An Optimal Markovian Quantization Algorithm for Multidimensional Stochastic Control Problems

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Abstract

We propose a probabilistic numerical method based on optimal quantization to solve some multidimensional stochastic control problems that arise, for example, in Mathematical Finance for portfolio optimization. We then consider some controlled diffusions with most control free components. The Euler scheme of the uncontrolled diffusion part is approximated by a discrete time process obtained by a nearest neighbor projection on some grids optimally fitted to its dynamics. The resulting process is also designed to preserve the Markov property with respect to the filtration of the Euler scheme. This Markovian quantization approach leads to an approximate control problem that can be solved numerically by the dynamic programming formula. This approach seems promising in higher dimension. *A priori* L^p-error bounds are stated and we show that the spatial discretization error term is minimal at some specific grids. A simple recursive algorithm is devised to compute these optimal grids by induction based on a Monte Carlo simulation. Some numerical illustrations are processed for solving a mean-variance hedging problem.

Key words: Stochastic Control, Markov chain, Euler scheme, Vector Quantization, Stochastic gradient descent, mean-variance hedging.

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1 Introduction

We consider the controlled diffusion system

$$(1.1)\qquad dX_t = \mu(X_t, Y_t, \alpha_t)dt + \vartheta(X_t, Y_t, \alpha_t)dY_t, \quad X_0 := x_0$$

$$(1.2)\qquad dY_t = \eta(Y_t)dt + \gamma(Y_t)dW_t, \quad Y_0 = y_0$$

where W is an m-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_t$. The set of control processes $\mathcal{A}$ is the set of all $\mathbb{F}$-adapted processes $\alpha = (\alpha_t)_t$ valued in A, subset of $\mathbb{R}^l$. The controlled process X is valued in $\mathbb{R}^q$ and the uncontrolled diffusion Y is valued in $\mathbb{R}^d$. The functions $\mu, \vartheta, \eta, \gamma$ satisfy conditions specified in the next section.

The above system admits another formulation in which X is directly written as a diffusion process

$$(1.3)\qquad dX_t = b(X_t, Y_t, \alpha_t)\, dt + \sigma(X_t, Y_t, \alpha_t)\, dW_t.$$

Our initial choice for the dynamics (1.1)-(1.2) is motivated by financial applications : if Y denotes a risky asset, the value X_t at time t of a self-financed portfolio containing ϑ_t units of in asset Y obeys the above system. Actually, we may reduce w.l.o.g. a dynamics (1.3)-(1.2) into a dynamics (1.1)-(1.2) by considering the uncontrolled part (Y, W) instead of Y.

Consider now the stochastic control problem in finite horizon:

$$(1.4)\qquad v(t, x, y) = \inf_{\alpha \in \mathcal{A}} E\left[\int_t^T f(X_u, Y_u, \alpha_u)du + g(X_T, Y_T) \middle| (X_t, Y_t) = (x, y) \right],$$

for $t \in [0, T]$, $(x, y) \in \mathbb{R}^q \times \mathbb{R}^d$, where f and g are some functions satisfying conditions specified later.

It is well-known (see *e.g.* [14]) that function v can be characterized by dynamic programming principle as solution (in the viscosity sense) of the Bellman equation:

$$(1.5)\qquad \frac{\partial v}{\partial t} + \inf_{a \in A} [\mathcal{L}^a v + f(x, y, a)] = 0,$$

together with the terminal condition:

$$(1.6)\qquad v(T, x, y) = g(x, y),$$

where $\mathcal{L}^a$ is the second order differential operator associated to the diffusion (X, Y):

$$\mathcal{L}^a v = b(x, y, a).D_x v + \eta(y).D_y v + \frac{1}{2}\mathrm{tr}(\sigma\sigma^*(x, a) D^2_{xx} v) + \frac{1}{2}\mathrm{tr}(\gamma\gamma^*(y) D^2_{yy} v) + \mathrm{tr}(\gamma(y)\sigma^*(x, a) D^2_{xy} v),$$

with $b(x, y, a) = \mu(x, y, a) + \vartheta(x, y, a)\eta(y)$ and $\sigma(x, y, a) = \vartheta(x, y, a)\gamma(y)$. Here, the sign $*$ denotes the transposition of a matrix and the sign . is the usual inner product.

The purpose of this paper is to solve numerically the stochastic control problem (1.4) and consequently the highly non linear P.D.E. (1.5)-(1.6).

There are two types of numerical methods for stochastic control problems.

– Purely deterministic methods provided by Numerical Analysis consisting in discretizing the partial differential equation (1.5): discretization by finite difference or finite element methods lead to an approximation of the value function at the points of the space-time grid. Computational methods of the discretized Bellman Equation are studied in Akian ([1]). For some illustrations in Financial problems, see *e.g.* Fitzpatrick and Fleming ([13]) or Tourin and Zariphopoulou ([25]).

– Probabilistic methods based on the dynamic programming principle for the discretized control problem. The Markov chain approximation method, introduced by Kushner in 1977 ([18], see also his more recent book with Dupuis [19]) consists in approximating the original continuous time controlled process by an appropriate controlled Markov chain on a lattice satisfying the so-called local consistency condition. The numerical problem is then to solve the stochastic control problem for the approximating Markov chain. The finite difference scheme is a typical example of a numerical scheme for an approximating Markov chain with nearest neighbor transitions. For stochastic control problems under partial observation, Runggaldier and his coauthors (see [6], [11] and [24] for a monograph on this subject) also consider finite state spatial approximating Markov chain. The spatial discretization of the Markov chain is obtained by the projection of the Markov chain on some fixed representative elements of a partition of the state space.

In both methods described above, the required stability condition may be very restrictive in the case of controls appearing in the variance term. On the other hand, in these methods, the lattice is fixed regardless of the structure of the Markov chain. Moreover, its size is growing exponentially with the dimension. From a theoretical viewpoint, estimate of the rate of convergence is not always available. From a computational viewpoint, their limits are the dimension of the state space of the system ($q+d$ in the above problem). So, although the Markov chain approximation method is easily implemented, calculations can be done in practice for quite low dimension, say 1 or 2. There is a challenge to solve efficiently numerical stochastic control problems in higher dimension.

In this paper we propose a probabilistic method based on optimal quantization, in order to solve numerically stochastic control problems in dimension larger than 3. Like in usual probabilistic methods, we start from a time discretization of the controlled problem: we consider a controlled problem similar to (1.4) in which the above process $(X_t, Y_t)_{t\in[0,T]}$ given by (1.1)-(1.2) has been replaced by its Euler scheme, denoted $(\bar{X}_k, \bar{Y}_k)$ at time kT/n (see (2.6) below). This turns out to be a consistent approximation of the value function of the original problem when the time discretization step T/n goes to 0 (see Proposition 2.1 below). Then, and in the spirit of the Markov chain approximation method, we approximate the Euler scheme at every date $k\in\{0,\ldots,n\}$ by a process $\widehat{X}_k$, taking finitely many states. Furthermore, we wish to preserve a Markov structure for the process $(\widehat{X}_k)_{0\le k\le n}$ *with respect to the filtration of the Euler scheme*: doing so, it is possible to make use of Control Theory techniques, especially the dynamic programming principle, to derive some *a priori* error bounds. We are thus naturally led to consider an approximating Markovian procedure of the form

$$(\widehat{X}_{k+1}, \widehat{Y}_{k+1}) = \mathrm{Proj}_{k+1}\left(L_h((\widehat{X}_k, \widehat{Y}_k), \varepsilon_{k+1})\right)$$

derived from (1.1)-(1.2), where $(\varepsilon_k)_k$ is a Gaussian white noise and Proj_{k+1} is a kind of closest neighbour projection on a finite subset of $\mathbb{R}^q \times \mathbb{R}^d$.

One natural idea to approximate a random vector Z by a random vector taking its values in a finite grid $\Gamma := \{z_1, \dots, z_N\}$ is to consider its projection $\pi^\Gamma(Z)$ on the grid following the closest neighbour rule. The induced mean L^p-error is given by $\|Z - \pi^\Gamma(Z)\|_p = \|\min_{1\le i\le N}|Z - z_i|\|_p$ and only depends on the distribution P_Z of Z and the grid Γ. For historical reasons, $\pi^\Gamma(Z)$ is often called the *quantization of the r.v.* Z by the grid Γ and the induced error, the L^p-mean quantization error (see [15]). This quantity has been extensively investigated in Signal Processing and Information Theory since the early 50's. Thus, one knows that this mean error reaches a minimum over all the possible grids Γ having at most N elements and this mean error behaves like $c(P_Z, p, d)N^{-\frac{1}{d}}$ as N goes to infinity.

Except in some particular cases of little numerical interest, no closed form is available for the optimal grids that achieve the minimal quantization error of a probability distribution. In fact, no rigorous result is available to precisely describe the geometric structure or "shape" of such an optimal grid. However, using the integral representation of $\|Z - \pi^\Gamma Z\|_p^p$ one may devise a stochastic gradient descent, based on simulations of Z, that converges towards some grids which are optimal (at least locally). This makes possible the practical use of the optimal quantizers of a distribution along with their companion parameters for numerical purpose (see, *e.g.* [21]). Simulations then confirm what could be *a priori* expected: the heavier an area is weighted by the quantized distribution, the more points it contains.

This idea of projecting Markov chain on some grids is similar to the one of Runggaldier et al. [24] with their representative elements. Those are fixed *a priori* and cannot be simply adapted to take into account the fine structure of the Markov dynamics. Their size is growing exponentially with the dimension and in practice calculations can be done only for quite low dimension. Here, the main novelty is the optimal quantization part : given a total number of points to be dispatched among all the grids, we show how to get optimal grids with respect to the Markov structure of the process. We also derive a rate of convergence for the value functions. As a byproduct, we may also construct approximate nearly optimal controls, as in [24]. This approach seems efficient in medium dimension, say possibly up to 10 dimension.

As we will see in the numerical illustrations at the end of the paper, it is possible in many situations *to implement the same (optimal) quantization* for solving the whole numerical process. Thus, if the dynamics of the diffusion Y satisfies $Y_t = \varphi(t, W_t)$ it is possible, at least for practical matter, to use an optimal quantization of the standard d-dimensional Brownian motion. Such quantizations, with the requested size, can be built almost instantly from stored optimal quantizations of the normal distributions. These were computed very accurately for any size and any dimension (up to $d = 10$) and are kept off line (see [23]).

Discretizing multi-dimensional Markov chains using optimal quantization to solve non linear problems was first investigated in [2] and [3] for Optimal Stopping with promising results, at least in medium dimensions. Here, we face a different problem for two main reasons, both related to the high non linearity of the problem: first we need to preserve the Markov property of $\widehat{X}_k$ to keep the benefit of the Control Theory machinery, secondly the presence of the control in the equation that drives the couple (X, Y) makes hopeless to implement a global optimal quantization since the quantization then depends on the control. At this stage, the exact structure and the interpretation of the investigated model (1.1) becomes crucial: for the financial applications we have in mind, the process Y_t represents the value at time t of a basket of d traded assets, possibly with stochastic volatility and X_t represents the value at time t of a (self-financed) portfolio made up with $\vartheta(X_t, Y_t, \alpha_t)$ units of asset Y and $\mu(X_t, Y_t, \alpha_t)$ units of the riskless asset. So, typically, Y is a multi-dimensional *uncontrolled* process whereas X is usually 1- (or 2-)dimensional *controlled process* (the simplest model is $\vartheta(x, y, \alpha) := \alpha$: the control α_t is then the quantity invested in the risky assets at time t).

Taking this into account, it is natural to discretize $(\bar{X}_k)$ and $(\bar{Y}_k)$ differently:

• The q-dimensional process $\bar{X}$ will be quantized using a *regular orthogonal* grid of $\mathbb{R}^q$, namely $\Gamma^X := (2\delta)\mathbb{Z}^q \cap B_{\ell^\infty}(0, R)$ and π^{Γ^X} is simply the ℓ^∞-closest neighbour projection onto this grid.

• The d-dimensional process $(\bar{Y}_k)$ will be quantized by an *optimal* grid Γ_k for every $k \in \{0, \ldots, n\}$ and π^{Γ_k} is simply the closest neighbour projection on Γ_k.

The paper is organized as follows: Section 2 is devoted to consistency of the time discretization of the controlled problem. In Section 3, we state in (Theorem 3.1) an *a priori* L^p-error bounds at every $x \in \Gamma^X$, between $\bar{v}_k(x, \bar{Y}_k)$ and $\widehat{v}_k(x, \widehat{Y}_k)$, value functions at epoch k of the Euler scheme and the quantized Markov chain respectively. This bounds upon the parameters of the spatial discretization error: δ, R and the (optimal) mean L^p-quantization errors of $\widehat{Y}_l$, $l \geq k$. Section 4 provides a (partially heuristic) procedure to design an optimal "quantization tree", given the parameters η, γ and the global number N of elementary quantizer to be used to quantize all the $\bar{Y}_k$'s. In Section 5, we discuss the rate of convergence of the value functions and construct approximate nearly optimal controls. Section 6 is devoted to a short discussion of possible variants of the method (starting values, discretizing schemes). In section 7, we provide some numerical illustrations for a mean-variance hedging problem. We also discuss the computational complexity of our algorithm. Finally, proof of Theorem 3.1 is postponed in Section 8.

2 Time discretization

We shall assume Lipschitz conditions on the coefficients of the controlled diffusion (X, Y) given in (1.1)-(1.2) and governed by:

$$\begin{aligned} dX_t &= b(X_t, Y_t, \alpha_t)dt + \sigma(X_t, Y_t, \alpha_t)dW_t \\ dY_t &= \eta(Y_t)dt + \gamma(Y_t)dW_t. \end{aligned}$$

(H1) There exist positive constants L_1, L_2 such that, for all $x, x' \in \mathbb{R}^q$, $y, y' \in \mathbb{R}^d$, $a \in A$,

$$\begin{aligned} |b(x,y,a) - b(x',y',a)| + |\sigma(x,y,a) - \sigma(x',y',a)| &\leq L_1\left(|x-x'| + |y-y'|\right) \\ |\eta(y) - \eta(y')| + |\gamma(y) - \gamma(y')| &\leq L_2|y-y'|. \end{aligned}$$

We shall assume a quadratic growth condition on the cost functions of the control problem.

(H2) There exists a positive constant K such that, for all $x, y, a \in \mathbb{R}^q \times \mathbb{R}^d \times A$,

$$|f(x,y,a)| + |g(x,y)| \leq K(1 + |x|^2 + |y|^2).$$

We shall also assume a continuity condition on f and g.

(H3a) g is continuous and f is continuous in (x, y) uniformly in $a \in A$, i.e.:

$$\sup_{a \in A} |f(x',y',a) - f(x,y,a)| \quad \rightarrow \quad 0,$$

as $(x', y') \rightarrow (x, y) \in \mathbb{R}^q \times \mathbb{R}^d$.

We first approximate the diffusion $(Y_t)_{0 \leq t \leq T}$ at the discrete times $t_0 = 0, \ldots, t_n = T$. We consider the Gaussian Euler scheme with step $h = T/n$, $h \leq 1$. We denote by $\bar{Y}_k$ this approximation of Y_{t_k}, $t_k = kh$, which is then defined by:

$$\bar{Y}_0 = Y_0 \tag{2.1}$$

$$\begin{aligned} \bar{Y}_{k+1} &= \bar{Y}_k + \eta(\bar{Y}_k)h + \gamma(\bar{Y}_k)\sqrt{h}\,\varepsilon_{k+1} \\ &=: G_h(\bar{Y}_k, \varepsilon_{k+1}), \quad k = 0, \ldots, n-1, \end{aligned} \tag{2.2}$$

where $\varepsilon_{k+1} = (W_{t_{k+1}} - W_{t_k})/\sqrt{h}$ is a centered Gaussian random variable in $\mathbb{R}^m$ with variance I_m, independent of $\bar{\mathcal{F}}_k := \mathcal{F}_{t_k}$. The Euler scheme $(\bar{Y}_k)_k$ is then a $\bar{\mathcal{F}}_k$-homogeneous Markov chain.

We denote by $\bar{\mathcal{A}}_n$ the set of all $\{\bar{\mathcal{F}}_k, k = 0, \ldots, n-1\}$-adapted processes $\bar{\alpha} = \{\bar{\alpha}_k, k = 0, \ldots, n-1\}$ valued in A. Given $\bar{\alpha} \in \bar{\mathcal{A}}_n$, we consider the approximation $(\bar{X}_k)$ of the controlled diffusion (X_t) at times t_k, and defined by:

$$\begin{aligned} \bar{X}_{k+1} &= \bar{X}_k + \mu(\bar{X}_k, \bar{Y}_k, \bar{\alpha}_k)h + \vartheta(\bar{X}_k, \bar{Y}_k, \bar{\alpha}_k)(\bar{Y}_{k+1} - \bar{Y}_k) \\ &= H_h(\bar{X}_k, \bar{Y}_k, \bar{\alpha}_k, \bar{Y}_{k+1}) \\ &= \bar{X}_k + b(\bar{X}_k, \bar{Y}_k, \bar{\alpha}_k)h + \sigma(\bar{X}_k, \bar{Y}_k, \bar{\alpha}_k)\sqrt{h}\varepsilon_{k+1} \\ &= F_h(\bar{X}_k, \bar{Y}_k, \bar{\alpha}_k, \varepsilon_{k+1}), \quad k = 0, \ldots, n-1. \end{aligned} \tag{2.3}$$

Here, the functions H_h and F_h are defined on $\mathbb{R}^q \times \mathbb{R}^d \times A \times \mathbb{R}^d$, respectively on $\mathbb{R}^q \times \mathbb{R}^d \times A \times \mathbb{R}^m$, by :

$$H_h(x,y,a,y') = x + \mu(x,y,a)h + \vartheta(x,y,a)(y'-y), \tag{2.4}$$

$$F_h(x,y,a,\varepsilon) = H_h(x,y,a,G_h(y,\varepsilon)). \tag{2.5}$$

We now consider the stochastic control problem in discrete time:

$$\bar{v}_k(x,y) = \inf_{\bar{\alpha} \in \bar{\mathcal{A}}_n} E\left[\sum_{j=k}^{n-1} hf(\bar{X}_j, \bar{Y}_j, \bar{\alpha}_j) + g(\bar{X}_n, \bar{Y}_n) \,\middle|\, (\bar{X}_k, \bar{Y}_k) = (x,y)\right], \tag{2.6}$$

for all $k = 0, \dots, n$ and $(x, y) \in \mathbb{R}^q \times \mathbb{R}^d$. We also denote

$$\bar{V}_k(x) \quad = \quad \bar{v}_k(x, \bar{Y}_k), \quad k = 0, \dots, n, \; x \in \mathbb{R}^q.$$

We have the following convergence result from the discrete-time stochastic control problem to the continuous one.

Proposition 2.1 *Assume that* **(H1)**, **(H2)** *and* **(H3a)** *hold. Then,*

$$\bar{v}_k(x', y') \quad \longrightarrow \quad v(t, x, y), \tag{2.7}$$

as h goes to zero, $t_k \to t$ and $(x', y') \to (x, y)$, uniformly on any compact of $\mathbb{R}^q \times \mathbb{R}^d$. Moreover, for every $p \geq 1$,

$$\|\bar{V}_k(x) - v(t, x, Y_t)\|_p \quad \longrightarrow \quad 0, \tag{2.8}$$

as h goes to zero and t_k goes to t, uniformly on compact sets of $\mathbb{R}^q$.

