# SPINORIAL REPRESENTATION OF SUBMANIFOLDS IN METRIC LIE GROUPS

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ABSTRACT. In this paper we give a spinorial representation of submanifolds of any dimension and codimension into Lie groups equipped with left invariant metrics. As applications, we get a spinorial proof of the Fundamental Theorem for submanifolds into Lie groups, we recover previously known representations of submanifolds in $\mathbb{R}^n$ and in the 3-dimensional Lie groups $S^3$ and $E(\kappa,\tau)$, and we get a new spinorial representation for surfaces in the 3-dimensional semi-direct products: this achieves the spinorial representations of surfaces in the 3-dimensional homogeneous spaces. We finally indicate how to recover a Weierstrass-type representation for CMC-surfaces in 3-dimensional metric Lie groups recently given by Meeks, Mira, Perez and Ros.

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## 1. Introduction

The purpose of this paper is to give a spinorial representation of an isometric immersion of a Riemannian manifold $M$ into a Lie group $G$ equipped with a left invariant metric. The result is roughly the following: if $M$ is a simply connected Riemannian manifold, $E$ is a real vector bundle on $M$ equipped with a fibre metric and a compatible connection, and $B : TM \times TM \to E$ is bilinear and symmetric, then an isometric immersion of $M$ into $G$ with normal bundle $E$ and second fundamental form $B$ is equivalent to a spinor field $\varphi$ solution of a Killing-type equation on $M$; the spinor bundle is constructed from the Clifford algebra of the metric Lie algebra $\mathcal{G}$ of the group, and the immersion is explicitly obtained by the integration of a $\mathcal{G}$-valued 1-form on $M$ defined in terms of the spinor field $\varphi$. A precise statement with the suitable necessary hypotheses is given in Section 3 of the paper.

The explicit representation formula of the immersion in terms of the spinor field may be considered as a generalized Weierstrass representation formula for manifolds into metric Lie groups.

We then give some applications of this result. We first obtain an easy proof of a theorem by Piccione and Tausk [18]: under suitable hypotheses, the necessary equations of Gauss, Codazzi and Ricci are also sufficient to obtain an immersion of a simply connected manifold into a metric Lie group. We then show how our general result permits to recover the known spinorial representation for submanifolds in $\mathbb{R}^n$ [4], and also obtain a new spinorial representation for submanifolds in $\mathbb{H}^n$ considered as a metric Lie group. We finally study more precisely the case of surfaces in a 3-dimensional metric Lie group: we recover the known spinorial representations in $S^3$ [16] and $E(\kappa,\tau)$ [19], and obtain a new spinorial representation of surfaces in a general semi-direct product; this especially includes the cases of surfaces into the groups $Sol_3$ and $\mathbb{H}^2 \times \mathbb{R}$, which achieves the spinorial representations of surfaces into the 3-dimensional homogeneous spaces initiated in [8, 16, 19]. We also deduce alternative proofs of the Fundamental Theorems for surfaces in $E(\kappa, \tau)$ by Daniel [7] and in $Sol_3$ by Lodovici [11]. We finish the paper showing how the general spinorial representation formula permits to recover the recent Weierstrass-type representation formula by Meeks, Mira, Perez and Ros [13, Theorem 3.12] concerning constant mean curvature surfaces in 3-dimensional metric Lie groups.

The main result of the paper thus gives a general framework for a variety of Weierstrass-type representation formulas existing in the literature, and is also a tool to get representation formulas in new contexts.

We quote the following related papers: Friedrich obtained in [8] a geometric spinorial representation of a surface in $\mathbb{R}^3$ showing that a surface in $\mathbb{R}^3$ may be represented by a constant spinor field of $\mathbb{R}^3$ restricted to the surface; this result was then extended to $S^3$ and $\mathbb{H}^3$ by Morel [16] and to other 3-dimensional homogeneous spaces by Roth [19]. It was then extended by Bayard, Lawn and Roth to surfaces in dimension 4 [3] and afterwards to manifolds in $\mathbb{R}^n$ [4]. Spinorial representation were also studied in pseudo-Riemannian spaces, by Lawn in $\mathbb{R}^{2,1}$ [9], Lawn and Roth in 3-dimensional Lorentzian space forms [10], Bayard in $\mathbb{R}^{3,1}$ [1], Bayard and Patty [5] and Patty [17] in $\mathbb{R}^{2,2}$. Close to the purpose of the paper, Berdinskii and Taimanov gave in [6] a spinorial representation for a surface in a 3-dimensional metric Lie group.

The outline of the paper is as follows: Section 2 is dedicated to preliminaries concerning notation and spin geometry of a submanifold in a metric Lie group, Section 3 to the statement and the proof of the main theorem, and Section 4 to a spinorial proof of the Fundamental Theorem for submanifolds in a metric Lie group. We then give further applications in Section 5: we study the cases of a submanifold in $\mathbb{R}^n$ and $\mathbb{H}^n$, and of a hypersurface in a general metric Lie group, specifying further to the cases of a surface in $S^3$, $E(\kappa, \tau)$ and a semi-direct product, as $Sol_3$ and $\mathbb{H}^2 \times \mathbb{R}$. We finally consider the case of a CMC-surface in a 3-dimensional metric Lie group. An appendix ends the paper concerning the links between the Clifford product and some natural operations on skew-symmetric operators.

## 2. Preliminaries

2.1. **Notations.** Let $G$ be a Lie group, endowed with a left invariant metric $\langle .,. \rangle$, and $\mathcal{G}$ its Lie algebra: $\mathcal{G}$ is the space of the left invariant vector fields on $G$, equipped with the Lie bracket $[.,.]$ and is identified to the linear space tangent to $G$ at the identity. We consider the Maurer-Cartan form $\omega_G \in \Omega^1(G, \mathcal{G})$ defined by

$$\omega_G(v) = L_{g^{-1}}{}_*(v) \qquad \in \ \mathcal{G} \tag{1}$$

for all $v \in T_gG$, where $L_{g^{-1}}$ denotes the left multiplication by $g^{-1}$ on $G$ and $L_{g^{-1}}{}_* : T_gG \to \mathcal{G}$ is its differential. This form induces a bundle isomorphism

$$\begin{array}{ccc} TG & \to & G \times \mathcal{G} \\ (g, v) & \mapsto & (g, \omega_G(v)). \end{array} \tag{2}$$

which preserves the fibre metrics. We note that a vector field $X \in \Gamma(TG)$ is left invariant if, by (2), $X : G \to \mathcal{G}$ is a constant map. Let us consider the Levi-Civita connection $\nabla^G$ of $(G, \langle .,. \rangle)$ and the linear map

$$\begin{array}{rccc} \Gamma: & \mathcal{G} & \rightarrow & \Lambda^2\mathcal{G} \\ & X & \mapsto & \Gamma(X) \end{array}$$

such that, for all $X, Y \in \mathcal{G}$

$$\nabla^G_X Y = \Gamma(X)(Y). \tag{3}$$

By the Koszul formula, $\Gamma$ is determined by the metric as follows: for all $X, Y, Z \in \mathcal{G}$,

$$\langle \Gamma(X)(Y), Z \rangle = \frac{1}{2}\langle [X,Y], Z\rangle + \frac{1}{2}\langle [Z,X], Y\rangle - \frac{1}{2}\langle [Y,Z], X\rangle. \tag{4}$$

Since $\nabla^G$ is without torsion, we have, for all $X, Y \in \mathcal{G}$,

$$\Gamma(X)(Y) - \Gamma(Y)(X) = [X,Y]. \tag{5}$$

We note that the curvature of $\nabla^G$ is given by

$$R^G(X,Y) = [\Gamma(X), \Gamma(Y)] - \Gamma([X,Y]) \qquad \in \Lambda^2\mathcal{G} \tag{6}$$

for all $X, Y \in \mathcal{G}$. In the formula the first brackets stand for the commutator of the endomorphisms.

