Strong stability results for solutions of elliptic equations with power-like lower order terms and measure data

Luigi Orsina¹ and Alain Prignet²

Abstract

Let $u_n$ be the sequence of solutions of

$$\begin{cases} 
-\text{div}(a(x, u_n, \nabla u_n)) + |u_n|^{q-1}u_n = f_n & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\Omega$ is a bounded set in $\mathbb{R}^N$ and $f_n$ is a sequence of functions which is strongly convergent to a function $f$ in $L^1_{\text{loc}}(\Omega \setminus K)$, with $K$ a compact in $\Omega$ of zero $r$-capacity; no assumptions are made on the sequence $f_n$ on the set $K$. We prove that if $a$ has growth of order $p-1$ with respect to $\nabla u$ ($p > 1$), and if $q > r(p-1)/(r-p)$, then $u_n$ converges to $u$, the solution of the same problem with datum $f$, thus extending to the nonlinear case a well-known result by H. Brezis.

1 Introduction

Let us recall the following result due to H. Brezis (see [4]).

**Theorem 1.1** Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $N > 2$, with $0 \in \Omega$, let $f$ be a function in $L^1(\Omega)$, and let $\{f_n\}$ be a sequence of $L^\infty(\Omega)$ functions such that

$$\lim_{n \to +\infty} \int_{\Omega \setminus B_\rho(0)} |f_n - f| \, dx = 0, \quad \forall \rho > 0. \quad (1.1)$$

Let $\{u_n\}$ be the sequence of solutions of the following nonlinear elliptic problems

$$\begin{cases} 
-\Delta u_n + |u_n|^{q-1}u_n = f_n & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}$$

¹Dipartimento di Matematica, Università di Roma “La Sapienza”, P.le A. Moro 5, 00185 Roma, Italia
²Mathématiques, Université d’Orléans, Rue de Chartres, 45067 Orléans cedex 2, France
with \( q \geq \frac{N}{N-2} \). Then \( u_n \) converges to the unique solution \( u \) of the equation

\[-\Delta u + |u|^{q-1}u = f.\]

Thus, even though there are no assumptions on \( f_n \) near the origin, the solution of the problem with datum \( f \) is “stable”, in the sense that it is recovered as limit of the sequence \( u_n \) of solutions of (1.2). The assumption on \( q \) is crucial to obtain such a result, and the limiting value \( N/(N-2) \) depends on the fact that the origin is a subset of \( \Omega \) of zero \( N \)-capacity, and on the fact the differential operator is defined on \( H^1_0(\Omega) \).

In this paper, we are going to extend this result to more general operators and sets. Before quoting our result, let us give the assumptions on the various objects we will deal with.

Let \( \Omega \) be a open, bounded subset of \( \mathbb{R}^N \), \( N > 2 \), and let \( p \) be a real number such that \( 1 < p < N \). Let \( a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) be a Carathéodory function (i.e., \( a(\cdot, s, \xi) \) is measurable on \( \Omega \) for every \((s, \xi)\) in \( \mathbb{R} \times \mathbb{R}^N \), and \( a(x, \cdot, \cdot) \) is continuous on \( \mathbb{R} \times \mathbb{R}^N \) for almost every \( x \) in \( \Omega \)) such that

\[ a(x, s, \xi) \cdot \xi \geq \alpha |\xi|^p, \tag{1.3} \]

for almost every \( x \) in \( \Omega \), for every \((s, \xi)\) in \( \mathbb{R} \times \mathbb{R}^N \), where \( \alpha \) is a positive constant;

\[ |a(x, s, \xi)| \leq \beta |\xi|^{p-1}, \tag{1.4} \]

for almost every \( x \) in \( \Omega \), for every \((s, \xi)\) in \( \mathbb{R} \times \mathbb{R}^N \), where \( \beta \) is a positive constant;

\[ (a(x, s, \xi) - a(x, s, \xi')) \cdot (\xi - \xi') > 0, \tag{1.5} \]

for almost every \( x \) in \( \Omega \), for every \( s \) in \( \mathbb{R} \), for every \( \xi \) and \( \xi' \) in \( \mathbb{R}^N \), with \( \xi \neq \xi' \).

Recall that (see [1]) if \( f \) belongs to \( L^1(\Omega) \) and \( q > 0 \), then there exists at least a solution \( u \) in the sense of distributions of the problem

\[
\begin{align*}
-\text{div}(a(x, u, \nabla u)) + |u|^{q-1}u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

If \( f \) belongs to \( L^\infty(\Omega) \), then any solution \( u \) of (1.6) is in \( W^{1,p}_0(\Omega) \) and belongs to \( L^\infty(\Omega) \) as well; this easily follows from the fact that the lower order term has the same sign of the solution, and from well-known regularity results on
the solutions of problem (1.6) without the lower order term (see [8] for the linear case, and [3] for the nonlinear one).

If \(1 < r \leq N\), and \(K\) is a compact subset of \(\Omega\), the \(r\)-capacity of \(K\) is defined as

\[
\text{cap}_r(K, \Omega) = \inf \left\{ \int_\Omega |\nabla \psi|^r \, dx : \psi \in C_c^\infty(\Omega), \psi \geq \chi_K \right\},
\]

where \(\chi_K\) is the characteristic function of the sets \(K\), and, as usual, \(\inf\emptyset = +\infty\).

If \(s\) is a real number, we define

\[
s^+ = \max(s, 0), \quad s^- = \max(-s, 0) = (-s)^+, \quad t(s) = \max(-k, \min(k, s))\]

so that \(s = s^+ - s^-\) and \(|s| = s^+ + s^-\). We also define

\[
T_k(s) = \max(-k, \min(k, s)), \quad G_k(s) = s - T_k(s).
\]

Since we will deal with functions \(u\) that may not belong to Sobolev spaces, we need to give, following [1], a suitable definition of “gradient”.