This convergence result of the value function is certainly well-known. It may be proved by probabilistic arguments, using weak convergence : indeed, the approximation scheme satisfy clearly the local consistency conditions of Kushner, see [18] or [19]. However, in these works, the costs functions f and g are assumed to be bounded. Since we could not find a direct reference in the literature for the more general case of quadratic growth condition **(H2)**, and for sake of completeness, we give in Appendix an alternative proof of Proposition 2.1, based on viscosity solutions approach.

Remark 2.1 We mention that recently, Krylov presents in [17] a different method for proving the convergence which also provides an estimate of the rate of convergence of finite difference approximations of Bellman's equation.

3 Markovian Quantization

We shall assume

(H0) The function $\vartheta(x, y, a)$ is bounded in $(x, y, a) \in \mathbb{R}^q \times \mathbb{R}^d \times A$.

We also strengthen condition **(H3a)** by assuming Lipschitz continuity of the costs functions:

(H3b) There exist positive constants $[f_x]$, $[g_x]$, $[f_y]$ and $[g_y]$, and $p_1 \in \mathbb{N}$ such that

$$\begin{aligned}
|f(x, y, a) - f(x', y, a)| \quad &\leq \quad [f_x]|x - x'|\,(1 + |y|^{p_1}) \\
|g(x, y) - g(x', y)| \quad &\leq \quad [g_x]|x - x'|\,(1 + |y|^{p_1}) \\
|f(x, y, a) - f(x, y', a)| \quad &\leq \quad [f_y]|y - y'|\,(1 + |x|^{p_1}) \\
|g(x, y) - g(x, y')| \quad &\leq \quad [g_y]|y - y'|\,(1 + |x|^{p_1})
\end{aligned}$$

for all $x, x' \in \mathbb{R}^q$, $y, y' \in \mathbb{R}^d$, $a \in A$.

Note than, when $p_1 = 0$, assumption **(H3b)** simply means that f and g are Lipschitz continuous. The main error bound in Theorem 3.1 below then becomes significantly simpler.

We first consider a fixed bounded uniform grid Γ^X on the state space $\mathbb{R}^q$ of the controlled process X, with step δ and size R. Namely we set $\Gamma^X := (2\delta)\mathbb{Z}^q \cap B_\infty(x_0, R) = \{x \in \mathbb{R}^q \ : \ x = 2\delta z, \text{ for some } z \in \mathbb{Z}^q, \text{ and } |x - x_0| \leq R\}$ where $X_0 = x_0$ and $|\,.\,|$ denotes the ℓ^∞ norm on $\mathbb{R}^q$ (this choice is motivated only by technical commodity). Furthermore, we will assume from now on that $x_0 = 0$ for notational simplicity. We denote by π^X the projection on the grid Γ^X according to the closest neighbour rule. The projection π^X satisfies

$$|x - \pi^X(x)| \leq \max(|x| - R, 0) + \delta \quad \text{for every } \ x \in \mathbb{R}^q. \tag{3.1}$$

At each discrete time t_k, we consider a grid $\Gamma_k = \{y_k^1, \ldots, y_k^{N_k}\}$ on the state space $\mathbb{R}^d$ of the uncontrolled process Y, to be determined later. We denote by π_k a projection on the grid Γ_k (*i.e.* following the closest neighbour rule). We now define as a *quantized Euler scheme* the following finite state space $(\bar{\mathcal{F}}_k)$-Markov chain:

$$\begin{aligned} \hat{Y}_0 &= \bar{Y}_0 \ (= Y_0 = y_0), \\ \hat{Y}_{k+1} &= \pi_{k+1}\left(G_h(\hat{Y}_k, \varepsilon_{k+1})\right), \quad k = 0, \ldots, n-1. \end{aligned}$$

Given $\bar{\alpha} = \{\bar{\alpha}_k, k = 0, \ldots, n-1\} \in \bar{\mathcal{A}}_n$, we consider the finite state space controlled $(\bar{\mathcal{F}}_k)$-Markov chain defined by:

$$\hat{X}_{k+1} = \pi^X\left(H_h(\hat{X}_k, \hat{Y}_k, \bar{\alpha}_k, \hat{Y}_{k+1})\right), \quad k = 0, \ldots, n-1. \tag{3.2}$$

We now consider the stochastic control problem in discrete time:

$$\hat{v}_k(x, y) = \inf_{\bar{\alpha} \in \bar{\mathcal{A}}_n} E\left[\sum_{j=k}^{n-1} hf(\hat{X}_j, \hat{Y}_j, \bar{\alpha}_j) + g(\hat{X}_n, \hat{Y}_n)\middle| (\hat{X}_k, \hat{Y}_k) = (x, y)\right], \tag{3.3}$$

for all $k = 0, \ldots, n$ and $(x, y) \in \Gamma^X \times \Gamma_k$. We also denote

$$\hat{V}_k(x) = \hat{v}_k(x, \hat{Y}_k), \quad k = 0, \ldots, n, \ x \in \Gamma^X.$$

By the dynamic programming principle, functions $\hat{v}_k$ can be computed recursively by a descent algorithm on the tree induced by the grids:

$$\begin{aligned} \hat{v}_n(x, y) &= g(x, y), \quad x \in \Gamma^X, y \in \Gamma_n \\ \hat{v}_k(x, y) &= \inf_{a \in A} E\left[hf(x, y, a) + \hat{v}_{k+1}(\hat{X}_{k+1}^{x,y,a}, \hat{Y}_{k+1}^y)\right], \quad k = 0, \ldots, n-1, \ x \in \Gamma^X, y \in \Gamma_k, \end{aligned}$$

where $\hat{X}_{k+1}^{x,y,a} = \pi^X(H_h(x, y, a, \hat{Y}_{k+1}^y))$ and $\hat{Y}_{k+1}^y = \pi_{k+1}(G_h(y, \varepsilon_{k+1}))$.

More precisely, functions $\hat{v}_k$ are computed whenever we know or estimate the transition matrix of the Markov chain $(\hat{Y}_k)_k$. Indeed, by denoting $\hat{p}_{ij}^k = P[\hat{Y}_{k+1} = y_{k+1}^j | \hat{Y}_k = y_k^i]$, we have:

$$\begin{aligned} \hat{v}_k(x, y_k^i) &= \inf_{a \in A} \sum_{j=1}^{N_{k+1}} \hat{p}_{ij}^k \left[hf(x, y_k^i, a) + \hat{v}_{k+1}(\pi^X(H_h(x, y_k^i, a, y_{k+1}^j)), y_{k+1}^j)\right], \\ & \qquad k = 0, \ldots, n-1, \ x \in \Gamma^X, \ y_k^i \in \Gamma_k. \end{aligned} \tag{3.4}$$

Our first main result provides an estimate for $\|\bar{V}_k(x) - \hat{V}_k(x)\|_p$ depending on the quantization error $\|\Delta_j\|$, $j \geq k$, defined by

$$\Delta_j := \hat{Y}_j - G_h(\hat{Y}_{j-1}, \varepsilon_j) \ = \ \pi_j \Big(G_h(\hat{Y}_{j-1}, \varepsilon_j) \Big) - G_h(\hat{Y}_{j-1}, \varepsilon_j). \tag{3.5}$$

Theorem 3.1 *Assume that* **(H0)**, **(H1)** *and* **(H3b)** *hold. Then for all* $p \geq 1$, $\bar{p} \geq 1$, $\hat{p} \geq 2$ *with* $\hat{p} > p$ *if* $p_1 > 0$ *and* $\hat{p} \geq p$ *if* $p_1 = 0$, *there exists a positive constant* $C_{p,\bar{p},\hat{p}}$ *(independent of* n*) such that for all* $k = 0, \ldots, n$, *for all* $x \in \Gamma^X$*:*

$$\begin{aligned}
\|\bar{V}_k(x) - \hat{V}_k(x)\|_p \ \leq \ & C_{p,\bar{p},\hat{p}} \left[\frac{[f_x]}{n} \sum_{j=k+1}^{n-1} \Big(1 + \|\bar{Y}_j\|_{q_1'}^{p_1} \Big) + [g_x] \Big(1 + \|\bar{Y}_n\|_{q_1'}^{p_1} \Big) \right] \\
& \left[\frac{1}{n^{\frac{1}{\bar{p}}}} \sum_{l=1}^{n-1} (n-l) \|\Delta_l\|_{\hat{p}} + \sum_{l=k+1}^{n} \|\Delta_l\|_{\hat{p}} + n\delta \right. \\
& \left. + \frac{n}{R^{\bar{p}-1}} \left(1 + (n\delta)^{\bar{p}} + |x|^{\bar{p}} + \|\bar{Y}_k\|_{\hat{p}\bar{p}}^{\bar{p}} + \left(n^{1-\frac{1}{\hat{p}\bar{p}}} \sum_{l=1}^{n} \|\Delta_l\|_{\hat{p}\bar{p}} \right)^{\bar{p}} \right) \right] \\
& + C_{p,\bar{p},\hat{p}} \left[\frac{[f_y]}{n} \sum_{l=1}^{n-1} (n-l) \|\Delta_l\|_{\hat{p}} + [g_y] \sum_{l=1}^{n} \|\Delta_l\|_{\hat{p}} \right] \\
& \left[1 + (n\delta)^{p_1} + |x|^{p_1} + \|\bar{Y}_k\|_{\bar{q}_1}^{p_1} + \left(n^{1-\frac{1}{\bar{q}_1 \vee 2}} \sum_{l=1}^{n} \|\Delta_l\|_{\bar{q}_1} \right)^{p_1} \right]
\end{aligned}$$

where $q_1' = pp_1\hat{p}/(\hat{p} - p)$, $\hat{q}_1 = p_1\hat{p}/(\hat{p} - 1)$ *and* $\bar{q}_1 = q_1' \vee \hat{q}_1 \vee 2$.

Remark 3.1 The estimation of Theorem 3.1 consists basically in two error terms. One term is due to the spatial discretization in Y :

$$\frac{1}{n^{\frac{1}{\bar{p}}}} \sum_{l=1}^{n-1} (n-l) \|\Delta_l\|_{\hat{p}} + \sum_{l=1}^{n} \|\Delta_l\|_{\hat{p}},$$

and the second one is due to the spatial discretization in X :

$$n\delta + \frac{n}{R^{\bar{p}-1}}.$$

Of course, one has also to bound the quantities appearing in factor of these two error terms. Classical estimates for the Euler scheme, see e.g. [16], yield that for any $r \geq 1$,

$$\sup_{k=0,\ldots,n} \|\bar{Y}_k\|_r \ \leq \ C,$$

for some positive constant C (depending on r, T, Y_0 and the coefficients of the diffusion Y) but independent of n. We shall also see in Section 4, Proposition 4.2, that for an optimal dispatching of the total number of points $\sum_{k=1}^{n} N_k$ in the space grid, with respect to the number n of points in the time grid, for any $r \geq 1$, the quantity

$$n^{1-\frac{1}{r}} \sum_{l=1}^{n} \|\Delta_l\|_r$$

remains bounded by a constant depending on r, T, d, and the coefficients of the diffusion Y. By choosing also a discretization parameter δ on X such that $n\delta$ goes to zero, we obtain from Theorem 3.1 an estimation of the rate of convergence for the quantization ruled by :

$$\begin{aligned} \|\bar{V}_k(x) - \hat{V}_k(x)\|_p \quad &\leq \quad C_1 \left[\frac{1}{n^{\frac{1}{\bar{p}}}} \sum_{l=1}^{n-1} (n-l)\|\Delta_l\|_{\hat{p}} + \sum_{l=1}^{n} \|\Delta_l\|_{\hat{p}} \right] \\ &\qquad + C_2 n\delta + C_3(\bar{p}) \frac{n}{R^{\bar{p}-1}}, \end{aligned} \tag{3.6}$$

for some constants C_1, C_2 and $C_3(\bar{p})$, with $C_3(\bar{p})$ increasing exponentially with $\bar{p} > 1$. We see in particular that the size R of the grid Γ^X must satisfy $n \ll R^{\bar{p}-1}$, or put in a more mathematical way: $n/R^{\bar{p}-1}$ goes to 0 as n goes infinity. On one hand, one has interest to take $\bar{p}$ large but on the other hand, recall that the constant $C_3(\bar{p})$ increases exponentially with $\bar{p}$. The first term in the r.h.s of (3.6), due to the quantization error on the spatial discretization of $\bar{Y}$, will be investigated further on in Proposition 4.2.

The methods for proving the estimation in Theorem 3.1 are more and less standard and technical. We postpone the proof of Theorem 3.1 in Section 8.

4 Optimization of the grids of the quantized Euler scheme

4.1 Basic facts about optimal quantization of a random vector Y

In this subsection we provide a short background about optimal quantization of random vectors. Optimal quantization has originally been conceived as a compressing data process. It was extensively investigated by specialists in Information Theory and Signal Processing since the early 50's. For a modern and mathematically rigorous overview of quantization of random vectors, one may consult [15] and the references therein.

Let $Y \in L^p(\Omega, \mathbb{R}^d)$ be a random vector. From a probabilistic point of view, L^p-quantization ($p \geq 1$) consists in studying the best L^p-approximation of Y by random vectors Y' taking at most N fixed values $y^1, \cdots, y^N \in \mathbb{R}^d$. Hence, Y' reads

$$Y' := \sum_{i=1}^{N} y^i \mathbf{1}_{A_i}(Y), \quad (A_i)_{1 \leq i \leq N} \text{ Borel partition of } \mathbb{R}^d.$$

One easily proves that, a N-grid $\Gamma := \{y^1, \cdots, y^N\}$ being fixed, the L^p-mean error $\|Y - Y'\|_p$ actually reaches a minimum if the partition $(A_i)_{1 \leq i \leq N}$ is a Voronoi tessellation $(C_i(\Gamma))_{1 \leq i \leq N}$ related to the grid Γ. Let us be more specific at this stage.

A partition $C_1(\Gamma), \ldots, C_N(\Gamma)$ of $\mathbb{R}^d$ is a Voronoi tessellation of the N-grid Γ if, for every $i \in \{1, \ldots, N\}$, $C_i(\Gamma)$ is a Borel set satisfying

$$C_i(\Gamma) \subset \left\{ y \in \mathbb{R}^d \,/\, |y^i - y| = \min_{\Gamma} |y - y^j| \right\}.$$

where $|\,.\,|$ (usually) denotes the canonical Euclidean norm. Set for every $y\in\mathbb{R}^d$, $\pi^\Gamma(y) := y^i$ if $y\in C_i(\Gamma)$. Then π^Γ is a closest neighbour projection on the grid $\Gamma := \{y^1,\dots,y^N\}$ and

$$\widehat{Y}^\Gamma := \pi^\Gamma(Y) = \sum_{i=1}^N y^i \mathbf{1}_{C_i(\Gamma)}(Y) \qquad (\widehat{Y}^\Gamma \text{ will often be denoted } \widehat{Y})$$

clearly satisfies

$$(4.1)\qquad \|Y-\widehat{Y}^\Gamma\|_p^p = \min\Big\{\|Y-Y'\|_p^p,\ Y' : (\Omega,\mathcal{B}) \to \Gamma\Big\}.$$

Next step is to find a tractable expression for this L^p*-quantization error* $\|Y-\widehat{Y}\|_p$ induced by the quantization of Y using the N-grid Γ. It is straightforward that

$$(4.2)\quad \|Y-\widehat{Y}\|_p^p = \sum_{i=1}^N E\left[\mathbf{1}_{C_i(\Gamma)}|Y-y^i|^p\right] = E\left[\min_{1\le i\le N}|Y-y^i|^p\right] = \int_{\mathbb{R}^d}\min_{1\le i\le N}|y^i-v|^p P_Y(dv)$$

where P_Y denotes the distribution of Y. Notice that the L^p-quantization error only depends on the distribution of Y whereas the random quantizer $\widehat{Y}$ only depends on Γ and the (Euclidean) distance on $\mathbb{R}^d$. The expression (4.2) will be the key for the algorithmic approach to optimal quantization.

Let us point out now a typical property of the L^p-mean quantization error that makes it attractive for our purpose [1]: on one hand for every $p\ge 1$

$$\|Y-\widehat{Y}\|_p^p = \max\Big\{\|\varphi(Y)-\varphi(\widehat{Y})\|_p^p,\ \varphi \text{ Lipschitz continuous, } [\varphi]_{Lip}\le 1\Big\}$$

(the equality holds for the function $\varphi : y \mapsto \min_\Gamma |y-y^i|$). On the other hand, from a numerical viewpoint, $E\varphi(\widehat{Y})$ reads

$$E\,\varphi(\widehat{Y}) = \int_{\mathbb{R}^d}\varphi d\pi^\Gamma(P_Y) = \sum_{i=1}^N P(Y\in C_i(\Gamma))\varphi(y^i)$$

Practical computation of such discrete integrals only requires to have access to

– the grid $\Gamma := \{y^1,\dots,y^N\}$ and to

– the related weights $p_i := P(Y\in C_i(\Gamma)) = P_Y(C_i(\Gamma))$.

Next step is to optimize the choice of the N-grid Γ so as to achieve the smallest possible quantization error and then to evaluate how fast this error goes to 0 as N goes to infinity.

4.1.1 Optimal quantization: existence and asymptotics

The expression (4.2) for the L^p-quantization error stresses that among other usual error bounds it behaves as a regular function of the quantizing grid $\Gamma := \{y^1,\dots,y^N\}$ (with the temporary convention that some elements are possibly equal so that $|\Gamma|\le N$). More precisely, as a *symmetric* function of the N-tuple $y := (y^1,\dots,y^N)$, the L^p-quantization error is Lipschitz continuous. One shows by induction on N that

$$\Gamma \mapsto \|Y-\widehat{Y}^\Gamma\|_p \text{ still reaches an absolute minimum at some grid } \Gamma^*\subset\mathbb{R}^d$$

[1]For a Lipschitz function $\varphi:\mathbb{R}^d\mapsto\mathbb{R}$, we set $[\varphi]_{Lip} = \sup_{x\neq y}\frac{|\varphi(x)-\varphi(y)|}{|x-y|}$.

and one may always assume that $\Gamma^* \subset \mathcal{H}(\text{supp}P_{_Y})$ ($\mathcal{H}$ is for convex hull) (see [21] or [15], among others). Furthermore, one shows the following simple facts (see [21] or [15] and references therein):

– If $\text{supp}P_{_Y}$ has an *infinite* support, any optimal grid Γ^* has N pairwise distinct elements.

– The minimal L^p-quantization error decreases to zero as $N \to \infty$ *i.e.*

$$\lim_N \min_{|\Gamma| \le N} \|Y - \widehat{Y}^\Gamma\|_{_p} = 0.$$

As a matter of fact, let $(z_k)_{k\in\mathbb{N}}$ denote an everywhere dense sequence in $\mathbb{R}^d$ and set $\Gamma_{_N} := \{z_1, \ldots, z_{_N}\}$. It is straightforward that $\|Y - \widehat{Y}^{\Gamma_N}\|_{_p}$ goes to zero by the Lebesgue Dominated Convergence Theorem. Then, $\min_{|\Gamma|\le N} \|Y - \widehat{Y}^\Gamma\|_{_p}$ goes to 0 as well.

