Isoperimetric Inequalities: Methods and Applications - Part I

Emanuel Milman

Technion - Israel Institute of Technology

Workshop on “Functional Inequalities and Discrete Spaces"
January 13, 2011
Marne-la-Vallée
Part I - Methods.
- Isoperimetric Inqs - Definition and Motivation.
- Connections to Sobolev and Concentration Inqs.

Part II - Applications.
- Transport-Entropy Inqs - Definition and Connections.
- Stability of Isop / Sobolev / TE Inqs under measure perturbation (e.g. extending Holley–Stroock lemma).
- Equivalence of Transport-Entropy Inqs with different costs.
(\Omega, d, \mu) - measure metric space; \(d\) - metric, \(\mu\) - Borel measure.

Assume: \(\Omega \subset (M^n, g)\) Riemannian manifold, \(d\) induced geodesic distance on \(M\), \(\mu = h \text{ vol}_M|_\Omega\).

Isoperimetric Inqs compare between \(\mu(A)\) and \(\mu^+(A)\) (Minkowski’s exterior boundary measure):

\[
\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^d_\varepsilon) - \mu(A)}{\varepsilon},
\]

\[A^d_\varepsilon := \{ x \in \Omega; d(x, A) < \varepsilon \}.
\]

**Isoperimetric profile:** \(I : [0, \mu(\Omega)] \ni v \mapsto \inf \{ \mu^+(A); \mu(A) = v \} \).

Typically \(\mu^+(A) = \mu^+(\Omega \setminus A)\); if \(\mu(\Omega) = 1\) then \(I(1 - v) = I(v)\).

So we restrict to \(\mu(A) \leq 1/2\), and \(I : [0, 1/2] \to \mathbb{R}_+\).
Isoperimetric Inequalities

$(\Omega, d, \mu)$ - measure metric space; $d$ - metric, $\mu$ - Borel measure.

Assume: $\Omega \subset (M^n, g)$ Riemannian manifold, $d$ induced geodesic distance on $M$, $\mu = h \operatorname{vol}_M|_{\Omega}$.

Isoperimetric Inqs compare between $\mu(A)$ and $\mu^+(A)$ (Minkowski’s exterior boundary measure):

$$\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^d_{\varepsilon}) - \mu(A)}{\varepsilon},$$

$$A^d_{\varepsilon} := \{x \in \Omega; d(x, A) < \varepsilon\}.$$ 

Isoperimetric profile: $\mathcal{I} : [0, \mu(\Omega)] \ni \nu \mapsto \inf \{\mu^+(A); \mu(A) = \nu\}$.

Typically $\mu^+(A) = \mu^+(\Omega \setminus A)$; if $\mu(\Omega) = 1$ then $\mathcal{I}(1 - \nu) = \mathcal{I}(\nu)$.

So we restrict to $\mu(A) \leq 1/2$, and $\mathcal{I} : [0, 1/2] \to \mathbb{R}_+$. 

Emanuel Milman  
Isoperimetric Inequalities: Methods and Applications
Isoperimetric Inequalities

$(\Omega, d, \mu)$ - measure metric space; $d$ - metric, $\mu$ - Borel measure.

Assume: $\Omega \subset (M^n, g)$ Riemannian manifold, $d$ induced geodesic distance on $M$, $\mu = h \ vol_M|_\Omega$.

Isoperimetric Inqs compare between $\mu(A)$ and $\mu^+(A)$ (Minkowski’s exterior boundary measure):

$$
\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^d_\varepsilon) - \mu(A)}{\varepsilon},
$$

$$
A^d_\varepsilon := \{ x \in \Omega; d(x, A) < \varepsilon \}.
$$

**Isoperimetric profile:** $I : [0, \mu(\Omega)] \ni \nu \mapsto \inf \{\mu^+(A); \mu(A) = \nu\}$.

Typically $\mu^+(A) = \mu^+(\Omega \setminus A)$; if $\mu(\Omega) = 1$ then $I(1 - \nu) = I(\nu)$. So we restrict to $\mu(A) \leq 1/2$, and $I : [0, 1/2] \to \mathbb{R}_+$. 

Emanuel Milman  
Isoperimetric Inequalities: Methods and Applications
(\Omega, d, \mu) - measure metric space; d - metric, \mu - Borel measure.

Assume: \Omega \subset (M^n, g) Riemannian manifold, d induced geodesic distance on M, \mu = h vol_\Omega |M|.

Isoperimetric Inqs compare between \mu(A) and \mu^+(A) (Minkowski’s exterior boundary measure):

\[
\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A_\varepsilon^d) - \mu(A)}{\varepsilon},
\]

\[
A_\varepsilon^d := \{x \in \Omega; d(x, A) < \varepsilon\}.
\]

\underline{Isoperimetric profile}: \mathcal{I} : [0, \mu(\Omega)] \ni \nu \mapsto \inf \{\mu^+(A); \mu(A) = \nu\}.

Typically \mu^+(A) = \mu^+(\Omega \setminus A); if \mu(\Omega) = 1 then \mathcal{I}(1 - \nu) = \mathcal{I}(\nu).

So we restrict to \mu(A) \leq 1/2, and \mathcal{I} : [0, 1/2] \to \mathbb{R}_+.
Examples

Classical isoperimetric inequality in \((\mathbb{R}^n, |\cdot|, \text{Leb})\):
Euclidean balls minimize boundary measure: \(I(v) = c_n v^{\frac{n-1}{n}}\).

Isoperimetric Problem: determine \(I\) (characterize minimizers).

Other known solutions to the isoperimetric problem:
- \((S^n, d, \text{Haar})\) (Lévy, Schmidt) - geodesic balls.
- \((\mathbb{R}^n, |\cdot|, \text{Gauss})\) (Sudakov–Tsirelson, Borell) - half spaces.
- \((B^n, |\cdot|, \text{Leb})\) (Burago–Maz’ya) - perpendicular caps.
- Other: \(H^n\), cones, variations, low-dim spaces.

Open: \([0, 1]^3, |\cdot|, \text{Leb}\), Flat Torus, Slabs, Heisenberg, etc.

Therefore: content in having good lower bounds \(I \geq J\).

Why should we care? (why are Isoperimetric Inqs important?)
Isoperimetric Inqs \(\Rightarrow\) Sobolev Inqs \(\Rightarrow\) Transport-Entropy Inqs \(\Rightarrow\) Concentration Inqs.
Examples

Classical isoperimetric inequality in \((\mathbb{R}^n, |\cdot|, \text{Leb})\):
Euclidean balls minimize boundary measure: \(I(\nu) = c_n v^{\frac{n-1}{n}}\).

Isoperimetric Problem: determine \(I\) (characterize minimizers).

Other known solutions to the isoperimetric problem:
- \((S^n, d, \text{Haar})\) (Lévy, Schmidt) - geodesic balls.
- \((\mathbb{R}^n, |\cdot|, \text{Gauss})\) (Sudakov–Tsirelson, Borell) - half spaces.
- \((B^n, |\cdot|, \text{Leb})\) (Burago–Maz'ya) - perpendicular caps.
- Other: \(H^n\), cones, variations, low-dim spaces.

Open: \([0, 1]^3, |\cdot|, \text{Leb}\), Flat Torus, Slabs, Heisenberg, etc.

Therefore: content in having good lower bounds \(I \geq J\).

Why should we care? (why are Isoperimetric Inqs important?)
Isoperimetric Inqs \(\Rightarrow\) Sobolev Inqs \(\Rightarrow\) Transport-Entropy Inqs \(\Rightarrow\) Concentration Inqs.
Examples

Classical isoperimetric inequality in \((\mathbb{R}^n, |\cdot|, \text{Leb})\): Euclidean balls minimize boundary measure: 

\[ I(\nu) = c_n \nu^{\frac{n-1}{n}}. \]

Isoperimetric Problem: determine \( I \) (characterize minimizers).

Other known solutions to the isoperimetric problem:

- \((S^n, d, Haar)\) (Lévy, Schmidt) - geodesic balls.
- \((\mathbb{R}^n, |\cdot|, Gauss)\) (Sudakov–Tsirelson, Borell) - half spaces.
- \((B^n, |\cdot|, Leb)\) (Burago–Maz’ya) - perpendicular caps.
- Other: \(H^n\), cones, variations, low-dim spaces.

Open: \([0, 1]^3, |\cdot|, \text{Leb}\), Flat Torus, Slabs, Heisenberg, etc.

Therefore: content in having good lower bounds \( I \geq J \).

Why should we care? (why are Isoperimetric Inqs important?)

Isoperimetric Inqs \( \Rightarrow \) Sobolev Inqs \( \Rightarrow \) Transport-Entropy Inqs \( \Rightarrow \) Concentration Inqs.
Examples

Classical isoperimetric inequality in \((\mathbb{R}^n, |\cdot|, \text{Leb})\):
Euclidean balls minimize boundary measure: \(I(\nu) = c_n \nu^{\frac{n-1}{n}}\).

Isoperimetric Problem: determine \(I\) (characterize minimizers).

Other known solutions to the isoperimetric problem:

- \((S^n, d, \text{Haar})\) (Lévy,Schmidt) - geodesic balls.
- \((\mathbb{R}^n, |\cdot|, \text{Gauss})\) (Sudakov–Tsirelson,Borell) - half spaces.
- \((B^n, |\cdot|, \text{Leb})\) (Burago–Maz’ya) - perpendicular caps.
- Other: \(H^n\), cones, variations, low-dim spaces.

Open: \(([0, 1]^3, |\cdot|, \text{Leb})\), Flat Torus, Slabs, Heisenberg, etc.

Therefore: content in having good lower bounds \(I \geq J\).

Why should we care? (why are Isoperimetric Inqs important?)
Isoperimetric Inqs \implies Sobolev Inqs \implies Transport-Entropy Inqs \implies Concentration Inqs.
Examples

Classical isoperimetric inequality in \((\mathbb{R}^n, |\cdot|, \text{Leb})\):
Euclidean balls minimize boundary measure: \(I(v) = c_n v^{\frac{n-1}{n}}\).

**Isoperimetric Problem**: determine \(I\) (characterize minimizers).

Other known solutions to the isoperimetric problem:

- \((S^n, d, \text{Haar})\) (Lévy,Schmidt) - geodesic balls.
- \((\mathbb{R}^n, |\cdot|, \text{Gauss})\) (Sudakov–Tsirelson,Borell) - half spaces.
- \((B^n, |\cdot|, \text{Leb})\) (Burago–Maz’ya) - perpendicular caps.
- Other: \(H^n\), cones, variations, low-dim spaces.

Open: \(([0, 1]^3, |\cdot|, \text{Leb})\), Flat Torus, Slabs, Heisenberg, etc.

**Therefore**: content in having good lower bounds \(I \geq J\).

Why should we care? (why are Isoperimetric Inqs important?)
Isoperimetric Inqs \(\Rightarrow\) Sobolev Inqs \(\Rightarrow\) Transport-Entropy Inqs
\(\Rightarrow\) Concentration Inqs.
Examples

Classical isoperimetric inequality in \((\mathbb{R}^n, |\cdot|, \text{Leb})\):
Euclidean balls minimize boundary measure: \(\mathcal{I}(\nu) = c_n \nu^{\frac{n-1}{n}}\).

Isoperimetric Problem: determine \(\mathcal{I}\) (characterize minimizers).

Other known solutions to the isoperimetric problem:
- \((S^n, d, \text{Haar})\) (Lévy, Schmidt) - geodesic balls.
- \((\mathbb{R}^n, |\cdot|, \text{Gauss})\) (Sudakov–Tsirelson, Borell) - half spaces.
- \((B^n, |\cdot|, \text{Leb})\) (Burago–Maz’ya) - perpendicular caps.
- Other: \(H^n\), cones, variations, low-dim spaces.

Open: \(([0, 1]^3, |\cdot|, \text{Leb})\), Flat Torus, Slabs, Heisenberg, etc.