2.2. **The spinor bundle of $G$.** Let us denote by $Cl(\mathcal{G})$ the Clifford algebra of $\mathcal{G}$ with its scalar product, and let us consider the representation

$$\begin{array}{rccc} \rho: & Spin(\mathcal{G}) & \rightarrow & GL(Cl(\mathcal{G})) \\ & a & \mapsto & \xi \mapsto a\xi. \end{array}$$

This representation is a real representation and is not irreducible in general: it is a sum of irreducible representations [12]. By (2) the principal bundle $Q_G$ of the positively oriented and orthonormal frames of $G$ is trivial

$$Q_G \simeq G \times SO(\mathcal{G}),$$

and we may consider the trivial spin structure

$$\tilde{Q}_G := G \times Spin(\mathcal{G})$$

and the corresponding spinor bundle

$$\Sigma := \tilde{Q}_G \times_\rho Cl(G) \simeq G \times Cl(\mathcal{G}).$$

We will say that a spinor field $\varphi \in \Gamma(\Sigma)$ is *left invariant* if it is constant as a map $G \rightarrow Cl(\mathcal{G})$. The covariant derivative of a left invariant spinor field is

$$\nabla^G_X \varphi = \frac{1}{2}\Gamma(X) \cdot \varphi \tag{7}$$

where $\Gamma(X) \in \Lambda^2\mathcal{G} \subset Cl(\mathcal{G})$ and the dot "$\cdot$" stands for the Clifford product.

2.3. **The spin representation of $Spin(p)\times Spin(q)$.** Let us assume that $p+q=n$, and fix an orthonormal basis $e_1^o, e_2^o, \ldots, e_n^o$ of $\mathcal{G}$; this gives a splitting $\mathcal{G}=\mathbb{R}^p\oplus\mathbb{R}^q$ (the first factor corresponds to the first $p$ vectors, and the second factor to the last $q$ vectors of the basis) and a natural map

$$Spin(p)\times Spin(q)\to Spin(\mathcal{G}),\qquad (a_p,a_q)\mapsto a:=a_p\cdot a_q$$

associated to the isomorphism

$$Cl(\mathcal{G})=Cl_p\hat{\otimes}Cl_q.$$

We thus also have a representation, still denoted by $\rho$,

$$\begin{array}{rrcl}\rho: & Spin(p)\times Spin(q) & \to & GL(Cl(\mathcal{G}))\\ & (a_p,a_q) & \mapsto & \xi\mapsto a\xi.\end{array}\tag{8}$$

2.4. **The twisted spinor bundle.** We consider a $p$-dimensional Riemannian manifold $M$ and a bundle $E\to M$ of rank $q$, with a fibre metric and a compatible connection. We assume that $E$ and $TM$ are oriented and spin, with given spin structures

$$\tilde{Q}_M\stackrel{2:1}{\to}Q_M\qquad\text{and}\qquad\tilde{Q}_E\stackrel{2:1}{\to}Q_E$$

where $Q_M$ and $Q_E$ are the bundles of positively oriented orthonormal frames of $TM$ and $E$, and we set

$$\tilde{Q}:=\tilde{Q}_M\times_M\tilde{Q}_E;$$

this is a $Spin(p)\times Spin(q)$ principal bundle. We define

$$\Sigma:=\tilde{Q}\times_\rho Cl(\mathcal{G})$$

and

$$U\Sigma:=\tilde{Q}\times_\rho Spin(\mathcal{G})\qquad\subset\ \Sigma$$

where $\rho$ is the representation (8). Similarly to the usual construction in spin geometry, if we consider the representation

$$Ad:\qquad Spin(p)\times Spin(q)\to Spin(\mathcal{G})\stackrel{2:1}{\to}SO(\mathcal{G})\to GL(Cl(\mathcal{G}))$$

and the Clifford bundle

$$Cl(TM\oplus E)=\tilde{Q}\times_{Ad}Cl(\mathcal{G}),$$

there is a Clifford action of $Cl(TM\oplus E)$ on $\Sigma$; this action will be denoted below by a dot " $\cdot$ ". The vector bundle $\Sigma$ is moreover equipped with the covariant derivative $\nabla$ naturally associated to the spinorial connections on $\tilde{Q}_M$ and $\tilde{Q}_E$. Let us denote by $\tau: Cl(\mathcal{G})\to Cl(\mathcal{G})$ the anti-automorphism of $Cl(\mathcal{G})$ such that

$$\tau(x_1\cdot x_2\cdots x_k)=x_k\cdots x_2\cdot x_1$$

for all $x_1,x_2,\ldots,x_k\in\mathcal{G}$, and set

$$\begin{array}{rrcl}\langle\langle.,.\rangle\rangle: & Cl(\mathcal{G})\times Cl(\mathcal{G}) & \to & Cl(\mathcal{G})\\ & (\xi,\xi') & \mapsto & \tau(\xi')\xi.\end{array}\tag{9}$$

This map is $Spin(\mathcal{G})-$invariant: for all $\xi,\xi'\in Cl(\mathcal{G})$ and $g\in Spin(\mathcal{G})$ we have

$$\langle\langle g\xi,g\xi'\rangle\rangle=\tau(g\xi')g\xi=\tau(\xi')\tau(g)g\xi=\tau(\xi')\xi=\langle\langle\xi,\xi'\rangle\rangle,$$

since $Spin(\mathcal{G}) \subset \{g \in Cl^0(\mathcal{G}) : \tau(g)g = 1\}$; this map thus induces a $Cl(\mathcal{G})-$valued map

$$\begin{array}{rccc} \langle\langle .,. \rangle\rangle : & \Sigma \times \Sigma & \rightarrow & Cl(\mathcal{G}) \\ & (\varphi, \varphi') & \mapsto & \langle\langle [\varphi], [\varphi'] \rangle\rangle \end{array} \tag{10}$$

where $[\varphi]$ and $[\varphi'] \in Cl(\mathcal{G})$ represent $\varphi$ and $\varphi'$ in some spinorial frame $\tilde{s} \in \tilde{Q}$.

**Lemma 2.1.** *The map $\langle\langle .,. \rangle\rangle : \Sigma \times \Sigma \rightarrow Cl(\mathcal{G})$ satisfies the following properties: for all $\varphi, \psi \in \Gamma(\Sigma)$ and $X \in \Gamma(TM \oplus E)$,*

$$\langle\langle \varphi, \psi \rangle\rangle = \tau \langle\langle \psi, \varphi \rangle\rangle \tag{11}$$

*and*

$$\langle\langle X \cdot \varphi, \psi \rangle\rangle = \langle\langle \varphi, X \cdot \psi \rangle\rangle. \tag{12}$$

*Proof.* We have

$$\langle\langle \varphi, \psi \rangle\rangle = \tau[\psi]\ [\varphi] = \tau(\tau[\varphi]\ [\psi]) = \tau \langle\langle \psi, \varphi \rangle\rangle$$

and

$$\langle\langle X \cdot \varphi, \psi \rangle\rangle = \tau[\psi]\ [X][\varphi] = \tau([X][\psi])[\varphi] = \langle\langle \varphi, X \cdot \psi \rangle\rangle$$

where $[\varphi]$, $[\psi]$ and $[X]\ \in Cl(\mathcal{G})$ represent $\varphi, \psi$ and $X$ in some given frame $\tilde{s} \in \tilde{Q}$. ☐

**Lemma 2.2.** *The connection $\nabla$ is compatible with the product $\langle\langle .,. \rangle\rangle$ :*

$$\partial_X \langle\langle \varphi, \varphi' \rangle\rangle = \langle\langle \nabla_X \varphi, \varphi' \rangle\rangle + \langle\langle \varphi, \nabla_X \varphi' \rangle\rangle$$

*for all $\varphi, \varphi' \in \Gamma(\Sigma)$ and $X \in \Gamma(TM)$.*

*Proof.* If $\varphi = [\tilde{s}, [\varphi]]$ is a section of $\Sigma = \tilde{Q} \times_\rho Cl(\mathcal{G})$, we have

$$\nabla_X \varphi = [\tilde{s}, \partial_X[\varphi] + \rho_*(\tilde{s}^*\alpha(X))([\varphi])], \qquad \forall X \in\ TM, \tag{13}$$

where $\rho$ is the representation (8) and $\alpha$ is the connection form on $\tilde{Q}$; the term $\rho_*(\tilde{s}^*\alpha(X))$ is an endomorphism of $Cl(\mathcal{G})$ given by the multiplication on the left by an element belonging to $\Lambda^2\mathcal{G} \subset Cl(\mathcal{G})$, still denoted by $\rho_*(\tilde{s}^*\alpha(X))$. Such an element satisfies

$$\tau\left(\rho_*(\tilde{s}^*\alpha(X))\right) = -\rho_*(\tilde{s}^*\alpha(X)),$$

and we have

$$\begin{array}{rcl} \langle\langle \nabla_X \varphi, \varphi' \rangle\rangle + \langle\langle \varphi, \nabla_X \varphi' \rangle\rangle & = & \tau\{[\varphi']\}\left(\partial_X[\varphi] + \rho_*(\tilde{s}^*\alpha(X))[\varphi]\right) \\ & & +\tau\left\{\partial_X[\varphi'] + \rho_*(\tilde{s}^*\alpha(X))[\varphi']\right\}[\varphi] \\ & = & \tau\{[\varphi']\}\partial_X[\varphi] + \tau\left\{\partial_X[\varphi']\right\}[\varphi] \\ & = & \partial_X \langle\langle \varphi, \varphi' \rangle\rangle. \end{array}$$