At which rate does this convergence hold turned out to be a much more challenging question. The answer was completed by several authors (Zador [26], Bucklew & Wise [8] and finally Graf & Luschgy [15]). It reads as follows

Theorem 4.1 *([15]) Assume that $E|Y|^{p+\eta} < +\infty$ for some $\eta > 0$. Then*

$$\lim_N \left(N^{\frac{p}{d}} \min_{|\Gamma|\le N} \|Y - \widehat{Y}^\Gamma\|^p_{_p} \right) = J_{p,d} \left(\int_{\mathbb{R}^d} \varphi^{\frac{d}{d+p}}(u)du \right)^{1+p/d} \tag{4.3}$$

where $P_{_Y}(dy) = \varphi(y)\,\lambda_d(dy) + \nu(dy)$, $\nu \perp \lambda_d$ (λ_d Lebesgue measure on $\mathbb{R}^d$). The constant $J_{p,d}$ corresponds to the case of the uniform distribution on $[0,1]^d$.

Little is known about the true value of the constant $J_{p,d}$ except in 1-dimension 1 ($J_{p,1} = \frac{1}{2^p(p+1)}$) and 2-dimension (*e.g.* $J_{2,2} = \frac{5}{18\sqrt{3}}$, see [15]). However, one shows using random quantization techniques that $J_{p,d} \sim (d/(2\pi e))^{p/2}$ as $d \to +\infty$ (see [15] or [9]).

Whatsoever, this theorem says that $\min_{|\Gamma|\le N} \|Y - \widehat{Y}^\Gamma\|_{_p} \sim C_{Y,p,d} N^{\frac{1}{d}}$ as $N \to +\infty$. This is the same *rate* as that obtained with uniform lattice grids (*i.e.* homothety-translation of $\{i, \ldots, m\}^d$ when $N = m^d \to \infty$. However, these grids are never optimal – except for the 1-dim uniform distribution – and optimal quantization produces significantly lower quantization error, not only asymptotically. Furthermore, it works for any N, not only d^{th} powers of integers: optimal quantization yields the best possible "grid method" for a given distribution μ.

Figure 1 below shows an optimal grid for the Normal distribution. It is obtained by the procedure described in the next section.

4.1.2 Optimal quantization: algorithmic aspects

Following (4.2), one sets for every N-tuple $y = (y^1, \ldots, y^N) \in (\mathbb{R}^d)^N$

$$Q^p_{_N}(y) := E\, q^p_{_N}(y, Y) \quad \text{where} \quad q^p_{_N}(y, v) := \min_{1\le i\le N} |v - y^i|^p.$$

so that $Q^p_{_N}(y) = \|Y - \widehat{Y}^\Gamma\|^p_{_p}$ with $\Gamma = \{y^1, \ldots, y^N\}$. Function $Q^p_{_N}(y)$ is symmetric, continuous since $\sqrt[p]{Q^p_n}$ is Lipschitz. For notational convenience, we will temporarily denote $C_i(y)$ for $C_i(\Gamma)$ (the i^{th} Voronoi tessel of y). One shows (see, *e.g.*, [15] when $p = 2$ or

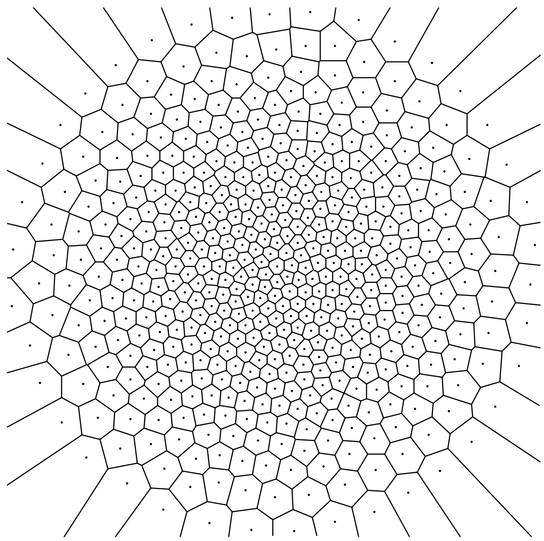

Figure 1: A L^2-optimal 500-grid for the Normal distribution $\mathcal{N}(0; I_2)$ with its Voronoi tesselation.

[21]) that, if $p > 1$, $Q^p_{_N}$ is continuously differentiable at every N-tuple $y \in (\mathbb{R}^d)^N$ satisfying $\forall\, i \neq j,\ y^i \neq y^j$ and $P_Y\left(\cup_{i=1}^N \partial C_i(y)\right) = 0$. Its gradient $\nabla Q^p_{_N}$ is obtained by formal differentiation (see [21] for a rigorous proof), that is

$$\nabla Q^p_{_N}(y) = E\left[\nabla_y\, q^p_{_N}(y, Y)\right],$$

where

$$\nabla_y\, q^p_{_N}(y, v) = \left(\frac{\partial\, q^p_{_N}}{\partial y^i}(y, v)\right)_{1 \le i \le n} := p\left(\frac{y^i - v}{|y^i - v|}|y^i - v|^{p-1}\mathbf{1}_{C_i(y)}(v)\right)_{1 \le i \le n}. \tag{4.4}$$

for every $v \in \cup_{1 \le i \le N} C_i(y)$. Note that then, $\nabla_y q^p_{_N}(y, v)$ *has exactly one non-zero component* $i(y, v)$ defined by $v \in C_{i(y,v)}(y)$. (If P_Y is continuous the result still holds for $p = 1$.)

So, the gradient of $Q^p_{_N}$ has an integral representation with respect to the distribution of Y. This strongly suggests to implement the stochastic gradient descent derived from this representation to approximate some (local) minimum of $Q^p_{_N}$ (when $d \ge 2$, the implementation of deterministic gradient descent is unrealistic since it would rely on the computation of many integrals with respect to P_Y). This stochastic gradient descent is defined as follows: let $(\xi^s)_{s \ge 1}$ be a sequence of i.i.d. P_Y-distributed random variables and let $(\delta_s)_{s \ge 1}$ be a sequence of positive steps satisfying

$$\sum_s \delta_s = +\infty \quad \text{and} \quad \sum_s \delta_s^2 < +\infty. \tag{4.5}$$

Then, starting from an initial grid Γ^0 with N pairwise distinct components, set

$$\Gamma^{s+1} = \Gamma^s - (\delta_{s+1}/p)\nabla\, q^p_{_N}(\Gamma^s, \xi^{s+1}) \tag{4.6}$$

(this formula *a.s.* grants by induction that Γ^t has pairwise distinct components). Unfortunately, the usual assumptions that ensure the *a.s.* convergence of the algorithm (see [12]) are not fulfilled by $Q^p_{_N}$ (see, *e.g.* [12] or [20] for an overview on stochastic approximation). To be more specific, let us stress that $Q^p_{_N}(y)$ does not go to infinity as $|y|$ goes to infinity in $(\mathbb{R}^d)^N$ and $\nabla Q^p_{_N}$ is clearly not Lipschitz continuous on $(\mathbb{R}^d)^N$. Some *a.s.* convergence results in the Kushner & Clark sense have been obtained in [21] for compactly supported absolutely continuous distributions P_Y, mainly in the quadratic case $p = 2$ (however, regular *a.s.* convergence is established when $d = 1$). In fact the quadratic case is the most commonly implemented for applications and is known as the Competitive Learning Vector Quantization (CLVQ) algorithm.

Formulae (4.6) and (4.4) can be developed as follows if one sets $\Gamma^s := \{y^{1,s}, \ldots, y^{N,s}\}$,

(4.7) COMPETITIVE PHASE : select $i(s+1) := i(\Gamma^s, \xi^{s+1}) \in \operatorname{argmin}_i |y^{i,s} - \xi^{s+1}|$

(4.8) LEARNING PHASE : $$\left\{\begin{array}{lcl} y^{i(s+1),s+1} &:=& y^{i(s+1),s} - \delta_{s+1} \frac{y^{i(s+1),s}-\xi^{s+1}}{|y^{i(s+1),s}-\xi^{s+1}|} |y^{i(s+1),s} - \xi^{s+1}|^{p-1} \\ *[.5em] y^{i,s+1} &:=& y^{i,s}, \qquad i \neq i(s+1). \end{array}\right.$$

The competitive phase (4.7) corresponds to selecting the closest point in Γ^s *i.e.* $i(s+1)$ such that $\xi^{s+1} \in C_{i(s+1)}(\Gamma^s)$. The learning phase (4.8) consists in updating the closest neighbour and leaving unchanged other components of the grid Γ^s.

Furthermore, it is established in [21] that, if $Y \in L^{p+\varepsilon}$ $(\varepsilon > 0)$, then the sequences $(Q^{p,s}_{_N})_{s\geq 1}$ and $(\pi^s_i)_{s\geq 1}$, $1 \leq i \leq N$, of random variables recursively defined by

$$Q^{p,s+1}_{_N} := Q^{p,s}_{_N} - \delta_{s+1}(Q^{p,s}_{_N} - |y^{i(s+1),s} - \xi^{s+1}|^p), \qquad Q^{p,0}_{_N} := 0, \tag{4.9}$$

$$\pi^{s+1}_i := \pi^s_i - \delta_{s+1}(\pi^s_i - \mathbf{1}_{\{i=i(s+1)\}}), \qquad \pi^0_i := 1/N, \quad 1 \leq i \leq N. \tag{4.10}$$

satisfy on the event $\{\Gamma^s \to \Gamma^*\}$

$$Q^{p,s}_{_N} \xrightarrow{a.s.} Q^p_{_N}(\Gamma^*) \quad \text{and} \quad \pi^s_i \xrightarrow{a.s.} P_Y(C_i(\Gamma^*)),\ 1 \leq i \leq N,\ \text{as } s \to \infty.$$

These "companion" – hence costless – procedures yield the parameters (weights of the Voronoi cells, L^p-mean quantization error of Γ^*) necessary to exploit the grid Γ^* for numerical purpose. Note that this holds whatever the limiting grid Γ^* is: this means that the procedure is consistent.

Concerning practical implementations of the algorithm, it is to be noticed that, when $p = 2$, at each step the grid Γ^{s+1} lives in the convex hull of Γ^s and ξ^{s+1} which has a stabilizing effect on the procedure. One checks on simulation that the CLVQ algorithms does behave better than its non-quadratic counterparts. This follows from the cooperative procedure (4.8) which then simply becomes a homothety centered at ξ^{s+1} with ratio $1-\delta_{s+1}$.

4.2 Optimal Markovian quantization of the Euler scheme of Y

Let Γ_k denote for every $k \in \{0, \ldots, n\}$ the state space of $\widehat{Y}_k$ and N_k denote the size of Γ_k. For simplicity of notation, we set $\pi_k := \pi^{\Gamma_k}$ the closest neighbour projection on Γ_k. The parameters n and N_k, $1 \leq k \leq n$ being fixed, a quantized Euler scheme $(\widehat{Y}_k)_{0\leq k\leq n}$ can be considered as optimal if, at each step, the L^p-error induced by the quantization effect is minimal. This leads to built up the grids Γ_k recursively by setting

$$\Gamma_1 \quad \in \quad \operatorname{argmin}\left\{\|G_h(y_0, \varepsilon_1) - \pi^\Gamma\left(G_h(y_0, \varepsilon_1)\right)\|_{_p},\ |\Gamma| = N_1\right\} \tag{4.11}$$

$$\Gamma_{k+1} \quad \in \quad \operatorname{argmin}\left\{\|G_h(\widehat{Y}_k, \varepsilon_1) - \pi^\Gamma\left(G_h(\widehat{Y}_k, \varepsilon_1)\right)\|_{_p},\ |\Gamma| = N_{k+1}\right\} \tag{4.12}$$

(where $\widehat{Y}_k := \pi_k\Big(G_h(\widehat{Y}_{k-1},\varepsilon_k)\Big)$, $h = T/n$ and G_h is the Euler scheme operator). The proposition below gives an estimate for the asymptotic local quantization error induced by the L^p-optimal grids.

Proposition 4.1 *Assume that γ is (simply) elliptic ($\gamma\gamma^*(y) > 0$ for every $y \in \mathbb{R}^d$). Let $p \geq 1$ and $k \in \{1,\dots,n\}$. Let Γ_k denote an L^p-optimal grid. Set $N_k := |\Gamma_k|$. Then*

$$\|\Delta_k\|_p = \|G_h(\hat{Y}_{k-1},\varepsilon_k) - \pi_k(G_h(\hat{Y}_{k-1},\varepsilon_k))\|_p \sim J_{p,d}^{\frac{1}{p}} \|\hat{f}_k\|_{\frac{d}{d+p}}^{\frac{1}{p}} \frac{1}{|N_k|^{\frac{1}{d}}} \quad \text{as} \quad |N_k| \to +\infty. \tag{4.13}$$

where $\widehat{f}_k$ is the p.d.f of $G_h(\hat{Y}_{k-1},\varepsilon_k)$ (see (4.14) *below).*

Proof: It is straightforward that $G_h(\hat{Y}_{k-1},\varepsilon_k) \in L^{p+1}$. It follows from (2.2) that he distribution of $G_h(\hat{Y}_{k-1},\varepsilon_k)$ is absolutely continuous with respect to λ_d since ε_k and Y_{k-1} are independent and ε_k has a Gaussian distribution. Furthermore, as $\gamma\gamma^*(y)$ is everywhere invertible, its p.d.f. $\hat{f}_k$ given by

$$\hat{f}_k(y) = E\left(\frac{\exp\left(-\frac{(y-\hat{Y}_{k-1}-h\eta(\hat{Y}_{k-1}))^*(\gamma\gamma^*(\hat{Y}_{k-1}))^{-1}(y-\hat{Y}_{k-1}-h\eta(\hat{Y}_{k-1}))}{2h}\right)}{(2\pi h)^{d/2}\sqrt{\det(\gamma\gamma^*)(\hat{Y}_{k-1})}}\right). \tag{4.14}$$

The result follows from the Bucklew and Wise Theorem 4.1. $\square$

The term "local" is motivated by the fact that the above rate holds when N_k goes to infinity, the distribution of $\widehat{Y}_{k-1}$ being settled.

4.3 Optimal dispatching of the grid sizes

Recall from Theorem 3.1 and Remark 3.1, that the effect of the quantization on the rate of convergence of the value function of the stochastic control problem is measured by

$$\frac{1}{n^{\frac{1}{\hat{p}}}}\sum_{k=1}^{n-1}(n-k)\|\Delta_k\|_{\hat{p}} + \sum_{k=1}^{n}\|\Delta_k\|_{\hat{p}} \quad \text{for some } \hat{p} \geq 1. \tag{4.15}$$

We also have to bound quantities in the form :

$$n^{1-\frac{1}{r}}\sum_{k=1}^{n}\|\Delta_k\|_r \quad \text{for some } r \geq 1. \tag{4.16}$$

Then, the question of interest is : having at hand a global stack of N points, how can we assign N_k points to layer k, $N_1 + \cdots + N_n = N$, so that, producing an $L^{\hat{p}}$-optimal N_k-grid Γ_k on each layer k (inducing a mean quantization error $\|\Delta_k\|_{\hat{p}}$), provides the smallest global quantization error (4.15)?

Furthermore, the N_k have to be settled *prior to* the optimization procedure of the grids to keep the complexity of the global algorithm reasonable.

This problem has already been tackled in [3] for the original non-Markovian quantization method. A rigorous and satisfactory solution has been provided when the underlying diffusion is uniformly elliptic with bounded smooth coefficient η and γ *i.e.* lying in $\mathcal{C}_b^\infty(\mathbb{R}^d)$.

In our framework it turns out that this approach no longer works *stricto sensu* for the distributions of $G_h(\hat{Y}_{k-1}, \varepsilon_k)$ cannot be properly dominated by affine transforms of a *fixed* distribution.

An obvious induction on $k \geq 1$ using on Inequality (8.2) of Lemma 8.2 shows that

$$\forall\, k \in \{1, \ldots, n\}, \quad \widehat{Y}_k \stackrel{L^{\hat{p}}}{\longrightarrow} \bar{Y}_k \quad \text{as} \quad \min_{1 \leq i \leq n} N_i \to +\infty.$$

Let $\bar{f}_k$ denote the p.d.f. of the Euler scheme at time $t_k = kh = kT/n$, $k \geq 1$. If furthermore, γ is *uniformly elliptic*, then equation (4.14) implies that

$$\forall\, k \in \{1, \ldots, n\}, \quad \hat{f}_k \stackrel{\lambda_d \text{a.e.} \& L^1(\lambda_d)}{\longrightarrow} \bar{f}_k \quad \text{as} \quad \min_{1 \leq i \leq n} N_i \to +\infty.$$

(the L^1 convergence follows from Scheffé's Theorem). In particular, $\widehat{Y}_k$ converges in variation toward $\bar{Y}_k$. Consequently,

$$\forall\, k \in \{1, \ldots, n\}, \quad \|\hat{f}_{k+1}\|_{\frac{d}{d+\hat{p}}} \longrightarrow \|\bar{f}_{k+1}\|_{\frac{d}{d+\hat{p}}}. \tag{4.17}$$

If, furthermore η, $\gamma \in \mathcal{C}_b^\infty$ (and γ still uniformly elliptic), then (see [4]), for every $n \geq 1$,

$$\forall\, k \in \{1, \ldots, n\}, \quad \bar{f}_k(y) \leq \lambda \frac{\exp -(\frac{|y-y_0|^2}{2\mu^2 t_k})}{(2\pi\mu^2 t_k)^{\frac{d}{2}}} \qquad (\text{if } Y_0 = y_0)$$

where the real constants λ and μ only depend on T, η and $\gamma\gamma^*$ (especially its uniform ellipticity coefficient). Hence, a standard computation shows that

$$\|\bar{f}_k(y)\|_{\frac{d}{d+\hat{p}}} \leq \lambda (2\pi\mu^2 t_k)^{\frac{\hat{p}}{2}} (1 + \hat{p}/d)^{\frac{1}{2}(\frac{1}{d}+\frac{1}{\hat{p}})}$$

so that

$$\|\bar{f}_k(y)\|^{\frac{1}{\hat{p}}}_{\frac{d}{d+\hat{p}}} \leq \bar{c}_{\eta,\gamma,T,d,\hat{p}} \sqrt{t_k} \tag{4.18}$$

with $\bar{c}_{\eta,\gamma,T,d,\hat{p}} \leq e^{\frac{1}{d}} \lambda^{\frac{1}{\hat{p}}} \sqrt{2\pi}\mu$. Combining (4.17) and (4.18) one may reasonably conjecture the existence of a real constant $\hat{c}_{\eta,\gamma,T,d}$ such that

$$\|\hat{f}_k(y)\|^{\frac{1}{\hat{p}}}_{\frac{d}{d+\hat{p}}} \leq \hat{c}_{\eta,\gamma,T,d} \sqrt{t_k} \tag{4.19}$$

(what is not rigorously established at this stage is that $\hat{c}_{\eta,\gamma,T,d}$ may be chosen not depending on k, N_k and n). Now, plugging (4.19) in (4.13) and then in (4.15) and (4.16) yields

$$\frac{1}{n^{\frac{1}{\hat{p}}}} \sum_{k=1}^{n-1} (n-k) \|\Delta_k\|_{\hat{p}} + \sum_{k=1}^{n} \|\Delta_k\|_{\hat{p}} \leq \hat{C}_{\eta,\gamma,T,d,\hat{p}}\, n^{1-\frac{1}{\hat{p}}} \sum_{k=1}^{n} \Big(T(1 + n^{\frac{1}{\hat{p}}-1}) - t_k\Big) \frac{\sqrt{t_k}}{N_k^{\frac{1}{d}}}, \tag{4.20}$$

$$n^{1-\frac{1}{r}} \sum_{k=1}^{n} \|\Delta_k\|_r \leq C_{\eta,\gamma,T,d,r}\, n^{1-\frac{1}{r}} \sum_{k=1}^{n} \frac{\sqrt{t_k}}{N_k^{\frac{1}{d}}}, \tag{4.21}$$

for some positive constants $\hat{C}_{\eta,\gamma,T,d,\hat{p}}$ and $C_{\eta,\gamma,T,d,\hat{r}}$.