Therefore: content in having good lower bounds \(\mathcal{I} \geq J\).

Why should we care? (why are Isoperimetric Inqs important?)
Isoperimetric Inqs \(\Rightarrow\) Sobolev Inqs \(\Rightarrow\) Transport-Entropy Inqs \(\Rightarrow\) Concentration Inqs.
Concentration (large deviation) Inqs (when $\mu(\Omega) = 1$):

$$\forall r > 0 \quad \forall A \subset \Omega \quad \mu(A) \geq 1/2 \quad \Rightarrow \quad \mu(\Omega \setminus A_r^d) \leq K(r).$$

$$\left[ \forall r > 0 \quad \forall \text{1-Lip } f \quad \mu \{ x; f(x) - \text{med}_\mu f \geq r \} \leq K(r) \right].$$

Capture how tightly $\mu$ is concentrated around 1/2-sets ("concentration of measure").

Used in Banach Space Theory, Asymptotic Geometric Analysis, Probability, Graph Theory, Combinatorics, Analysis ...

**Important**: only meaningful in the large ($r \gg 0$), do not provide infinitesimal information, contrary to Isoperimetric Inqs.
Concentration (large deviation) Inqs (when $\mu(\Omega) = 1$):

\[ \forall r > 0 \ \forall A \subset \Omega \ \mu(A) \geq 1/2 \ \Rightarrow \ \mu(\Omega \setminus A_r^d) \leq K(r). \]

\[ [ \ \forall r > 0 \ \forall 1\text{-Lip } f \ \mu \{ x; f(x) - \text{med}_\mu f \geq r \} \leq K(r). \ ] \]

Capture how tightly $\mu$ is concentrated around $1/2$-sets ("concentration of measure").

Used in Banach Space Theory, Asymptotic Geometric Analysis, Probability, Graph Theory, Combinatorics, Analysis ...

Important: only meaningful in the large ($r \gg 0$), do not provide infinitesimal information, contrary to Isoperimetric Inqs.
Concentration (large deviation) Inqs (when $\mu(\Omega) = 1$):

$$\forall r > 0 \quad \forall A \subset \Omega \quad \mu(A) \geq 1/2 \implies \mu(\Omega \setminus A^d_r) \leq K(r).$$

[ $\forall r > 0 \quad \forall 1$-Lip $f \quad \mu \{x; f(x) - \text{med}_\mu f \geq r\} \leq K(r).$ ]

Capture how tightly $\mu$ is concentrated around $1/2$-sets ("concentration of measure").

Used in Banach Space Theory, Asymptotic Geometric Analysis, Probability, Graph Theory, Combinatorics, Analysis ...

Important: only meaningful in the large ($r \gg 0$), do not provide infinitesimal information, contrary to Isoperimetric Inqs.
Isoperimetric Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Concentration Inqs.

\[
\forall r > 0 \quad \forall A \subset \Omega \quad \mu(A) \geq 1/2 \quad \Rightarrow \quad \mu(\Omega \setminus A^d_r) \leq \mathcal{K}(r)
\]

Examples: (Federer–Fleming, Maz’ya; Cheeger, Maz’ya; Gromov–V. Milman; Ledoux, Beckner; Herbst)

\[
\mathcal{I}(v) \geq c_n v^{\frac{n-1}{n}} \quad \Leftrightarrow \quad \|\nabla f\|_1 \geq c_n \|f\|^{\frac{n-1}{n}} \quad \Rightarrow \quad \cdots
\]

(Exponential Conc)

\[
\mathcal{I}(v) \geq Dv \quad \Rightarrow \quad \|\nabla f\|_2 \geq \frac{D}{2} \left\| f - \int f \right\|_2 \quad \Rightarrow \quad \mathcal{K}(r) \leq \exp(-cDr)
\]

(Expanding, $H^n$, log-concave)

\[
\mathcal{I}(v) \geq Dv \sqrt{\log 1/v} \quad \Rightarrow \quad \|\nabla f\|_2 \geq c_1 D \sqrt{\text{Ent}(f^2)} \quad \Rightarrow \quad \mathcal{K}(r) \leq \exp(-c_2 Dr^2)
\]

(Gauss, log-Sobolev)

\[
[\text{Ent}_\mu(g) := \mu(g \log g) - \mu(g) \log \mu(g)]
\]
Isoperimetry $\Rightarrow$ Concentration

$$[ \forall r > 0 \ \forall A \subset \Omega \ \mu(A) \geq 1/2 \ \Rightarrow \ \mu(\Omega \setminus A_r^d) \leq \mathcal{K}(r) ]$$

**Easy - Isoperimetric Inq $\Rightarrow$ Concentration Inq, since**

$$\mu^+(A) \geq I(\mu(A)) \ \Rightarrow \ -\mathcal{K}'(r) := \liminf_{\varepsilon \to 0^+} \frac{\mathcal{K}(r) - \mathcal{K}(r + \varepsilon)}{\varepsilon} \geq I(\mathcal{K}(r)).$$

$$I(v) \geq v \gamma(\log 1/v) \ \forall v \in [0, 1] \ \Rightarrow$$

$$\mathcal{K}(r) \leq \exp(-\alpha(r)) \ \forall r \geq 0 \ \text{where} \ \alpha^{-1}(x) = \int_{\log 2}^x \frac{dy}{\gamma(y)}.$$

$$I(v) \geq Dv \ \Rightarrow \ \||\nabla f||_2 \geq \frac{D}{2} \||f - \int f||_2 \ \Rightarrow \ \mathcal{K}(r) \leq \exp(-cDr)$$

(Expanders, $H^n$, log-concave) (Poincaré, Spectral-Gap) (Exponential Conc)

$$I(v) \geq Dv \sqrt{\log 1/v} \ \Rightarrow \ \||\nabla f||_2 \geq c_1 D \sqrt{\text{Ent}(f^2)} \ \Rightarrow \ \mathcal{K}(r) \leq \exp(-c_2 Dr^2)$$

(Gauss) (log-Sobolev) (Gaussian Conc)
$$\forall r > 0 \ \forall A \subset \Omega \ \mu(A) \geq 1/2 \ \Rightarrow \ \mu(\Omega \setminus A^d_r) \leq \mathcal{K}(r)$$

**Easy - Isoperimetric Inq ⇒ Concentration Inq**, since

$$\mu^+(A) \geq I(\mu(A)) \ \Rightarrow \ -\mathcal{K}'(r) := \lim \inf_{\epsilon \to 0^+} \frac{\mathcal{K}(r) - \mathcal{K}(r+\epsilon)}{\epsilon} \geq I(\mathcal{K}(r)).$$

$$I(v) \geq v\gamma(\log 1/v) \ \forall v \in [0, 1] \ \Rightarrow$$

$$\mathcal{K}(r) \leq \exp(-\alpha(r)) \ \forall r \geq 0 \ \ \text{where} \ \alpha^{-1}(x) = \int_{\log 2}^x \frac{dy}{\gamma(y)}.$$

$$I(v) \geq Dv \ \Rightarrow \ \|\nabla f\|_2 \geq \frac{D}{2} \left\| f - \int f \right\|_2 \ \Rightarrow \ \mathcal{K}(r) \leq \exp(-cDr)$$

(Expanders, $H^n$, log-concave)

(Poincaré, Spectral-Gap)

(Exponential Conc)

$$I(v) \geq Dv\sqrt{\log 1/v} \ \Rightarrow \ \|\nabla f\|_2 \geq c_1 D\sqrt{\text{Ent}(f^2)} \ \Rightarrow \ \mathcal{K}(r) \leq \exp(-c_2 Dr^2)$$

(Gauss)

(log-Sobolev)

(Gaussian Conc)
\[
\forall r > 0 \ \forall A \subset \Omega \quad \mu(A) \geq 1/2 \ \Rightarrow \ \mu(\Omega \setminus A_r^d) \leq \mathcal{K}(r)
\]

**Easy** - Isoperimetric Inq \(\Rightarrow\) Concentration Inq, since

\[
\mu^+(A) \geq \mathcal{I}(\mu(A)) \ \Rightarrow \ -\mathcal{K}'(r) := \liminf_{\varepsilon \to 0^+} \frac{\mathcal{K}(r) - \mathcal{K}(r + \varepsilon)}{\varepsilon} \geq \mathcal{I}(\mathcal{K}(r)).
\]

\[
\mathcal{I}(v) \geq v\gamma(\log 1/v) \ \forall v \in [0, 1] \ \Rightarrow \ \mathcal{K}(r) \leq \exp(-\alpha(r)) \ \forall r \geq 0 \ \text{where} \ \alpha^{-1}(x) = \int_1^x \frac{dy}{\log 2\gamma(y)}.
\]

\[
\mathcal{I}(v) \geq Dv \ \Rightarrow \ \|\nabla f\|_2 \geq \frac{D}{2} \|f - \int f\|_2 \ \Rightarrow \ \mathcal{K}(r) \leq \exp(-cDr) \ \text{(Exponential Conc)}
\]

\[
\mathcal{I}(v) \geq Dv\sqrt{\log 1/v} \ \Rightarrow \ \|\nabla f\|_2 \geq c_1 D\sqrt{\text{Ent}(f^2)} \ \Rightarrow \ \mathcal{K}(r) \leq \exp(-c_2Dr^2) \ \text{(Gaussian Conc)}
\]

---

Emanuel Milman  
*Isoperimetric Inequalities: Methods and Applications*
Remark: reverse implications are in general false due to bottlenecks in space (geometry of $(\Omega, d)$ or measure $\mu$).

$$(\mathbb{R}, | \cdot |, \exp(-V(x))dx), \ V(x) = x^2 + p(x).$$
Conclusion (hopefully): Isoperimetric Inqs are useful.
Sobolev, Concentration (and Transport-Entropy) Inqs will reappear soon.

Now: methods for obtaining Isoperimetric Inqs.
How to obtain isoperimetric inqs $I = I(\Omega, d, \mu) \geq \ldots$ ?

Riemannian setting: $(M^n, g)$, $d$, $\Omega \subset M$ convex, $\mu(\Omega) = 1$.

- **Constant curvature** $(E^n, S^n, H^n)$ - symmetrization.
- **Strictly positive curvature** - compare to model space.
  - Constant density $\mu = \text{vol}_M|_{\Omega}$.
    Under Ricci Curvature condition $Ric_g \geq \lambda g$, $\lambda > 0$
    Gromov–Lévy: $I \geq I(S^n_\lambda, d, \text{vol}_{S^n_\lambda})$.
  - Manifold-with-density $\mu = \exp(-\psi)\text{vol}_M|_{\Omega}$.
    Under Bakry–Émery condition $Ric_g + \text{Hess}_g\psi \geq \lambda g$, $\lambda > 0$
    Bakry–Ledoux, Morgan: $I \geq I(\mathbb{R}, |\cdot|, \text{Gauss}_\lambda)$.

- **Curvature lower-bound** $Ric_g + \text{Hess}_g\psi \geq -\kappa g$, $\kappa \geq 0$.
  No comparison model space + need additional information:
  - Diameter bound (Bérard, Besson, Gallot, Li, Yau,...).
  - $\int_{\Omega} \exp(\beta(d(x, x_0)))d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov).
  Lead to inherently dimension-dependent bounds.
How to obtain isoperimetric inqs $\mathcal{I} = \mathcal{I}(\Omega, d, \mu) \geq ...$ ?

Riemannian setting: $(M^n, g), d, \Omega \subset M$ convex, $\mu(\Omega) = 1$.