**Definition 1.2** Let \(u\) be a measurable function on \(\Omega\) which is finite almost everywhere, and is such that \(T_k(u) \in W^{1,p}_0(\Omega)\) for every \(k > 0\). Then (see [1], Lemma 2.1) there exists a unique measurable function \(v : \Omega \to \mathbb{R}^N\) such that

\[
\nabla T_k(u) = v \chi_{\{|u| \leq k\}}, \quad \text{almost everywhere in} \ \Omega, \ \text{for every} \ k > 0.
\]

We will define the gradient of \(u\) as the function \(v\), and we will denote it by \(v = \nabla u\). If \(u\) belongs to \(W^{1,1}_0(\Omega)\), then this gradient coincides with the usual gradient in distributional sense.

We can now state the main result of this paper.

**Theorem 1.3** Let \(p < r \leq N\), let \(f = f^+ - f^-\) be a function in \(L^1(\Omega)\) and let \(K^+\) and \(K^-\) be two disjoint compact subsets of \(\Omega\) of zero \(r\)-capacity. Let \(f_n^\oplus\) and \(f_n^\ominus\) be two sequences of nonnegative \(L^\infty(\Omega)\) functions such that

\[
\lim_{n \to +\infty} \int_{\Omega \setminus K^+} |f_n^\oplus - f^+| \, dx = 0, \quad \lim_{n \to +\infty} \int_{\Omega \setminus K^-} |f_n^\ominus - f^-| \, dx = 0, \quad (1.7)
\]
for every neighbourhood $I(K^+) \text{ of } K^+$, and $I(K^-) \text{ of } K^-$. Let us denote $f_n = f_n^\oplus - f_n^\ominus$. Let

$$q > \frac{r(p - 1)}{r - p},$$

(1.8)

and let $u_n$ be a solution of

$$\begin{cases}
-\text{div} (a(x, u_n, \nabla u_n)) + |u_n|^{q-1} u_n = f_n \quad \text{in } \Omega, \\
u_n = 0 \quad \text{on } \partial \Omega.
\end{cases}$$

(1.9)

Then, up to subsequences still denoted by $u_n$, $u_n$ converges to a solution in the sense of distributions of the problem

$$\begin{cases}
-\text{div} (a(x, u, \nabla u)) + |u|^{q-1} u = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega.
\end{cases}$$

(1.10)

Moreover, $u$ belongs to $L^q(\Omega)$, and $|\nabla u|^{p-1}$ belongs to $L^r(\Omega)$ ($\nabla u$ is defined in Definition 1.2).

Some remarks are in order.

**Remark 1.4** The sequences $f_n^\oplus$ and $f_n^\ominus$ are not necessarily the positive and negative parts of $f_n$, since their supports may not be disjoint. This explains the “nonstandard” notation adopted.

Observe that the meaning of (1.7) is that the sequences $f_n^\oplus$ and $f_n^\ominus$ are strongly convergent in $L^1_{\text{loc}}(\Omega\setminus K^+)$ and $L^1_{\text{loc}}(\Omega\setminus K^-)$, respectively, so that we do not impose conditions on them near $K^+$ or $K^-$ (apart from positiveness).

If, for example, $f \equiv 0$, $K^+ = \{x^+\}$ and $K^- = \{x^-\}$, with $x^+$ and $x^-$ two distinct points in $\Omega$, and $r = N$, then admissible sequences of functions are

$$f_n^\oplus = n^n \chi_{B(x^+,1/n)}, \quad f_n^\ominus = n^n \chi_{B(x^-,1/n)},$$

where $B(x,\delta)$ is the ball of center $x$ and radius $\delta$. These sequences are not bounded in $L^1(\Omega)$, so that one cannot expect to have a priori estimates on the sequence $u_n$. Anyway, the result of Theorem 1.3 states in this case that the solution $u$ of problem (1.10) (which is $u \equiv 0$) is “stable” due to the fact that the sequence $f_n = f_n^\oplus - f_n^\ominus$ “explodes” on sets of zero $N$-capacity, and to the presence of the power-like lower order term.
Remark 1.5 If $p = 2$, $a(x, s, \xi) = \xi$, $K^+ = \{0\}$, $K^- = \emptyset$, and $r = N$, the result of Theorem 1.3 can be compared with that of Theorem 1.1. Observe that, in this case, our result is weaker, since we have sign assumptions on the sequence $f_n$ (as explained before), and we require $q$ to be strictly larger than $N/(N - 2)$.

Remark 1.6 A particular case of Theorem 1.3 is when the sequence $f_n^\omega$ is convergent to $\lambda^+$ in the tight topology of measures, with $\lambda^+$ a bounded Radon measure concentrated on a set $K^+$ of zero $r$-capacity (and similar assumptions for $f_n^\ominus$). In this case, Theorem 1.3 states that the sequence $u_n$ converges to zero. This is exactly the same result which has been proved (under the same assumptions on $q$ and $r$) in [7].

Remark 1.7 The result of Theorem 1.3 can also be seen as a result of removable singularities for problem (1.10). Indeed it states that sets of zero $r$-capacity are not “seen” by the equation if $q$ is large enough. There are several results in the literature concerning the problem of removable singularities for problem (1.10), among which we quote again [4]; see also [9] (for zero $r$-capacity sets in the linear case) and [10] (for zero $N$-capacity sets in the nonlinear case).

2 Proof of the main theorem

Before giving the proof of Theorem 1.3, we recall the following result which has been proved (in a slightly different version) in [5], Lemma 5.1.