☐

We finally note that there is a natural action of $Spin(\mathcal{G})$ on $U\Sigma$, by right multiplication: for $\varphi = [\tilde{s}, [\varphi]] \in U\Sigma = \tilde{Q} \times_\rho Spin(\mathcal{G})$ and $a \in Spin(\mathcal{G})$ we set

$$\varphi \cdot a := [\tilde{s}, [\varphi] \cdot a]\ \ \in U\Sigma. \tag{14}$$

2.5. **The spin geometry of a submanifold of $G$.** We keep the notation of the previous section, assuming moreover here that $M$ is a submanifold of a Lie group $G$ and that $E \to M$ is its normal bundle. If we consider spin structures on $TM$ and on $E$ whose sum is the trivial spin structure of $TM \oplus E$ [15], we have

$$\Sigma = \tilde{Q} \times_\rho Cl(\mathcal{G}) \simeq M \times Cl(\mathcal{G}),$$

where the last bundle is the spinor bundle of $G$ restricted to $M$. Two connections are thus defined on $\Sigma$, the connection $\nabla$ and the connection $\nabla^G$; they satisfy the following Gauss formula:

$$\nabla^G_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi \tag{15}$$

for all $\varphi \in \Gamma(\Sigma)$ and all $X \in \Gamma(TM)$, where $B : TM \times TM \to E$ is the second fundamental form of $M$ into $G$ and $e_1, \ldots, e_p$ is an orthonormal basis of $TM$. We refer to [1] for the proof (in a slightly different context). Since the covariant derivative of a left invariant spinor field is given by (7), the restriction to $M$ of such a spinor field satisfies

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi + \frac{1}{2} \Gamma(X) \cdot \varphi \tag{16}$$

for all $X \in TM$.

## 3. Main result

We consider a $p$-dimensional Riemannian manifold $M$ and a bundle $E \to M$ of rank $q$, with a fibre metric and a compatible connection. We assume that $E$ and $TM$ are oriented and spin, with given spin structures, and consider the spinor bundles $\Sigma$ and $U\Sigma$ introduced in the previous section. We suppose that a bilinear and symmetric map $B : TM \times TM \to E$ is given, and we moreover do the following two assumptions:

(1) There exists a bundle isomorphism

$$f : \ TM \oplus E \to M \times \mathcal{G} \tag{17}$$

which preserves the metrics; this mapping permits to define a bundle map

$$\Gamma : \ TM \oplus E \to \Lambda^2(TM \oplus E) \tag{18}$$

such that, for all $X, Y \in \Gamma(TM \oplus E)$,

$$f(\Gamma(X)(Y)) = \Gamma(f(X))(f(Y)) \tag{19}$$

where on the right-hand side $\Gamma$ is the map defined on $\mathcal{G}$ by (3), together with the following notion: a section $Z \in \Gamma(TM \oplus E)$ will be said to be left invariant if $f(Z) : M \to \mathcal{G}$ is a constant map.

(2) The covariant derivative of a left invariant section $Z \in \Gamma(TM \oplus E)$ is given by

$$\nabla_X Z = \Gamma(X)(Z) - B(X, Z^T) + B^*(X, Z^N) \tag{20}$$

for all $X \in TM$, where $Z = Z^T + Z^N$ in $TM \oplus E$ and $B^* : TM \times E \to TM$ is the bilinear map such that for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(E)$

$$\langle B(X, Y), N \rangle = \langle Y, B^*(X, N) \rangle.$$

**Remark 1.** *These two assumptions are equivalent to the assumptions made in* [11, 18]*: they are necessary to write down the equations of Gauss, Codazzi and Ricci in a general metric Lie group, and to obtain a Fundamental Theorem for immersions in that context; see Section 4.*

**Remark 2.** *Sometimes it is convenient to write these assumptions in some local frames. For sake of simplicity, we assume that $E$ is a trivial line bundle, oriented by a unit section $\nu$. Let $(e_1^o, e_2^o, \ldots, e_n^o)$ be an orthonormal basis of $\mathcal{G}$ and $\Gamma_{ij}^k \in \mathbb{R}$, $1 \le i, j, k \le n$, be such that*

$$\Gamma(e_i^o)(e_j^o) = \sum_{k=1}^{n} \Gamma_{ij}^k \ e_k^o.$$

*We set, for $i = 1, \ldots, n$, $\underline{e}_i \in \Gamma(TM \oplus E)$ such that $f(\underline{e}_i) = e_i^o$, and $f_i \in C^\infty(M)$, $T_i \in \Gamma(TM)$ such that $\underline{e}_i = T_i + f_i\nu$. Since $f$ preserves the metrics, the vectors $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n$ are orthonormal, and we have*

$$\langle T_i, T_j\rangle + f_i f_j = \delta_{ij} \tag{21}$$

*for all $i, j = 1, \ldots, n$. The assumption (20) then reads as follows: for all $X \in TM$ and $j = 1, \ldots, n$,*

$$\nabla_X T_j = \sum_{i,k} \Gamma_{ij}^k \langle X, T_i\rangle T_k + f_j S(X), \tag{22}$$

$$df_j(X) = \sum_{i,k} \Gamma_{ij}^k f_k \langle X, T_i\rangle - h(X, T_j) \tag{23}$$

*where $S(X) = B^*(X, \nu)$ and $h(X, Y) = \langle B(X, Y), \nu\rangle$. Conversely, if vector fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$, $1 \le i \le n$, are given such that (21), (22) and (23) hold, we may define a bundle isomorphism $f : TM \oplus E \to M \times \mathcal{G}$ preserving the metrics and such that (20) holds: setting $\underline{e}_i = T_i + f_i\nu$, we define $f$ such that $f(\underline{e}_i) = e_i^o$, $i = 1, \ldots, n$.*

We state the main result of the paper:

**Theorem 1.** *We moreover assume that $M$ is simply connected. The following statements are equivalent:*

1. *(1) There exists a section $\varphi \in \Gamma(U\Sigma)$ such that*

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi + \frac{1}{2}\Gamma(X) \cdot \varphi \tag{24}$$

   *for all $X \in TM$.*
2. *(2) There exists an isometric immersion $F : M \to G$ with normal bundle $E$ and second fundamental form $B$.*

*More precisely, if $\varphi$ is a solution of (24), replacing $\varphi$ by $\varphi \cdot a$ for some $a \in Spin(\mathcal{G})$ if necessary, and considering the $\mathcal{G}-$valued 1-form $\xi$ defined by*

$$\xi(X) := \langle\langle X \cdot \varphi, \varphi\rangle\rangle \tag{25}$$

*for all $X \in TM$, the formula $F = \int \xi$ defines an isometric immersion in $G$ with normal bundle $E$ and second fundamental form $B$. Here $\int$ stands for the Darboux integral, i.e. $F = \int \xi : M \to G$ is such that $F^*\omega_G = \xi$, where $\omega_G \in \Omega^1(G, \mathcal{G})$ is the Maurer-Cartan form of $G$ defined in (1). Reciprocally, an isometric immersion*

*$M \to G$ with normal bundle $E$ and second fundamental form $B$ may be written in that form.*

The formula $F = \int \xi$ where $\xi$ is defined by (25) may be regarded as a generalized Weierstrass representation formula.

This theorem generalizes the main result of [4] to a Lie group equipped with a left invariant metric (see Section 5).