- **Constant curvature** $(E^n, S^n, H^n)$ - symmetrization.
- **Strictly positive curvature** - compare to model space.
  - Constant density $\mu = \overline{vol}_M|_\Omega$.
  Under Ricci Curvature condition $Ric_g \geq \lambda g, \lambda > 0$
  Gromov–Lévy: $\mathcal{I} \geq \mathcal{I}(S^n_\lambda, d, \overline{vol}_{S^n_\lambda})$.
  - Manifold-with-density $\mu = \exp(-\psi)\overline{vol}_M|_\Omega$.
  Under Bakry–Émery condition $Ric_g + Hess_g\psi \geq \lambda g, \lambda > 0$
  Bakry–Ledoux, Morgan: $\mathcal{I} \geq \mathcal{I}(\mathbb{R}, |\cdot|, Gauss_\lambda)$.

- **Curvature lower-bound** $Ric_g + Hess_g\psi \geq -\kappa g, \kappa \geq 0$.

No comparison model space + need additional information:

- Diameter bound (Bérard, Besson, Gallot, Li, Yau,...).
- $\int_\Omega \exp(\beta(d(x, x_0)))d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov).

Lead to inherently dimension-dependent bounds.
How to obtain isoperimetric inqs \( I = I(\Omega, d, \mu) \geq \ldots \)?

Riemannian setting: \((M^n, g), d, \Omega \subset M\) convex, \(\mu(\Omega) = 1\).

- **Constant curvature** \((E^n, S^n, H^n)\) - symmetrization.
- **Strictly positive curvature** - compare to model space.
  - Constant density \(\mu = \widetilde{vol}_M|_{\Omega}\).
    Under Ricci Curvature condition \(Ric_g \geq \lambda g, \lambda > 0\)
    Gromov–Lévy: \(I \geq I(S^n_\lambda, d, \widetilde{vol}_{S^n_\lambda})\).
  - Manifold-with-density \(\mu = \exp(-\psi)\widetilde{vol}_M|_{\Omega}\).
    Under Bakry–Émery condition \(Ric_g + Hess_g\psi \geq \lambda g, \lambda > 0\)
    Bakry–Ledoux, Morgan: \(I \geq I(\mathbb{R}, |.|, Gauss_\lambda)\).

- **Curvature lower-bound** \(Ric_g + Hess_g\psi \geq -\kappa g, \kappa \geq 0\).

No comparison model space + need additional information:

- Diameter bound (Bérard, Besson, Gallot, Li, Yau, ...).
- \(\int_{\Omega} \exp(\beta(d(x, x_0)))d\mu(x) < \infty\) (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov).

Lead to inherently dimension-dependent bounds.
How to obtain isoperimetric inqs $I = I(\Omega, d, \mu) \geq \ldots$?

Riemannian setting: $(M^n, g)$, $d$, $\Omega \subset M$ convex, $\mu(\Omega) = 1$.

- Constant curvature $(E^n, S^n, H^n)$ - symmetrization.
- Strictly positive curvature - compare to model space.
  - Constant density $\mu = \text{vol}_M|_{\Omega}$.
  - Under Ricci Curvature condition $\text{Ric}_g \geq \lambda g$, $\lambda > 0$
  - Gromov–Lévy: $I \geq I(S^n_\lambda, d, \text{vol}_{S^n_\lambda})$.

- Manifold-with-density $\mu = \exp(-\psi)\text{vol}_M|_{\Omega}$.
  - Under Bakry–Émery condition $\text{Ric}_g + \text{Hess}_g \psi \geq \lambda g$, $\lambda > 0$
  - Bakry–Ledoux, Morgan: $I \geq I(\mathbb{R}, |\cdot|, \text{Gauss}_\lambda)$.

- Curvature lower-bound $\text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g$, $\kappa \geq 0$.

No comparison model space + need additional information:

- Diameter bound (Bérard, Besson, Gallot, Li, Yau,...).
- $\int_{\Omega} \exp(\beta(d(x, x_0)))d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov).

Lead to inherently dimension-dependent bounds.
Riemannian setting: \((M^n, g), d, \Omega \subset M\) convex, \(\mu(\Omega) = 1\).

- **Constant curvature** \((E^n, S^n, H^n)\) - symmetrization.
- **Strictly positive curvature** - compare to model space.
  - Constant density \(\mu = \overline{vol}_M|_{\Omega}\).
  - Under Ricci Curvature condition \(Ric_g \geq \lambda g, \lambda > 0\)
  - Gromov–Lévy: \(I \geq I(S^n_\lambda, d, \overline{vol}_{S^n_\lambda})\).
- Manifold-with-density \(\mu = \exp(-\psi)vol_M|_{\Omega}\).
  - Under Bakry–Émery condition \(Ric_g + Hess_g \psi \geq \lambda g, \lambda > 0\)
  - Bakry–Ledoux, Morgan: \(I \geq I(\mathbb{R}, |\cdot|, Gauss_\lambda)\).

- **Curvature lower-bound** \(Ric_g + Hess_g \psi \geq -\kappa g, \kappa \geq 0\).
  - No comparison model space + need additional information:
    - Diameter bound (Bérard, Besson, Gallot, Li, Yau, ...).
    - \(\int_{\Omega} \exp(\beta(d(x, x_0)))d\mu(x) < \infty\) (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov).

  Lead to inherently dimension-dependent bounds.
How to obtain isoperimetric inqs $I = I(\Omega, d, \mu) \geq \ldots$ ?

Riemannian setting: $(M^n, g)$, $d$, $\Omega \subset M$ convex, $\mu(\Omega) = 1$.

- Constant curvature $(E^n, S^n, H^n)$ - symmetrization.
- Strictly positive curvature - compare to model space.
  - Constant density $\mu = \text{vol}_M|_\Omega$.
    Under Ricci Curvature condition $\text{Ric}_g \geq \lambda g$, $\lambda > 0$
    Gromov–Lévy: $I \geq I(S^n_\lambda, d, \text{vol}_{S^n_\lambda})$.
  - Manifold-with-density $\mu = \exp(-\psi) \text{vol}_M|_\Omega$.
    Under Bakry–Émery condition $\text{Ric}_g + \text{Hess}_g \psi \geq \lambda g$, $\lambda > 0$
    Bakry–Ledoux, Morgan: $I \geq I(\mathbb{R}, |\cdot|, \text{Gauss}_\lambda)$.

- Curvature lower-bound $\text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g$, $\kappa \geq 0$.
  No comparison model space + need additional information:
  - Diameter bound (Bérard, Besson, Gallot, Li, Yau,...).
  - $\int_{\Omega} \exp(\beta(d(x, x_0))) d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov).
  Lead to inherently dimension-dependent bounds.
How to obtain isoperimetric inqs $I = I(\Omega, d, \mu) \geq \ldots$?

Riemannian setting: $\left( M^n, g \right)$, $d$, $\Omega \subset M$ convex, $\mu(\Omega) = 1$.

- **Constant curvature** ($E^n$, $S^n$, $H^n$) - symmetrization.
- **Strictly positive curvature** - compare to model space.
  - Constant density $\mu = \overline{vol}_M|_{\Omega}$.
    - Under Ricci Curvature condition $Ric_g \geq \lambda g$, $\lambda > 0$
    - Gromov–Lévy: $I \geq I(S^n_\lambda, d, \overline{vol}_{S^n_\lambda})$.
  - Manifold-with-density $\mu = \exp(-\psi)\overline{vol}_M|_{\Omega}$.
    - Under Bakry–Émery condition $Ric_g + Hess_g\psi \geq \lambda g$, $\lambda > 0$
    - Bakry–Ledoux, Morgan: $I \geq I(\mathbb{R}, |\cdot|, Gauss_\lambda)$.

- **Curvature lower-bound** $Ric_g + Hess_g\psi \geq -\kappa g$, $\kappa \geq 0$.
  - No comparison model space + need additional information:
    - Diameter bound (Bérard, Besson, Gallot, Li, Yau,...).
    - $\int_\Omega \exp(\beta(d(x, x_0)))d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov).
  - Lead to inherently dimension-dependent bounds.

Emanuel Milman

Isoperimetric Inequalities: Methods and Applications
How to obtain isoperimetric inqs $\mathcal{I} = \mathcal{I}(\Omega, d, \mu) \geq ...$?

Riemannian setting: $(M^n, g)$, $d$, $\Omega \subset M$ convex, $\mu(\Omega) = 1$.

- **Constant curvature** $(E^n, S^n, H^n)$ - symmetrization.
- **Strictly positive curvature** - compare to model space.
  - Constant density $\mu = \overline{vol}_M|_{\Omega}$.
    - Under Ricci Curvature condition $Ric_g \geq \lambda g$, $\lambda > 0$
  - Gromov–Lévy: $\mathcal{I} \geq \mathcal{I}(S^n_\lambda, d, \overline{vol}_{S^n_\lambda})$.
  - Manifold-with-density $\mu = \exp(-\psi)\overline{vol}_M|_{\Omega}$.
    - Under Bakry–Émery condition $Ric_g + Hess_g\psi \geq \lambda g$, $\lambda > 0$
    - Bakry–Ledoux, Morgan: $\mathcal{I} \geq \mathcal{I}(\mathbb{R}, |\cdot|, Gauss_\lambda)$.

- **Curvature lower-bound** $Ric_g + Hess_g\psi \geq -\kappa g$, $\kappa \geq 0$.
  - No comparison model space + need additional information:
    - Diameter bound (Bérard, Besson, Gallot, Li, Yau,...).
    - $\int_\Omega \exp(\beta(d(x, x_0)))d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov).

Lead to inherently dimension-dependent bounds.
How to obtain isoperimetric inqs $I = I(\Omega, d, \mu) \geq \ldots$?

Riemannian setting: $(M^n, g)$, $d$, $\Omega \subset M$ convex, $\mu(\Omega) = 1$.

- **Constant curvature** $(E^n, S^n, H^n)$ - symmetrization.
- **Strictly positive curvature** - compare to model space.
  - Constant density $\mu = \widetilde{\text{vol}}_M|_\Omega$.
    Under Ricci Curvature condition $Ric_g \geq \lambda g$, $\lambda > 0$
    Gromov–Lévy: $I \geq I(S^n_\lambda, d, \widetilde{\text{vol}}_{S^n_\lambda})$.
  - Manifold-with-density $\mu = \exp(-\psi)\text{vol}_M|\Omega$.
    Under Bakry–Émery condition $Ric_g + Hess_g\psi \geq \lambda g$, $\lambda > 0$
    Bakry–Ledoux, Morgan: $I \geq I(\mathbb{R}, |\cdot|, \text{Gauss}_\lambda)$.

- **Curvature lower-bound** $Ric_g + Hess_g\psi \geq -\kappa g$, $\kappa \geq 0$.
  No comparison model space + need **additional** information:
  - Diameter bound (Bérard, Besson, Gallot, Li, Yau,...).
  - $\int_\Omega \exp(\beta(d(x, x_0)))d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov).
  Lead to inherently dimension-dependent bounds.
How to obtain isoperimetric inqs $I = I(\Omega, d, \mu) \geq \ldots$ ?

- **Curvature lower-bound** $Ric_g + Hess_g \psi \geq -\kappa g$, $\kappa \geq 0$.

  No comparison model space + need **additional** information:

  - Diameter bound (Bérard, Besson, Gallot, Li, Yau, ...).
  - $\int_{\Omega} \exp(\beta(d(x, x_0)))d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe–Kolesnikov).

  Lead to inherently dimension-dependent bounds.

Hierarchy: Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Conc Inqs.

Cannot be reversed in general due to bottlenecks.
But we assume curvature lower-bound...

- Sobolev inqs (Buser, Ledoux, M.) - Dimension independent!
- Concentration inqs (M.) - “Dim-indep. Hierarchy Reversal”.

Assumption: $Ric_g + Hess_g \psi \geq -\kappa g$, two cases:

  - $\kappa = 0$ - “convex case" (e.g. $(\mathbb{R}^n, |\cdot|, \mu)$ log-concave).
  - $\kappa > 0$ - “semi-convex case" (e.g. double-well potentials).

Survey methods of obtaining isop. inqs in these scenarios.
How to obtain isoperimetric inqs $I = I(\Omega, d, \mu) \geq \ldots$?