Lemma 2.1 Let $K^+$ and $K^-$ be two disjoint compact subsets of $\Omega$ of zero $r$-capacity, with $p < r \leq N$. Then, for every $\delta > 0$ there exist $A^\delta_+$ and $A^\delta-$, two disjoint open subsets of $\Omega$, and $\psi^+_\delta$ and $\psi^-_\delta$ in $C^\infty_c(\Omega)$ such that

\begin{align}
0 &\leq \psi^+_\delta \leq 1, \quad 0 \leq \psi^-_\delta \leq 1, \quad \text{in } \Omega, \tag{2.1} \\
\psi^+_\delta &\equiv 1 \quad \text{on } K^+, \quad \psi^-_\delta &\equiv 1 \quad \text{on } K^-, \tag{2.2} \\
\text{supp}(\psi^+_\delta) &\subseteq A^\delta_+, \quad \text{supp}(\psi^-_\delta) \subseteq A^\delta-, \tag{2.3} \\
\int_\Omega |\nabla \psi^+_\delta|^r \, dx &\leq \delta, \quad \int_\Omega |\nabla \psi^-_\delta|^r \, dx \leq \delta, \tag{2.4} \\
\text{meas } (A^\delta_+) &\leq \delta, \quad \text{meas } (A^\delta-) \leq \delta. \tag{2.5}
\end{align}
If $A_+^\delta$, $A_-^\delta$, $\psi^+_\delta$ and $\psi^-_\delta$ are as in Lemma 2.1, we define
\[
A_\delta = A_+^\delta \cup A_-^\delta, \quad \psi_\delta = \psi^+_\delta + \psi^-_\delta.
\]

**Proof of Theorem 1.3.** The proof is divided in several steps.

**Step 1.** Let $k > 0$ be fixed. Let $s > 1$ to be fixed later, and choose $T_k(u_n)(1 - \psi_\delta)^s$ as test function in (1.9). We obtain
\[
\begin{align*}
\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n) (1 - \psi_\delta)^s \, dx & \quad \text{(A)} \\
- s \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla \psi_\delta T_k(u_n) (1 - \psi_\delta)^{s-1} \, dx & \quad \text{(B)} \\
+ \int_\Omega |u_n|^{q-1} u_n T_k(u_n) (1 - \psi_\delta)^s \, dx & \quad \text{(C)} \\
= \int_\Omega f_n^\ominus T_k(u_n)(1 - \psi_\delta)^s \, dx & \quad \text{(D)} \\
- \int_\Omega f_n^\ominus T_k(u_n)(1 - \psi_\delta)^s \, dx. & \quad \text{(E)}
\end{align*}
\]

We now define $\mu_\delta = (1 - \psi_\delta)^s \, dx$, that is to say the measure such that
\[
\mu_\delta(E) = \int_E (1 - \psi_\delta)^s \, dx,
\]
for every Borel set $E$ in $\Omega$.

Using (1.3), we have
\[
(A) \geq \alpha \int_\Omega |\nabla T_k(u_n)|^p (1 - \psi_\delta)^s \, dx.
\]
Moreover, since the integrand function in (C) is nonnegative (recall that $T_k(s)$ has the same sign as $s$),
\[
(C) \geq \int_{\{|u_n| \geq k\}} |u_n|^{q-1} T_k(u_n)(1 - \psi_\delta)^s \, dx \geq k^{q+1} \mu_\delta(\{|u_n| \geq k\}).
\]
We now define
\[
c_\delta(n) = \int_\Omega (f_n^\ominus + f_n^\ominus)(1 - \psi_\delta)^s \, dx. \tag{2.6}
\]
Thus,
\[
|(D)| + |(E)| \leq k c_\delta(n).
\]

6
Finally, for (B) we have, using (1.4) and Young inequality,

\[ |(B)| \leq sk \int_\Omega |\nabla u_n|^{p-1} (|\nabla \psi^+_t| + |\nabla \psi^-_s|) (1 - \psi_\delta)^{s-1} \, dx \]
\[ \leq c_1 k \int_\Omega (|\nabla \psi^+_t| + |\nabla \psi^-_s|) \, dx \]
\[ + c_2 k \int_\Omega |\nabla u_n|^{(p-1)r'} (1 - \psi_\delta)^{(s-1)r'} \, dx. \]  

(2.7)

If we define

\[ I_{n, \delta} = \int_\Omega |\nabla u_n|^{(p-1)r'} (1 - \psi_\delta)^{(s-1)r'} \, dx, \]  

(2.8)

putting together the results on (A)–(D), we have, recalling (2.4),

\[ \int_\Omega |\nabla T_k(u_n)|^p (1 - \psi_\delta)^s \, dx + k^{q+1} \mu_\delta(\{|u_n| \geq k\}) \leq c k (\delta + c_\delta(n) + I_{n, \delta}). \]  

(2.9)

Let now \( \rho > 0 \) be fixed. Then we have, using twice (2.9)

\[ \mu_\delta(\{|\nabla u_n| \geq \rho\}) = \mu_\delta(\{|\nabla u_n| \geq \rho \}) + \mu_\delta(\{|u_n| \geq k\}) \]
\[ \leq \mu_\delta(\{|\nabla T_k(u_n)| \geq \rho\}) + \mu_\delta(\{|u_n| \geq k\}) \]
\[ \leq \frac{1}{\rho^p} \int_\Omega |\nabla T_k(u_n)|^p (1 - \psi_\delta)^s \, dx + \frac{c k (\delta + c_\delta(n) + I_{n, \delta})}{k^{q+1}} \]
\[ \leq c (\delta + c_\delta(n) + I_{n, \delta}) \left( \frac{k}{\rho^p} + \frac{1}{k^q} \right). \]

We now minimize the right hand side on \( k > 0 \), to find that there exists a nonnegative constant \( c_q \) such that

\[ \mu_\delta(\{|\nabla u_n| \geq \rho\}) \leq \frac{c_q (\delta + c_\delta(n) + I_{n, \delta})}{\rho^{\frac{pq}{q+1}}}. \]  