At this stage, it is possible to choose the N_k's in order to minimize the sum appearing at the r.h.s of (4.20) : one starts from the lemma below whose proof is left to the reader.

Lemma 4.1 *Let $a_1,\dots,a_n\in\mathbb{R}_+$ and $L\in\mathbb{R}_+$. Then*

$$\min\left\{\sum_{k=1}^n a_k x_k^{-\frac{1}{d}},\ (x_1,\dots,x_n)\in\mathbb{R}_+^n,\ x_1+\cdots+x_n\le L\right\}=\left(\sum_{k=1}^n a_k^{\frac{d}{d+1}}\right)^{1+\frac{1}{d}}L^{-\frac{1}{d}}.$$

It is reached by setting $x_k:=a_k^{\frac{d}{d+1}}\left(\sum_{l=1}^n a_l^{\frac{d}{d+1}}\right)^{-1}$, $k\in\{1,\dots,n\}$.

Following the above lemma, one sets for every $k\in\{1,\dots,n\}$,

$$N_k \quad = \quad \left\lceil \frac{\left(\left(T(1+n^{\frac{1}{\hat p}-1})-t_k\right)\sqrt{t_k}\right)^{\frac{d}{d+1}}N}{\sum_{j=1}^n\left(\left(T(1+n^{\frac{1}{\hat p}-1})-t_j\right)\sqrt{t_j}\right)^{\frac{d}{d+1}}}\right\rceil \tag{4.22}$$

with $\lceil x\rceil:=\min\{k\in\mathbb{N},\ k\ge x\}$. Doing so, we have $N_k\ge 1$ for all k and $N\le N_1+\cdots+N_k+\cdots+N_n\le N+n$. Moreover, the r.h.s. of (4.20) is estimated by :

$$\begin{aligned} n^{1-\frac{1}{\hat p}}\sum_{k=1}^n\left(T(1+n^{\frac{1}{\hat p}-1})-t_k\right)\sqrt{t_k}\,\frac{1}{N_k^{\frac{1}{d}}} &\le \frac{n^{2-\frac{1}{\hat p}+\frac{1}{d}}}{N^{\frac{1}{d}}}\left(\frac{1}{n}\sum_{k=1}^n\left((T(1+n^{\frac{1}{\hat p}-1})-t_k)\sqrt{t_k}\right)^{\frac{d}{d+1}}\right)^{1+1/d}\\ &\le \frac{n^{2-\frac{1}{\hat p}+\frac{1}{d}}(1+o(n))}{N^{\frac{1}{d}}}\left(\frac{1}{T}\int_0^T((T-u)\sqrt{u})^{\frac{d}{d+1}}du\right)^{1+1/d}, \end{aligned}$$

while the r.h.s. of (4.21) is bounded by :

$$\begin{aligned} & n^{1-\frac{1}{r}}\sum_{k=1}^n\sqrt{t_k}\frac{1}{N_k^{\frac{1}{d}}}\\ \le\ & \frac{n^{2-\frac{1}{r}+\frac{1}{d}}}{N^{\frac{1}{d}}}\frac{1}{n}\sum_{k=1}^n\frac{(\sqrt{t_k})^{\frac{1}{d+1}}}{\left(\left(T(1+n^{\frac{1}{\hat p}-1})-t_k\right)\sqrt{t_k}\right)^{\frac{d}{d+1}}}\left(\frac{1}{n}\sum_{k=1}^n\left((T(1+n^{\frac{1}{\hat p}-1})-t_k)\sqrt{t_k}\right)^{\frac{d}{d+1}}\right)^{1+1/d}\\ \le\ & \frac{n^{2-\frac{1}{r}+\frac{1}{d}}(1+o(n))}{N^{\frac{1}{d}}}\frac{1}{T}\int_0^T\frac{(\sqrt{u})^{\frac{1}{d+1}}}{((T-u)\sqrt{u})^{\frac{d}{d+1}}}du\left(\frac{1}{T}\int_0^T((T-u)\sqrt{u})^{\frac{d}{d+1}}du\right)^{1+1/d}. \end{aligned}$$

We then deduce the following estimation and bound results.

Proposition 4.2 *Let $\hat p, r\ge 1$. Assume that $\eta,\gamma\in\mathcal{C}_b^\infty$ and that γ is uniformly elliptic. If one considers $N+n$ points optimally dispatched among the n times layers following* (4.22) *into N_k-optimal grids, then :*

$$\frac{1}{n^{\frac{1}{\hat p}}}\sum_{k=1}^{n-1}(n-k)\|\Delta_k\|_{\hat p}+\sum_{k=1}^n\|\Delta_k\|_{\hat p} \quad\le\quad \hat C_{\eta,\gamma,T,d,\hat p}\frac{n^{2-\frac{1}{\hat p}}}{(N/n)^{\frac{1}{d}}}, \tag{4.23}$$

$$n^{1-\frac{1}{r}}\sum_{k=1}^n\|\Delta_k\|_r \quad\le\quad C_{\eta,\gamma,T,d,r}\frac{n^{2-\frac{1}{r}}}{(N/n)^{\frac{1}{d}}} \tag{4.24}$$

for some positive constants $\hat C_{\eta,\gamma,T,d,\hat p}$ and $C_{\eta,\gamma,T,d,r}$.

Moreover, when N, n and N/n get large, one has the following equivalent for the N_k's:

$$\text{If} \quad \hat{p} > 1 \quad N_k \sim \frac{\varphi_d(\frac{k}{n})}{\int_0^1 \varphi_d(v)\,dv}\,\frac{N}{n},\ 1 \le k \le n-1, \qquad \text{with} \qquad \varphi_d(v) = ((1-v)\sqrt{v})^{\frac{d}{d+1}},$$

$$N_n \sim \frac{1}{\int_0^1 \psi_d(v)\,dv}\,\frac{1}{n^{\frac{d(\hat{p}-1)}{\hat{p}(d+1)}}}\,\frac{N}{n},$$

$$\text{If} \quad \hat{p} = 1 \quad N_k \sim \frac{\psi_d(\frac{k}{n})}{\int_0^1 \psi_d(v)\,dv}\,\frac{N}{n},\ 1 \le k \le n, \qquad \text{with} \qquad \psi_d(v) = ((2-v)\sqrt{v})^{\frac{d}{d+1}}.$$

Remark 4.1 (about (4.23)): • One must keep in mind that the optimization of the dispatching procedure of the N_k's described above is only a way to obtain the *smallest possible real constant* $\hat{C}_{\eta,\gamma,T,d,\hat{p}}$ in (4.23). If one is simply interested in obtaining the upper-bound $O(\frac{n^{2-\frac{1}{\hat{p}}}}{(N/n)^{\frac{1}{d}}})$, it suffices to set $N_k = \lceil N/n \rceil + 1$ for every $1 \le k \le n$.

• Note that functions $u \mapsto (1-u)\sqrt{u}$ and $u \mapsto (2-u)\sqrt{u}$ respectively reach their maximum at $u_{\max} = 1/3$ and $u_{\max} = 2/3$. This means that in this dispatching rule, for large enough n, either $T/3$ or $2T/3$ is the more accurately quantized period of time according to the value of $\hat{p}$.

4.4 How to get the optimal grids Γ_k and the transition matrices?

To implement the above method, one needs to have, for every $k \in \{0, \dots, n\}$,

- a grid Γ_k which minimizes the L^p-quantization error $\|G(\hat{Y}_{k-1}, \varepsilon_k) - \pi_k(G(\hat{Y}_{k-1}, \varepsilon_k))\|_p$. This optimization is necessarily inductive since $\hat{Y}_{k-1}$ depends on the grid Γ_{k-1}, etc;
- the transition matrices $(\hat{p}^k_{ij})$ of the (non-homogenous) Markov chain $\hat{Y}_k$, $0 \le k \le n$. Namely, if one denotes $\Gamma_k := \{y^1_k, \dots, y^{N_k}_k\}$

  $$\hat{p}^k_{ij} := P(\hat{Y}_{k+1} = y^j_{k+1} \,|\, \hat{Y}_k = y^i_k), \quad 1 \le i \le N_k,\ 1 \le j \le N_{k+1}$$

  For numerical purpose, one introduces the joint distribution matrix $(\hat{\beta}^k_{ij})$ and the marginal distribution vector $(\hat{p}^k_i)$

  $$\begin{aligned} \hat{\beta}^k_{ij} &:= P(\hat{Y}_{k+1} = y^j_{k+1} \,\text{and}\, \hat{Y}_k = y^i_k), \quad 1 \le i \le N_k,\ 1 \le j \le N_{k+1}, \\ \hat{p}^k_i &:= P(\hat{Y}_k = y^i_k), \qquad 1 \le i \le N_k. \end{aligned}$$

  so that $\hat{p}^k_{ij} = \dfrac{\hat{\beta}^k_{ij}}{\hat{p}^k_i}$, $1 \le i \le N_k,\ 1 \le j \le N_{k+1}$.
- The L^p-quantization error $\|\Delta_k\|_p = \|G_h(\hat{Y}_{k-1}, \varepsilon_k) - \hat{Y}_k\|_p$, $1 \le k \le n$.

We present below one method to achieve this program by simulation. It is a global approach which carries out all the tasks simultaneously. The algorithm is designed for a general L^p setting, $p \ge 1$.

The optimal quantized Euler scheme (*i.e.* the grids that satisfy (4.12)) is obtained by a cascade of stochastic gradient descents (4.6) (or (4.7) & (4.8)) derived from the successive

L^p-mean quantization error functions. The companion parameters are computed recursively as a by-product following (4.9) and (4.10). This algorithm appears as an adaptation of the procedure developed in [3]. The gain parameter sequences $(\delta^k_s)_{s\geq 1}$ of the gradient descents may depend on the layer k (but will always satisfy the regular "decreasing step" assumption (4.5)). In practice, one sets $\delta^k_s := \frac{A^k}{B^k+s}$, $s \geq 1$. Theoretical optimal choice for A^k and B^k do exist (see, *e.g.*, [12]). They are usually out of reach on practical situations (they rely on the computation of the Hessian of the potential function, here the successive L^p-mean quantization errors $Q^p_{_N}$).

One starts from some initial grids Γ^0_k, $0 \leq k \leq n$ at step $s = 0$, and let us denote by $\Gamma^s_k := \{y^{i,s}_k,\ 1 \leq i \leq N_k\}$ the k^{th} grid at time s during the optimization phase ($\Gamma^s_0 := \{y^1_0\}$ for every $s \in \mathbb{N}$). Let us denote by $\hat\beta^{k,s}_{i,j}$ and $\hat p^{k,s}_i$ the recursive estimator at step s of $\hat\beta^k_{i,j}$ and $\hat p^k_i$ and by $Q^{p,k,s}$ the recursive estimator of the L^p-quantization error at step s. Set $\hat\beta^{k,0}_{i,j} = 1/N_{k+1}$, $\hat p^{k,0}_i = 1/N_k$ and $Q^{p,k,0} = 0$. The updating procedure of the above quantities from $s-1$ to s is as follows

- *Updating* Γ^s_1

  1. Simulate a new trial $\xi^{1,s}$ of $G_h(y^1_0, \varepsilon_1) = y^1_0 + h\,\eta(y^1_0) + \sqrt{h}\,\gamma(y^1_0)\varepsilon_1$, $\varepsilon_1 \sim \mathcal{N}(0; I_m)$.
  2. Select the nearest neighbour in the current grid Γ^{s-1}_1 *i.e.* (the lowest integer) $j_1(s) \in \{1, \ldots, N_1\}$ such that

  $$|y^{j_1(s),s-1}_1 - \xi^{1,s}| = \min_{y^{j,s-1}_1 \in \Gamma^{s-1}_1} |y^{j,s-1}_1 - \xi^{1,s}| \tag{4.25}$$

  3. Update the grid Γ^{s-1}_1 by moving $y^{j_1(s),s-1}_1$ by an homothety centered at $\xi^{1,s}$ with ratio $1 - \delta^1_s |y^{j_1(s),s-1}_1 - \xi^{1,s}|^{p-2}$ whereas other components are left unchanged (note that when $p = 2$ the ratio of the homothety is simply $1 - \delta_s$):

  $$y^{j_1(s),s}_1 = y^{j_1(s),s-1}_1 - \delta^1_s \frac{y^{j_1(s),s-1}_1 - \xi^{1,s}}{|y^{j_1(s),s-1}_1 - \xi^{1,s}|} |y^{j_1(s),s-1}_1 - \xi^{1,s}|^{p-1}, \quad y^{j,s}_1 = y^{j,s-1}_1,\ j \neq j_1(s). \tag{4.26}$$

  4. Update the joint distribution matrix $[\hat\beta^{0,s-1}_{1,j}]_{1\leq j\leq N_1}$:

  $$\hat\beta^{0,s}_{1,j_1(s)} := \hat\beta^{0,s-1}_{1,j_1(s)} - \delta^1_s(\hat\beta^{0,s-1}_{1\,j_1(s)} - 1),\ \hat\beta^{0,s}_{1\,j} := \hat\beta^{0,s-1}_{1\,j} \text{ if } \quad j \neq j_1(s) \tag{4.27}$$

  $$\hat p^{0,s}_1 := 1. \tag{4.28}$$

  5. Update the quantization error (like in (4.9))

  $$Q^{p,1,s} = Q^{p,1,s-1} - \delta^1_s(Q^{p,1,s-1} - |y^{j_1(s),s-1}_1 - \xi^{1,s}|^p). \tag{4.29}$$

- *Updating* Γ^s_k, $2 \leq k \leq n$:

  1. Simulate a new trial $\xi^{k,s}$ of $G_h(y^{j_{k-1}(s),s}_{k-1}, \varepsilon_k)$.

2. Select the nearest neighbour in the current grid Γ_k^{s-1} *i.e.* (the lowest integer) $j_k(s) \in \{1, \ldots, N_k\}$ such that

$$|y_k^{j_k(s),s-1} - \xi^{k,s}| = \min_{y_k^{j,s-1} \in \Gamma_k^{s-1}} |y_k^{j,s-1} - \xi^{k,s}| \tag{4.30}$$

3. Update the grid Γ_k^{s-1} by moving $y_k^{j_k(s),s-1}$ by an homothety centered at $\xi^{k,s}$ with ratio $1 - \delta_s^k$ whereas other components are left unchanged :

$$y_k^{j_k(s),s} = y_k^{j_k(s),s-1} - \delta_s^k \frac{y_k^{j_k(s),s-1} - \xi^{k,s}}{|y_k^{j_k(s),s-1} - \xi^{k,s}|} |y_k^{j_k(s),s-1} - \xi^{k,s}|^{p-1}, \; y_k^{j,s} = y_k^{j,s-1}, \; j \neq j_k(s). \tag{4.31}$$

4. Update the joint distribution matrix $[\hat{\beta}_{i\,j}^{k-1,s-1}]_{1 \le i \le N_{k-1}, 1 \le j \le N_k}$ and the marginal distribution vector $[\hat{p}_j^{k,s-1}]_{1 \le j \le N_k}$

$$\hat{\beta}_{j_{k-1}(s)\,j_k(s)}^{k-1,s} = \hat{\beta}_{j_{k-1}(s)\,j_k(s)}^{k-1,s-1} - \delta_s^k (\hat{\beta}_{j_{k-1}(s)\,j_k(s)}^{k-1,s-1} - 1), \tag{4.32}$$

$$\hat{p}_{j_k(s)}^{k,s} = \hat{p}_{j_k(s)}^{k,s-1} - \delta_s^k (\hat{p}_{j_k(s)}^{k,s-1} - 1) \tag{4.33}$$

5. Update the quantization error

$$Q^{p,k,s} = Q^{p,k,s-1} - \delta_s^k (Q^{p,k,s-1} - |y_k^{j_k(s),s-1} - \xi^{k,s}|^p). \tag{4.34}$$

- *Computation of the transition matrix after M trials*

$$\hat{p}_{i\,j}^{k-1,M} := \frac{\hat{\beta}_{i\,j}^{k-1,M}}{\hat{p}_i^{k-1,M}}, \; 1 \le i \le N_{k-1}, \; 1 \le j \le N_k, \; 1 \le k \le n.$$

This optimization algorithm can easily be processed on $M = 10^6$ or more trials in a few minutes (less than 10).

Remark 4.2 One possible drawback of the above algorithm is that it carries out simultaneously the optimization of all the grids: Γ_k is optimized before its "ancestor" Γ_{k-1} is settled. This may cause numerical instability since Γ_k quantizes $G_h(\hat{Y}_{k-1}, \varepsilon_k)$ where $\hat{Y}_{k-1}$ is Γ_{k-1}-supported. A possible alternative consists in proceeding sequentially by freezing successively all the layers: this means stopping the optimization procedure of Γ_k before starting that of Γ_{k+1}. The adaptation of the procedure is easy.

5 Convergence of the value function and approximate controls

Fix the initial conditions (x_0, y_0) of the continuous state process (X, Y). Given an admissible control $\alpha \in \mathcal{A}$, we denote by $J(\alpha)$ the cost function

$$J(\alpha) = E\left[\int_0^T f(X_u, Y_u, \alpha_u) du + g(X_T, Y_T)\right] \tag{5.1}$$

and by $v_0 = v(0, x_0, y_0)$ the value function at time 0 of the continuous control problem :

$$v_0 \;=\; \inf_{\alpha \in \mathcal{A}} J(\alpha).$$

Given a time discretization parameter n, and a control $\bar{\alpha} \in \bar{\mathcal{A}}_n$, we denote by $\bar{J}^n(\bar{\alpha})$ the cost function associated to the control problem (2.6) at the initial date 0 :

$$\bar{J}^n(\bar{\alpha}) \;=\; E\left[\sum_{j=0}^{n-1} hf(\bar{X}_j, \bar{Y}_j, \bar{\alpha}_j) + g(\bar{X}_n, \bar{Y}_n)\right],$$

and by $\bar{v}_0^n = \bar{v}_0(x_0, y_0)$ its value function :

$$\bar{v}_0^n \;=\; \inf_{\bar{\alpha} \in \bar{\mathcal{A}}_n} \bar{J}^n(\bar{\alpha}).$$

Now, for Y-spatial discretization parameter N, X-spatial discretization parameters R and δ, we denote for any control $\bar{\alpha} \in \bar{\mathcal{A}}_n$, by $\hat{J}^{n,N,R,\delta}(\bar{\alpha})$ the cost function associated to the control problem (3.3) at the initial date 0 :

$$\hat{J}^{n,N,R,\delta}(\bar{\alpha}) \;=\; E\left[\sum_{j=0}^{n-1} hf(\hat{X}_j, \hat{Y}_j, \bar{\alpha}_j) + g(\hat{X}_n, \hat{Y}_n)\right].$$

We denote by $\hat{v}_0^{n,N,R,\delta} = \hat{v}_0(x_0, y_0)$ its value function :

$$\hat{v}_0^{n,N,R,\delta} \;=\; \inf_{\bar{\alpha} \in \bar{\mathcal{A}}_n} \hat{J}^{n,N,R,\delta}(\bar{\alpha}).$$

We know from Proposition 4.2 that when the $N + n$ points of the Y-grid are optimally dispatched among the n layers in time, for any $r \geq 1$, we have the following bound :

$$n^{1-\frac{1}{r}} \sum_{k=1}^{n} \|\Delta_k\|_r \;\leq\; C \frac{n^{2-\frac{1}{r}}}{\left(\frac{N}{n}\right)^{\frac{1}{d}}}$$

for some positive constant C. Now, with the notations of Theorem 3.1, choose $p = 1$, $\hat{p} = \max(2, p_1 + 1)$ (so that $\hat{p} = \max(\hat{q}_1, 2)$), $\bar{p} > 1$, and n, N such that

$$N \;\geq\; C n^{1+2d-\frac{d}{\hat{p}\bar{p}}} \tag{5.2}$$

for some positive constant C. Hence, from the estimate of Theorem 3.1 (see also Remark 3.1), and Proposition 4.2, we have the following rate of convergence for the value functions :

$$\left|\bar{v}_0^n - \hat{v}_0^{n,N,R,\delta}\right| \;\leq\; C_1 \frac{n^{2-\frac{1}{\hat{p}}}}{(N/n)^{\frac{1}{d}}} + C_2 n\delta + C_3(\bar{p}) \frac{n}{R^{\bar{p}-1}}, \tag{5.3}$$

with C_1, C_2 and $C_3(\bar{p})$ are positive constants depending on $d, T, \hat{p}$ and the coefficients of Y.