- **Curvature lower-bound** $Ric_g + Hess_g \psi \geq -\kappa g$, $\kappa \geq 0$.

  No comparison model space + need **additional** information:
  
  - Diameter bound (Bérard, Besson, Gallot, Li, Yau, ...).
  - $\int_{\Omega} \exp(\beta(d(x, x_0))) d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe–Kolesnikov).

  Lead to inherently dimension-dependent bounds.

  **Hierarchy:** Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Conc Inqs.

  Cannot be reversed in general due to bottlenecks.

  But we assume curvature lower-bound...

- Sobolev inqs (Buser, Ledoux, M.) - **Dimension independent!**
- Concentration inqs (M.) - "Dim-indep. Hierarchy Reversal".

**Assumption:** $Ric_g + Hess_g \psi \geq -\kappa g$, two cases:

- $\kappa = 0$ - "convex case" (e.g. $(\mathbb{R}^n, | \cdot |, \mu)$ log-concave).
- $\kappa > 0$ - "semi-convex case" (e.g. double-well potentials).

Survey methods of obtaining isop. inqs in these scenarios.
How to obtain isoperimetric inqs $I = I(\Omega, d, \mu) \geq \ldots$ ?

- **Curvature lower-bound** $\text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g$, $\kappa \geq 0$.
  No comparison model space + need additional information:
  - Diameter bound (Bérard, Besson, Gallot, Li, Yau, ...).
  - $\int_{\Omega} \exp(\beta(d(x, x_0)))d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe–Kolesnikov).

  Lead to inherently dimension-dependent bounds.

  Hierarchy: Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Conc Inqs.
  Cannot be reversed in general due to bottlenecks.
  But we assume curvature lower-bound...

- Sobolev inqs (Buser, Ledoux, M.) - Dimension independent!
- Concentration inqs (M.) - “Dim-indep. Hierarchy Reversal”.

Assumption: $\text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g$, two cases:
  - $\kappa = 0$ - “convex case” (e.g. $(\mathbb{R}^n, | \cdot |, \mu)$ log-concave).
  - $\kappa > 0$ - “semi-convex case” (e.g. double-well potentials).

Survey methods of obtaining isop. inqs in these scenarios.
Curvature lower-bound $Ric_g + Hess_g \psi \geq -\kappa g$, $\kappa \geq 0$.
No comparison model space + need additional information:
- Diameter bound (Bérard, Besson, Gallot, Li, Yau, ...).
- $\int_{\Omega} \exp(\beta(d(x, x_0)))d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe–Kolesnikov).

Lead to inherently dimension-dependent bounds.

Hierarchy: Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Conc Inqs.
Cannot be reversed in general due to bottlenecks.
But we assume curvature lower-bound...

Sobolev inqs (Buser, Ledoux, M.) - Dimension independent!
Concentration inqs (M.) - “Dim-indep. Hierarchy Reversal”.

Assumption: $Ric_g + Hess_g \psi \geq -\kappa g$, two cases:
- $\kappa = 0$ - “convex case” (e.g. $(\mathbb{R}^n, |\cdot|, \mu)$ log-concave).
- $\kappa > 0$ - “semi-convex case” (e.g. double-well potentials).

Survey methods of obtaining isop. inqs in these scenarios.
How to obtain isoperimetric inequalities $\mathcal{I} = \mathcal{I}(\Omega, d, \mu) \geq ...$?

- **Curvature lower-bound** $\text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g$, $\kappa \geq 0$.

  No comparison model space + need **additional** information:
  - Diameter bound (Bérard, Besson, Gallot, Li, Yau, ...).
  - $\int_{\Omega} \exp(\beta(d(x, x_0))) d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe–Kolesnikov).

  Lead to inherently dimension-dependent bounds.

Hierarchy: Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Conc Inqs.
Cannot be reversed in general due to bottlenecks.
But we assume **curvature lower-bound**...

- Sobolev inqs (Buser, Ledoux, M.) - Dimension independent!
- Concentration inqs (M.) - “Dim-indep. Hierarchy Reversal”.

Assumption: $\text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g$, two cases:
  - $\kappa = 0$ - “convex case” (e.g. $(\mathbb{R}^n, \cdot), \mu$) log-concave).
  - $\kappa > 0$ - “semi-convex case” (e.g. double-well potentials).

Survey methods of obtaining isop. inqs in these scenarios.
How to obtain isoperimetric inqs $\mathcal{I} = \mathcal{I}(\Omega, d, \mu) \geq ...$?

Curvature lower-bound $Ric_g + Hess_g \psi \geq -\kappa g$, $\kappa \geq 0$.

No comparison model space + need additional information:

- Diameter bound (Bérard, Besson, Gallot, Li, Yau, ...).
- $\int_{\Omega} \exp(\beta(d(x, x_0)))d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe–Kolesnikov).

Lead to inherently dimension-dependent bounds.

Hierarchy: Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Conc Inqs.
Cannot be reversed in general due to bottlenecks.
But we assume curvature lower-bound...

- Sobolev inqs (Buser, Ledoux, M.) - Dimension independent!
- Concentration inqs (M.) - “Dim-indep. Hierarchy Reversal”.

Assumption: $Ric_g + Hess_g \psi \geq -\kappa g$, two cases:

- $\kappa = 0$ - “convex case" (e.g. $(\mathbb{R}^n, |\cdot|, \mu)$ log-concave).
- $\kappa > 0$ - “semi-convex case" (e.g. double-well potentials).

Survey methods of obtaining isop. inqs in these scenarios.
How to obtain isoperimetric inqs $\mathcal{I} = \mathcal{I}(\Omega, d, \mu) \geq \ldots$ ?

- **Curvature lower-bound** $Ric_g + Hess_g \psi \geq -\kappa g$, $\kappa \geq 0$.

  No comparison model space + need **additional** information:
  - Diameter bound (Bérard, Besson, Gallot, Li, Yau, ...).
  - $\int_{\Omega} \exp(\beta(d(x, x_0))) \, d\mu(x) < \infty$ (Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe–Kolesnikov).

  Lead to inherently dimension-dependent bounds.

Hierarchy: Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Conc Inqs.
Cannot be reversed in general due to bottlenecks.
But we assume curvature lower-bound...

- Sobolev Inqs (Buser, Ledoux, M.) - Dimension independent!
- Concentration Inqs (M.) - “Dim-indep. Hierarchy Reversal”.

Assumption: $Ric_g + Hess_g \psi \geq -\kappa g$, two cases:
  - $\kappa = 0$ - “convex case” (e.g. $(\mathbb{R}^n, | \cdot |, \mu)$ log-concave).
  - $\kappa > 0$ - “semi-convex case” (e.g. double-well potentials).

Survey methods of obtaining isop. Inqs in these scenarios.
Semi-group method: \( \text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g \)

\((p, q)\) Poincaré inequality: \( \forall f \ D\|f - \mu(f)\|_{L^p(\mu)} \leq \|\nabla f\|_{L^q(\mu)} \).

Ledoux \((q = 2)\): implies back the “right” isoperimetric inequality.

M. 08: generalized to arbitrary \(1 \leq q \leq \infty\) (and Orlicz norms):

\[ \Rightarrow \mathcal{I}(v) \geq \min(cD, c_{p,q}D^r\kappa^{-\frac{r-1}{2}})v^{1+\frac{1}{p}-\frac{1}{q}}, \quad r = \max(q, 2). \]

Idea (Bakry–Émery–Ledoux):
\( \Delta_{\mu} := \Delta - \langle \nabla \psi, \nabla \rangle \) generator of diffusion process with invariant measure \( \mu \), \( P_t := \exp(t\Delta_{\mu}) \). Curvature lower bound implies contractivity of \( P_t \) (only state \( \kappa = 0, q \geq 2 \) case below):

\[ \|\nabla P_t f\|_{L^q(\mu)} \leq \frac{1}{\sqrt{2t}} \|f\|_{L^q(\mu)}, \quad \|\nabla f\|_{L^1(\mu)} \geq \frac{1}{\sqrt{2t}} \|f - P_t f\|_{L^1(\mu)}. \]

Apply to \( f = 1_A \):

\( \mu^+(A) \geq \frac{1}{\sqrt{2t}} (\|1_A - \mu(A)\|_{L^1(\mu)} - \|P_t(1_A) - \mu(A)\|_{L^1(\mu)}) \).

Use Hölder, \((p, q)\) inq for \( P_t(1_A) \), smoothing, and optimize in \( t \).

\[ \|P_t(1_A) - \mu(A)\|_{L^p(\mu)} \leq \frac{1}{D} \|\nabla P_t(1_A)\|_{L^q(\mu)} \leq \frac{1}{D\sqrt{2t}} \|1_A\|_{L^q(\mu)}. \]
Semi-group method: \( \text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g \)

\((p, q)\) Poincaré inequality: \( \forall f \ D \| f - \mu(f) \|_{L^p(\mu)} \leq \| \nabla f \|_{L^q(\mu)} \).

Ledoux \((q = 2)\): implies back the "right" isoperimetric inequality.

M. 08: generalized to arbitrary \( 1 \leq q \leq \infty \) (and Orlicz norms):

\[ \Rightarrow \mathcal{I}(\nu) \geq \min(cD, c_{p,q}D^r \kappa^{\frac{r-1}{2}}) \nu^{1 + \frac{1}{p} - \frac{1}{q}} , \quad r = \max(q, 2) . \]

Idea (Bakry–Émery–Ledoux):

\[ \Delta_\mu := \Delta - \langle \nabla \psi, \nabla \rangle \] generator of diffusion process with invariant measure \( \mu, \ P_t := \exp(t\Delta_\mu) \). Curvature lower bound implies contractivity of \( P_t \) (only state \( \kappa = 0, q \geq 2 \) case below):

\[ \| \nabla P_t f \|_{L^q(\mu)} \leq \frac{1}{\sqrt{2t}} \| f \|_{L^q(\mu)} , \quad \| \nabla f \|_{L^1(\mu)} \geq \frac{1}{\sqrt{2t}} \| f - P_t f \|_{L^1(\mu)} . \]

Apply to \( f = 1_A \):

\[ \mu^+(A) \geq \frac{1}{\sqrt{2t}} (\| 1_A - \mu(A) \|_{L^1(\mu)} - \| P_t(1_A) - \mu(A) \|_{L^1(\mu)} ) . \]

Use Hölder, \((p, q)\) inq for \( P_t(1_A) \), smoothing, and optimize in \( t \).

\[ \| P_t(1_A) - \mu(A) \|_{L^p(\mu)} \leq \frac{1}{D} \| \nabla P_t(1_A) \|_{L^q(\mu)} \leq \frac{1}{D\sqrt{2t}} \| 1_A \|_{L^q(\mu)} . \]
(p, q) Poincaré inequality: \( \forall f \ D\|f - \mu(f)\|_{L^p(\mu)} \leq \|\nabla f\|_{L^q(\mu)}. \)

Ledoux (q = 2): implies back the “right” isoperimetric inequality.