(2.10)

Let \( \theta \) be such that

\[ (p - 1) r' < \theta < \frac{pq}{q+1}. \]  

(2.11)

It is easy to see that assumption (1.8) on \( q \) implies that such a \( \theta \) exists. Starting from (2.10) we then have

\[ \int_\Omega |\nabla u_n|^\theta (1 - \psi_\delta)^s \, dx \leq c_\theta c_q (c_\Omega + \delta + c_\delta(n) + I_{n, \delta}), \]  

(2.12)
for some nonnegative constant \( c_\theta \) and \( c_\Omega \). Since \( \theta > (p - 1)r' \), we have, by H"older inequality,
\[
I_{n,\delta} = \int_\Omega |\nabla u_n|^{(p-1)r'} (1 - \psi_\delta)^{(s-1)r'} dx \\
\leq c_\Omega \left( \int_\Omega |\nabla u_n|^\theta (1 - \psi_\delta)^{(s-1)\frac{\theta}{r+1}} dx \right)^{\frac{(p-1)r'}{\theta}}.
\]

We now choose \( s \) such that \( \frac{(s-1)\theta}{p-1} = s \), that is
\[
s = \frac{\theta}{\theta - p + 1}. \tag{2.13}
\]

Since \( \theta > p - 1 \) by assumption (2.11), then \( s > 1 \). We thus have, by (2.12),
\[
I_{n,\delta} \leq c_\Omega \left( \int_\Omega |\nabla u_n|^\theta (1 - \psi_\delta)^s dx \right)^{\frac{(p-1)r'}{\theta}} \\
\leq c_\Omega (c_\theta c_q (c_\Omega + \delta + c_\delta(n) + I_{n,\delta}))^{\frac{(p-1)r'}{\theta}}.
\]

Since \( \frac{(p-1)r'}{\theta} < 1 \), the previous estimate implies that there exists a continuous function \( \Psi \), depending on \( \Omega, \theta \) and \( q \), but not on \( n \), such that
\[
I_{n,\delta} = \int_\Omega |\nabla u_n|^{(p-1)r'} (1 - \psi_\delta)^{(s-1)r'} dx \leq \Psi(\delta + c_\delta(n)). \tag{2.14}
\]

We now observe that by (2.2) and (2.3), \( 1 - \psi_\delta \) is zero both on a neighbourhood of \( K^+ \) and of \( K^- \). Thus, by assumption (1.7), for \( \delta \) fixed, \( c_\delta(n) \) (defined in (2.6)) is bounded with respect to \( n \); this implies that \( I_{n,\delta} \) is bounded with respect to both \( n \) and \( \delta \). Hence, from (2.9), (2.7) and (2.14) it follows that
\[
\int_\Omega |\nabla T_k(u_n)|^p (1 - \psi_\delta)^s dx \leq c_\delta k, \tag{2.15}
\]
\[
\int_\Omega |u_n|^{q-1} u_n T_k(u_n) (1 - \psi_\delta)^s dx \leq c_\delta k, \tag{2.16}
\]
and
\[
\int_\Omega |\nabla u_n|^{p-1} (|\nabla \psi_\delta^+| + |\nabla \psi_\delta^-|) (1 - \psi_\delta)^{r-1} dx \leq c_\delta, \tag{2.17}
\]
for some nonnegative constant \( c_\delta \) independent on \( n \) and \( k \). We now choose as test functions \( T_k(u_n^+)(1 - \psi_\delta^+)^s \) and \( -T_k(u_n^-)(1 - \psi_\delta^-)^s \); performing the same
estimates, we obtain
\[
\int_{\Omega} |\nabla T_k(u_n^+)|^p (1 - \psi_\delta^+)^s \, dx \leq c_\delta k, \\
\int_{\Omega} |\nabla T_k(u_n^-)|^p (1 - \psi_\delta^-)^s \, dx \leq c_\delta k, 
\]
and
\[
\int_{\Omega} |u_n^+|^{q-1} u_n^+ T_k(u_n^+) (1 - \psi_\delta^+)^s \, dx \leq c_\delta k, \\
\int_{\Omega} |u_n^-|^{q-1} u_n^- T_k(u_n^-) (1 - \psi_\delta^-)^s \, dx \leq c_\delta k, 
\]
(2.19)
since the only change is that now (E) \leq 0 or (D) \leq 0 respectively.

**Step 2.** We fix \( k > 0 \), let \( s \) as in (2.13), and choose
\[
(k - T_k(u_n^+)) (1 - (1 - \psi_\delta^+)^s)
\]
as test function in (1.9). We obtain
\[
- \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n^+) (1 - (1 - \psi_\delta^+)^s) \, dx \quad \text{ (A)}
\]
\[
+ s \int_{\Omega} a(x, u_n, \nabla u_n) \nabla \psi_\delta^+(k - T_k(u_n^+)) (1 - \psi_\delta^+)^{s-1} \, dx \quad \text{ (B)}
\]
\[
+ \int_{\Omega} |u_n|^{q-1} u_n (k - T_k(u_n^+)) (1 - (1 - \psi_\delta^+)^s) \, dx \quad \text{ (C)}
\]
\[
= \int_{\Omega} f_n^\oplus(k - T_k(u_n^+)) (1 - (1 - \psi_\delta^+)^s) \, dx \quad \text{ (D)}
\]
\[
- \int_{\Omega} f_n^\ominus(k - T_k(u_n^+)) (1 - (1 - \psi_\delta^+)^s) \, dx. \quad \text{ (E)}
\]