**Remark 3.** *If $\varphi$ is a solution of (24) and a belongs to $Spin(\mathcal{G})$, $\varphi' := \varphi \cdot a$ is also a solution of (24) (see (14) for the definition of $\varphi \cdot a$). Moreover the associated 1-forms $\xi_\varphi$ and $\xi_{\varphi'}$ are linked by*

$$\xi_{\varphi'} \;=\; \tau(a)\ \xi_\varphi\ a = Ad(a^{-1}) \circ \xi_\varphi. \tag{26}$$

*Let us recall that a 1-form $\xi \in \Omega^1(M, \mathcal{G})$ is Darboux integrable if and only if it satisfies the structure equation $d\xi + [\xi, \xi] = 0$ ($M$ is simply connected). The theorem thus says that if $\varphi$ is a solution of (24), it is possible to find an other solution $\varphi'$ of this equation such that $\xi_{\varphi'}$ is Darboux integrable and $F = \int \xi_{\varphi'}$ is an immersion with normal bundle $E$ and second fundamental form $B$. The proof of (1) $\Rightarrow$ (2) in the theorem will in fact follow these lines. See also Remark 5 below.*

**Remark 4.** *Setting*

$$\vec{H} = \frac{1}{2}\sum_{j=1}^{p} B(e_j, e_j)\ \in E \quad \text{and} \quad \gamma = \frac{1}{2}\sum_{j=1}^{p} e_j \cdot \Gamma(e_j)\ \in Cl(TM \oplus E)$$

*where $e_1, \ldots, e_p$ is an orthonormal basis of $TM$, a solution $\varphi$ of (24) is a solution of the Dirac equation*

$$D\varphi := \sum_{j=1}^{p} e_j \cdot \nabla_{e_j}\varphi = \left(\vec{H} + \gamma\right) \cdot \varphi. \tag{27}$$

*This equation will be especially interesting for the representation of a surface in a 3-dimensional Lie group (see Section 5).*

We now prove the theorem: (1) $\Rightarrow$ (2) will be a consequence of Propositions 3.1 and 3.2 below, and (2) $\Rightarrow$ (1) will be proved at the end of the section.

**Proposition 3.1.** *Assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (24) and define $\xi$ by (25). Then*

1. *$\xi$ takes its values in $\mathcal{G} \subset Cl(\mathcal{G})$;*
2. *there exists $T \in SO(\mathcal{G})$ such that $\xi = T \circ f$;*
3. *replacing $\varphi$ by $\varphi \cdot a$ where $a \in Spin(\mathcal{G})$ is such that $Ad(a) = T$, we have $\xi = f$, and $\xi$ satisfies the structure equation*

$$d\xi + [\xi, \xi] = 0. \tag{28}$$

*Proof.* (1). By the very definition of $\xi$, we have

$$\xi(X) = \tau[\varphi][X][\varphi]$$

for all $X \in TM$, where $[X]$ and $[\varphi]$ represent $X$ and $\varphi$ in a given frame $\tilde{s}$ of $\tilde{Q}$. Since $[X]$ belongs to $\mathcal{G} \subset Cl(\mathcal{G})$ and $[\varphi]$ is an element of $Spin(\mathcal{G})$, $\xi(X)$ belongs to $\mathcal{G}$.

(2). Let us first show that for every left invariant section $Z \in \Gamma(TM \oplus E)$, the map

$\xi(Z):M\rightarrow\mathcal{G}$ is constant: if $Z\in\Gamma(TM\oplus E)$ is left invariant, we compute, for $X\in TM$,

$$\partial_X\ \xi(Z)=\langle\langle\nabla_X Z\cdot\varphi,\varphi\rangle\rangle+\langle\langle Z\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle Z\cdot\varphi,\nabla_X\varphi\rangle\rangle.$$

But, by (24),

$$\begin{aligned}\langle\langle Z\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle Z\cdot\varphi,\nabla_X\varphi\rangle\rangle &= \langle\langle[-\Gamma(X)+\sum_{j=1}^p e_j\cdot B(X,e_j),Z]\cdot\varphi,\varphi\rangle\rangle\\ &= \langle\langle\left\{-\Gamma(X)(Z)+B(X,Z^T)-B^*(X,Z^N)\right\}\cdot\varphi,\varphi\rangle\rangle \qquad (29)\end{aligned}$$

where the brackets $[.,.]$ stand here for the commutator in $Cl(TM\oplus E)$ and where we use Lemmas A.1 and A.3 in the last step. Thus $\partial_X\ \xi(Z)=0$ by (20), and $\xi(Z):M\rightarrow\mathcal{G}$ is constant. Now, if $(e_1^o,\ldots,e_n^o)$ is a fixed orthonormal basis of $\mathcal{G}$ and denoting by $\underline{e}_1,\ldots,\underline{e}_n$ the left invariant sections of $TM\oplus E$ such that $f(\underline{e}_i)=e_i^o$, $i=1,\ldots,n$, we have, for all section $Z=\sum_i Z_i\underline{e}_i\in\Gamma(TM\oplus E)$,

$$\xi(Z)=\sum_{i=1}^n Z_i\ \xi(\underline{e}_i)$$

where $(\xi(\underline{e}_1),\ldots,\xi(\underline{e}_n))$ is a constant orthonormal basis of $\mathcal{G}$. Considering the orthogonal transformation $T:\mathcal{G}\rightarrow\mathcal{G}$ such that $T(e_i^o)=\xi(\underline{e}_i)$, $i=1,\ldots,n$, we get

$$\xi(Z)=\sum_{i=1}^n Z_i\ T(e_i^o)=T\left(\sum_{i=1}^n Z_ie_i^o\right)=T(f(Z)),$$

i.e. $\xi=T\circ f$.

(3). For all $a\in Spin(\mathcal{G})$ and $X\in TM$, we have

$$\begin{aligned}\langle\langle X\cdot(\varphi\cdot a),\varphi\cdot a\rangle\rangle &= \tau([\varphi]a)[X][\varphi]a\\ &= \tau(a)\ \langle\langle X\cdot\varphi,\varphi\rangle\rangle\ a\\ &= Ad(a^{-1})(\xi(X))\\ &= Ad(a^{-1})(T\circ f(X));\end{aligned}$$

thus, replacing $\varphi$ by $\varphi\cdot a$ where $a\in Spin(\mathcal{G})$ is such that $Ad(a)=T$ we get $\xi=f$. By the computation in (29), we have, for $X,Y\in\Gamma(TM)$ such that $\nabla X=\nabla Y=0$ at $x_0$,

$$\begin{aligned}\partial_X\ \xi(Y) &= \langle\langle Y\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle Y\cdot\varphi,\nabla_X\varphi\rangle\rangle\\ &= \langle\langle\{-\Gamma(X)(Y)+B(X,Y)\}\cdot\varphi,\varphi\rangle\rangle\end{aligned}$$

and thus

$$\begin{aligned}d\xi(X,Y) &= \partial_X\ \xi(Y)-\partial_Y\ \xi(X)\\ &= -\langle\langle\{\Gamma(X)(Y)-\Gamma(Y)(X)\}\cdot\varphi,\varphi\rangle\rangle\\ &= -\xi(\Gamma(X)(Y)-\Gamma(Y)(X))\\ &= -[\xi(X),\xi(Y)],\end{aligned}$$

since $\xi=f$, $\Gamma$ satisfies (19), and by (5). □

We keep the assumption and notation of Proposition 3.1, and moreover assume that $M$ is simply connected; we consider

$$F:M\rightarrow G$$

such that $F^*\omega_G=\xi$ (assuming that $\varphi$ is chosen in such a way that $\xi$ satisfies the structure equation (28)). The next proposition follows from the properties of the Clifford product:

**Proposition 3.2.** *1. The map $F:M\to G$ is an isometry.*
*2. The map*

$$\begin{array}{rccl}\Phi_E: & E & \to & F(M)\times\mathcal{G}\\ & X\in E_m & \mapsto & (F(m),\xi(X))\end{array}$$

*is an isometry between $E$ and the normal bundle of $F(M)$ into $G$, preserving connections and second fundamental forms. Here, for $X\in E$, $\xi(X)$ still stands for the quantity $\langle\langle X\cdot\varphi,\varphi\rangle\rangle$.*

*Proof.* For $X,Y\in\Gamma(TM\oplus E)$, we have

$$\begin{aligned}\langle\xi(X),\xi(Y)\rangle &= -\frac{1}{2}\left(\xi(X)\xi(Y)+\xi(Y)\xi(X)\right)\\ &= -\frac{1}{2}\left(\tau[\varphi][X][\varphi]\tau[\varphi][Y][\varphi]+\tau[\varphi][Y][\varphi]\tau[\varphi][X][\varphi]\right)\\ &= -\frac{1}{2}\tau[\varphi]\left([X][Y]+[Y][X]\right)[\varphi]\\ &= \langle X,Y\rangle,\end{aligned}$$

since $[X][Y]+[Y][X]=-2\langle[X],[Y]\rangle=-2\langle X,Y\rangle$. This implies that $F$ is an isometry, and that $\Phi_E$ is a bundle map between $E$ and the normal bundle of $F(M)$ into $G$ which preserves the metrics of the fibers. Let us denote by $B_F$ and $\nabla'^F$ the second fundamental form and the normal connection of the immersion $F$; the aim is now to prove that

$$\xi(B(X,Y))=B_F(\xi(X),\xi(Y))\quad\text{and}\quad\xi(\nabla'_XN)=\nabla'^F_{\xi(X)}\xi(N)\tag{30}$$

for $X,Y\in\Gamma(TM)$ and $N\in\Gamma(E)$. First,

$$B_F(\xi(X),\xi(Y))=(\nabla^G_{\xi(X)}\xi(Y))^N=\{\partial_X\ \xi(Y)+\Gamma(\xi(X))(\xi(Y))\}^N$$

where the superscript $N$ means that we consider the component of the vector which is normal to the immersion. We fix a point $x_0\in M$, assume that $\nabla Y=0$ at $x_0$, and compute, using (29):

$$\begin{aligned}\partial_X\ \xi(Y) &= \langle\langle Y\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle Y\cdot\varphi,\nabla_X\varphi\rangle\rangle\\ &= \langle\langle B(X,Y)\cdot\varphi,\varphi\rangle\rangle-\langle\langle\Gamma(X)(Y)\cdot\varphi,\varphi\rangle\rangle.\end{aligned}$$