M. 08: generalized to arbitrary \( 1 \leq q \leq \infty \) (and Orlicz norms):

\[ \Rightarrow \mathcal{I}(v) \geq \min\left(cD, c_p, q D^r \kappa^{-\frac{r-1}{2}}\right)v^{1 + \frac{1}{p} - \frac{1}{q}}, \quad r = \max(q, 2). \]

Idea (Bakry–Émery–Ledoux):
\[
\Delta_\mu := \Delta - \langle \nabla \psi, \nabla \rangle \text{ generator of diffusion process with invariant measure } \mu, \quad P_t := \exp(t\Delta_\mu). \text{ Curvature lower bound implies contractivity of } P_t \text{ (only state } \kappa = 0, q \geq 2 \text{ case below):}
\]

\[
\|\nabla P_t f\|_{L^q(\mu)} \leq \frac{1}{\sqrt{2t}} \|f\|_{L^q(\mu)}, \quad \|\nabla f\|_{L^1(\mu)} \geq \frac{1}{\sqrt{2t}} \|f - P_t f\|_{L^1(\mu)}. 
\]

Apply to \( f = 1_A \):
\[
\mu^+(A) \geq \frac{1}{\sqrt{2t}} (\|1_A - \mu(A)\|_{L^1(\mu)} - \|P_t(1_A) - \mu(A)\|_{L^1(\mu)}).
\]

Use Hölder, \((p, q)\) inq for \(P_t(1_A)\), smoothing, and optimize in \(t\).
\[
\|P_t(1_A) - \mu(A)\|_{L^p(\mu)} \leq \frac{1}{D} \|\nabla P_t(1_A)\|_{L^q(\mu)} \leq \frac{1}{D\sqrt{2t}} \|1_A\|_{L^q(\mu)}. 
\]
Semi-group method: $Ric_g + Hess_g \psi \geq -\kappa g$

$(p, q)$ Poincaré inequality: $\forall f \quad D\|f - \mu(f)\|_{L^p(\mu)} \leq \|\nabla f\|_{L^q(\mu)}$.

Ledoux ($q = 2$): implies back the “right” isoperimetric inequality.

M. 08: generalized to arbitrary $1 \leq q \leq \infty$ (and Orlicz norms):

$$\Rightarrow \quad I(v) \geq \min(cD, c_{p,q}D'r\kappa^{\frac{r-1}{2}})v^{1+\frac{1}{p}-\frac{1}{q}}, \quad r = \max(q, 2).$$

Idea (Bakry–Émery–Ledoux):

$\Delta_\mu := \Delta - \langle \nabla \psi, \nabla \rangle$ generator of diffusion process with invariant measure $\mu$, $P_t := \exp(t\Delta_\mu)$. Curvature lower bound implies contractivity of $P_t$ (only state $\kappa = 0$, $q \geq 2$ case below):

$$\|\nabla P_t f\|_{L^q(\mu)} \leq \frac{1}{\sqrt{2t}} \|f\|_{L^q(\mu)}, \quad \|\nabla f\|_{L^1(\mu)} \geq \frac{1}{\sqrt{2t}} \|f - P_t f\|_{L^1(\mu)}.$$

Apply to $f = 1_A$:

$$\mu^+(A) \geq \frac{1}{\sqrt{2t}} (\|1_A - \mu(A)\|_{L^1(\mu)} - \|P_t(1_A) - \mu(A)\|_{L^1(\mu)}).$$

Use Hölder, $(p, q)$ inq for $P_t(1_A)$, smoothing, and optimize in $t$.

$$\|P_t(1_A) - \mu(A)\|_{L^p(\mu)} \leq \frac{1}{D} \|\nabla P_t(1_A)\|_{L^q(\mu)} \leq \frac{1}{D\sqrt{2t}} \|1_A\|_{L^q(\mu)}.$$

Emanuel Milman
Isoperimetric Inequalities: Methods and Applications
Semi-group method: $\text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g$

$(p, q)$ Poincaré inequality: $\forall f \quad D\|f - \mu(f)\|_{L^p(\mu)} \leq \|\nabla f\|_{L^q(\mu)}$.

Ledoux ($q = 2$): implies back the “right” isoperimetric inequality.

M. 08: generalized to arbitrary $1 \leq q \leq \infty$ (and Orlicz norms):

$$\Rightarrow \quad \mathcal{I}(\nu) \geq \min(cD, c_p,q D^r \kappa^{\frac{r-1}{2}}) \nu^{1+\frac{1}{p}-\frac{1}{q}}, \quad r = \max(q, 2).$$

Idea (Bakry–Émery–Ledoux):

$$\Delta_\mu := \Delta - \langle \nabla \psi, \nabla \rangle$$ generator of diffusion process with invariant measure $\mu$, $P_t := \exp(t\Delta_\mu)$. Curvature lower bound implies contractivity of $P_t$ (only state $\kappa = 0$, $q \geq 2$ case below):

$$\|\nabla P_t f\|_{L^q(\mu)} \leq \frac{1}{\sqrt{2t}} \|f\|_{L^q(\mu)}, \quad \|\nabla f\|_{L^1(\mu)} \geq \frac{1}{\sqrt{2t}} \|f - P_t f\|_{L^1(\mu)}.$$

Apply to $f = 1_A$:

$$\mu^+(A) \geq \frac{1}{\sqrt{2t}} (\|1_A - \mu(A)\|_{L^1(\mu)} - \|P_t(1_A) - \mu(A)\|_{L^1(\mu)}).$$

Use Hölder, $(p, q)$ inq for $P_t(1_A)$, smoothing, and optimize in $t$.

$$\|P_t(1_A) - \mu(A)\|_{L^p(\mu)} \leq \frac{1}{D} \|\nabla P_t(1_A)\|_{L^q(\mu)} \leq \frac{1}{D \sqrt{2t}} \|1_A\|_{L^q(\mu)}.$$
Isoperimetric profile method: $\text{Ric}_g + \text{Hess}_g \psi \geq 0$

$(1, \infty)$ Poincaré inequality: $\forall f \ D \| f - \mu(f) \|_{L^1(\mu)} \leq \| \nabla f \|_{L^\infty(\mu)}$, i.e. $L^1$ bound on centered Lipschitz functions.

$\Rightarrow$ (curvature lower bound + Thm) $\mathcal{I}(v) \geq cDv^2 \ \forall v \in [0, 1/2]$.

**Thm** (Bobkov, Sternberg–Zumbrun, Kuwert, Bayle–Rosales): $\mathcal{I}$ is concave on $[0, 1]$.

Weaker and easier (M. 09): $v \mapsto \mathcal{I}(v)/v$ is non-increasing.

**Cor:** $\min_{v \in [0, 1/2]} \mathcal{I}(v)/v$ attained at $v = 1/2$.

$\Rightarrow \mathcal{I}(v) \geq \frac{c}{2}Dv \ \forall v \in [0, 1/2]

\Rightarrow \forall 1 \leq p \leq q \leq \infty \ C_{p,q}D \| f - \mu(f) \|_{L^p(\mu)} \leq \| \nabla f \|_{L^q(\mu)}$.

Useful: to prove spectral-gap ($p = q = 2$) estimate, it is enough to bound $L^1$ norm of centered Lipschitz functions.

Applications (Barthe, Wolff, Cordero–Erausquin, Huet, Fleury).

Open: find conditions on $G = (V, E)$ for analogous statement.
Isoperimetric profile method: $Ric_g + Hess_g \psi \geq 0$

$(1, \infty)$ Poincaré inequality: $\forall f \ D\|f - \mu(f)\|_{L^1(\mu)} \leq \|\nabla f\|_{L^\infty(\mu)}$, i.e. $L^1$ bound on centered Lipschitz functions.

$\Rightarrow$ (curvature lower bound + Thm) $I(v) \geq cDv^2 \ \forall v \in [0, 1/2]$.

**Thm** (Bobkov, Sternberg–Zumbrun, Kuwert, Bayle–Rosales): $I$ is concave on $[0, 1]$.

Weaker and easier (M. 09): $v \mapsto I(v)/v$ is non-increasing.

**Cor**: $\min_{v \in [0, 1/2]} I(v)/v$ attained at $v = 1/2$.

$\Rightarrow \ I(v) \geq \frac{c}{2}Dv \ \forall v \in [0, 1/2]$.

$\Rightarrow \ \forall 1 \leq p \leq q \leq \infty \ C_{p,q}D\|f - \mu(f)\|_{L^p(\mu)} \leq \|\nabla f\|_{L^q(\mu)}$.

Useful: to prove spectral-gap ($p = q = 2$) estimate, it is enough to bound $L^1$ norm of centered Lipschitz functions.

Applications (Barthe, Wolff, Cordero–Erausquin, Huet, Fleury).

**Open**: find conditions on $G = (V, E)$ for analogous statement.
Isoperimetric profile method: $Ric_g + Hess_g \psi \geq 0$

$(1, \infty)$ Poincaré inequality: $\forall f \quad D\|f - \mu(f)\|_{L^1(\mu)} \leq \||\nabla f||_{L^\infty(\mu)}$, i.e. $L^1$ bound on centered Lipschitz functions.

$\Rightarrow$ (curvature lower bound + Thm) $\mathcal{I}(v) \geq cDv^2 \quad \forall v \in [0, 1/2]$.

**Thm** (Bobkov, Sternberg–Zumbrun, Kuwert, Bayle–Rosales): $\mathcal{I}$ is concave on $[0, 1]$.

Weaker and easier (M. 09): $v \mapsto \mathcal{I}(v)/v$ is non-increasing.

**Cor:** $\min_{v \in [0, 1/2]} \mathcal{I}(v)/v$ attained at $v = 1/2$.

$\Rightarrow \mathcal{I}(v) \geq \frac{c}{2}Dv \quad \forall v \in [0, 1/2]$.

$\Rightarrow \quad \forall 1 \leq p \leq q \leq \infty \quad C_{p,q}D\|f - \mu(f)\|_{L^p(\mu)} \leq \||\nabla f||_{L^q(\mu)}$.

Useful: to prove spectral-gap ($p = q = 2$) estimate, it is enough to bound $L^1$ norm of centered Lipschitz functions.

Applications (Barthe, Wolff, Cordero–Erausquin, Huet, Fleury).

**Open:** find conditions on $G = (V, E)$ for analogous statement.
Isoperimetric profile method: $\text{Ric}_g + \text{Hess}_g \psi \geq 0$

$(1, \infty)$ Poincaré inequality: $\forall f \ D\|f - \mu(f)\|_{L^1(\mu)} \leq \|\nabla f\|_{L^\infty(\mu)}$, i.e. $L^1$ bound on centered Lipschitz functions.

$\Rightarrow$ (curvature lower bound + Thm) $\mathcal{I}(\nu) \geq cD\nu^2 \ \forall \nu \in [0, 1/2].$

**Thm** (Bobkov, Sternberg–Zumbrun, Kuwert, Bayle–Rosales): $\mathcal{I}$ is concave on $[0, 1]$. Weaker and easier (M. 09): $\nu \mapsto \mathcal{I}(\nu)/\nu$ is non-increasing.

**Cor:** $\min_{\nu \in [0, 1/2]} \mathcal{I}(\nu)/\nu$ attained at $\nu = 1/2$.  

$\Rightarrow \mathcal{I}(\nu) \geq \frac{c}{2}D\nu \ \forall \nu \in [0, 1/2]$

$\Rightarrow \ \forall 1 \leq p \leq q \leq \infty \ C_{p,q}D\|f - \mu(f)\|_{L^p(\mu)} \leq \|\nabla f\|_{L^q(\mu)}.$

Useful: to prove spectral-gap ($p = q = 2$) estimate, it is enough to bound $L^1$ norm of centered Lipschitz functions.

Applications (Barthe, Wolff, Cordero–Erausquin, Huet, Fleury).

Open: find conditions on $G = (V, E)$ for analogous statement.
(1, ∞) Poincaré inequality: \( \forall f \quad D \| f - \mu(f) \|_{L^1(\mu)} \leq \| |\nabla f| \|_{L^\infty(\mu)}, \)
i.e. \( L^1 \) bound on centered Lipschitz functions.

\( \Rightarrow \) (curvature lower bound + Thm) \( \mathcal{I}(v) \geq cDv^2 \quad \forall v \in [0, 1/2]. \)

**Thm** (Bobkov, Sternberg–Zumbrun, Kuwert, Bayle–Rosales): \( \mathcal{I} \) is concave on \([0, 1] \).
Weaker and easier (M. 09): \( v \mapsto \mathcal{I}(v)/v \) is non-increasing .