Therefore, by combining this last estimate with the convergence result of Proposition 2.1, we deduce the convergence of the value function approximated by optimal quantization :

$$\hat{v}_0^{n,N,R,\delta} \;\longrightarrow\; v_0, \tag{5.4}$$

as n goes to infinity, (5.2) holds, $n\delta$, and $n/R^{\bar{p}-1}$ go to zero.

The value function $\hat{v}_0^{n,N,R,\delta}$ is computed recursively by the dynamic programming formula (3.4). Moreover, this scheme allows us to compute at each step k an optimal control in $\bar{\mathcal{A}}_n$ by taking the infimum in (3.4) :

$$\hat{\alpha}_k \quad = \quad \hat{\alpha}_k(\hat{Y}_k), \quad k = 0, \dots, n.$$

By misuse of notation, we still denote by $\hat{\alpha}$ the stepwise constant control process defined on $[0, T)$ by :

$$\hat{\alpha}_t \quad = \quad \hat{\alpha}_k, \quad \text{for } t \in [t_k, t_{k+1}), \quad k = 0, \dots, n-1. \tag{5.5}$$

Hence, $\hat{\alpha}$ lies in $\mathcal{A}$ and may be applied to the original continuous time control problem v_0. Similarly as in (5.5), any control $\alpha \in \bar{\mathcal{A}}_n$ may be interpolated into a continuous control in $\mathcal{A}$. With this continuous interpolation, we have $\bar{\mathcal{A}}_n \subset \mathcal{A}$. We now check that $\hat{\alpha}$ is ε-optimal for the original problem. This problem of approximating control has been extensively studied by Runggaldier et al. [24], and we closely follow their arguments. Denote by $\bar{Z}_t^n = (\bar{X}_t^n, \bar{Y}_t^n)$, $0 \leq t \leq T$, the continuous Euler scheme associated to the diffusion (X, Y) (X is controlled by $\alpha \in \bar{\mathcal{A}}_n$): it is defined for all $t_k \leq t < t_{k+1}$, $k = 0, \dots, n-1$ by

$$\begin{aligned} \bar{X}_t^n &= \bar{X}_k^n + b(\bar{X}_k^n, \bar{Y}_k^n, \bar{\alpha}_k)(t - t_k) + \sigma(\bar{X}_k^n, \bar{Y}_k^n, \bar{\alpha}_k)(W_t - W_{t_k}) \\ \bar{Y}_t^n &= \bar{Y}_k^n + \eta(\bar{Y}_k^n)(t - t_k) + \gamma(\bar{Y}_k^n)(W_t - W_{t_k}). \end{aligned}$$

Using the Lipschitz continuous assumption **(H1)** (uniform with respect to the control term a), and mimicking standard methods to get L^p-error estimates on the Euler scheme (see e.g. [16]) yield for every $p \geq 1$:

$$\sup_{\bar{\alpha} \in \mathcal{A}_n} E\left[\sup_{t \in [0,T]} \left|\bar{Z}_t^n - Z_t\right|^{2p}\right] \quad \leq \quad \frac{C_{p,T,b,\sigma}}{n^p},$$

for some positive constant $C_{p,T,b,\sigma}$. Under assumption **(H3b)**, one easily checks that :

$$\begin{aligned} \sup_{\bar{\alpha} \in \bar{\mathcal{A}}_n} \left|\bar{J}^n(\bar{\alpha}) - J(\bar{\alpha})\right| \quad &\leq \quad \sup_{\bar{\alpha} \in \bar{\mathcal{A}}_n} E\left[\int_0^T |f(\bar{Z}_t^n, \bar{\alpha}_t) - f(Z_t, \bar{\alpha}_t)|dt + |g(\bar{Z}_T^n) - g(Z_T)|\right] \\ &\longrightarrow \quad 0, \end{aligned}$$

as n goes to infinity. Moreover, by same arguments as in the proof of the convergence (5.3), we also have :

$$\sup_{\bar{\alpha} \in \bar{\mathcal{A}}_n} \left|\hat{J}^{n,N,R,\delta}(\bar{\alpha}) - J(\bar{\alpha})\right| \quad \leq \quad C_1 \frac{n^{2-\frac{1}{\bar{p}}}}{(N/n)^{\frac{1}{d}}} + C_2 n\delta + C_3(\bar{p}) \frac{n}{R^{\bar{p}-1}}. \tag{5.6}$$

Therefore, for any $\varepsilon > 0$, one can find n, N, R, δ, such that :

$$\sup_{\bar{\alpha} \in \bar{\mathcal{A}}_n} \left|\hat{J}^{n,N,R,\delta}(\bar{\alpha}) - J(\bar{\alpha})\right| \leq \varepsilon/2 \quad \text{and} \quad \left|\hat{v}_0^{n,N,R,\delta} - v_0\right| \leq \varepsilon/2.$$

Now, by definition of $\hat{\alpha}$, we have $\hat{v}_0^{n,N,R,\delta} = \hat{J}^{n,N,R,\delta}(\hat{\alpha})$. We then deduce that

$$J(\hat{\alpha}) \quad \leq \quad v_0 + \varepsilon,$$

which means that $\hat{\alpha}$ is ε-optimal for the original control problem, provided n, N are sufficiently large such that (5.2) holds and $n\delta$, $n/R^{\bar{p}-1}$ are sufficiently small.

6 Possible variants: a short discussion

To illustrate how optimal quantization can be used as a numerical method for stochastic control, some features of the problem have been specified. Some cannot be easily modified – at least from a theoretical point of view – some can without damage. Thus, to get some true *a priori* error bounds needs to quantize only the uncontrolled Y part of the whole process (X, Y). The main interest of quantization is to compress and store some information on the distribution of a Markov chain after an optimization/estimation phase ($CLVQ$). So the set of optimal quantizers and companion parameters must not depend upon the control. (In fact, as far as numerics are concerned, some quantized grid can probably help but not as they are used here).

Let us pass to less crucial specifications. In the present paper, the starting value of the uncontrolled process has been settled to a deterministic value and the quantization has been processed accordingly. This is natural given the applications we had in mind in which Y is an asset price process which value at time 0 is known, but this may appear as an important drawback for other models in which the value function needs to be computed at several (closed) points. There are many ways to overcome this restriction. First, any random starting value Y_0 can be considered as well as its induced quantization grid Γ_0. This will yield some proxies of the value functions $v(0, y_0)$ for different points y_0. Doing so maintains the complexity of the estimation procedure but induces a loss of accuracy. On the other hand one may also implement on appropriate computers a parallel version of the $CLVQ$ procedure starting from different points (one can then take advantage from the fact that these optimal quantizations will remain closed during the closest neighbour search phase). Finally, in some more specific examples where the diffusion Y_t can be written as a function $\varphi(t, W_t)$ like in the Black & Scholes model, it is much more convenient to use an optimal quantization of the standard d-dim Brownian motion $(W_t)_{t\in[0,T]}$ which is a universal object computed for once and stored.

The choice of the Euler scheme can also be justified. This is a first order scheme. In 1-dim, the Milshtein scheme is defined as follows

$$\bar{Y}_{k+1} \;=\; \bar{Y}_k + \eta(\bar{Y}_k)h + \gamma(\bar{Y}_k)\sqrt{h}\,\varepsilon_{k+1} + \frac{1}{2}\gamma'(\bar{Y}_k)\gamma(\bar{Y}_k)h\left(\varepsilon_{k+1}^2 - 1\right) \;:=\; G_h^{Mil}(\bar{Y}_k, \varepsilon_{k+1}),$$

where $\sqrt{h}\varepsilon_k := W_{kh} - W_{(k-1)h}$ is the k^{th} Brownian increment of time step h and γ' is for the derivative of γ. There is no doubt that Theorem 3.1 can be established using the Milshtein scheme instead of the Euler scheme (in any dimension). But when one looks carefully at the proof, it appears that *the rate of convergence* of the scheme $\bar{Y}_k$ toward the diffusion Y_y is not involved : only the L^p-Lipschitz property of the discretization functional $y \mapsto \bar{Y}_1^y = G_h(y, \varepsilon_1)$ (resp. $G_h^{Mil}(y, \varepsilon_1)$) and some L^p-boundedness properties do. These properties are shared by both schemes (with much more technicalities for the second one).

The main advantage of the Milshtein scheme lies in its L^p-rate of convergence toward the underlying diffusion since – under some commutativity assumptions involving γ and its partial derivatives in higher dimensions (see [16]) – it converges at a $O(h)$-rate (instead of $O(\sqrt{h})$-rate for the Euler scheme). In fact, the rate of convergence of the scheme possibly has an influence on the rate of convergence of the time discretized control problem toward the continuous time control problem. But this aspect is not investigated here (only the convergence is), one reason being that, in this phase quantization plays no rôle.

7 Numerical simulations

We numerically apply in this section the algorithms given in Section 3 to the portfolio optimization problems. More precisely, we focuse on the replication of an European Put option using self-financing strategies in a mean-variance hedging context. A stochastic volatility model (SVM) is considered in which the process Y (see Eq. (1.2)) is 2-dimensional and denotes the asset price together with the volatility. The process X (see Eq. (1.1)) is the wealth process. In the SVM model, the portfolio is traded with the asset price component of Y.

Let us emphasize that, since Y does not depend on X, the quantization of Y are in the two cases computed *off-line* using methods introduced in Section 4. It means that the discretized semi-group ($\{\hat{p}_{i,j}\}$ in (3.4)) and the optimal grids $\{y_k^i\}$ are kept off-line once the optimal quantization has been achieved. Discretization of the wealth process $\{x_\ell\}_{-n_X \le \ell \le n_X}$ is done according an uniform mesh in dimension 1 of step Δ. We will compute the value function given by (3.4) together with the optimal control at the points (x_ℓ, y_k^i) for $\ell = -n_X, \ldots, +n_X$, $i = 1, \ldots, N_k$, $k = 0, \ldots, n$.

For the sake of completeness, let us now describe the SVM model. Let $S := (S_t)_{t\in[0,T]}$ be a traded "risky" asset whose price process is driven by a diffusion process with stochastic volatility given by

$$\text{(7.1)} \qquad \begin{cases} dS_t &= \sigma_t S_t dW_t^1, \quad S_0 := s_0, \\ \sigma_t &= |Z_t + \sigma_0|, \\ dZ_t &= -\eta\, Z_t\, dt + \beta dW_t^2, \quad Z_0 = 0, \end{cases}$$

where $W := (W^1, W^2)$ is a standard 2-dimensional Wiener process. For notational convenience, set $Y_t := (S_t, \sigma_t)$. (We assume for the sake of simplicity that the "riskless" interest rate is 0 and S is already a martingale under P).

We are looking for the optimal quadratic risk minimization of a *European Put* option on S (maturity $T > 0$ and strike price K) using some self-financing ($\underline{\mathcal{F}}^S$-predictable) strategies. Let α_t denote the quantity of assets S held in the portfolio at time t. In order to bound the risk induced by the authorized strategies, we define the set $\mathcal{A}$ of admissible self-financing ($\underline{\mathcal{F}}^S$-progressively measurable) strategies $(\alpha_t)_{t\in[0,T]}$ by

$$\mathcal{A} := \left\{(\alpha_t)_{t\in[0,T]} \,/\, \forall\, t \in [0,T], \quad \alpha_t \in A := \mathbb{R}\right\}.$$

Let x_0 be the initial wealth to be invested in the replication procedure, $(\alpha_t)_{t\in[0,T]} \in \mathcal{A}$ and $X_t^{x_0,\alpha}$ the value of the resulting portfolio at time t. The process $(X_t^{x_0,\alpha})_{t\in[0,T]}$ satisfies

$$dX_t^{x,\alpha} = \alpha_t\, dS_t = \begin{pmatrix} \alpha_t \\ 0 \end{pmatrix} .dY_t, \qquad X_0 = x_0 > 0.$$

This leads to the following stochastic control problem with horizon T :

$$v_0(x_0, s_0, \sigma_0) = \inf_{(\alpha_t)\in\mathcal{A}} E\left(\left(X_T^{x_0,\alpha} - (K - S_T)_+\right)^2\right)$$

We are interested in approximating by quantization $E[v_0(x_0, s_0, \sigma_0)]$ and the optimal control α^{opt}. On the other hand, we have access to a closed form for both quantities.

Let $P(t, s, \sigma) := E[(K - S_T)_+ \mid (S_t, \sigma_t) = (s, \sigma)]$. Using that $P(t, S_t, \sigma_t)$, one gets using Itô's formula

$$(K - S_T)_+ = E((K - S_T)_+) + \int_0^T \frac{\partial P}{\partial S}(t, S_t, \sigma_t) dS_t + Z_T.$$

with $Z_T = \beta \int_0^T \frac{\partial P}{\partial \sigma}(t, S_t, \sigma_t) dW_t^2$. Hence

$$E\left(\left(X_T^{x_0,\alpha} - (K - S_T)_+\right)^2\right) = (x_0 - E((K - S_T)_+))^2 + E\left(\int_0^T \left(\alpha_t - \frac{\partial P}{\partial S}(t, S_t, \sigma_t)\right)^2 S_t^2 dt\right) + \mathbb{E}(Z_T^2)$$

Consequently

$$(7.2)\ v_0(x_0, s_0, \sigma_0) = (x_0 - E((K - S_T)_+)^2 + C \quad \text{and} \quad \alpha^{opt}(t, S_t, \sigma_t) = \frac{\partial P}{\partial S}(t, S_t, \sigma_t)$$

In particular the function $x_0 \mapsto v_0(x_0, s_0, \sigma_0)$ reaches its minimum at $x_{\min} = E((K - S_T)_+)$ and the optimal control α^{opt} does not depend upon the initial wealth x_0 (but depends on s_0 and σ_0. A lower bound for $x_{\min}$ is available since, the volatility process σ_t being independent of W^1, Jensen inequality yields

$$x_{\min} = E((K - S_T)_+) > \mathrm{Put}_{\mathrm{B\&S}}(s_0, K, T, \sigma_0).$$

Let us pass to the numerical treatment of the problem. In a first time, algorithm (3.4) is rewritten as

$$v_n^{i,\ell} = \left(x_\ell - \left(K - s_n^i\right)_+\right)^2, \quad -n_X \le \ell \le +n_X,\ 1 \le i \le N_n \tag{7.3}$$

$$v_k^{i,\ell} = \inf_{\alpha \in \mathcal{A}} \sum_{j=1}^{N_{k+1}} \hat{p}_{i,j}^k\ v_{k+1}^{j,m}, \quad -n_X \le \ell \le +n_X,\ 1 \le i \le N_k \tag{7.4}$$

where

$$m = m(i, j, \ell, \alpha) = \left[\ell + \frac{\alpha\left(s_{k+1}^j - s_k^i\right)}{\Delta} + 0.5\right]. \tag{7.5}$$

The optimal control is given by

$$\alpha_k^{i,\ell} = \mathrm{argmin}_{\alpha \in \mathcal{A}} \sum_{j=1}^{N_{k+1}} \hat{p}_{i,j}^k v_{k+1}^{j,m(i,j,\ell,\alpha)}. \tag{7.6}$$

Let us note that, since we have to deal with fixed arrays, we would need to have

$$-n_X \le m(i, j, \ell, a) \le +n_X,$$

for any $(i,j) \in N_k \times N_{k+1}$ and $a \in [-1,1]$ where $\ell(i,j,a)$ is defined by (7.5). We can not do this because it would be very space consumer unless we find suitable "boundary conditions" for the values of $v_{k+1}^{j,\ell}$ with $|\ell| > n_X$. It is why we will work with a shifted wealth process

$$x_k^{i,\ell} = x_\ell - P_k^i, \quad 1 \le i \le N_k, \quad 0 \le k \le n, \quad -n_X \le \ell \le n_X, \tag{7.7}$$

where P_k^i is the quantized approximation of the European price. Hence, Algorithm (7.3)–(7.5) is rewritten:

$$v_n^{i,\ell} = \left(x_n^{i,\ell}\right)^2, \quad -n_X \le \ell \le +n_X,\ 1 \le i \le N_n \tag{7.8}$$

$$v_k^{i,\ell} = \inf_{\alpha \in \mathcal{A}} \sum_{j=1}^{N_{k+1}} \hat{p}_{i,j}^k \, v_{k+1}^{j,m}, \quad -n_X \le \ell \le +n_X,\ 1 \le i \le N_k \tag{7.9}$$

$$P_n^i = (K - s_n^i)_+, \quad 1 \le i \le N_n \tag{7.10}$$

$$P_k^i = \sum_{j=1}^{N_{k+1}} \hat{p}_{i,j}^k \, P_{k+1}^j, \quad 1 \le i \le N_k \tag{7.11}$$

where

$$m = m(i,j,\ell,\alpha) = \left[\ell + \frac{P_k^i - P_{k+1}^j + \alpha\left(s_{k+1}^j - s_k^i\right)}{\Delta} + 0.5 \right]. \tag{7.12}$$

This choice is motivated by the expression of the optimal control in (7.2). This explains why the term

$$-(P_{k+1}^j - P_k^i) + \alpha(s_{k+1}^j - s_k^i)$$

should not be too large near the optimum. Nevertheless, when $|m|$ becomes greater than n_X in (7.9)–(7.12), we set

$$v_{k+1}^{j,m} = \left(x_{k+1}^{j,m}\right)^2. \tag{7.13}$$

The following computations are done with the algorithm (7.8)–(7.12) and (7.13).

Off–line computations.

The parameters of the process $Y_t = (S_t, \sigma_t)$ (see (7.1)) used for the simulation are $(s_0, \overline{\sigma}, \eta, \beta) = (100, 0.2, 1.0, 0.2)$. Let us note the value of the asymptotic square mean deviation of σ_t:

$$\nu = \frac{\beta}{\sqrt{2\eta}} \approx 0.14.$$

The main task of theses off–line computations consists in the optimal quantization of the couple $(\ln(S_t), Z_t)$. This is achieved owing to the numerical methods described in Section 4. We use for this $M = 10^6$ independant trials of the Euler scheme applied to (7.1) (where $\ln(S_t)$ is simulated instead of S_t) with n time steps on the time interval $[0,T]$. Here $(n,T) = (25,1)$.