Since $\langle\langle B(X,Y)\cdot\varphi,\varphi\rangle\rangle=\xi(B(X,Y))$ is normal to the immersion, we get

$$\{\partial_X\ \xi(Y)\}^N=\xi(B(X,Y))-\langle\langle\Gamma(X)(Y)\cdot\varphi,\varphi\rangle\rangle^N,$$

and thus

$$\begin{aligned}B_F(\xi(X),\xi(Y)) &= \xi(B(X,Y))-\langle\langle\Gamma(X)(Y)\cdot\varphi,\varphi,\rangle\rangle^N+\Gamma(\xi(X))(\xi(Y))^N\\ &= \xi(B(X,Y))\end{aligned}$$

since

$$\begin{aligned}\langle\langle\Gamma(X)(Y)\cdot\varphi,\varphi\rangle\rangle &= \xi(\Gamma(X)(Y))\\ &= f(\Gamma(X)(Y))\\ &= \Gamma(f(X))(f(Y))\qquad\text{(by definition of }\Gamma\text{ on }TM\oplus E)\\ &= \Gamma(\xi(X))(\xi(Y)).\end{aligned}$$

We finally show the second identity in (30): we have

$$\begin{aligned}\nabla'^F_{\xi(X)}\xi(N) &= (\nabla^G_{\xi(X)}\xi(N))^N\\ &= (\partial_X\ \xi(N)+\Gamma(\xi(X))(\xi(N)))^N\\ &= \langle\langle\nabla'_X N\cdot\varphi,\varphi\rangle\rangle^N+\langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle^N+\langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle^N\\ &\quad +\Gamma(\xi(X))(\xi(N))^N.\end{aligned}$$

The first term in the right-hand side is $\xi(\nabla'_X N)$, and we only need to show that

$$\langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle^N+\langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle^N+\Gamma(\xi(X))(\xi(N)))^N=0. \tag{31}$$

From (29), we have

$$\langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle=-\langle\langle B^*(X,N)\cdot\varphi,\varphi\rangle\rangle-\langle\langle\Gamma(X)(N)\cdot\varphi,\varphi\rangle\rangle,$$

which gives (31) since $\langle\langle B^*(X,N)\cdot\varphi,\varphi\rangle\rangle$ is tangent to the immersion ($B^*(X,N)$ belongs to $TM$) and

$$\langle\langle\Gamma(X)(N)\cdot\varphi,\varphi\rangle\rangle=\Gamma(\xi(X))(\xi(N))$$

(see the first part of the proof above). □

We finally show the converse statement (2) ⇒ (1) : we suppose that $F: M\to G$ is an isometric immersion with normal bundle $E$ and second fundamental form $B$, we consider the orthonormal frame $s_o=\mathit{1}_{SO(\mathcal{G})}$ of $\mathcal{G}$, and the spinor frame $\tilde{s}_o=\mathit{1}_{Spin(\mathcal{G})}$ (recall that $Q_G=G\times SO(\mathcal{G})$ and $\tilde{Q}_G=G\times Spin(\mathcal{G})$; see Section 2). The spinor field $\varphi=[\tilde{s}_o,\mathit{1}_{Cl(\mathcal{G})}]$ satisfies (24) as a consequence of the Gauss formulas (15)-(16); moreover, its associated 1-form is, for all $X\in TM$,

$$\xi(X)=\langle\langle F_*X\cdot\varphi,\varphi\rangle\rangle=\tau[\varphi]\ [F_*X]\ [\varphi]=[F_*X],$$

where $[F_*X]\in\mathcal{G}$ represents $F_*X$ in $s_o$, that is $[F_*X]=\omega_G(F_*X)$ ($\omega_G\in\Omega^1(G,\mathcal{G})$ is the Maurer-Cartan form of $G$). Thus $\xi=F^*\omega_G$, that is $F=\int\xi$.

**Remark 5.** *We proved in Proposition 3.2 that if $\varphi\in\Gamma(U\Sigma)$ is a solution of (24) such that $\xi_\varphi$ satisfies the structure equation (28) then $F=\int\xi_\varphi$ is an immersion with normal bundle $E$ and second fundamental form $B$. By (26) it is clear that if $a\in Spin(\mathcal{G})$ is such that $Ad(a^{-1}):\mathcal{G}\to\mathcal{G}\ \in SO(\mathcal{G})$ is an automorphism of Lie algebra, then $\xi_{\varphi\cdot a}$ satisfies the structure equation too; in fact, the corresponding immersions $F_\varphi=\int\xi_\varphi$ and $F_{\varphi\cdot a}=\int\xi_{\varphi\cdot a}$ are linked by the following formula: if $\Phi_a:G\to G$ is the automorphism of $G$ such that $d(\Phi_a)_e=Ad(a^{-1})$, then $\Phi_a$ is also an isometry for the left invariant metric, and*

$$F_{\varphi\cdot a}=L_b\circ\Phi_a\circ F_\varphi \tag{32}$$

*for some $b$ belonging to $G$. This relies on the following formula: if $\Phi:G\to G$ is an automorphism, $\omega_G\in\Omega^1(G,\mathcal{G})$ is the Maurer-Cartan form of $G$ and $F:M\to G$ is a smooth map, then*

$$(\Phi\circ F)^*\omega_G=d(\Phi)_e\circ(F^*\omega_G).$$

*This formula applied to $\Phi=\Phi_a$ and $F=F_\varphi$ shows that $\Phi_a\circ F_\varphi$ is a solution of the Darboux equation associated to the form $\xi_{\varphi\cdot a}$; thus, by uniqueness of a solution of the Darboux equation, (32) holds for some $b$ belonging to $G$.*

## 4. An application: the Fundamental Theorem for immersions in a metric Lie group

We now show that the equations of Gauss, Ricci and Codazzi on $B$ are exactly the integrability conditions of (24). We recall these equations for immersions in the metric Lie group $G$: if $R^G$ denotes the curvature tensor of $(G,\langle .,.\rangle)$, and if $R^T$ and $R^N$ stand for the curvature tensors of the connections on $TM$ and on $E$ ($M$ is a submanifold of $G$ and $E$ is its normal bundle), then we have, for all $X,Y,Z\in\Gamma(TM)$ and $N\in\Gamma(E)$,

(1) the Gauss equation

$$(R^G(X,Y)Z)^T=R^T(X,Y)Z-B^*(X,B(Y,Z))+B^*(Y,B(X,Z)), \tag{33}$$

(2) the Ricci equation

$$(R^G(X,Y)N)^N=R^N(X,Y)N-B(X,B^*(Y,N))+B(Y,B^*(X,N)), \tag{34}$$

(3) the Codazzi equation

$$(R^G(X,Y)Z)^N=\tilde\nabla_XB(Y,Z)-\tilde\nabla_YB(X,Z); \tag{35}$$

in the last equation, $\tilde\nabla$ denotes the natural connection on $T^*M\otimes T^*M\otimes E$.