The term (D) is nonnegative since \( f_n^\oplus \geq 0 \) and by assumption (2.1) on \( \psi_\delta^+ \); for (E) we have, by (1.7),
\[
-(E) \leq c_\delta k,
\]
for some nonnegative constant \( c_\delta \) independent on \( n \), since the function \( 1 - (1 - \psi_\delta^+)^s \) is zero on a neighbourhood of \( K^- \) by (2.2) and (2.3). As for (C) we have, since \( k - T_k(u_n^+) \) is zero on the set \( \{ u_n > k \} \), is bounded by \( k \) and is nonnegative,
\[
(C) \leq \int_{\{ 0 \leq u_n \leq k \}} |u_n|^{q-1} u_n (k - T_k(u_n^+)) (1 - (1 - \psi_\delta^+)^s) \, dx \leq c_\delta k^{q+1}.
\]
For (B) we have, using (1.4), using that $\psi_{\delta}^{+} = \psi_{\delta}$ on the support of $\psi_{\delta}^{+}$ (see (2.3), and (2.17)

$$|(B)| \leq c_1 k \int_{\Omega} |\nabla u_n|^{|p-1|} |\nabla \psi_{\delta}^{+}|(1 - \psi_{\delta})^{s-1} \, dx \leq c_\delta k,$$

with $c_\delta$ independent on both $n$ and $k$. Since, using (1.3),

$$-(A) \geq \alpha \int_{\Omega} |\nabla T_k(u_n^+)|^p (1 - (1 - \psi_{\delta}^+)^s) \, dx,$$

we thus have proved that

$$\int_{\Omega} |\nabla T_k(u_n^+)|^p (1 - (1 - \psi_{\delta}^+)^s) \, dx \leq c_\delta k^{q+1},$$

(2.20)

for some nonnegative constant $c_\delta$ independent on $n$ and $k$. Using

$$(k + T_k(u_n^-))(1 - (1 - \psi_{\delta}^-)^s)$$

in (1.9), we can similarly prove that

$$\int_{\Omega} |\nabla T_k(u_n^-)|^p (1 - (1 - \psi_{\delta}^-)^s) \, dx \leq c_\delta k^{q+1},$$

(2.21)

for some nonnegative constant $c_\delta$ independent on $n$ and $k$. Putting (2.20), (2.21) together with (2.18), and then choosing $\delta = 1$ (for example), we have

$$\int_{\Omega} |\nabla T_k(u_n)|^p \, dx \leq ck^{q+1}.$$  

(2.22)

Thus, $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for every $k > 0$, and this means that, for every $k$ fixed, there exists a subsequence of $T_k(u_n)$ which is weakly convergent in $W_0^{1,p}(\Omega)$ to some function $v_k$.

**Step 3.** Consider now the set $\{|u_n| \geq k\}$, which we split as the union of two sets:

$$\{|u_n| \geq k\} \cap A_\delta \quad \text{and} \quad \{|u_n| \geq k\} \cap A_\delta^c.$$

The Lebesgue measure of $\{|u_n| \geq k\} \cap A_\delta$ is smaller than the measure of $A_\delta$, which is smaller than $2\delta$ by (2.5). We now estimate the measure of
\{ |u_n| \geq k \} \cap A_\delta^c. We have, since \( 1 - \psi_\delta \equiv 1 \) on \( A_\delta^c \), and by Poincaré inequality,

\[
\text{meas}(\{ |u_n| \geq k \} \cap A_\delta^c) \leq \frac{1}{k^p} \int_{\{ |u_n| \geq k \} \cap A_\delta^c} |T_k(u_n)|^p \, dx
\]
\[
= \frac{1}{k^p} \int_{\{ |u_n| \geq k \} \cap A_\delta^c} |T_k(u_n)|^p \left( 1 - \psi_\delta \right)^s \, dx
\]
\[
\leq \frac{1}{k^p} \int_\Omega |T_k(u_n) \left( 1 - \psi_\delta \right)^s \, dx
\]
\[
\leq \frac{c}{k^p} \int_\Omega \left| \nabla T_k(u_n) \left( 1 - \psi_\delta \right)^s \right|^p \, dx
\]
\[
\leq \frac{c}{k^p} \int_\Omega \left| \nabla \psi_\delta \right|^p \left( 1 - \psi_\delta \right)^s - p \left| T_k(u_n) \right|^p \, dx
\]
\[
\leq \frac{c}{k^p} \int_\Omega \left| \nabla \psi_\delta \right|^p \left( 1 - \psi_\delta \right)^{s - p} \left| T_k(u_n) \right|^p \, dx
\]

Observing that \( s > p \) by (2.13) (as it is easily seen starting from assumption (2.11) which implies \( \theta < p \)), we have

\[
\frac{c}{k^p} \int_\Omega \left| \nabla \psi_\delta \right|^p \left( 1 - \psi_\delta \right)^{s - p} \left| T_k(u_n) \right|^p \, dx \leq c \int_\Omega \left| \nabla \psi_\delta \right|^p \, dx.
\]

Since \( r > p \) by assumption, Hölder inequality yields

\[
\frac{c}{k^p} \int_\Omega \left| \nabla \psi_\delta \right|^p \left( 1 - \psi_\delta \right)^{s - p} \left| T_k(u_n) \right|^p \, dx \leq c \left( \int_\Omega \left| \nabla \psi_\delta \right|^r \, dx \right)^{\frac{p}{r}} \leq c \delta^{\frac{p}{r}}.
\]

Recalling (2.15), we have

\[
\text{meas}(\{ |u_n| \geq k \} \cap A_\delta^c) \leq \frac{c_\delta}{k^{p-1}} + c \delta^{\frac{p}{r}}.
\]

Thus

\[
\text{meas}(\{ |u_n| \geq k \}) \leq 2 \delta + \frac{c_\delta}{k^{p-1}} + \delta^{\frac{p}{r}},
\]

which implies

\[
\lim_{k \to +\infty} \sup_{n \in \mathbb{N}} \text{meas}(\{ |u_n| \geq k \}) \leq 2 \delta + \delta^{\frac{p}{r}}.
\]