Spatial discretization is done with $N = 1 + \ldots + N_n = 5746$ according the dispatching rule described in subsection 4.3.

This gives us the grids $\{(\ln(s_k^i), z_k^i)\}$ together with the weights $\{\hat{p}_{i,j}\}$ where $\sigma_k^i = |\sigma_0 + z_k^i|$. Figure 2 depicts such an optimal quantization $(\ln(s_n^i), z_n^i)$ of the random variable $(\ln(S_T), Z_T)$ with $N_n = 300$ points. The CPU time needed for the quantization of the whole process with $M = 10^6$ independant trials of the Euler Scheme took 1580 seconds on a Pentium III 500MHz.

On–line computations.

Algorithm (7.8)–(7.12) and (7.13) is then applied. The numerical results presented here have been obtained with the parameters $n_X \in \{50, 100, 200\}$, $\Delta = 5/n_X$ and $K = 100 = s_0$.

In Figure 3 is depicted the value function $v_0(x_0, s_0, \sigma_0)$ as a function of x_0 (See (7.2)). Its shape is those of a parabola as expected and its minimum located at $x_{min} = 8.55$, a good approximation of the price computed by Monte-Carlo (see Table 1). Let us recall that the B&S price with $\sigma_t = \sigma_0$ is 7.96.

In Table 2 are summed up the location of the minimum of v_0 and the optimal control associated for different values of n_X together with the CPU time needed for the computations. As regards the prices, we see that a convergence towards the right value (8.55) seems to arise when n_X becomes large. Nevertheless the different values of the optimal control remain good approximations of the value obtained by Monte-Carlo (in Table 1) even for low discretization.

Then, in order to test the trajectorial behaviour of the algorithm, we simulate one trajectory of the process Y and compute at each time–step an approximation of the wealth process X_{t_k} by the rule

$$X_{k+1} = X_k + \alpha_k^{i(\omega),\ell(\omega)} (S_{k+1} - S_k), \quad X_0 = x_0,$$

where S_k is the Euler approximation of S_{t_k} and $i(\omega), \ell(\omega)$ the indices of the nearest neighbour on the grid of (X_k, S_k, σ_k) at the time t_k of the simulation. The initial value x_0 has been chosen according to the values of Table 2. Figure 4 shows the trajectories of both the wealth process and the contingent claim obtained with $n_X = 200$. Figure 5 shows the evolution of the associated optimal control $k \mapsto \alpha_k^{i(\omega),\ell(\omega)}$.

Table 1: Monte-Carlo price and delta at $t = 0$ with the quantized price.

Quantized price	8.55
Monte–Carlo price	8.51
Monte–Carlo delta	**-0.457**

Table 2: Minimum value of mean-variance $x_0 \mapsto v_0(x_0, s_0, \sigma_0)$ (see 7.2) at time $t = 0$. See also Figure 3.

n_X	$\min_\ell v_0^{0,\ell}$	$\alpha_0^{0,\star}$	CPU time (sec.)
50	8.158	-0.4453	319
100	8.439	-0.4609	587
200	8.550	-0.4336	1127

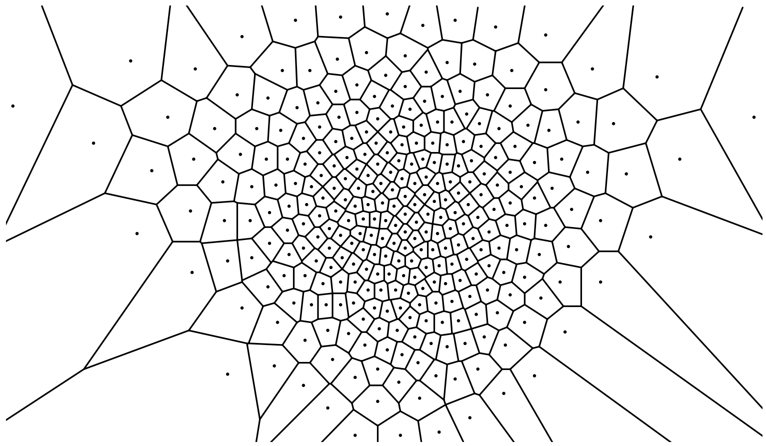

Figure 2: L^2 Optimal quanization 300-grid for the distribution of $(\ln(S_T), Z_T)$ described by (7.1) at $T = 1$ with its Voronoi tesselation.

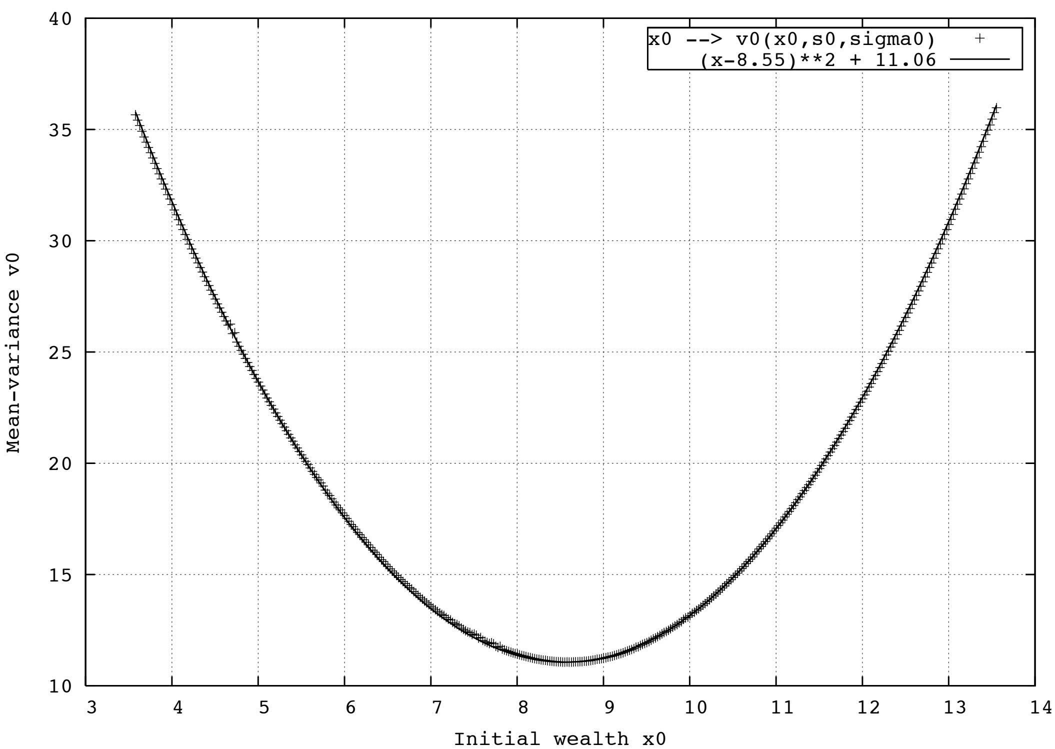

Figure 3: Mean-variance v_0 as a function of x_0 the initial wealth. Here $n_X = 200$ (see Table 2).

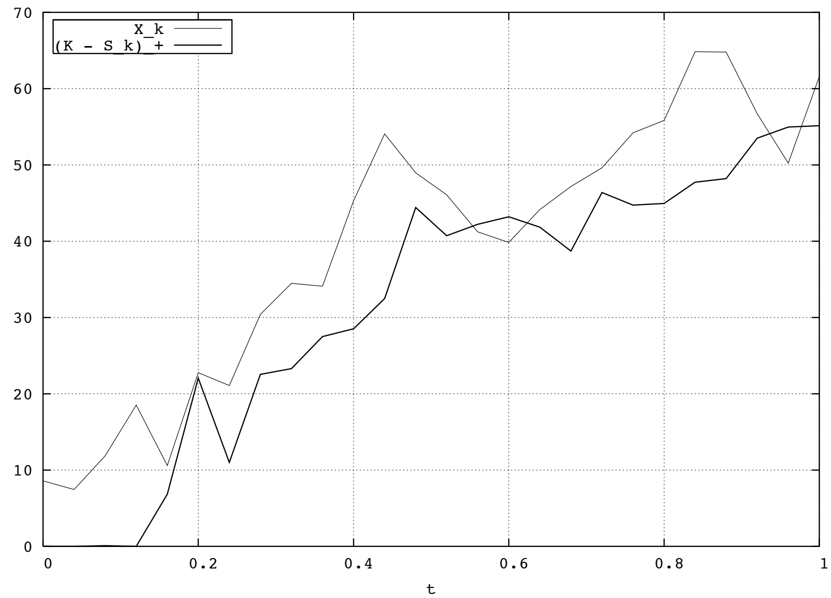

Figure 4: One trajectory of the wealth process $t \mapsto X_t$ computed with the numerical optimal control together with the associated trajectory of the pay-off function $t \mapsto (K - S_t)_+$.

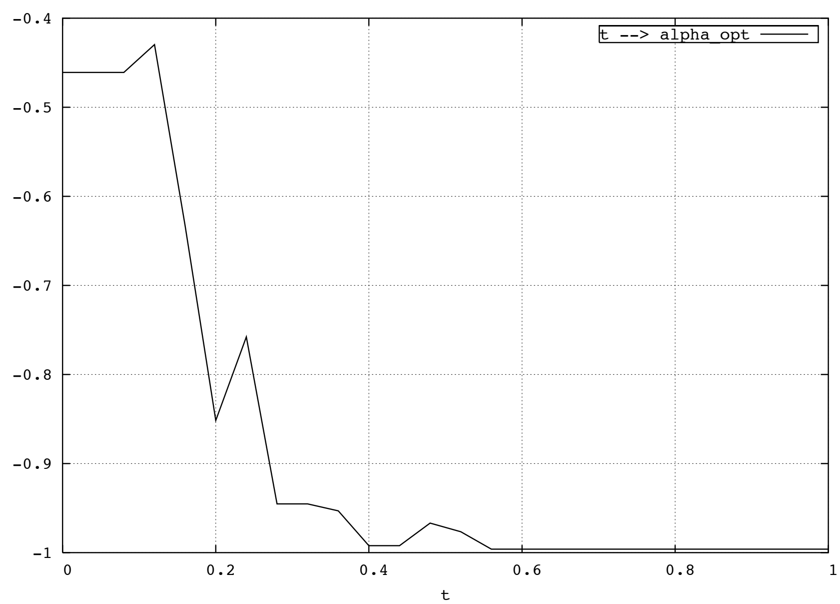

Figure 5: The same trajectory of the associated optimal control (see Fig. 4).

8 Proof of Theorem 3.1

The proof of Theorem 3.1 is divided into several lemmata. Let us introduce some notations. We denote E_k the conditional expectation given $\bar{\mathcal{F}}_k$ and P_k the associated conditional probability. Given $\bar\alpha \in \bar{\mathcal{A}}_n$, we also denote $\bar Z_k = (\bar X_k, \bar Y_k)$, $\hat Z_k = (\hat X_k, \hat Y_k)$, and $\bar X_j^x$ (resp. $\hat X_j^x$), $j \geq k$, the solution to (2.3) (resp. (3.2)) starting from x at time t_k, i.e. $\bar X_k^x = x$ (resp. $\hat X_k^x = x$).

Lemma 8.1 *Assume that* **(H1)** *holds. Then for every $p \in [1, +\infty)$, the Euler scheme operator G_h is Lipschitz, namely*

$$\forall y, y' \in \mathbb{R}^d,\ \|G_h(y,\varepsilon_1) - G_h(y',\varepsilon_1)\|_p \leq [G_h]_p |y-y'| \tag{8.1}$$

$$\text{with} \quad [G_h]_p = 1 + hC_{\eta,\gamma,p} + O(h^2) \quad \text{where} \quad C_{\eta,\gamma,p} = L_2(1 + L_2 E|\varepsilon_1|^2 \max(\lceil p/2\rceil - \frac{1}{2}, 1))$$

where $\lceil x\rceil$ denotes the smallest integer not lower than x.

Proof. We will simply need the random vector ε_1 to be symmetrical and in L^{2r}, $p \leq 2r$ and $r \in \mathbb{N}^*$. First set $\widetilde G_h$ the Euler scheme operator associated to $\eta \equiv 0$ and γ. Let $y, y' \in \mathbb{R}^d$ and $u \in \mathbb{R}^m$.

$$\begin{aligned}\frac{|\widetilde G_h(y,u) - \widetilde G_h(y',u)|^2}{|y-y'|^2} &= 1 + 2\sqrt{h}\left(\frac{\gamma(y)-\gamma(y')}{|y-y'|}u\right)^* \frac{y-y'}{|y-y'|} + h\left|\frac{\gamma(y)-\gamma(y')}{|y-y'|}u\right|^2 \\ &\leq 1 + L_2^2 h|u|^2 + 2\sqrt{h}\left(\frac{\gamma(y)-\gamma(y')}{|y-y'|}u|\frac{y-y'}{|y-y'|}\right).\end{aligned}$$

Hence

$$\frac{|\widetilde G_h(y,u) - \widetilde G_h(y',u)|^{2r}}{|y-y'|^{2r}} \leq \sum_{i=0}^{r}\binom{r}{i}(1+L_2^2h|u|^2)^{r-i}(2\sqrt{h})^i\left(\frac{\gamma(y)-\gamma(y')}{|y-y'|}u|\frac{y-y'}{|y-y'|}\right)^i.$$

Using the fact that ε_1 and $-\varepsilon_1$ have the same distribution yields :

$$\begin{aligned}&E\left[\frac{|\widetilde G_h(y,\varepsilon_1) - \widetilde G_h(y',\varepsilon_1)|^{2r}}{|y-y'|^{2r}}\right] \\ \leq\ & E\left[\sum_{i=0,\, i\equiv 0\ [2]}^{r}\binom{r}{i}(1+L_2^2h|\varepsilon_1|^2)^{r-i}(2\sqrt{h})^i\left(\frac{\gamma(y)-\gamma(y')}{|y-y'|}\varepsilon_1|\frac{y-y'}{|y-y'|}\right)^i\right] \\ \leq\ & E\left[\sum_{i=0,\, i\equiv 0\ [2]}^{r}\binom{r}{i}\left(1+L_2^2h|\varepsilon_1|^2\right)^{r-i}\left(2\sqrt{h}L_2|\varepsilon_1|\right)^i\right] \\ =\ & \frac{1}{2}E\left[(1+\sqrt{h}L_2|\varepsilon_1|)^{2r} + (1-\sqrt{h}L_2|\varepsilon_1|)^{2r}\right].\end{aligned}$$

Then

$$\begin{aligned}\|\widetilde G_h(y,\varepsilon_1) - \widetilde G_h(y',\varepsilon_1)\|_p &\leq \left(1 + r(2r-1)hL_2^2\, E|\varepsilon_1|^2 + O(L_2^4h^2)\right)^{\frac{1}{2r}}|y-y'| \\ &\leq \left(1 + (r-1/2)hL_2^2\, E|\varepsilon_1|^2 + O(L_2^4h^2)\right)|y-y'|\end{aligned}$$

since $(1+u)^{1/(2r)} \leq 1 + u/(2r)$ for every $u \geq 0$. The conclusion follows from the identity $G_h(y,u) = h\,\eta(y) + \widetilde G_h(y,u)$. □

Remark 8.1 The key fact in the above lemma is that the Lipschitz coefficient $[G_h]_p^k$ of the iterate G_h^k of G_h does not explode when $h \to 0$ and $kh \to t \in [0,T]$. As a matter of fact it converges toward $\exp(C_{\eta,\gamma,p} t)$.

Lemma 8.2 *Assume that* **(H1)** *holds. Then for all* $p \in [1,+\infty)$, *there exists a positive constant* C_p *(independent of* n*) such that for every* $j \in \{1,\dots,n\}$,

$$\|\bar{Y}_j - \hat{Y}_j\|_p \;\leq\; C_p \sum_{l=1}^{j} \|\Delta_l\|_p \tag{8.2}$$

and so

$$\sum_{l=1}^{j} \|\bar{Y}_l - \hat{Y}_l\|_p \;\leq\; C_p \sum_{l=1}^{j} (j+1-l)\|\Delta_l\|_p \tag{8.3}$$

Proof. It follows from the above Lemma 8.1 and the obvious equality

$$\bar{Y}_{j+1} - \hat{Y}_{j+1} = G_h(\bar{Y}_j, \varepsilon_{j+1}) - G_h(\hat{Y}_j, \varepsilon_{j+1}) + G_h(\hat{Y}_j, \varepsilon_{j+1}) - \pi_{j+1}(G_h(\hat{Y}_j, \varepsilon_{j+1}))$$

that $\qquad \|\bar{Y}_{j+1} - \hat{Y}_{j+1}\|_p \leq [G_h]_p \|\bar{Y}_j - \hat{Y}_j\|_p + \|\Delta_{j+1}\|_p.$

Remark 8.1 and backward induction complete the proof. □

Lemma 8.3 *Assume that* **(H0)** *and* **(H1)** *hold. Then for any* $p \geq 1$, *there exists a positive constant* C_p *(independent of* n*) such that for all* $k = 0,\dots,n-1$, $j = k,\dots,n-1$, $x \in \mathbb{R}^q$*:*

$$\left\| \Big(\mathrm{esssup}_{\bar{\alpha} \in \bar{\mathcal{A}}_n} E_k |\hat{X}^x_{j+1}|^{2p} \Big)^{\frac{1}{2p}} \right\|_{p'} + \left\| \Big(\mathrm{esssup}_{\bar{\alpha} \in \bar{\mathcal{A}}_n} E_k |H_h(\hat{X}^x_j, \hat{Y}_j, \bar{\alpha}_j, \hat{Y}_{j+1})|^{2p} \Big)^{\frac{1}{2p}} \right\|_{p'}$$

$$\leq C_p \left[1 + |x| + \|\bar{Y}_k\|_{p'} + n\delta + n^{1-\frac{1}{2p}} \sum_{l=1}^{j+1} \|\Delta_l\|_{p'} \right] \; \textit{where } p' \geq 2p. \tag{8.4}$$

Proof. Let $\bar{\alpha} \in \bar{\mathcal{A}}_n$ and the controlled process defined on $[t_j, t_{j+1}]$, $0 \leq j \leq n-1$ by :

$$\tilde{X}_t \;=\; \hat{X}_j + b(\hat{Z}_j, \bar{\alpha}_j)(t - t_j) + \sigma(\hat{Z}_j, \bar{\alpha}_j)(W_t - W_{t_j}).$$

Applying Itô's formula to $|\tilde{X}|^{2p}$ between t_j and t, standard computations as for the estimation of L^p-moments of continuous Euler scheme, see e.g Chapter 5 in [7], show the existence of a positive constant C_p dependent of p such that:

$$E_k|\tilde{X}_t|^{2p} \;\leq\; E_k|\hat{X}_j|^{2p} + C_p \int_{t_j}^{t} E_k|\tilde{X}_u|^{2p} + E_k\left[|b(\hat{Z}_j, \bar{\alpha}_j)|^{2p} + |\sigma(\hat{Z}_j, \bar{\alpha}_j)|^{2p} \right] du.$$