These equations make sense if $M$ is an abstract manifold and $E\rightarrow M$ is an abstract bundle as in Section 3, if we assume the existence of the bundle map $f$ in (17), since $f$ permits to define $\Gamma$ on $TM\oplus E$ by (19), and $R^G$ may be written in terms of $\Gamma$ only (see (5)-(6)). We prove the following:

**Proposition 4.1.** *We assume that $M$ is simply connected. There exists $\varphi\in\Gamma(U\Sigma)$ solution of (24) if and only if $B:TM\times TM\rightarrow E$ satisfies the Gauss, Ricci and Codazzi equations.*

*Proof.* We first prove that the Gauss, Ricci and Codazzi equations are necessary if we have a non-trivial solution of (24). We assume that $\varphi\in\Gamma(U\Sigma)$ is a solution of (24) and compute the curvature

$$R(X,Y)\varphi=\nabla_X\nabla_Y\varphi-\nabla_Y\nabla_X\varphi-\nabla_{[X,Y]}\varphi.$$

We fix a point $x_0\in M$, and assume that $\nabla X=\nabla Y=0$ at $x_0$. We have

$$\begin{aligned}\nabla_X\nabla_Y\varphi &= -\frac{1}{2}\sum_{j=1}^{p}e_j\cdot\left(\tilde\nabla_XB(Y,e_j)\cdot\varphi+B(Y,e_j)\cdot\nabla_X\varphi\right)\\ &\quad+\frac{1}{2}\left(\nabla_X\Gamma(Y)\cdot\varphi+\Gamma(Y)\cdot\nabla_X\varphi\right)\\ &= -\frac{1}{2}\sum_{j=1}^{p}e_j\cdot\tilde\nabla_XB(Y,e_j)\cdot\varphi-\frac{1}{4}\sum_{j,k=1}^{p}e_j\cdot e_k\cdot B(Y,e_j)\cdot B(X,e_k)\\ &\quad-\frac{1}{4}\sum_{j=1}^{p}e_j\cdot B(Y,e_j)\cdot\Gamma(X)\cdot\varphi+\frac{1}{2}\nabla_X\Gamma(Y)\cdot\varphi-\frac{1}{4}\Gamma(Y)\cdot\sum_{j=1}^{p}e_j\cdot B(X,e_j)\cdot\varphi\\ &\quad+\frac{1}{4}\Gamma(Y)\cdot\Gamma(X)\cdot\varphi.\end{aligned}$$

Thus

$$
\begin{aligned}
R(X,Y)\varphi \;=\; & -\frac{1}{2}\sum_{j=1}^{p} e_j\cdot\left(\tilde\nabla_X B(Y,e_j)-\tilde\nabla_Y B(X,e_j)\right)\cdot\varphi \\
& \underbrace{+\frac{1}{4}\sum_{j\neq k} e_j\cdot e_k\cdot\left(B(X,e_j)\cdot B(Y,e_k)-B(Y,e_j)\cdot B(X,e_k)\right)\cdot\varphi}_{\mathcal{A}} \\
& \underbrace{-\frac{1}{4}\sum_{j=1}^{p}\left(B(X,e_j)\cdot B(Y,e_j)-B(Y,e_j)\cdot B(X,e_j)\right)\cdot\varphi}_{\mathcal{B}} \\
& \underbrace{+\frac{1}{2}\left[\sum_{j=1}^{p} e_j\cdot B(X,e_j),\Gamma(Y)\right]\cdot\varphi}_{\mathcal{C}_1} + \underbrace{-\frac{1}{2}\left[\sum_{j=1}^{p} e_j\cdot B(Y,e_j),\Gamma(X)\right]\cdot\varphi}_{\mathcal{C}_2} \\
& \underbrace{+\frac{1}{2}\left(\nabla_X\Gamma(Y)-\nabla_Y\Gamma(X)\right)\cdot\varphi}_{\mathcal{C}_3} + \underbrace{-\frac{1}{2}[\Gamma(X),\Gamma(Y)]\cdot\varphi}_{\mathcal{C}_4}
\end{aligned} \tag{36}
$$

where the brackets stand for the commutator in the Clifford bundle $Cl(TM\oplus E)$ : $\forall\eta,\xi\ \in Cl(TM\oplus E)$,

$$[\eta,\xi]=\frac{1}{2}\left(\eta\cdot\xi-\xi\cdot\eta\right).$$

We computed the second and the third terms in [4]; we only recall the result here:

**Lemma 4.2.** [4] *We have*

$$\mathcal{A}=\frac{1}{2}\sum_{j<k}\left\{\langle B^*(X,B(Y,e_j)),e_k\rangle-\langle B^*(Y,B(X,e_j)),e_k\rangle\right\}e_j\cdot e_k$$

*and*

$$\mathcal{B}=\frac{1}{2}\sum_{k<l}\langle B(X,B^*(Y,n_k))-B(Y,B^*(X,n_k)),n_l\rangle\, n_k\cdot n_l,$$

*where $e_1,\ldots,e_p$ and $n_1,\ldots,n_q$ are orthonormal bases of $TM$ and $E$.*

We now compute the other terms in (36). We first compute the covariant derivative of $\Gamma$, considering $\Gamma$ as a map

$$\Gamma:\qquad TM\oplus E\to End(TM\oplus E).$$

**Lemma 4.3.** *If $X,Y\in TM$ and $Z\in TM\oplus E$,*

$$
\begin{aligned}
(\nabla_X\Gamma)(Y)Z \;=\; & \left\{\Gamma(X)\circ\Gamma(Y)-\Gamma(Y)\circ\Gamma(X)\right\}(Z)-\Gamma(\Gamma(X)Y)(Z)+\Gamma(B(X,Y))(Z) \\
& -B(X,(\Gamma(Y)Z)^T)+B^*(X,(\Gamma(Y)Z)^N)+\Gamma(Y)(B(X,Z^T)-B^*(X,Z^N)).
\end{aligned}
$$

*Proof.* Since the expression is tensorial, we may assume that $X,Y,Z\in\Gamma(TM\oplus E)$ are left invariant vector fields. By definition,

$$(\nabla_X\Gamma)(Y)Z=\nabla_X(\Gamma(Y)Z)-\Gamma(\nabla_XY)Z-\Gamma(Y)(\nabla_XZ). \tag{37}$$

Since $X,Y$ and $Z$ are left invariant vector fields, so are $\Gamma(Y)Z$, $\nabla_XY$ and $\nabla_XZ$, and, by (20),

$$\nabla_X(\Gamma(Y)Z)=\Gamma(X)(\Gamma(Y)Z)-B(X,(\Gamma(Y)Z)^T)+B^*(X,(\Gamma(Y)Z)^N),$$

$$\Gamma(Y)(\nabla_X Z) = \Gamma(Y)(\Gamma(X)Z) - \Gamma(Y)B(X, Z^T) + \Gamma(Y)B^*(X, Z^N)$$

and

$$\Gamma(\nabla_X Y)(Z) = \Gamma(\Gamma(X)Y)Z - \Gamma(B(X, Y^T))Z + \Gamma(B^*(X, Y^N))Z.$$

Plugging these formulas in (37) and using finally that $Y$ belongs to $TM$ (ie $Y^T = Y$ and $Y^N = 0$), we get the result. $\square$

We now regard $\Gamma$ as a map

$$\Gamma : \quad TM \oplus E \to \Lambda^2(TM \oplus E) \subset Cl(TM \oplus E),$$

and compute the term $\mathcal{C}_3$ in (36). According to Lemma A.1, for all $X, Y \in TM \oplus E$,

$$\Gamma(X)(Y) = [\Gamma(X), Y]\,.$$

**Lemma 4.4.** *If $X, Y \in TM$,*

$$\begin{aligned}\frac{1}{2}\left((\nabla_X\Gamma)(Y) - (\nabla_Y\Gamma)(X)\right) &= [\Gamma(X),\Gamma(Y)] - \frac{1}{2}\Gamma([\Gamma(X), Y] - [\Gamma(Y), X]) \\ &\quad -\frac{1}{2}\left[\sum_{j=1}^{p} e_j \cdot B(X, e_j), \Gamma(Y)\right] + \frac{1}{2}\left[\sum_{j=1}^{p} e_j \cdot B(Y, e_j), \Gamma(X)\right].\end{aligned}$$

*Here the brackets stand for the commutator in $Cl(TM \oplus E)$.*

*Proof.* By Lemmas A.1 and A.2 in the appendix, the linear maps $\Gamma(X) \circ \Gamma(Y) - \Gamma(Y) \circ \Gamma(X)$, $Z \mapsto \Gamma(\Gamma(X)Y)Z$ and $Z \mapsto \Gamma(B(X, Y))Z$ appearing in Lemma 4.3 are respectively represented by the bivectors $[\Gamma(X), \Gamma(Y)]$, $\Gamma([\Gamma(X), Y])$ and $\Gamma(B(X, Y))$. Moreover, by Lemma A.4 applied to the linear maps $B(X, .) : TM \to E$ and $\Gamma(Y) : TM \oplus E \to TM \oplus E$, the map

$$Z \mapsto -B^*(X, (\Gamma(Y)Z)^N) + \Gamma(Y)(B^*(X, Z^N)) + B(X, (\Gamma(Y)Z)^T) - \Gamma(Y)(B(X, Z^T))$$

is represented by the bivector

$$\left[\sum_{j=1}^{p} e_j \cdot B(X, e_j), \Gamma(Y)\right] \quad \in Cl(TM \oplus E).$$