Since \( \delta \) is arbitrary, we have proved that

\[
\lim_{k \to +\infty} \sup_{n \in \mathbb{N}} \text{meas}(\{ |u_n| \geq k \}) = 0.
\] (2.23)
From (2.23) it easily follows that \( u_n \) is a Cauchy sequence in measure. Indeed, if \( n \) and \( m \) belong to \( \mathbb{N} \), and \( \varepsilon > 0 \) we have, for every \( k > 0 \),

\[
\{ |u_n - u_m| \geq \varepsilon \} = \{ |u_n - u_m| \geq \varepsilon \} \cup \{ |u_n - u_m| \geq \varepsilon \} \\
= \{ |u_n - u_m| \geq \varepsilon \} \cup \{ |u_n| \leq k, |u_m| \leq k \} \cup \{ |u_n - u_m| \geq \varepsilon \} \\
\subseteq \{ |u_n| > k \} \cup \{ |T_k(u_n) - T_k(u_m)| > \varepsilon \} \cup \{ |u_m| > k \}.
\]

We first choose \( \bar{k} \) such that the measure of \( \{ |u_n| > \bar{k} \} \) and \( \{ |u_m| > \bar{k} \} \) is small (this can be done by (2.23)). Once \( \bar{k} \) is fixed, since the sequence \( T_{\bar{k}}(u_n) \) is bounded in \( W^{1,p}_0(\Omega) \), there exists a subsequence (still denoted by \( T_{\bar{k}}(u_n) \)) which is strongly compact in \( L^p(\Omega) \). This means, in particular, that the sequence \( T_{\bar{k}}(u_n) \) is a Cauchy sequence in measure. We then choose \( n \) and \( m \), such that the measure of the set \( \{ |T_k(u_n) - T_k(u_m)| > \varepsilon \} \) is small.

We thus have that (up to subsequences, still denoted by \( u_n \)) \( u_n \) converges almost everywhere in \( \Omega \) to some function \( u \). As a consequence of this fact, and since \( T_k(s) \) is continuous, the weak limit of \( T_k(u_n) \) in \( W^{1,p}_0(\Omega) \) is \( T_k(u) \) for every \( k > 0 \). From now on, we will always consider this subsequence \( u_n \).

We now use (2.9): by (2.14), since \( T_k(u_n) \) converges weakly to \( T_k(u) \) in \( W^{1,p}_0(\Omega) \) and \( u_n \) is almost everywhere convergent to \( u \), we have (observing that \( c_{\delta}(n) \), defined in (2.6), converges to \( \int_{\Omega} |f| (1 - \psi_\delta)^s \, dx \))

\[
\int_{\Omega} |\nabla T_k(u)|^p (1 - \psi_\delta)^s \, dx + k^{p+1} \mu_\delta(\{ |u| \geq k \}) \\
\leq c k \left( \delta + \int_{\Omega} |f| (1 - \psi_\delta)^s \, dx + \Psi \left( \delta + \int_{\Omega} |f| (1 - \psi_\delta)^s \, dx \right) \right) = c_{\delta} k.
\]

Starting from this inequality, and working as in Step 1, we obtain

\[
\int_{\Omega} |\nabla u|^{(p-1)r'} (1 - \psi_\delta)^{(s-1)r'} \, dx \leq c_{\delta},
\]

which then implies (letting \( \delta \) tend to zero) that

\[
\int_{\Omega} |\nabla u|^{(p-1)r'} \, dx \leq c,
\]

that is, \( |\nabla u|^{p-1} \) belongs to \( L^{r'}(\Omega) \). Furthermore, starting from (2.16), and applying twice Fatou lemma, we have that \( u \) belongs to \( L^q(\Omega) \).
Step 4. Let $k$ and $h$ be fixed, let $s$ be as in (2.13), and choose $T_k(u_n - T_h(u)) (1 - \psi_\delta)^s$ as test function in (1.9). We obtain

\[
\int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - T_h(u)) (1 - \psi_\delta)^s \, dx \\
- s \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla \psi_\delta T_k(u_n - T_h(u)) (1 - \psi_\delta)^{s-1} \, dx \\
+ \int_{\Omega} |u_n|^{q-1} u_n T_k(u_n - T_h(u)) (1 - \psi_\delta)^s \, dx \\
= \int_{\Omega} f T_k(u_n - T_h(u)) (1 - \psi_\delta)^s \, dx \tag{D}
\]

We now define by $\omega(n)$, $\omega(h)$ and $\omega(n, h)$ any quantity such that

\[
\lim_{n \to +\infty} \omega(n) = 0, \quad \lim_{h \to +\infty} \omega(h) = 0, \quad \lim_{h \to +\infty, n \to +\infty} \omega(n, h) = 0,
\]

By assumption (1.7), by (2.2) and (2.3), and by the results on the sequence $u_n$, we have

\[
(D) = \int_{\Omega} f T_k(u - T_h(u)) (1 - \psi_\delta)^s \, dx + \omega(n) = \omega(n, h).
\]

We then have, since $u_n T_k(u_n - T_h(u)) \geq 0$ on the set \{|$u_n$| > $h$\},

\[
(C) = \int_{\{|u_n| \leq h\}} |u_n|^{q-1} u_n T_k(u_n - T_h(u)) (1 - \psi_\delta)^s \, dx \\
+ \int_{\{|u_n| > h\}} |u_n|^{q-1} u_n T_k(u_n - T_h(u)) (1 - \psi_\delta)^s \, dx \\
\geq \int_{\{|u_n| \leq h\}} |u_n|^{q-1} u_n T_k(u_n - T_h(u)) (1 - \psi_\delta)^s \, dx \\
= \int_{\{|u| \leq h\}} |u|^{q-1} u T_k(u - T_h(u)) (1 - \psi_\delta)^s \, dx + \omega(n) = \omega(n).
\]

