $B_3$ of the same function tends to zero when $h$ goes to infinity. Therefore we obtain
\[(4.57) \quad (I_3) = - \int_{B_0 \cup B_1} b a(-\gamma(u_1)n) n \gamma(T_k(u_1 - T_h(u_2))) d\sigma + \omega(h).\]

Moreover we observe that, since $u_1 - T_h(u_2) = T_h(u_1) - T_h(u_2)$ in $\{ |u_1| < h \}$, property 6) of Proposition 3.6 implies that
\[
\gamma(u_1 - T_h(u_2)) = \gamma(T_h(u_1) - T_h(u_2)) \quad \text{in } \{ |\gamma(u_1)| < h \}.
\]

Therefore if $\sigma \in \{ |\gamma(u_1)| < h \} \cap \{ |\gamma(u_2)| < h \}$, we get, by using property 5) of Proposition 3.6
\[(4.58) \quad \gamma(u_1 - T_h(u_2)) = \gamma(T_h(u_1)) - \gamma(T_h(u_2)) = T_h(\gamma(u_1)) - T_h(\gamma(u_2))
\]
\[
= \gamma(u_1) - \gamma(u_2).
\]

We deduce that, for $\sigma \in B_0$,
\[
\gamma(T_k(u_1 - T_h(u_2))) = T_k(\gamma(u_1) - T_h(u_2)) = \gamma(u_1) - T_h(u_2) = \gamma(u_1) - \gamma(u_2).
\]

By (4.57), (4.58) and the above identity, we get
\[(4.59) \quad (I_3) = - \int_{B_0} b a(-\gamma(u_1)n) n (\gamma(u_1) - \gamma(u_2)) d\sigma \]
\[
- \int_{B_1 \cap \{\gamma(u_1) - \gamma(u_2) > 0\}} b a(-\gamma(u_1)n) n T_k(\gamma(u_1) - \gamma(u_2)) d\sigma \\
- \int_{B_1 \cap \{\gamma(u_1) - \gamma(u_2) < 0\}} b a(-\gamma(u_1)n) n T_k(\gamma(u_1) - \gamma(u_2)) d\sigma \\
+ \omega(h).
\]

By using the same arguments, we evaluate $(I_4)$
\[(4.60) \quad (I_4) = \int_{B_0} b a(-\gamma(u_2)n) n (\gamma(u_1) - \gamma(u_2)) d\sigma \]
\[
+ \int_{B_1 \cap \{\gamma(u_1) - \gamma(u_2) > 0\}} b a(-\gamma(u_2)n) n T_k(\gamma(u_1) - \gamma(u_2)) d\sigma \\
+ \int_{B_1 \cap \{\gamma(u_1) - \gamma(u_2) < 0\}} b a(-\gamma(u_2)n) n T_k(\gamma(u_1) - \gamma(u_2)) d\sigma \\
+ \omega(h).
\]
Now we add (4.59) and (4.60):

$$\begin{align*}
(I_3) + (I_4) \\
= - \int_{B_0} b \left[ a(-\gamma(u_1)n) n - a(-\gamma(u_2)n) n \right] (\gamma(u_1) - \gamma(u_2)) \, d\sigma \\
- \int_{B_1 \cap \{ \gamma(u_1) - \gamma(u_2) > 0 \}} b \left[ a(-\gamma(u_1)n) n - a(-\gamma(u_2)n) n \right] \times \\
\times T_k (\gamma(u_1) - \gamma(u_2)) \, d\sigma \\
- \int_{B_1 \cap \{ \gamma(u_1) - \gamma(u_2) < 0 \}} b \left[ a(-\gamma(u_1)n) n - a(-\gamma(u_2)n) n \right] \times \\
\times T_k (\gamma(u_1) - \gamma(u_2)) \, d\sigma + \omega(h) \\
= J_1 + J_2 + J_3 + \omega(h). 
\end{align*}$$

(4.61)

By assumption (2.7) on the monotonicity of $a$,

$$- [a(-\gamma(u_1)n) n - a(-\gamma(u_2)n) n] (\gamma(u_1) - \gamma(u_2)) \geq 0.$$ 

(4.62)

Therefore $J_1$ in (4.61) is nonnegative. Moreover, since $\gamma(u_1) - \gamma(u_2) > 0$ in $B_1 \cap \{ \gamma(u_1) - \gamma(u_2) > 0 \}$, we get by using (2.7)

$$- [a(-\gamma(u_1)n) n - a(-\gamma(u_2)n) n] > 0, \quad \text{in } B_1 \cap \{ \gamma(u_1) - \gamma(u_2) > 0 \},$$

and then $J_2 > 0$. Anologously it results $J_3 > 0$ and this concludes the proof of (4.61).

Combining (4.50), (4.51), (4.55), (4.53), and (4.54), we have

$$\int_{A_0} [a(\nabla u_1) - a(\nabla u_2)] (\nabla u_1 - \nabla u_2) \leq \omega(h),$$

with $A_0$ and $\omega(h)$ depending on both $h$ and $k$. By letting $h$ and $k$ tend to infinity successively and by using assumption (2.7) on the monotonicity of $a$, this implies that $u_1 = u_2$ a.e. in $\Omega$. 

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5 Proof of Theorem 2.6

By assumption (2.25), the \textit{a priori} estimates of Section 4.1 are still valid, as well as the results of Sections 4.2 and 4.3. Moreover in this case it is easy to pass to the limit in (4.19). It remains to prove that, under the assumption (2.25), $\gamma(u) = 0$.

By Hölder inequality,

$$
\int_{\partial \Omega} |\gamma(T_k(u^\varepsilon))| d\mathcal{H}_{N-1}(\sigma) = \int_{\partial \Omega} (b^\varepsilon)^{-\frac{1}{p'}} (b^\varepsilon)^{\frac{1}{p'}} |\gamma(T_k(u^\varepsilon))| d\mathcal{H}_{N-1}(\sigma)
$$

(5.1)

$$
\leq \left( \int_{\partial \Omega} \left( \frac{1}{b^\varepsilon} \right)^{p'-1} d\mathcal{H}_{N-1}(\sigma) \right)^{\frac{1}{p'}} \left( \int_{\partial \Omega} b^\varepsilon |\gamma(T_k(u^\varepsilon))|^p d\mathcal{H}_{N-1}(\sigma) \right)^{\frac{1}{p}}.
$$

But from (4.13) and (4.10), we can prove that the last integral is bounded:

$$
\int_{\partial \Omega} b^\varepsilon |\gamma(T_k(u^\varepsilon))|^p d\mathcal{H}_{N-1}(\sigma)
$$

(5.2)

$$
\leq \int_{\partial \Omega} b^\varepsilon(\sigma)(h^\varepsilon(\sigma))^{p-1} \int_0^\lambda |\nabla T_k(u^\varepsilon)(\sigma + t\sigma)|^p dt d\mathcal{H}_{N-1}(\sigma)
\leq C \int_{\Omega^\varepsilon} |\nabla T_k(u^\varepsilon)|^p dx \leq C.
$$

Then it follows from (5.1), (5.2), (2.26) and (4.17) that

$$
\int_{\partial \Omega} |\gamma(T_k(u))| d\mathcal{H}_{N-1}(\sigma) = \lim_{\varepsilon \to 0} \int_{\partial \Omega} |\gamma(T_k(u^\varepsilon))| d\mathcal{H}_{N-1}(\sigma) = 0,
$$

for any $k$. By definition of the trace, this implies that

$$
\gamma(u) = \lim_{k \to \infty} \gamma(T_k(u)) = 0.
$$

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References