Now, by the linear growth condition on $b(x,y,a)$ and $\sigma(x,y,a)$ uniformly in a (condition **(H1)**), there exists a positive constant C_p (independent of α) such that:

$$E_k|\tilde{X}_t|^{2p} \;\leq\; (1 + C_p h) E_k|\hat{X}_j|^{2p} + C_p h \left(1 + E_k|\hat{Y}_j|^{2p} \right) + C_p \int_{t_j}^{t} E_k|\tilde{X}_u|^{2p} du.$$

Here and in the sequel, C_p denotes a generic positive constant depending on p but independent of α and n, and possibly different along the lines. By Gronwall's lemma and noting that $\tilde{X}_{t_{j+1}} = F_h(\hat{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1})$, we obtain:

$$E_k|F_h(\hat{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1})|^{2p} \;\leq\; e^{C_p h} E_k|\hat{X}_j|^{2p} + C_p h \left(1 + E_k|\hat{Y}_j|^{2p} \right). \tag{8.5}$$

Similarly as above (using Itô's formula and Gronwall's lemma), we have the following classical estimate for the Euler scheme (2.2):

$$E_k|\bar{Y}_{j+1}|^{2p} \leq e^{C_p h} E_k|\bar{Y}_j|^{2p} + C_p h,$$

and so by backward induction:

$$E_k|\bar{Y}_j|^{2p} \leq C_p(1 + |\bar{Y}_k|^{2p}). \tag{8.6}$$

Writing that $|\hat{Y}_l| \leq |\hat{Y}_l - \bar{Y}_l| + |\bar{Y}_l|$, and noting that $(a+b)^{2p} \leq C_p(a^{2p} + b^{2p})$, we have from (8.5)-(8.6):

$$E_k|F_h(\hat{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1})|^{2p} \leq e^{C_p h} E_k|\hat{X}_j|^{2p} + C_p h \left(1 + |\bar{Y}_k|^{2p} + E_k|\hat{Y}_j - \bar{Y}_j|^{2p}\right). \tag{8.7}$$

By definition of the functions H_h and F_h in (2.4)-(2.5), we have:

$$H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1}) = F_h(\hat{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1}) + \vartheta(\hat{Z}_j, \bar{\alpha}_j)\Delta_{j+1},$$

so that by **(H0)**, $|H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})| \leq |F_h(\hat{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1})| + C|\Delta_{j+1}|$, for some positive constant C. The computations below rely on the following convexity inequality: for every $u \in \mathbb{R}_+$, $h \in [0,1]$ and $p \geq 1/2$,

$$(1 + hu)^{2p} \leq 1 + 2^{2p} ph(1 + u^{2p}) \leq 1 + hC_p + C_p h u^{2p}$$

so that, for every $a, b \in \mathbb{R}_+$ and every $h \in [0,1]$,

$$(a + hb)^{2p} \leq (1 + hC_p)\, a^{2p} + C_p h\, b^{2p}.$$

Then
$$\begin{aligned}
|H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})|^{2p} &\leq \left(|F_h(\hat{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1})| + Ch\frac{|\Delta_{j+1}|}{h}\right)^{2p}, \\
&\leq (1 + C_p h)|F_h(\hat{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1})|^{2p} + C_p h \frac{|\Delta_{j+1}|^{2p}}{h^{2p}}. \qquad (8.8)
\end{aligned}$$

By definition of the projection π^X, we have:

$$|\hat{X}_{j+1}| \leq |H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})| + \delta.$$

Then, by the same arguments as above, we have (recall that $h = T/n$):

$$|\hat{X}_{j+1}|^{2p} \leq (1 + C_p h)|H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})|^{2p} + C_p h (n\delta)^{2p}.$$

By using (8.8) and (8.7), we then get:

$$\begin{aligned}
E_k|\hat{X}_{j+1}|^{2p} &\leq (1 + C_p h)^2 E_k|F_h(\hat{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1})|^{2p} + C_p h\left[(n\delta)^{2p} + \frac{E_k|\Delta_{j+1}|^{2p}}{h^{2p}}\right] \\
&\leq e^{C_p h} E_k|\hat{X}_j|^{2p} + C_p h\left[1 + |\bar{Y}_k|^{2p} + E_k|\hat{Y}_j - \bar{Y}_j|^{2p} + (n\delta)^{2p} + \frac{E_k|\Delta_{j+1}|^{2p}}{h^{2p}}\right].
\end{aligned}$$

By backward induction, this yields:

$$E_k|\hat{X}_{j+1}|^{2p}\le C_p\left[1+|\hat{X}_k|^{2p}+|\bar{Y}_k|^{2p}+(n\delta)^{2p}+h\sum_{l=k}^{j}E_k|\hat{Y}_l-\bar{Y}_l|^{2p}+h\sum_{l=k}^{j}\frac{E_k|\Delta_{l+1}|^{2p}}{h^{2p}}\right]. \tag{8.9}$$

Using (8.7) and (8.8), we have the same estimate:

$$\begin{aligned} E_k|H_h(\hat{Z}_j,\bar{\alpha}_j,\hat{Y}_{j+1})|^{2p} &\le C_p\Big[1+|\hat{X}_k|^{2p}+|\bar{Y}_k|^{2p}+(n\delta)^{2p} \\ &\quad +h\sum_{l=k}^{j}E_k|\hat{Y}_l-\bar{Y}_l|^{2p}+h\sum_{l=k}^{j}\frac{E_k|\Delta_{l+1}|^{2p}}{h^{2p}}\Big]. \end{aligned} \tag{8.10}$$

By noting that $(a+b)^{\frac{1}{2p}}\le a^{\frac{1}{2p}}+b^{\frac{1}{2p}}$, and by Jensen's inequality together with the law of iterated conditional expectations, we have by (8.9):

$$\begin{aligned} &\left\|\left(\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}_n}E_k|\hat{X}^x_{j+1}|^{2p}\right)^{\frac{1}{2p}}\right\|_{2p} \\ \le\; & C_p\left[1+|x|+\|\bar{Y}_k\|_{p'}+n\delta+h^{\frac{1}{2p}}\sum_{l=k}^{j}\|\hat{Y}_l-\bar{Y}_l\|_{2p}+h^{\frac{1}{2p}}\sum_{l=k}^{j}\frac{\|\Delta_{l+1}\|_{2p}}{h}\right]. \end{aligned}$$

Now, by estimation (8.3) in Lemma 8.2, we have:

$$\sum_{l=k}^{j}\|\bar{Y}_l-\hat{Y}_l\|_{2p} \le C_p\sum_{l=1}^{j}\frac{\|\Delta_l\|_p}{h},$$

and this implies inequality (8.4). The same estimate holds for $\|\mathrm{ess\,sup}E_k|H_h(\hat{Z}_j,\bar{\alpha}_j,\hat{Y}_{j+1})|^{2p}\|_{2p}$ by (8.10). □

Lemma 8.4 *Assume that* **(H0)**, **(H1)** *and* **(H3b)** *hold. Then for all p, $\bar{p}\ge 1$, there exists a positive constant $C_{p,\bar{p}}$ (independent of n) such that for all $k=0,\ldots,n-1$, $j=k,\ldots,n-1$, for all $x\in\mathbb{R}^q$:*

$$\begin{aligned} &\left\|\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}_n}\left(E_k|\hat{X}^x_{j+1}-\bar{X}^x_{j+1}|^{2p}\right)^{\frac{1}{2p}}\right\|_{2p} \\ \le\; & C_{p,\bar{p}}\Bigg[n\delta+\frac{1}{n^{\frac{1}{2p}}}\sum_{l=1}^{j}(j+1-l)\|\Delta_l\|_{2p}+\sum_{l=k+1}^{j+1}\|\Delta_l\|_{2p} \\ &+\frac{n}{R^{\bar{p}-1}}\left(1+|x|^{\bar{p}}+\|\bar{Y}_k\|^{\bar{p}}_{2p\bar{p}}+(n\delta)^{\bar{p}}+\left(n^{1-\frac{1}{2p\bar{p}}}\sum_{l=1}^{j+1}\|\Delta_l\|_{2p\bar{p}}\right)^{\bar{p}}\right)\Bigg]. \end{aligned} \tag{8.11}$$

Proof. Let $\bar{\alpha}\in\bar{\mathcal{A}}_n$ and the controlled processes defined on $[t_j,t_{j+1}]$, $j=0,\ldots,n-1$ by :

$$\begin{aligned} \bar{X}_t &= \bar{X}_j+b(\bar{Z}_j,\bar{\alpha}_j)(t_j-t)+\sigma(\bar{Z}_j,\bar{\alpha}_j)(W_t-W_{t_j}), \\ \tilde{X}_t &= \hat{X}_j+b(\hat{Z}_j,\bar{\alpha}_j)(t-t_j)+\sigma(\hat{Z}_j,\bar{\alpha}_j)(W_t-W_{t_j}), \end{aligned}$$

and let us denote $D_t = \bar{X}_t - \tilde{X}_t$. Applying Itô's formula to $|D|^{2p}$ between t_j and t, standard computation shows the existence of a positive constant C_p dependent of p such that:

$$
\begin{aligned}
E_k|D_t|^{2p} &\leq E_k|\bar{X}_j - \hat{X}_j|^{2p} + C_p \int_{t_j}^{t} E_k|D_u|^{2p} du \\
&+ C_p \int_{t_j}^{t} E_k \left[|b(\bar{Z}_j, \bar{\alpha}_j) - b(\hat{Z}_j, \bar{\alpha}_j)|^{2p} + |\sigma(\bar{Z}_j, \bar{\alpha}_j) - \sigma(\hat{Z}_j, \bar{\alpha}_j)|^{2p} \right] du.
\end{aligned}
$$

Now, from the Lipschitz condition on $b(x, y, a)$ and $\sigma(x, y, a)$ uniformly in a (condition **(H1)**), there exists a positive constant C_p (independent of α) such that:

$$
E_k|D_t|^{2p} \leq (1 + C_p h) E_k|\bar{X}_j - \hat{X}_j|^{2p} + C_p h E_k|\bar{Y}_j - \hat{Y}_j|^{2p} + C_p \int_{t_j}^{t} E_k|D_u|^{2p} du.
$$

By Gronwall's lemma, noting that $D_{t_{j+1}} = F_h(\bar{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1}) - F_h(\hat{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1})$, we get:

$$
E_k|F_h(\bar{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1}) - F_h(\hat{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1})|^{2p} \leq e^{C_p h} E_k|\bar{X}_j - \hat{X}_j|^{2p} + C_p h E_k|\bar{Y}_j - \hat{Y}_j|^{2p}. \tag{8.12}
$$

By definition of the functions H_h and F_h in (2.4)-(2.5), and by **(H0)**, we have:

$$
|H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1}) - H_h(\bar{Z}_j, \bar{\alpha}_j, \bar{Y}_{j+1})| \leq |F_h(\hat{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1}) - F_h(\bar{Z}_j, \bar{\alpha}_j, \varepsilon_{j+1})| + C|\Delta_{j+1}|,
$$

for some positive constant C. By (8.12) and Minkowski inequality, we then get:

$$
\begin{aligned}
&\Big(E_k|H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1}) - H_h(\bar{Z}_j, \bar{\alpha}_j, \bar{Y}_{j+1})|^{2p} \Big)^{\frac{1}{2p}} \\
&\qquad \leq e^{C_p h} \Big(E_k|\bar{X}_j - \hat{X}_j|^{2p} \Big)^{\frac{1}{2p}} + C_p h^{\frac{1}{2p}} \Big(E_k|\bar{Y}_j - \hat{Y}_j|^{2p} \Big)^{\frac{1}{2p}} + C_p \left(E_k|\Delta_{j+1}|^{2p} \right)^{\frac{1}{2p}}.
\end{aligned} \tag{8.13}
$$

On the other hand, using the π^X projection inequality (3.1) on the grid Γ^X yields:

$$
\left| H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1}) - \pi^X \left(H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1}) \right) \right| \leq \delta + \Big[|H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})| - R \Big] 1_{|H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})| \geq R}.
$$

This implies by Minkowski inequality:

$$
\begin{aligned}
&\left(E_k \left| H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1}) - \pi^X \left(H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1}) \right) \right|^{2p} \right)^{\frac{1}{2p}} \\
&\leq \delta + \left(E_k \left[|H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})|^{2p} 1_{|H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})| \geq R} \right] \right)^{\frac{1}{2p}} \\
&\leq \delta + \frac{\Big(E_k|H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})|^{2p\bar{p}} \Big)^{\frac{1}{2p}}}{R^{\bar{p}-1}},
\end{aligned} \tag{8.14}
$$

where we used Tchebychev inequality (with exponent $2p(\bar{p}-1) \geq 0$) in the second inequality. Now, writing that $|\bar{X}_{j+1} - \hat{X}_{j+1}| \leq |H_h(\bar{Z}_j, \bar{\alpha}_j, \bar{Y}_{j+1}) - H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})| + |H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1}) - \pi^X(H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1}))|$, and using again Minkowski inequality, we have by (8.13) and (8.14):

$$
\Big(E_k|\bar{X}_{j+1} - \hat{X}_{j+1}|^{2p} \Big)^{\frac{1}{2p}} \leq \Big(E_k|H_h(\bar{Z}_j, \bar{\alpha}_j, \bar{Y}_{j+1}) - H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})|^{2p} \Big)^{\frac{1}{2p}}
$$

$$
\begin{aligned}
& + \left(E_k |H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1}) - \pi^X \left(H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1}) \right) |^{2p} \right)^{\frac{1}{2p}} \\
\leq \;& e^{C_p h} \left(E_k |\bar{X}_j - \hat{X}_j|^{2p} \right)^{\frac{1}{2p}} + C_p h^{\frac{1}{2p}} \left(E_k |\bar{Y}_j - \hat{Y}_j|^{2p} \right)^{\frac{1}{2p}} \\
& + C_p \left(E_k |\Delta_{j+1}|^{2p} \right)^{\frac{1}{2p}} + \delta + \frac{\left(E_k |H_h(\hat{Z}_j, \bar{\alpha}_j, \hat{Y}_{j+1})|^{2p\bar{p}} \right)^{\frac{1}{2p}}}{R^{\bar{p}-1}}.
\end{aligned}
$$

By backward induction, this yields:

$$
\begin{aligned}
\left(E_k |\bar{X}^x_{j+1} - \hat{X}^x_{j+1}|^{2p} \right)^{\frac{1}{2p}} \leq \;& C_p \Bigg[h^{\frac{1}{2p}} \sum_{l=k}^{j} \left(E_k |\bar{Y}_l - \hat{Y}_l|^{2p} \right)^{\frac{1}{2p}} + \sum_{l=k}^{j} \left(E_k |\Delta_{l+1}|^{2p} \right)^{\frac{1}{2p}} \\
& + n\delta + \frac{1}{R^{\bar{p}-1}} \sum_{l=k}^{j} \left(E_k |H_h(\hat{X}^x_l, \hat{Y}_l, \bar{\alpha}_l, \hat{Y}_{l+1})|^{2p\bar{p}} \right)^{\frac{1}{2p}} \Bigg].
\end{aligned}
$$

We then get by the law of iterated conditional expectations and Hölder inequality:

$$
\begin{aligned}
& \left\| \mathrm{esssup}_{\bar{\alpha} \in \bar{\mathcal{A}}_n} \left(E_k |\bar{X}^x_{j+1} - \hat{X}^x_{j+1}|^{2p} \right)^{\frac{1}{2p}} \right\|_{2p} \\
\leq \;& C_p \Bigg[h^{\frac{1}{2p}} \sum_{l=k}^{j} \|\bar{Y}_l - \hat{Y}_l\|_{2p} + \sum_{l=k}^{j} \|\bar{\Delta}_{l+1}\|_{2p} + n\delta + \\
& + \frac{1}{R^{\bar{p}-1}} \sum_{l=k}^{j} \left\| \left(\mathrm{esssup}_{\bar{\alpha} \in \bar{\mathcal{A}}_n} E_k |H_h(\hat{X}^x_l, \hat{Y}_l, \bar{\alpha}_l, \hat{Y}_{l+1})|^{2p\bar{p}} \right)^{\frac{1}{2p\bar{p}}} \right\|^{\bar{p}}_{2p\bar{p}} \Bigg].
\end{aligned}
$$

We conclude with the estimations (8.3) in Lemma 8.2 and (8.4) in Lemma 8.3. □

Proof of Theorem 3.1
By (2.6) and (3.3), we have:

$$
|\bar{V}_k(x) - \hat{V}_k(x)| \leq \mathrm{esssup}_{\bar{\alpha} \in \bar{\mathcal{A}}_n} E_k \left[\sum_{j=k}^{n-1} h |f(\bar{X}^x_j, \bar{Y}_j, \bar{\alpha}_j) - f(\hat{X}^x_j, \hat{Y}_j, \bar{\alpha}_j)| + |g(\bar{X}^x_n, \bar{Y}_n) - g(\hat{X}^x_n, \hat{Y}_n)| \right].
$$

By condition **(H3b)**, this implies:

$$
\begin{aligned}
|\bar{V}_k(x) - \hat{V}_k(x)| \leq \;& \mathrm{esssup}_{\bar{\alpha} \in \bar{\mathcal{A}}_n} E_k \Bigg[[f_x] h \sum_{j=k+1}^{n-1} |\bar{X}^x_j - \hat{X}^x_j| (1 + |\bar{Y}_j|^{p_1}) + [g_x] |\bar{X}^x_n - \hat{X}^x_n| (1 + |\bar{Y}_n|^{p_1}) \\
& + [f_y] h \sum_{j=k}^{n-1} |\bar{Y}_j - \hat{Y}_j| (1 + |\hat{X}^x_j|^{p_1}) + [g_y] |\bar{Y}_n - \hat{Y}_n| (1 + |\hat{X}^x_n|^{p_1}) \Bigg] \\
\leq \;& [f_x] h \sum_{j=k+1}^{n-1} \mathrm{esssup}_{\bar{\alpha} \in \bar{\mathcal{A}}_n} \left(E_k |\bar{X}^x_j - \hat{X}^x_j|^{\hat{p}} \right)^{\frac{1}{\hat{p}}} \left[1 + \left(E_k |\bar{Y}_j|^{\hat{q}_1} \right)^{\frac{p_1}{\hat{q}_1}} \right] \\
& + [g_x] \mathrm{esssup}_{\bar{\alpha} \in \bar{\mathcal{A}}_n} \left(E_k |\bar{X}^x_n - \hat{X}^x_n|^{\hat{p}} \right)^{\frac{1}{\hat{p}}} \left[1 + \left(E_k |\bar{Y}_n|^{\hat{q}_1} \right)^{\frac{p_1}{\hat{q}_1}} \right]
\end{aligned}
$$

$$
\begin{aligned}
&+ [f_y]h\sum_{j=k}^{n-1}\left(E_k|\bar{Y}_j-\hat{Y}_j|^{\hat{p}}\right)^{\frac{1}{\hat{p}}}\left[1+\left(\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}_n}E_k|\hat{X}_j^x|^{\hat{q}_1}\right)^{\frac{p_1}{\hat{q}_1}}\right]\\
&+ [g_y]\left(E_k|\bar{Y}_n-\hat{Y}_n|^{\hat{p}}\right)^{\frac{1}{\hat{p}}}\left[1+\left(\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}_n}E_k|\hat{X}_n^x|^{\hat{q}_1}\right)^{\frac{p_1}{\hat{q}_1}}\right],
\end{aligned}
$$

where we used Hölder inequality. Using again Hölder inequality with p' integer to be chosen below and $1/p' + 1/q' = 1$, we obtain:

$$
\begin{aligned}
\left\|\bar{V}_k(x)-\hat{V}_k(x)\right\|_p \leq\ & [f_x]h\sum_{j=k+1}^{n-1}\left\|\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}_n}\left(E_k|\bar{X}_j^x-\hat{X}_j^x|^{\hat{p}}\right)^{\frac{1}{\hat{p}}}\right\|_{pp'}\left[1+\left\|\left(E_k|\bar{Y}_j|^{\hat{q}_1}\right)^{\frac{1}{\hat{q}_1}}\right\|_{p_1pq'}^{p_1}\right]\\
&+ [g_x]\left\|\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}}\left(E_k|\bar{X}_n^x-\hat{X}_n^x|^{\hat{p}}\right)^{\frac{1}{\hat{p}}}\right\|_{pp'}\left[1+\left\|\left(E_k|\bar{Y}_n|^{\hat{q}_1}\right)^{\frac{1}{\hat{q}_1}}\right\|_{p_1pq'}^{p_1}\right]\\
&+ [f_y]\sum_{j=k}^{n-1}\left\|\left(E_k|\bar{Y}_j-\hat{Y}_j|^{\hat{p}}\right)^{\frac{1}{\hat{p}}}\right\|_{pp'}\left[1+\left\|\left(\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}}E_k|\hat{X}_j^x|^{\hat{q}_1}\right)^{\frac{1}{\hat{q}_1}}\right\|_{p_1pq'}^{p_1}\right]\\
&+ [g_y]\left\|\left(E_k|\bar{Y}_n-\hat{Y}_n|^{\hat{p}}\right)^{\frac{1}{\hat{p}}}\right\|_{pp'}\left[1+\left\|\left(\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}}E_k|\hat{X}_n^x|^{\hat{q}_1}\right)^{\frac{1}{\hat{q}_1}}\right\|_{p_1pq'}^{p_1}\right].
\end{aligned}
$$