Finally, using (1.4),

\[
|\text{(B)}| \leq \int_{\Omega} |\nabla u_n|^{p-1} (1 - \psi_\delta)^{s-1} |\nabla \psi_\delta| |T_k(u_n - T_h(u))| \, dx.
\]

The sequence $|\nabla u_n|^{p-1} (1 - \psi_\delta)^{s-1}$ is bounded in $L'(\Omega)$ by (2.14) (hence it weakly converges to some $\sigma$ in $L'(\Omega)$), while $|\nabla \psi_\delta| |T_k(u_n - T_h(u))|$ is strongly convergent in $L'(\Omega)$ to $|\nabla \psi_\delta| |T_k(u - T_h(u))|$. Thus

\[
|\text{(B)}| \leq \int_{\Omega} \sigma |\nabla \psi_\delta| |T_k(u - T_h(u))| \, dx + \omega(n) = \omega(n, h).
\]
Putting together the results obtained for (B)–(D), we have

$$\int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla T_k(u_n - T_h(u)) \ (1 - \psi_\delta)^s \ dx = \omega(n, h).$$

(2.25)

Define now

$$Q_n(x) = (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)) \cdot \nabla (u_n - u) \ (1 - \psi_\delta)^s.$$  

By assumption (1.5), $Q_n$ is a nonnegative function. Furthermore, by (1.4), we have

$$Q_n \leq c \left[ |\nabla u_n|^p + |\nabla u|^p + |\nabla u_n|^{p-1} |\nabla u| + |\nabla u_n| |\nabla u|^{p-1} \right] (1 - \psi_\delta)^s.$$  

(2.26)

Recalling (2.14) and (2.24), there exists $\beta < 1$ such that

$$Q_n^{\beta} \text{ is bounded in } L^t(\Omega), \text{ for some } t > 1.$$  

We then fix $k$ and $h$, and start considering

$$\int_{\Omega} Q_n^{\beta} \ dx = \int_{\{|u| \leq h\}} Q_n^{\beta} \ dx + \int_{\{|u| > h\}} Q_n^{\beta} \ dx$$

$$\leq \int_{\{|u| \leq h\}} Q_n^{\beta} \ dx + \|Q_n^{\beta}\|_{L^t(\Omega)} \ \text{meas\{\{|u| > h\}\}}^{\frac{1}{t}}$$

$$= \int_{\{|u| \leq h\}} \left\{ \{|u| \leq \h \} \right\} Q_n^{\beta} \ dx + \int_{\left\{ \{|u| \leq h\} \right\}} Q_n^{\beta} \ dx + \omega(h)$$

$$\leq \int_{\left\{ \{|u| \leq h\} \right\}} Q_n^{\beta} \ dx + \|Q_n^{\beta}\|_{L^t(\Omega)} \ \text{meas} \left( \left\{ \{|u| \leq h\} \right\} \right)^{\frac{1}{t}} + \omega(h)$$

$$= \int_{\left\{ \{|u| \leq h\} \right\}} Q_n^{\beta} \ dx + \omega(n) + \omega(h)$$

$$\leq c_{\Omega} \left( \int_{\left\{ \{|u| \leq h\} \right\}} Q_n \ dx \right)^{\beta} + \omega(n) + \omega(h).$$

To perform the previous steps, we have used that $u$ belongs to $L^q(\Omega)$ (so that the measure of the set $\{|u| > h\}$ tends to zero as $h$ tends to infinity), and the fact that $u_n$ converges to $u$ in measure (so that the measure of the set $\{|u_n - u| > k\}$ tends to zero as $n$ tends to infinity). Recalling the definition
of $Q_n$, we have
\[
\int_{\{|u| \leq h\} \cap \{|u_n - u| \leq k\}} Q_n \, dx \\
= \int_{\{|u| \leq h\}} (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla T_h(u))) \cdot \nabla T_k(u_n - T_h(u)) \, (1 - \psi_s)^s \, dx \\
\leq \int_{\Omega} (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla T_h(u))) \cdot \nabla T_k(u_n - T_h(u)) \, (1 - \psi_s)^s \, dx.
\]

Since $a(x, u_n, \nabla T_h(u))$ converges to $a(x, u, \nabla T_h(u))$ strongly in $(L^p(\Omega))^N$ as $n$ tends to infinity, and since $a(x, u, \nabla T_h(u)) \cdot \nabla T_k(u - T_h(u)) \equiv 0$, we have
\[
\int_{\Omega} a(x, u_n, \nabla T_h(u)) \cdot \nabla T_k(u_n - T_h(u)) \, (1 - \psi_s)^s \, dx \\
= \int_{\Omega} a(x, u, \nabla T_h(u)) \cdot \nabla T_k(u - T_h(u)) \, (1 - \psi_s)^s \, dx + \omega(n) = \omega(n).
\]

Thus, recalling (2.25), we have
\[
\int_{\Omega} Q_n^\beta \, dx = \omega(n) + \omega(h) + \omega(n, h). \tag{2.27}
\]

From this relation, and reasoning as in [2], we obtain
\[
\nabla u_n (1 - \psi_s)^s \to \nabla u (1 - \psi_s)^s \quad \text{almost everywhere in } \Omega. \tag{2.28}
\]

**Step 5.** Let $\varepsilon > 0$, let $k > 0$ be fixed; let $s$ as in (2.13), and choose $S_{k,\varepsilon}(u_n) (1 - \psi_s)^s$ as test function in (1.9), where
\[
S_{k,\varepsilon}(t) = \frac{1}{\varepsilon} T'_{\varepsilon}(G_{k-\varepsilon}(t)).
\]