**Cor:** \( \min_{v \in [0, 1/2]} \mathcal{I}(v)/v \) attained at \( v = 1/2 \).

\[ \Rightarrow \mathcal{I}(v) \geq \frac{c}{2}Dv \quad \forall v \in [0, 1/2] \]

\[ \Rightarrow \quad \forall 1 \leq p \leq q \leq \infty \quad C_{p,q}D \| f - \mu(f) \|_{L^p(\mu)} \leq \| |\nabla f| \|_{L^q(\mu)}. \]

Useful: to prove spectral-gap \( (p = q = 2) \) estimate, it is enough to bound \( L^1 \) norm of centered Lipschitz functions.

Applications (Barthe, Wolff, Cordero–Erausquin, Huet, Fleury).

Open: find conditions on \( G = (V, E) \) for analogous statement.
Isoperimetric profile method: $\text{Ric}_g + \text{Hess}_g \psi \geq 0$

$(1, \infty)$ Poincaré inequality: $\forall f \ D\| f - \mu(f) \|_{L^1(\mu)} \leq \|[\nabla f]\|_{L^\infty(\mu)}$, i.e. $L^1$ bound on centered Lipschitz functions.

$\Rightarrow$ (curvature lower bound + Thm) $I(v) \geq cDv^2 \ \forall v \in [0, 1/2]$. 

**Thm** (Bobkov, Sternberg–Zumbrun, Kuwert, Bayle–Rosales): $I$ is concave on $[0, 1]$.

Weaker and easier (M. 09): $\nu \mapsto I(\nu)/\nu$ is non-increasing .

**Cor:** $\min_{\nu \in [0, 1/2]} I(\nu)/\nu$ attained at $\nu = 1/2$.

$$\Rightarrow I(\nu) \geq \frac{c}{2}D\nu \ \forall \nu \in [0, 1/2]$$

$$\Rightarrow \forall 1 \leq p \leq q \leq \infty \ C_{p,q}D\| f - \mu(f) \|_{L^p(\mu)} \leq \|[\nabla f]\|_{L^q(\mu)}.$$ 

Useful: to prove spectral-gap ($\rho = q = 2$) estimate, it is enough to bound $L^1$ norm of centered Lipschitz functions.

Applications (Barthe, Wolff, Cordero–Erausquin, Huet, Fleury).

**Open:** find conditions on $G = (V, E)$ for analogous statement.
(1, ∞) Poincaré inequality: \( \forall f \ D\|f - \mu(f)\|_{L^1(\mu)} \leq \|\nabla f\|_{L^\infty(\mu)}, \) i.e. \( L^1 \) bound on centered Lipschitz functions.

\( \Rightarrow \) (curvature lower bound + Thm) \( I(v) \geq cDv^2 \ \forall v \in [0, 1/2]. \)

**Thm** (Bobkov, Sternberg–Zumbrun, Kuwert, Bayle–Rosales): \( I \) is concave on \([0, 1]\).

Weaker and easier (M. 09): \( v \mapsto I(v)/v \) is non-increasing .

**Cor**: \( \min_{v \in [0, 1/2]} I(v)/v \) attained at \( v = 1/2. \)

\[ \Rightarrow I(v) \geq \frac{c}{2}Dv \ \forall v \in [0, 1/2] \]

\[ \Rightarrow \forall 1 \leq p \leq q \leq \infty \ C_{p,q}D\|f - \mu(f)\|_{L^p(\mu)} \leq \|\nabla f\|_{L^q(\mu)}. \]

Useful: to prove spectral-gap \( (p = q = 2) \) estimate, it is enough to bound \( L^1 \) norm of centered Lipschitz functions.

Applications (Barthe, Wolff, Cordero–Erausquin, Huet, Fleury).

**Open**: find conditions on \( G = (V, E) \) for analogous statement.
Saw that \((1, \infty)\) Poincaré inq \(\Rightarrow\) \(I(v) \geq c Dv\).

\[
\forall f \quad D \| f - \mu(f) \|_{L^1(\mu)} \leq \| \nabla f \|_{L^\infty(\mu)} \quad " \Leftrightarrow "
\]

\[
\forall 1\text{-Lip functions} \quad \mu(|f - \mu(f)| \geq r) \leq 1/(Dr) \quad " \Leftrightarrow "
\]

\[
\forall A \quad \mu(A) \geq 1/2 \Rightarrow \mu(\Omega \setminus A^d_r) \leq 1/(Dr) \quad .
\]

"weakest concentration implies linear isop. (hence exponential conc.)"

**Thm** (M. 08,09,10): If \(\exists \lambda_0 \in (0, 1/2) \exists r_0 > 0\) so that:

\[
\forall A \quad \mu(A) \geq 1/2 \Rightarrow \mu(\Omega \setminus A^d_{r_0}) \leq \lambda_0 .
\]

Then:

\[
I(v) \geq \frac{1 - 2\lambda_0}{r_0} v \quad \forall v \in [0, 1/2] .
\]

Remark: sharp dependence on \(\lambda_0, r_0\) \(([0, 1]^{n-1} \times [0, M], M \to \infty)\).

Next Thm:

"stronger than exp concentration implies stronger than linear isop."
Saw that \((1, \infty)\) Poincaré inq \(\Rightarrow \ I(v) \geq c \ Dv.\)

\[
\forall f \ D \|f - \mu(f)\|_{L^1(\mu)} \leq \|\nabla f\|_{L^\infty(\mu)} \quad " \Leftrightarrow "
\]

\[
\forall \text{1-Lip functions } \mu(|f - \mu(f)| \geq r) \leq 1/(Dr) \quad " \Leftrightarrow "
\]

\[
\forall A \ \mu(A) \geq 1/2 \Rightarrow \mu(\Omega \setminus A^d_r) \leq 1/(Dr) .
\]

"weakest concentration implies linear isop. (hence exponential conc.)"

**Thm** (M. 08,09,10): If \(\exists \lambda_0 \in (0, 1/2) \ \exists r_0 > 0\) so that:

\[
\forall A \ \mu(A) \geq 1/2 \Rightarrow \mu(\Omega \setminus A^d_{r_0}) \leq \lambda_0 .
\]

Then:

\[
I(v) \geq \frac{1 - 2\lambda_0}{r_0} v \quad \forall v \in [0, 1/2] .
\]

Remark: sharp dependence on \(\lambda_0, r_0 ([0, 1]^{n-1} \times [0, M], M \to \infty).\)

Next Thm:

"stronger than exp concentration implies stronger than linear isop."
Saw that \((1, \infty)\) Poincaré inq \(\Rightarrow\) \(\mathcal{I}(v) \geq c \, Dv\).

\[
\forall f \, D \| f - \mu(f) \|_{L^1(\mu)} \leq \| \nabla f \|_{L^\infty(\mu)} \quad \text{“} \Leftrightarrow \text{”}
\]

\[
\forall 1\text{-Lip functions } \mu(|f - \mu(f)| \geq r) \leq 1/(Dr) \quad \text{“} \Leftrightarrow \text{”}
\]

\[
\forall A \, \mu(A) \geq 1/2 \Rightarrow \mu(\Omega \setminus A^d_r) \leq 1/(Dr) \quad .
\]

“\textit{weakest concentration implies linear isop. (hence exponential conc.)}"

**Thm** (M. 08,09,10): If \(\exists \lambda_0 \in (0, 1/2) \, \exists r_0 > 0\) so that:

\[
\forall A \, \mu(A) \geq 1/2 \Rightarrow \mu(\Omega \setminus A^d_r) \leq \lambda_0 .
\]

Then:

\[
\mathcal{I}(v) \geq \frac{1 - 2\lambda_0}{r_0} \cdot v \quad \forall v \in [0, 1/2] .
\]

Remark: sharp dependence on \(\lambda_0, r_0\) \([0, 1]^{n-1} \times [0, M], M \to \infty\).

Next Thm:

“\textit{stronger than exp concentration implies stronger than linear isop.}"
Ric_g + Hess_g \psi \geq 0 - Hierarchy Reversal

Saw that \((1, \infty)\) Poincaré inq \(\Rightarrow I(v) \geq c \, Dv\).

\[
\forall f \, D\|f - \mu(f)\|_{L^1(\mu)} \leq \|\nabla f\|_{L^\infty(\mu)} \quad "\iff"
\]
\[
\forall 1\text{-Lip functions } \mu(|f - \mu(f)| \geq r) \leq 1/(Dr) \quad "\iff"
\]
\[
\forall A \, \mu(A) \geq 1/2 \Rightarrow \mu(\Omega \setminus A_r^d) \leq 1/(Dr)
\]

“weakest concentration implies linear isop. (hence exponential conc.)"

**Thm** (M. 08,09,10): If \(\exists \lambda_0 \in (0, 1/2) \, \exists r_0 > 0\) so that:

\[
\forall A \, \mu(A) \geq 1/2 \Rightarrow \mu(\Omega \setminus A_r^d) \leq \lambda_0.
\]

Then:

\[
I(v) \geq \frac{1 - 2\lambda_0}{r_0} v \quad \forall v \in [0, 1/2].
\]

Remark: sharp dependence on \(\lambda_0, r_0\) \(([0, 1]^{n-1} \times [0, M], M \to \infty\).

Next Thm:

“stronger than exp concentration implies stronger than linear isop."
"stronger than exp concentration implies stronger than linear isop."

**Thm** (M. 09):

\[ K(r) \leq \exp(-\alpha(r)) \Rightarrow I(v) \geq \min\left( cv \frac{\log 1/v}{\alpha^{-1}(\log 1/v)}, c_\alpha \right) \quad \forall v \in [0, 1/2]. \]

Remarks: dim-independent constants.

concentration implies back “right” isoperimetric inq.

**I(v) ≥ Dv**

(Expanders, \( H^n \), log-concave)

**I(v) ≥ Dv \sqrt{\log 1/v}**

(Gauss)

**\( I(v) \geq Dv \)\]

(Expanders, \( H^n \), log-concave)

**\( I(v) \geq Dv \sqrt{\log 1/v} \)**

(Gauss)

**\( K(r) \leq \exp(-cDr) \)**

(Exponential Conc)

**\( K(r) \leq \exp(-c_2 Dr^2) \)**

(Gaussian Conc)
“stronger than exp concentration implies stronger than linear isop.”

**Thm** (M. 09):

\[ \mathcal{K}(r) \leq \exp(-\alpha(r)) \Rightarrow \mathcal{I}(v) \geq \min(c v \frac{\log 1/v}{\alpha^{-1}(\log 1/v)}, c_{\alpha}) \quad \forall v \in [0, 1/2]. \]

Remarks: dim-independent constants.

concentration implies back “right” isoperimetric inq.

\[ \mathcal{I}(v) \geq Dv \quad \Rightarrow \quad \|\nabla f\|_2 \geq \frac{D}{2} \|f - \int f\|_2 \Rightarrow \mathcal{K}(r) \leq \exp(-cDr) \quad \text{(Exponential Conc)} \]

\[ \mathcal{I}(v) \geq Dv \sqrt{\log 1/v} \quad \Rightarrow \quad \|\nabla f\|_2 \geq c_1 D \sqrt{\operatorname{Ent}(f^2)} \Rightarrow \mathcal{K}(r) \leq \exp(-c_2 D r^2) \quad \text{(Gaussian Conc)} \]
"stronger than Gaussian conc. implies stronger than Gaussian isop."

**Thm** (M. 09): Assume that the following growth condition is satisfied:

$$\exists \delta_0 > 1/2 \quad \exists r_0 \geq 0 \quad \forall r \geq r_0 \quad \alpha(r) \geq \delta_0 \kappa r^2 .$$

Then:

$$\mathcal{K}(r) \leq \exp(-\alpha(r)) \quad \Rightarrow \quad I(v) \geq \min(c_{\delta_0} v \frac{\log 1/v}{\alpha^{-1}(\log 1/v)}, c_{\alpha,\kappa}) \quad \forall v \in [0, 1/2] .$$

Remark: growth condition is necessary even in 1-D case (Chen–Wang 07 - $\delta_0 = 1/2$ is a sharp threshold).