We now choose $p' = \hat{p}/p \geq 1$. Then $p_1pq' = q_1' \geq \hat{q}_1$. By Jensen's inequality, we then have:

$$
\begin{aligned}
\left\|\bar{V}_k(x)-\hat{V}_k(x)\right\|_p \leq\ & [f_x]h\sum_{j=k+1}^{n-1}\left\|\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}}\left(E_k|\bar{X}_j^x-\hat{X}_j^x|^{\hat{p}}\right)^{\frac{1}{\hat{p}}}\right\|_{\hat{p}}\left[1+\|\bar{Y}_j\|_{q_1'}^{p_1}\right]\\
&+[g_x]\left\|\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}}\left(E_k|\bar{X}_n^x-\hat{X}_n^x|^{\hat{p}}\right)^{\frac{1}{\hat{p}}}\right\|_{\hat{p}}\left[1+\|\bar{Y}_n\|_{q_1'}^{p_1}\right]\\
&+[f_y]h\sum_{j=k}^{n-1}\left\|\bar{Y}_j-\hat{Y}_j\right\|_{\hat{p}}\left[1+\left\|\left(\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}}E_k|\hat{X}_j^x|^{\hat{q}_1}\right)^{\frac{1}{\hat{q}_1}}\right\|_{q_1'}^{p_1}\right]\\
&+ [g_y]\left\|\bar{Y}_n-\hat{Y}_n\right\|_{\hat{p}}\left[1+\left\|\left(\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}}E_k|\hat{X}_n^x|^{\hat{q}_1}\right)^{\frac{1}{\hat{q}_1}}\right\|_{q_1'}^{p_1}\right].
\end{aligned}
$$

Here, we use the convention that when $p_1 = 0$ (and so $\hat{p}_1 = 0$), $1^\infty = 1$. If $p_1 = 0$, then the asserted result is proved by using estimation in Lemma 8.4. If $p_1 > 0$, then we have by Hölder inequality:

$$
\begin{aligned}
\left\|\bar{V}_k(x)-\hat{V}_k(x)\right\|_p \leq\ & [f_x]h\sum_{j=k+1}^{n-1}\left\|\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}}\left(E_k|\bar{X}_j^x-\hat{X}_j^x|^{\hat{p}}\right)^{\frac{1}{\hat{p}}}\right\|_{\hat{p}}\left[1+\|\bar{Y}_j\|_{q_1'}^{p_1}\right]\\
&+[g_x]\left\|\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}}\left(E_k|\bar{X}_n^x-\hat{X}_n^x|^{\hat{p}}\right)^{\frac{1}{\hat{p}}}\right\|_{\hat{p}}\left[1+\|\bar{Y}_n\|_{q_1'}^{p_1}\right]\\
&+[f_y]h\sum_{j=k}^{n-1}\left\|\bar{Y}_j-\hat{Y}_j\right\|_{\hat{p}}\left[1+\left\|\left(\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}}E_k|\hat{X}_j^x|^{\hat{q}_1\vee 2}\right)^{\frac{1}{\hat{q}_1\vee 2}}\right\|_{\bar{q}_1}^{p_1}\right]\\
&+ [g_y]\left\|\bar{Y}_n-\hat{Y}_n\right\|_{\hat{p}}\left[1+\left\|\left(\mathrm{esssup}_{\bar{\alpha}\in\bar{\mathcal{A}}}E_k|\hat{X}_n^x|^{\hat{q}_1\vee 2}\right)^{\frac{1}{\hat{q}_1\vee 2}}\right\|_{\bar{q}_1}^{p_1}\right],
\end{aligned}
$$

where $\bar{q}_1 = q_1 \vee \hat{q}_1 \vee 2$ (recall that $\hat{p} > p$ and so $q_1', \bar{q}_1 < +\infty$). We conclude with the estimations in Lemmas 8.3 and 8.4.

Appendix : Proof of Proposition 2.1

We shall use viscosity solutions methods introduced by Barles and Souganidis in [5] to prove the convergence result. First, notice that under **(H1)** and by classical estimates for Euler schemes, for all $p \geq 1$, there exists a positive constant C (independent of h) such that for all $k = 0, \ldots, n$, $j = k, \ldots, n$, $x, y \in \mathbb{R}^q \times \mathbb{R}^d$, we have:

$$\text{(A.1)} \qquad \sup_{\bar{\alpha} \in \bar{\mathcal{A}}} E\left[|\bar{X}_j|^{2p} + |\bar{Y}_j|^{2p} \big| (\bar{X}_k, \bar{Y}_k) = (x, y) \right] \leq C(1 + |x|^{2p} + |y|^{2p}).$$

Under the quadratic growth condition **(H2)**, this implies that there exists a positive constant C independent of h such that for all $k = 0, \ldots, n$, $x, y \in \mathbb{R}^q \times \mathbb{R}^d$:

$$\text{(A.2)} \qquad |\bar{v}_k(x, y)| \leq C(1 + |x|^2 + |y|^2).$$

In particular, the function $\bar{v}_k$ belongs to the set $\mathcal{B}_2(\mathbb{R}^q \times \mathbb{R}^d)$ of Borel functions defined on $\mathbb{R}^q \times \mathbb{R}^d$ with quadratic growth.

On the other hand, by the dynamic programming principle, the discrete time control problem (2.6) provides an approximation scheme backward in time by:

$$\text{(A.3)} \qquad \bar{v}_n(x, y) = g(x, y), \quad x \in \mathbb{R}^q, y \in \mathbb{R}^d$$

$$\text{(A.4)} \qquad \bar{v}_k(x, y) = \inf_{a \in A} E\left[h f(x, y, a) + \bar{v}_{k+1}(\bar{X}_{k+1}^{x,y,a}, \bar{Y}_{k+1}^{y}) \right], \quad k = 0, \ldots, n-1,\ x \in \mathbb{R}^q, y \in \mathbb{R}^d,$$

where $\bar{X}_{k+1}^{x,y,a} = F_h(x, y, a, \varepsilon_{k+1})$ and $\bar{Y}_{k+1}^{y} = G_h(y, \varepsilon_{k+1})$. This can be rewritten as:

$$\bar{v}_k(x, y) = \mathcal{G}_h(x, y, \bar{v}_{k+1}(.,.)), \quad k = 0, \ldots, n-1,\ x \in \mathbb{R}^q, y \in \mathbb{R}^d,$$

together with $\bar{v}_n(x, y) = g(x, y)$, where $\mathcal{G}_h$ is the operator on $\mathbb{R}^q \times \mathbb{R}^d \times \mathcal{B}_2(\mathbb{R}^q \times \mathbb{R}^d)$ defined by (A.4).

Lemma A.1 *Assume that* **(H1)**, **(H2)** *and* **(H3a)** *hold. Then the discrete time approximation scheme* (A.3)-(A.4) *is*
(i) monotone: for all $\varphi_1, \varphi_2 \in \mathcal{B}_2(\mathbb{R}^q \times \mathbb{R}^d)$ *with* $\varphi_1 \leq \varphi_2$,

$$\mathcal{G}_h(x, y, \varphi_1) \leq \mathcal{G}_h(x, y, \varphi_2), \quad x, y \in \mathbb{R}^q \times \mathbb{R}^d.$$

(ii) stable: the function $\bar{v}_k$ *is locally bounded on* $\mathbb{R}^q \times \mathbb{R}^d$ *with a bound independent of* h *and* $k = 0, \ldots, n$.
(iii) consistent:

- *for all* $t, x, y \in [0, T) \times \mathbb{R}^q \times \mathbb{R}^d$, *for all* φ *smooth on* $[0, T) \times \mathbb{R}^q \times \mathbb{R}^d$ *(say in* $\mathcal{C}^3$ *with compact support),*

$$\text{(A.5)} \quad \frac{1}{h}\left[\mathcal{G}_h(x', y', \varphi(t_{k+1}, ., .)) - \varphi(t_k, ., .) \right] \to \frac{\partial \varphi}{\partial t}(t, x, y) + \inf_{a \in A}\left[\mathcal{L}^a \varphi(t, x, y) + f(x, y, a) \right],$$

as h goes to zero, $t_k \to t$ and $(x', y') \to (x, y)$, for all $k = 0, \ldots, n-1$.

- *for all $x, y \in \mathbb{R}^q \times \mathbb{R}^d$,*

$$\lim \bar{v}_{n-1}(x', y') \quad = \quad g(x, y), \tag{A.6}$$

as h goes to zero and $(x', y') \to (x, y)$.

Proof. The monotonicity is obvious from the definition of $\mathcal{G}_h$ while the stability follows directly from relation (A.2).

For any $k = 0, \ldots, n-1$, $z' = (x', y') \in \mathbb{R}^q \times \mathbb{R}^d$, $a \in A$, define the random variable:

$$Z_{k+1}^{z',a} \quad = \quad z' + h \begin{pmatrix} b(z', a) \\ \eta(y') \end{pmatrix} + \sqrt{h} \begin{pmatrix} \sigma(z', a) \\ \gamma(y') \end{pmatrix} \varepsilon_{k+1}, \tag{A.7}$$

so that,

$$\mathcal{G}_h(x', y', \varphi(t_{k+1}, ., .)) - \varphi(t_k, ., .) \quad = \quad \inf_{a \in A} \left\{ h f(x', y', a) + E\left[\varphi(t_{k+1}, Z_{k+1}^{z',a}) - \varphi(t_k, z') \right] \right\}.$$

Now, by Taylor-Young formula, we have (here the sign . denotes the inner product) :

$$\begin{aligned} \varphi(t_{k+1}, Z_{k+1}^{z',a}) \quad &= \quad \varphi(t_k, z') + h \frac{\partial \varphi}{\partial t}(t_k, z') + D_z \varphi(t_k, z').(Z_{k+1}^{z',a} - z') \\ &\quad + \frac{1}{2} D_{zz}^2 \varphi(t_k, z')(Z_{k+1}^{z',a} - z').(Z_{k+1}^{z',a} - z') + \zeta_{k+1} |Z_{k+1}^{z',a} - z'|^3 \end{aligned}$$

where ζ_{k+1} is a bounded random variable. By (A.7), we have

$$\begin{aligned} E\left[Z_{k+1}^{z',a} - z' \right] \quad &= \quad h \begin{pmatrix} b(z', a) \\ \eta(y') \end{pmatrix} \\ E\left[\left(Z_{k+1}^{z',a} - z' \right) \left(Z_{k+1}^{z',a} - z' \right)' \right] \quad &= \quad h \begin{pmatrix} \sigma(z', a)\sigma^*(z', a) & \sigma(z', a)\gamma^*(y') \\ \gamma(y')\sigma^*(z', a) & \gamma(y')\gamma^*(y') \end{pmatrix} \\ &\quad + h^2 \begin{pmatrix} b(z', a)b^*(z', a) & b(z', a)\eta^*(y') \\ \eta(y')b^*(z', a) & \eta(y')\eta^*(y') \end{pmatrix}. \end{aligned}$$

Under **(H1)**, $b(z', a)$, $\sigma(z', a)$ are bounded in $a \in A$. Hence,

$$\begin{aligned} & \frac{1}{h} \left[\mathcal{G}_h(x', y', \varphi(t_{k+1}, ., .)) - \varphi(t_k, ., .) \right] \\ = \quad & \frac{\partial \varphi}{\partial t}(t_k, z') + \inf_{a \in A} \left[\mathcal{L}^a \varphi(t_k, z') + f(z', a) \right] + O_{a,z'}(h) \quad \text{with } |O_{a,z'}(h)| \le C\, h(1 + |z'|^2) \\ & \frac{\partial \varphi}{\partial t}(t, z) + \inf_{a \in A} \left[\mathcal{L}^a \varphi(t, z) + f(z, a) \right], \end{aligned} \tag{A.8}$$

by condition **(H3a)**. The consistency (A.5) in $t < T$ is proved.

We now prove the consistency (A.6) in T. Let us first suppose that g is smooth and consider the function:

$$w(t, z) \quad = \quad K(1 + \psi(z))(T - t) + g(z), \quad t \in [0, T], z = (x, y) \in \mathbb{R}^q \times \mathbb{R}^d,$$

where $\psi(z) = |z|^3$ and K is a positive constant chosen later. We then have:

$$\begin{aligned}
\mathcal{G}_h(z, w(T,.)) &= \inf_{a\in A} E\left[hf(x,y,a) + g(\bar{X}_n^{x,y,a}, \bar{Y}_n^{y,a})\right] \\
&= w(T-h,z) - h\left\{K(1+\psi(z)) - \inf_{a\in A} E\left[f(x,y,a) + \frac{g(\bar{X}_n^{x,y,a}, \bar{Y}_n^{y,a}) - g(x,y)}{h}\right]\right\} \\
&= w(T-h,z) - h\left\{K(1+\psi(z)) - \inf_{a\in A}\left[f(z,a) + \mathcal{L}^a g(z)\right]\right\} + o(h),
\end{aligned}$$

where the last equality follows by same arguments as in (A.8). Now from the quadratic growth condition on f and g and the linear growth condition on b, σ, η and γ, uniformly in $a \in A$, and by the third order polynomial growth condition on ψ, we can choose K large enough so that:

$$K(1+\psi(z)) \quad \geq \quad \sup_{a\in A}\left[|f(z,a)| + |\mathcal{L}^a g(z)|\right], \quad \forall z = (x,y) \in \mathbb{R}^q \times \mathbb{R}^d.$$

We then deduce that for h small enough,

$$\mathcal{G}_h w(T,.) \quad \leq \quad w(T-h,z), \quad \forall z = (x,y) \in \mathbb{R}^q \times \mathbb{R}^d.$$

Since $w(T,.) = g(.) = \bar{v}_n(.)$, this implies by one backward induction:

$$\bar{v}_{n-1}(z) \quad \leq \quad w(T-h,z), \quad \forall z = (x,y) \in \mathbb{R}^q \times \mathbb{R}^d,$$

for h small enough. Similarly, by considering the function $\tilde{w}(t,z) = -K(1+\psi(z))(T-t) + g(z)$, we obtain the relation $\tilde{w}(T-h,z) \leq \bar{v}_{n-1}(z)$. Therefore, we get the inequality

$$\begin{aligned}
|\bar{v}_{n-1}(z') - g(z)| \quad &\leq \quad |\bar{v}_{n-1}(z') - g(z')| + |g(z') - g(z)| \\
&\leq \quad K(1+\psi(z'))h + |g(z') - g(z)|,
\end{aligned}$$

which implies (A.6). Finally, if g is continuous with quadratic growth, a standard smoothing technique gives for any $\beta > 0$ a function $\tilde{g}$ such that $|\tilde{g}(z) - g(z)| \leq \beta(1+|z|^2)$. Let $\tilde{\bar{v}}_k$ the corresponding value functions for the discrete time control problem. Using (A.1), we have:

$$|\bar{v}_{n-1}(z) - \tilde{\bar{v}}_{n-1}(z)| \quad \leq \quad C\,\beta(1+|z|^2),$$

for all $z = (x,y) \in \mathbb{R}^q \times \mathbb{R}^d$, $k = 0, \ldots, n$ (the real constant C does not depend on β since it comes from (A.2). We have shown that

$$\lim_{h\to 0, z'\to z} \tilde{\bar{v}}_{n-1}(z') \quad = \quad \tilde{g}(z).$$

Since β is arbitrary, the same is true for $\bar{v}_{n-1}$ and g, as required in (A.6). □

Proof of Proposition 2.1

We just indicate the main steps and refer to Barles and Souganidis (1991) for complete arguments.

1) Under the stability property of the scheme $\mathcal{G}_h$ proved in Lemma A.1, we may define

$$\begin{aligned}
v_\star(t,x,y) \quad &= \quad \liminf \bar{v}_k(x', y') \\
v^\star(t,x,y) \quad &= \quad \limsup \bar{v}_k(x', y'),
\end{aligned}$$

as h goes to zero, $t_k \to t$, $(x', y') \to (x, y)$. By the monotonicity and the consistency (A.5) of the approximation scheme $\mathcal{G}_h$, $v^\star$ and $v_\star$ are respectively viscosity sub and supersolutions of (1.5). By the consistency result (A.6), we also have $v_\star(T, x, y) = v^\star(T, x, y) = g(x, y)$.

2) By the strong comparison result for viscosity solutions with quadratic growth (see Crandall and Lions 1990), we have

$$v^\star \quad \leq \quad v_\star, \quad \text{on } [0, T] \times \mathbb{R}^q \times \mathbb{R}^d.$$

3) Finally, by definition $v_\star \leq v^\star$, and therefore

$$v^\star \quad = \quad v_\star, \quad \text{on } [0, T] \times \mathbb{R}^q \times \mathbb{R}^d.$$

This equality implies the local uniform convergence of $\bar{v}_k$ to $v = v^\star = v_\star$, which is the unique continuous viscosity solution of (1.5)-(1.6) with quadratic growth. Hence assertion (2.7) is proved.

Moreover, it is well known that under **(H1)**, we have a pathwise convergence result for the Euler scheme (2.1)-(2.2):

$$\bar{Y}_k \quad \to \quad Y_t, \quad a.s.,$$

as $t_k \to t$ when $n \to \infty$. By 2.7, this implies:

$$\bar{v}_k(x, \bar{Y}_k) \quad \to \quad v(t, x, Y_t), \quad a.s.,$$

as $t_k \to t$ when $n \to \infty$, uniformly on compact sets of $\mathbb{R}^q$. Using (A.1) and (A.2), we obtain (2.8) by the dominated convergence theorem.

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