We obtain
\[
\frac{1}{\varepsilon} \int_{\{|k - \varepsilon \leq |u_n| \leq k\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n (1 - \psi_s)^s \, dx \\
- s \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla \psi_s S_{k,\varepsilon}(u_n) (1 - \psi_s)^{s-1} \, dx \\
+ \int_{\Omega} |u_n|^{q-1} u_n S_{k,\varepsilon}(u_n) (1 - \psi_s)^s \, dx \\
= \int_{\Omega} f_n S_{k,\varepsilon}(u_n) (1 - \psi_s)^s \, dx. \tag{D}
\]
Dropping (A), which is nonnegative, and then letting $\varepsilon$ tend to zero, we obtain, using (1.4) and (2.14),
\[
\int_{\{|u_n|\geq k\}} |u_n|^q (1 - \psi_\delta)^s \, dx \leq \int_{\{|u_n|\geq k\}} (f_n^\oplus + f_n^\ominus) (1 - \psi_\delta)^s \, dx \\
+ s \int_{\{|u_n|\geq k\}} a(x, u_n, \nabla u_n) \nabla \psi_\delta (1 - \psi_\delta)^{s-1} \, dx \\
\leq \int_{\{|u_n|\geq k\}} (f_n^\oplus + f_n^\ominus)(1 - \psi_\delta)^s \, dx \\
+ c \left( \int_{\{|u_n|\geq k\}} |\nabla \psi_\delta|^r \, dx \right)^{\frac{1}{r}}.
\]
Since $(f_n^\oplus + f_n^\ominus)(1 - \psi_\delta)^s$ is compact in $L^1(\Omega)$ (as a consequence of the assumptions on $f_n^\oplus$, $f_n^\ominus$ and $\psi_\delta$), $|\nabla \psi_\delta|^r$ belongs to $L^1(\Omega)$, and since (2.23) holds, Vitali theorem yields
\[
\lim_{k \to +\infty} \sup_{n \in \mathbb{N}} \int_{\{|u_n|\geq k\}} |u_n|^q (1 - \psi_\delta)^s \, dx = 0. \tag{2.29}
\]

**Step 6.** We now fix $s$ as in (2.13), fix a function $v$ in $C_c(\Omega)$, and take $v (1 - \psi_\delta)^s$ as test function in (1.9). We get
\[
\begin{align*}
\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v (1 - \psi_\delta)^s \, dx & \quad \text{(A)} \\
& - s \int_\Omega a(x, u_n, \nabla u_n) \nabla \psi_\delta v (1 - \psi_\delta)^{s-1} \, dx \quad \text{(B)} \\
& + \int_\Omega |u_n|^{q-1} u_n v (1 - \psi_\delta)^s \, dx \quad \text{(C)} \\
& = \int_\Omega f_n v (1 - \psi_\delta)^s \, dx. \quad \text{(D)}
\end{align*}
\]
Setting $\omega(n, \delta)$ any quantity such that
\[
\lim_{\delta \to 0^+} \lim_{n \to +\infty} \omega(n, \delta) = 0,
\]
and using (2.28) we have
\[
\text{(A)} = \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v (1 - \psi_\delta)^s \, dx + \omega(n) = \int_\Omega a(x, u_n, \nabla u_n) \nabla v \, dx + \omega(n, \delta).
\]
Using again (2.28), and the assumptions on $\psi_\delta$,
\[
\text{(B)} = -s \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla \psi_\delta v (1 - \psi_\delta)^{s-1} \, dx + \omega(n) = \omega(n, \delta).
\]
Using (1.7), (2.2) and (2.3),

\[(D) = \int_{\Omega} f v (1 - \psi \delta)^s \, dx + \omega(n) = \int_{\Omega} f v \, dx + \omega(n, \delta),\]

so that we only have to deal with (C). We begin proving that the sequence $|u_n|^q (1 - \psi \delta)^s$ is equiintegrable on $\Omega$ with respect to $n$. We fix $E$ a measurable subset of $\Omega$, and write

\[
\int_{E} |u_n|^q (1 - \psi \delta)^s \, dx = \int_{E \cap \{|u_n| < k\}} |u_n|^q (1 - \psi \delta)^s \, dx \\
+ \int_{E \cap \{|u_n| \geq k\}} |u_n|^q (1 - \psi \delta)^s \, dx \\
\leq k^q \text{meas}(E) + \int_{\{|u_n| \geq k\}} |u_n|^q (1 - \psi \delta)^s \, dx.
\]

Let $\varepsilon > 0$ be fixed; using (2.29), we fix $k_\varepsilon$ large so that

\[
\int_{\{|u_n| \geq k_\varepsilon\}} |u_n|^q (1 - \psi \delta)^s \, dx \leq \varepsilon,
\]

and then take $\delta_\varepsilon > 0$ such that $\text{meas}(E) < \delta_\varepsilon$ implies

\[
k^q \text{meas}(E) \leq \varepsilon.
\]

We have thus proved that the sequence $|u_n|^q (1 - \psi \delta)^s$ is equiintegrable on $\Omega$ with respect to $n$. Thus, by the almost everywhere convergence of $u_n$ to $u$, by Vitali theorem, and by the fact that $u$ belongs to $L^q(\Omega)$, we have

\[(C) = \int_{\Omega} |u|^{q-1} u v (1 - \psi \delta)^s \, dx + \omega(n) = \int_{\Omega} |u|^{q-1} u v \, dx + \omega(n, \delta).
\]

Putting together the results obtained on (A)–(D) we have

\[
\int_{\Omega} a(x, u, \nabla u) \cdot \nabla v \, dx + \int_{\Omega} |u|^{q-1} u v \, dx = \int_{\Omega} f v \, dx,
\]

for every $v$ in $C^\infty_c(\Omega)$, so that $u$ is a distributional solution of (1.10).
References