Intuitively, $\alpha(r) > \frac{1}{2} \kappa r^2$ needed to compensate for $-\kappa$ curvature (second derivative).
"stronger than Gaussian conc. implies stronger than Gaussian isop."

**Thm** (M. 09): Assume that the following growth condition is satisfied:

$$\exists \delta_0 > 1/2 \quad \exists r_0 \geq 0 \quad \forall r \geq r_0 \quad \alpha(r) \geq \delta_0 \kappa r^2.$$  

Then:

$$K(r) \leq \exp(-\alpha(r)) \implies I(v) \geq \min(c_{\delta_0} v \frac{\log 1/v}{\alpha^{-1}(\log 1/v)}, c_{\alpha, \kappa}) \quad \forall v \in [0, 1/2].$$

Remark: growth condition is necessary even in 1-D case (Chen–Wang 07 - $\delta_0 = 1/2$ is a sharp threshold).

Intuitively, $\alpha(r) > \frac{1}{2} \kappa r^2$ needed to compensate for $-\kappa$ curvature (second derivative).
“stronger than Gaussian conc. implies stronger than Gaussian isop.”

**Thm** (M. 09): Assume that the following growth condition is satisfied:

\[ \exists \delta_0 > 1/2 \quad \exists r_0 \geq 0 \quad \forall r \geq r_0 \quad \alpha(r) \geq \delta_0 \kappa r^2. \]

Then:

\[ K(r) \leq \exp(-\alpha(r)) \quad \Rightarrow \quad I(v) \geq \min(c_{\delta_0} v \frac{\log 1/v}{\alpha^{-1}(\log 1/v)}, c_{\alpha,\kappa}) \quad \forall v \in [0, 1/2]. \]

Remark: growth condition is necessary even in 1-D case (Chen–Wang 07 - \( \delta_0 = 1/2 \) is a sharp threshold).

Intuitively, \( \alpha(r) > \frac{1}{2} \kappa r^2 \) needed to compensate for \( -\kappa \) curvature (second derivative).
"stronger than Gaussian conc. implies stronger than Gaussian isop."

**Thm** (M. 09): Assume that the following growth condition is satisfied:

\[ \exists \delta_0 > \frac{1}{2} \quad \exists r_0 \geq 0 \quad \forall r \geq r_0 \quad \alpha(r) \geq \delta_0 \kappa r^2. \]

Then:

\[ K(r) \leq \exp(-\alpha(r)) \Rightarrow I(v) \geq \min(c_{\delta_0} v \frac{\log 1/v}{\alpha^{-1}(\log 1/v)}, c_{\alpha,\kappa}) \quad \forall v \in [0, 1/2]. \]

Remark: growth condition is necessary even in 1-D case (Chen–Wang 07 - \( \delta_0 = 1/2 \) is a sharp threshold).

Intuitively, \( \alpha(r) > \frac{1}{2} \kappa r^2 \) needed to compensate for \(-\kappa\) curvature (second derivative).
Existence & regularity of isop minimizers (Geometric Measure Th.:
Almgren, Bombieri, De Giorgi, Federer, Fleming, Giusti, Gonzalez–Massari–Tamanini, Morgan, Simons)

- $\partial A \cap \Omega = \partial_r A \cup \partial_s A$, $\mathcal{H}^{n-8}(\partial_s A) < \infty$, $\partial_r A$ is as smooth as $\mu$ and meets $\partial \Omega$ orthogonally.
- Normal variations of $\partial_r A$; calculate variations of $t \mapsto \mu^+(A_t^d)$.
- 1st variation: $\partial_r A$ has constant (gen.) mean-curvature $H_\mu(\partial_r A)$.
- Remark - 2nd variation: if $\kappa = 0$, a tedious calculation shows that $\mu(A_t^d) \mapsto \mu^+(A_t^d)$ is concave, and hence so is $v \mapsto I(v)$. 

Emanuel Milman
Isoperimetric Inequalities: Methods and Applications
Existence & regularity of isop minimizers (Geometric Measure Th.:
Almgren, Bombieri, De Giorgi, Federer, Fleming, Giusti, Gonzalez–Massari–Tamanini, Morgan, Simons)

- $\partial A \cap \Omega = \partial_r A \cup \partial_s A$, $H^{n-8}(\partial_s A) < \infty$, $\partial_r A$ is as smooth as $\mu$ and meets $\partial \Omega$ orthogonally.

- Normal variations of $\partial_r A$; calculate variations of $t \mapsto \mu^+(A^d_t)$.

- 1st variation: $\partial_r A$ has constant (gen.) mean-curvature $H_\mu(\partial_r A)$.

- Remark - 2nd variation: if $\kappa = 0$, a tedious calculation shows that $\mu(A^d_t) \mapsto \mu^+(A^d_t)$ is concave, and hence so is $\nu \mapsto I(\nu)$. 
Existence & regularity of isop minimizers (Geometric Measure Th.: Almgren, Bombieri, De Giorgi, Federer, Fleming, Giusti, Gonzalez–Massari–Tamanini, Morgan, Simons)

- $\partial A \cap \Omega = \partial_r A \cup \partial_s A$, $\mathcal{H}^{n-8}(\partial_s A) < \infty$, $\partial_r A$ is as smooth as $\mu$ and meets $\partial \Omega$ orthogonally.
- Normal variations of $\partial_r A$; calculate variations of $t \mapsto \mu^+(A_t^d)$.
- 1st variation: $\partial_r A$ has constant (gen.) mean-curvature $H_\mu(\partial_r A)$.
- Remark - 2nd variation: if $\kappa = 0$, a tedious calculation shows that $\mu(A_t^d) \mapsto \mu^+(A_t^d)$ is concave, and hence so is $v \mapsto \mathcal{I}(v)$. 

Emanuel Milman
Isoperimetric Inequalities: Methods and Applications
Existence & regularity of isop minimizers (Geometric Measure Th.: Almgren, Bombieri, De Giorgi, Federer, Fleming, Giusti, Gonzalez–Massari–Tamanini, Morgan, Simons)

- \( \partial A \cap \Omega = \partial_r A \cup \partial_s A, \mathcal{H}^{n-8}(\partial_s A) < \infty, \partial_r A \) is as smooth as \( \mu \) and meets \( \partial \Omega \) orthogonally.

- Normal variations of \( \partial_r A \); calculate variations of \( t \mapsto \mu^+(A_t^d) \).

- 1st variation: \( \partial_r A \) has constant (gen.) mean-curvature \( H_{\mu}(\partial_r A) \).

- Remark - 2nd variation: if \( \kappa = 0 \), a tedious calculation shows that \( \mu(A_t^d) \mapsto \mu^+(A_t^d) \) is concave, and hence so is \( v \mapsto I(v) \).
Proof: Geometric Method (Gromov)

Existence & regularity of isop minimizers (Geometric Measure Th.: Almgren, Bombieri, De Giorgi, Federer, Fleming, Giusti, Gonzalez–Massari–Tamanini, Morgan, Simons)

- \( \partial A \cap \Omega = \partial_r A \cup \partial_s A \), \( \mathcal{H}^{n-8}(\partial_s A) < \infty \), \( \partial_r A \) is as smooth as \( \mu \) and meets \( \partial \Omega \) orthogonally.

- Normal variations of \( \partial_r A \); calculate variations of \( t \mapsto \mu^+(A_t^d) \).

- 1st variation: \( \partial_r A \) has constant (gen.) mean-curvature \( H_\mu(\partial_r A) \).

- Remark - 2nd variation: if \( \kappa = 0 \), a tedious calculation shows that \( \mu(A_t^d) \mapsto \mu^+(A_t^d) \) is concave, and hence so is \( \nu \mapsto \mathcal{I}(\nu) \).
Thm (F. Morgan, generalized Heintze–Karcher)

Consider \((M^n, d, \mu = \exp(-\psi)d\text{vol}_M)\). If \(Ric_g + \text{Hess}_g\psi \geq -\kappa g\) and \(\partial A\) is smooth then:

\[
\mu(A_r \setminus A) \leq \int_{\partial A} \int_0^r \exp(H_{\mu}(\partial A, x)t + \kappa t^2/2)dt \exp(-\psi(x))dx.
\]

False even in the presence of a single-point singularity.

Observation (Gromov 1980)

Heintze–Karcher Thm valid for isoperimetric minimizers, since "closest point in \(\partial A\) is always regular".

Observation

Also valid for \((\Omega \subset M, d, \mu)\) when \(\Omega\) is geodesically convex, since minimizers meet \(\partial \Omega\) orthogonally.
Thm (F. Morgan, generalized Heintze–Karcher)

Consider \((M^n, d, \mu = \exp(-\psi) \text{dvol}_M)\). If \(\text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g\) and \(\partial A\) is smooth then:

\[
\mu(A^d_r \setminus A) \leq \int_{\partial A} \int_0^r \exp(H_{\mu, (\partial A, x)} t + \kappa t^2/2) dt \exp(-\psi(x)) dx .
\]

False even in the presence of a single-point singularity.

Observation (Gromov 1980)

Heintze–Karcher Thm valid for isoperimetric minimizers, since “closest point in \(\partial A\) is always regular”.

Observation

Also valid for \((\Omega \subset M, d, \mu)\) when \(\Omega\) is geodesically convex, since minimizers meet \(\partial \Omega\) orthogonally.
Thm (F. Morgan, generalized Heintze–Karcher)
Consider \((M^n, d, \mu = \exp(-\psi)\text{dvol}_M)\). If \(\text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g\) and \(\partial A\) is smooth then:

\[
\mu(A^d_r \setminus A) \leq \int_{\partial A} \int_0^r \exp(H_{\mu}(\partial A, x)t + \kappa t^2/2)dt \exp(-\psi(x))dx.
\]

False even in the presence of a single-point singularity.

Observation (Gromov 1980)
Heintze–Karcher Thm valid for isoperimetric minimizers, since “closest point in \(\partial A\) is always regular”.

Observation
Also valid for \((\Omega \subset M, d, \mu)\) when \(\Omega\) is geodesically convex, since minimizers meet \(\partial \Omega\) orthogonally.
Proof II - Comparison Theorems

**Thm (F. Morgan, generalized Heintze–Karcher)**

Consider \((M^n, d, \mu = \exp(-\psi)dvol_M)\). If \(\text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g\) and \(\partial A\) is smooth then:

\[
\mu(A^d_r \setminus A) \leq \int_{\partial A} \int_0^r \exp(H_{\mu}(\partial A, x)t + \kappa t^2/2)dt \exp(-\psi(x))dx.
\]

False even in the presence of a single-point singularity.

**Observation (Gromov 1980)**

Heintze–Karcher Thm valid for isoperimetric minimizers, since “closest point in \(\partial A\) is always regular”.