The result follows. $\square$

We readily deduce the sum of the last four terms in (36):

**Lemma 4.5.** *Let us set, for $X, Y \in TM$,*

$$R^G(X, Y) = [\Gamma(X), \Gamma(Y)] - \Gamma\left\{[\Gamma(X), Y] - [\Gamma(Y), X]\right\} \ \in \Lambda^2(TM \oplus E),$$

*the curvature tensor of $G$, pulled-back to $TM \oplus E$ by the bundle isomorphism $f$ introduced in (17). Then*

$$\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 = \frac{1}{2}R^G(X, Y).$$

We thus get from (36) the formula

$$\begin{aligned}R(X,Y)\varphi &= -\frac{1}{2}\sum_{j=1}^{p} e_j \cdot \left(\tilde{\nabla}_X B(Y, e_j) - \tilde{\nabla}_Y B(X, e_j)\right) \cdot \varphi \\ &\quad + \mathcal{A} \cdot \varphi + \mathcal{B} \cdot \varphi + \frac{1}{2}R^G(X, Y) \cdot \varphi \end{aligned} \tag{38}$$

where $\mathcal{A}$ and $\mathcal{B}$ are computed in Lemma 4.2 and $R^G$ may be conveniently written in the form

$$\begin{aligned} R^G(X,Y) &= \sum_{1\leq j<k\leq p}\langle R^G(X,Y)(e_j),e_k\rangle e_j\cdot e_k \\ &\quad+\sum_{j=1}^{p}\sum_{r=1}^{q}\langle R^G(X,Y)(e_j),n_r\rangle e_j\cdot n_r \\ &\quad+\sum_{1\leq r<s\leq q}\langle R^G(X,Y)(n_s),n_r\rangle n_r\cdot n_s. \end{aligned}$$

On the other hand, the curvature of the spinorial connection is given by

$$R(X,Y)\varphi = \frac{1}{2}\left(\sum_{1\leq j<k\leq p}\langle R^T(X,Y)(e_j),e_k\rangle\ e_j\cdot e_k + \sum_{1\leq r<s\leq q}\langle R^N(X,Y)(n_r),n_s\rangle\ n_r\cdot n_s\right)\cdot\varphi. \tag{39}$$

We now compare the expressions (38) and (39): since in a given frame $\tilde{s}$ belonging to $\tilde{Q}$, $\varphi$ is represented by an element which is invertible in $Cl(\mathcal{G})$ (it is in fact represented by an element belonging to $Spin(\mathcal{G})$), we may identify the coefficients and get

$$\langle R^T(X,Y)(ej),e_k\rangle=\langle B^*(X,B(Y,e_j)),e_k\rangle-\langle B^*(Y,B(X,e_j)),e_k\rangle+\langle R^G(X,Y)(e_j),e_k\rangle,$$

$$\langle R^N(X,Y)(n_r),n_s\rangle=\langle B(X,B^*(Y,n_r)),n_s\rangle-\langle B(Y,B^*(X,n_r)),n_s\rangle+\langle R^G(X,Y)(n_r),n_s\rangle$$

and

$$\langle\tilde{\nabla}_XB(Y,e_j)-\tilde{\nabla}_YB(X,e_j),n_r\rangle=\langle R^G(X,Y)(e_j),n_r\rangle$$

for all the indices. These equations are the equations of Gauss, Ricci and Codazzi.

We now prove that the equations of Gauss, Ricci and Codazzi are also sufficient to get a solution of (24). The calculations above in fact show that the connection on $\Sigma$ defined by

$$\nabla'_X\varphi:=\nabla_X\varphi+\frac{1}{2}\sum_{j=1}^{p}e_j\cdot B(X,e_j)\cdot\varphi-\frac{1}{2}\Gamma(X)\cdot\varphi \tag{40}$$

for all $\varphi\in\Gamma(\Sigma)$ and $X\in\Gamma(TM)$ is flat if and only if the equations of Gauss, Ricci and Codazzi hold. But if this connection is flat there exists a solution $\varphi\in\Gamma(U\Sigma)$ of (24); this is because $\nabla'$ may be also interpreted as a connection on $U\Sigma$ regarded as a principal bundle (of group $Spin(\mathcal{G})$, acting on the right): indeed, $\nabla$ defines such a connection (since it comes from a connection on $\tilde{Q}$), and the right hand side term in (40) defines a linear map

$$\begin{aligned} TM &\to \chi_V^{inv}(U\Sigma) \\ X &\mapsto \varphi\mapsto\frac{1}{2}\sum_{j=1}^{p}e_j\cdot B(X,e_j)\cdot\varphi-\frac{1}{2}\Gamma(X)\cdot\varphi \end{aligned}$$

from $TM$ to the vector fields on $U\Sigma$ which are vertical and invariant under the action of the group (these vector fields are of the form $\varphi\mapsto\eta\cdot\varphi$, $\eta\in\Lambda^2(TM\oplus E)\subset$

$Cl(TM \oplus E)$). Assuming that the equations of Gauss, Codazzi and Ricci hold, we thus get a solution $\varphi \in \Gamma(U\Sigma)$ of (24). □

The considerations above give a spinorial proof of the Fundamental Theorem of submanifold theory in the metric Lie group $G$ (see [18] for another proof). We keep the hypotheses and notation of the beginning of Section 3.

**Corollary 1.** *We moreover assume that $M$ is simply connected and that $B : TM \times TM \to E$ satisfies the equations of Gauss, Codazzi and Ricci (33)-(35). Then there is an isometric immersion of $M$ into $G$ with normal bundle $E$ and second fundamental form $B$. The immersion is unique up to a rigid motion in $G$, that is up to a transformation of the form*

$$L_b \circ \Phi_a : \begin{array}{ccc} G & \to & G \\ g & \mapsto & b\Phi_a(g) \end{array} \tag{41}$$

*where $a \in Spin(\mathcal{G})$ is such that $Ad(a) : \mathcal{G} \to \mathcal{G}$ is an automorphism of Lie algebra, $\Phi_a : G \to G$ is the group automorphism such that $d(\Phi_a)_e = Ad(a)$, and $b$ belongs to $G$.*

*Proof.* The equations of Gauss, Codazzi and Ricci are the integrability conditions of (24). We thus get a solution $\varphi \in \Gamma(U\Sigma)$ of (24); with such a spinor field at hand, $F = \int \xi$ where $\xi$ is defined in (25) is the immersion. Finally, a solution of (24) is unique up to the right action of an element of $Spin(\mathcal{G})$; the right multiplication of $\varphi$ by $a \in Spin(\mathcal{G})$ and the left multiplication by $b \in G$ in the last integration give also an immersion, if $Ad(a) : \mathcal{G} \to \mathcal{G}$ is moreover an automorphism of Lie algebra. This immersion is obtained from the immersion defined by $\varphi$ by a rigid motion, as described in (41). □

**Remark 6.** *In $\mathbb{R}^n$, a rigid motion as in (41) is a transformation of the form*

$$\begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & ax + b, \end{array}$$

*with $a \in SO(n)$ and $b \in \mathbb{R}^n$.*

## 5. Special cases

5.1. **Submanifolds in $\mathbb{R}^n$.** If the metric Lie group is $\mathbb{R}^n$ with its natural metric, we recover the main result of [4]. We suppose that $M$ is a $p$-dimensional Riemannian manifold, $E \to M$ a bundle of rank $q$, with a fibre metric and a compatible connection. We assume that $TM$ and $E$ are oriented and spin with given spin structures, and that $B : TM \times TM \to E$ is bilinear and symmetric.