**Observation**

Also valid for \((\Omega \subset M, d, \mu)\) when \(\Omega\) is geodesically convex, since minimizers meet \(\partial \Omega\) orthogonally.
Generalized Heintze-Karcher:

$$\mu(A^d_r \setminus A) \leq \int_{\partial A} \int_0^r \exp(H_{\mu}(\partial A, x)t + \kappa t^2/2) dt \exp(-\psi(x)) dx .$$

Let \( v \in (0, 1/2) \), apply to isop. minimizer \( A (\mu(A) = v, \mu^+(A) = I(v)) \):

$$\mu(A^d_{r_1} \setminus A) \leq \mu^+(A) \int_0^{r_1} \exp(H_{\mu}(\partial r A)t + \kappa t^2/2) dt .$$
Generalized Heintze-Karcher:

\[ \mu(A_d \setminus A) \leq \int_{\partial A} \int_0^r \exp(H_{\mu}(\partial A, x)t + \kappa t^2/2)dt \exp(-\psi(x))dx . \]

Let \( v \in (0, 1/2) \), apply to isop. minimizer \( A (\mu(A) = v, \mu^+(A) = I(v)) \):

\[ \mu(A_d \setminus A) \leq \mu^+(A) \int_0^{r_1} \exp(H_{\mu}(\partial r A)t + \kappa t^2/2)dt , \]
Proof III - using everything

Generalized Heintze-Karcher:

\[ \mu(A_r^d \setminus A) \leq \int_{\partial A} \int_0^r \exp(H_\mu(\partial A, x)t + \kappa t^2/2)dt \exp(-\psi(x))dx . \]

Let \( v \in (0, 1/2) \), apply to isop. minimizer \( A (\mu(A) = v, \mu^+(A) = I(v)) \):

\[ K(r_1) \leq v \Rightarrow \frac{1}{2} - v \leq \mu(A_{r_1}^d \setminus A) \leq \mu^+(A) \int_0^{r_1} \exp(H_\mu(\partial_r A)t + \kappa t^2/2)dt , \]
Proof III - using everything

Generalized Heintze-Karcher:

$$\mu(A_r^d \setminus A) \leq \int_{\partial A} \int_0^T \exp(H_{\mu}(\partial A, x) t + \kappa t^2/2) dt \exp(-\psi(x)) dx .$$

Let $\nu \in (0, 1/2)$, apply to isop. minimizer $A$ ($\mu(A) = \nu$, $\mu^+(A) = I(\nu)$):

apply to $B = \Omega \setminus A$ ($\mu(B) = 1 - \nu$, $\mu^+(B) = \mu^+(A)$)

$$K(r_1) \leq \nu \Rightarrow \frac{1}{2} - \nu \leq \mu(A_{r_1}^d \setminus A) \leq \mu^+(A) \int_0^{r_1} \exp(H_{\mu}(\partial \Gamma A) t + \kappa t^2/2) dt ,$$

$$K(r_2) \leq \frac{\nu}{2} \Rightarrow \frac{\nu}{2} \leq \mu(B_{r_2}^d \setminus B) \leq \mu^+(A) \int_0^{r_2} \exp(-H_{\mu}(\partial \Gamma A) t + \kappa t^2/2) dt .$$
Proof III - using everything

Generalized Heintze-Karcher:

\[ \mu(A_r \setminus A) \leq \int_{\partial A} \int_0^r \exp(H_\mu(\partial A, x)t + \kappa t^2/2)dt \exp(-\psi(x))dx . \]

Let \( v \in (0, 1/2) \), apply to isop. minimizer \( A \) \((\mu(A) = v, \mu^+(A) = \mathcal{I}(v))\):

apply to \( B = \Omega \setminus A \) \((\mu(B) = 1 - v, \mu^+(B) = \mu^+(A))\)

\[ \mathcal{K}(r_1) \leq v \implies \frac{1}{2} - v \leq \mu(A_{r_1} \setminus A) \leq \mu^+(A) \int_0^{r_1} \exp(H_\mu(\partial A)t + \kappa t^2/2)dt , \]

\[ \mathcal{K}(r_2) \leq \frac{v}{2} \implies \frac{v}{2} \leq \mu(B_{r_2} \setminus B) \leq \mu^+(A) \int_0^{r_2} \exp(-H_\mu(\partial A)t + \kappa t^2/2)dt . \]

Last useful fact for handling big sets: \( H_\mu(\partial A) = \left. \frac{d}{dv}\mathcal{I}(v) \right|_v \).
Main Thms \((Ric_{g,\mu} \geq -\kappa)\):

1. \(\kappa = 0\) "weakest conc. \(\Rightarrow\) linear isop."
2. \(\kappa = 0\) "stronger than exp conc. \(\Rightarrow\) stronger than linear isop."
3. \(\kappa > 0\) "stronger than Gauss conc. \(\Rightarrow\) stronger than Gauss isop."

Our proof is entirely Geometric.
Recently, two new proofs of 2,3 have been obtained (perhaps with some additional minor technical assumptions):

- Ledoux - Semi-Group method (recovers 2 and most of 3).
- Gozlan–Roberto–Samson - Transport-Entropy Inqs (prototypical examples of Conc. Inq \(\Rightarrow\) Sobolev Inq, non-optimal \(\delta_0\) in 3) - extends to mm length-spaces with \(Ric \geq -\kappa\).

**Open**: extend and sharpen to more general settings (length-spaces, discrete)?

**Open**: none of the methods manage to obtain 1 without using concavity of \(I\) (or \(I(\nu)/\nu\) non-increasing), so currently confined to manifold-with-density setting.
Applications Overview

Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ TE Inqs $\Rightarrow$ Conc Inqs.

\[
\begin{cases}
\text{Under } \text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g, \kappa \geq 0 \\
\text{additional growth condition if } \kappa > 0
\end{cases}
\]

- Recover all previous dim-dependent results:
  - Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov:
    \[
    \int_\Omega \exp(\beta(d(x,x_0)))d\mu(x) < \infty \Rightarrow K(r) \leq \text{Markov}.
    \]
  - Bérard, Besson, Gallot, Li, Yau, . . . :
    \[
    \text{diameter}(\Omega) < D \Rightarrow K(r) = 0 \quad \forall r > D.
    \]
  - Obtain best possible dependence on $\kappa, D$.
  - Generalize everything to Riemannian setting.

- Stability of Isoperimetric, Sobolev and Transport-Entropy inqs under measure perturbation w.r.t. $d_{TV}, W_1, H(\mu_2|\mu_1),...$

- Equivalence of Transport-Entropy Inqs, tightening weak inqs, ...
Applications Overview

Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ TE Inqs $\Rightarrow$ Conc Inqs.

\[
\begin{cases}
\text{Under } \text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g, \kappa \geq 0 \\
\text{additional growth condition if } \kappa > 0
\end{cases}
\]

- Recover all previous dim-dependent results:
  - Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov:
    \[
    \int_{\Omega} \exp(\beta(d(x, x_0))) d\mu(x) < \infty \Rightarrow \mathcal{K}(r) \leq \text{Markov}.
    \]
  - Bérard, Besson, Gallot, Li, Yau, . . . :
    \[
    \text{diameter}(\Omega) < D \Rightarrow \mathcal{K}(r) = 0 \quad \forall r > D.
    \]
    Obtain best possible dependence on $\kappa, D$.
  - Generalize everything to Riemannian setting.

- Stability of Isoperimetric, Sobolev and Transport-Entropy inqs under measure perturbation w.r.t. $d_{TV}, W_1, H(\mu_2|\mu_1)$,

- Equivalence of Transport-Entropy Inqs, tightening weak inqs, ...
Applications Overview

Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ TE Inqs $\Rightarrow$ Conc Inqs.

\[ \begin{cases} 
\text{Under } \text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g, \quad \kappa \geq 0 \\
\text{additional growth condition if } \kappa > 0
\end{cases} \]

- Recover all previous dim-dependent results:
  - Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov:
    \[
    \int_{\Omega} \exp(\beta(d(x, x_0))) d\mu(x) < \infty \Rightarrow K(r) \leq \text{Markov}.
    \]
  - Bérard, Besson, Gallot, Li, Yau, . . . :
    diameter(\Omega) < D $\Rightarrow$ $K(r) = 0$ $\forall r > D$.
    Obtain best possible dependence on $\kappa, D$.

- Generalize everything to Riemannian setting.

- **Stability** of Isoperimetric, Sobolev and Transport-Entropy inqs under measure perturbation w.r.t. $d_{TV}, W_1, H(\mu_2|\mu_1),...$

- Equivalence of Transport-Entropy Inqs, tightening weak inqs, ...
Applications Overview

Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ TE Inqs $\Rightarrow$ Conc Inqs.

\[
\begin{aligned}
\left\{ \begin{array}{l}
\text{Under } R_{\text{ic}} + H_{\text{ess}} \psi \geq -\kappa g, \kappa \geq 0 \\
\text{additional growth condition if } \kappa > 0
\end{array} \right.
\end{aligned}
\]

- Recover all previous dim-dependent results:
  - Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov:
    \[ \int_{\Omega} \exp(\beta(d(x, x_0))) \, d\mu(x) < \infty \Rightarrow \mathcal{K}(r) \leq \text{Markov}. \]
  - Bérard, Besson, Gallot, Li, Yau, . . .:
    \[ \text{diameter}(\Omega) < D \Rightarrow \mathcal{K}(r) = 0 \quad \forall r > D. \]
    Obtain best possible dependence on \( \kappa, D \).
  - Generalize everything to Riemannian setting.

- Stability of Isoperimetric, Sobolev and Transport-Entropy inqs under measure perturbation w.r.t. \( d_{TV}, W_1, H(\mu_2|\mu_1), \ldots \)

- Equivalence of Transport-Entropy Inqs, tightening weak inqs, ...
Applications Overview

Isop Inqs ⇒ Sobolev Inqs ⇒ TE Inqs ⇒ Conc Inqs.

\[ \begin{cases} \text{Under } \text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g, \quad \kappa \geq 0 \\ \text{additional growth condition if } \kappa > 0 \end{cases} \]

- Recover all previous dim-dependent results:
  - Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov:
    \[ \int_\Omega \exp(\beta(d(x, x_0))) \, d\mu(x) < \infty \Rightarrow \mathcal{K}(r) \leq \text{Markov}. \]
  - Bérard, Besson, Gallot, Li, Yau, . . . :
    \[ \text{diameter}(\Omega) < D \Rightarrow \mathcal{K}(r) = 0 \quad \forall r > D. \]
    Obtain best possible dependence on \( \kappa, D \).
  - Generalize everything to Riemannian setting.

- **Stability** of Isoperimetric, Sobolev and Transport-Entropy inqs under measure perturbation w.r.t. \( d_{TV}, W_1, H(\mu_2|\mu_1), \ldots \)

- Equivalence of Transport-Entropy Inqs, tightening weak inqs, ...
Applications Overview

Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ TE Inqs $\Rightarrow$ Conc Inqs.

\[
\begin{cases}
\text{Under } \text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g, \kappa \geq 0 \\
\text{additional growth condition if } \kappa > 0
\end{cases}
\]

- Recover all previous dim-dependent results:
  - Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov:
    \[
    \int_{\Omega} \exp(\beta(d(x, x_0))) d\mu(x) < \infty \Rightarrow K(r) \leq \text{Markov}.
    \]
  - Bérard, Besson, Gallot, Li, Yau, . . . :
    \[
    \text{diameter}(\Omega) < D \Rightarrow K(r) = 0 \quad \forall r > D.
    \]
    Obtain best possible dependence on $\kappa, D$.
  - Generalize everything to Riemannian setting.

- **Stability** of Isoperimetric, Sobolev and Transport-Entropy inqs under measure perturbation w.r.t. $d_{TV}, W_1, H(\mu_2|\mu_1),...$

- Equivalence of Transport-Entropy Inqs, tightening weak inqs, ...
Applications Overview

Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ TE Inqs $\Rightarrow$ Conc Inqs.

\[
\begin{cases}
\text{Under } \text{Ric}_g + \text{Hess}_g \psi \geq -\kappa g, \kappa \geq 0 \\
\text{additional growth condition if } \kappa > 0 
\end{cases}
\]

- Recover all previous dim-dependent results:
  - Kannan–Lovász–Simonovits, Wang, Bobkov, Barthe, Barthe–Kolesnikov:
    \[
    \int_{\Omega} \exp(\beta(d(x, x_0))) d\mu(x) < \infty \Rightarrow K(r) \leq \text{Markov}.
    \]
  - Bérard, Besson, Gallot, Li, Yau, . . .:
    \[
    \text{diameter} (\Omega) < D \Rightarrow K(r) = 0 \quad \forall r > D.
    \]
  - Obtain best possible dependence on $\kappa, D$.
  - Generalize everything to Riemannian setting.

- **Stability of Isoperimetric, Sobolev and Transport-Entropy inqs** under measure perturbation w.r.t. $d_{TV}, W_1, H(\mu_2|\mu_1), ...$

- **Equivalence of Transport-Entropy Inqs**, tightening weak inqs, ...