**Theorem 2.** [4] *We moreover assume that $M$ is simply connected. The following statements are equivalent:*

*(1) There exists a section $\varphi \in \Gamma(U\Sigma)$ such that*

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi \tag{42}$$

*for all $X \in TM$.*

*(2) There exists an isometric immersion $F:\ M\rightarrow\mathbb{R}^n$ with normal bundle $E$ and second fundamental form $B$.*
*Moreover, $F=\int\xi$ where $\xi$ is the $\mathbb{R}^n-$valued 1-form defined by*

$$\xi(X):=\langle\langle X\cdot\varphi,\varphi\rangle\rangle \tag{43}$$

*for all $X\in TM$.*

*Proof.* We only prove (1) $\Rightarrow$ (2). This will be a consequence of Theorem 1 if we may define a bundle map $f$ as in (17) such that (20) holds. We assume that $\varphi$ is a solution of (42), and set

$$\begin{array}{rccc} f: & TM\oplus E & \rightarrow & M\times\mathbb{R}^n \\ & Z & \mapsto & \langle\langle Z\cdot\varphi,\varphi\rangle\rangle. \end{array}$$

The map $\Gamma$ defined by (19) is $\Gamma=0$. We now show that (20) is satisfied for every $Z\in\Gamma(TM\oplus E)$ such that $f(Z):M\rightarrow\mathbb{R}^n$ is a constant map: for all $X\in TM$, we have $\partial_X\{f(Z)\}=0$, which reads

$$\langle\langle\nabla_XZ\cdot\varphi,\varphi\rangle\rangle+\langle\langle Z\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle Z\cdot\varphi,\nabla_X\varphi\rangle\rangle=0.$$

But (42) gives

$$\langle\langle Z\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle Z\cdot\varphi,\nabla_X\varphi\rangle\rangle=\langle\langle\{B(X,Z^T)-B^*(X,Z^N)\}\cdot\varphi,\varphi\rangle\rangle$$

(see the computations in (29) with $\Gamma=0$). Thus

$$\langle\langle\nabla_XZ\cdot\varphi,\varphi\rangle\rangle=\langle\langle\{-B(X,Z^T)+B^*(X,Z^N)\}\cdot\varphi,\varphi\rangle\rangle$$

and

$$\nabla_XZ=-B(X,Z^T)+B^*(X,Z^N),$$

which is (20) with $\Gamma=0$. $\square$

5.2. **Submanifolds in $\mathbb{H}^n$.** Spinor representations of submanifolds in $\mathbb{H}^n$ with its natural metric were already given in [16, 3, 4]. We give here another representation using the group structure of $\mathbb{H}^n$, with an arbitrary left invariant metric. Let us set

$$\mathbb{H}^n=\{a=(a',a_n)\ \in\mathbb{R}^n:\ a_n>0\},$$

and, for $a\in\mathbb{H}^n$, the transformation

$$\begin{array}{rccc} \varphi_a: & \mathbb{R}^{n-1} & \rightarrow & \mathbb{R}^{n-1} \\ & x & \mapsto & a_nx+a'; \end{array}$$

$\varphi_a$ is an homothety composed by a translation. The homotheties composed by translations naturally form a group under composition, and the bijection

$$\begin{array}{rccl} \varphi: & \mathbb{H}^n & \rightarrow & \{\text{homotheties-translations } \mathbb{R}^{n-1}\rightarrow\mathbb{R}^{n-1}\} \\ & a & \mapsto & \varphi_a \end{array}$$

induces a group structure on $\mathbb{H}^n$ : it is such that

$$ab=(a_nb'+a',a_nb_n) \tag{44}$$

for all $a,b\in\mathbb{H}^n$; the identity element is $e=(0,1)\in\mathbb{H}^n$. Let us denote by $(e_1^o,e_2^o,\ldots,e_n^o)$ the canonical basis of $T_e\mathbb{H}^n=\mathbb{R}^n$ and keep the same letters to denote the corresponding left invariant vector fields on $\mathbb{H}^n$. The Lie bracket may be easily seen to be given by

$$[e_i^o,e_j^o]=0\quad\text{ and }\quad[e_n^o,e_i^o]=e_i^o$$

for $i, j = 1, \ldots, n-1$. This may also be written in the form

$$[X, Y] = l(X)Y - l(Y)X \tag{45}$$

for all $X, Y \in \mathbb{R}^n$, where $l : \mathbb{R}^n \to \mathbb{R}$ is the linear form such that $l(e_i^o) = 0$ if $i \leq n-1$ and $l(e_n^o) = 1$. This property implies that every left invariant metric on $\mathbb{H}^n$ has constant negative curvature $-|l|^2$ [14, 13].

We suppose that a left invariant metric $\langle ., . \rangle$ is given on $\mathbb{H}^n$, and consider the vector $U_o \in T_e\mathbb{H}^n$ such that $l(X) = \langle U_o, X \rangle$ for all $X \in T_e\mathbb{H}^n$. We have $|U_o| = |l|$, and, by the Koszul formula (4),

$$\Gamma(X)(Y) = -\langle Y, U_o \rangle X + \langle X, Y \rangle U_o \tag{46}$$

for all $X, Y \in T_e\mathbb{H}^n$.

We keep the hypotheses made at the beginning of Section 5.1. We suppose moreover that $U \in \Gamma(TM \oplus E)$ is given such that $|U| = |l|$ and, for all $X \in TM$,

$$\nabla_X U = -|U|^2 X + \langle X, U \rangle U - B(X, U^T) + B^*(X, U^N). \tag{47}$$

We set, for $X \in TM$ and $Y \in TM \oplus E$,

$$\Gamma(X)(Y) = -\langle Y, U \rangle X + \langle X, Y \rangle U. \tag{48}$$

**Remark 7.** *Equation (47) implies the following:*

*(1) $U$ is a solution of (20), with the definition (48) of $\Gamma$.*
*(2) The norm of $U$ is constant, since, by a straightforward computation,*

$$d|U|^2(X) = 2\langle \nabla_X U, U \rangle = 0$$

*for all $X \in TM$. The additional hypothesis $|U| = |l|$ is thus not very restrictive.*

*We note that it is not necessary to assume the existence of $U$ solution of (47) to get a spinor representation of a submanifold in $\mathbb{H}^n$ if $\mathbb{H}^n$ is regarded as the set of unit vectors in Minkowski space $\mathbb{R}^{n,1}$ [16, 3, 4]. Nevertheless, this hypothesis seems necessary if we consider $\mathbb{H}^n$ as a group, since the group structure introduces an anisotropy: the vector $e_n \in T_e\mathbb{H}^n$ is indeed a special direction for the group structure.*

Let us construct the spinor bundles $\Sigma$ and $U\Sigma$ on $M$ as in Section 2.4 with here $\mathcal{G} = T_e\mathbb{H}^n$.

**Theorem 3.** *We assume that $M$ is simply connected. The following statements are equivalent:*

*(1) There exists a spinor field $\varphi \in \Gamma(U\Sigma)$ solution of (24) where $\Gamma$ is defined by (48).*
*(2) There exists an isometric immersion $M \to \mathbb{H}^n$ with normal bundle $E$ and second fundamental form $B$.*

*Proof.* We assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (24) where $\Gamma$ is defined by (48), and define $f : TM \oplus E \to M \times T_e\mathbb{H}^n$ by

$$f(Z) = \langle\langle Z \cdot \varphi, \varphi \rangle\rangle$$

for all $Z\in TM\oplus E$. Let us first observe that if $Z$ is a vector field solution of (20), then $f(Z)$ is constant: we have, for all $X\in TM$,

$$\partial_X f(Z)=\langle\langle\nabla_X Z\cdot\varphi,\varphi\rangle\rangle+\langle\langle Z\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle Z\cdot\varphi,\nabla_X\varphi\rangle\rangle;$$

this is 0, by (20), (24) and the computation (29). Since $U$ is a solution of (20) (see Remark 7), we deduce that $f(U)\in T_e\mathbb{H}^n$ is a constant, and, since $|f(U)|=|U|=|U_o|$, replacing $\varphi$ by $\varphi\cdot a$ for some $a\in Spin(T_e\mathbb{H}^n)$ if necessary, we may suppose that $f(U)=U_o$. Since $\Gamma$ is defined on $T_e\mathbb{H}^n$ by (46) and on $TM\oplus E$ by (48), and since $f$ preserves the metrics, it is straightforward to see that $f(\Gamma(X)(Y))=\Gamma(f(X))(f(Y))$ for all $X,Y\in TM\oplus E$. Finally, (20) holds for all $Z\in\Gamma(TM\oplus E)$ such that $f(Z)$ is constant: this is the same argument as in the proof of Theorem 2 in Section 5.1, just adding the term $\Gamma$. The result then follows from Theorem 1. □

5.3. **Hypersurfaces in a metric Lie group.** We assume that $G$ is a simply connected $n$-dimensional metric Lie group, $M$ is a $p$-dimensional Riemannian manifold, $n=p+1$, and $E$ is the trivial line bundle on $M$, oriented by a unit section $\nu\in\Gamma(E)$. We moreover suppose that $M$ is simply connected and that $h:TM\times TM\to\mathbb{R}$ is a given symmetric bilinear form, and that the hypotheses (1) and (2) of Section 3 with $B=h\nu$ hold. According to Theorem 1, an isometric immersion of $M$ into $G$ with second fundamental form $h$ is equivalent to a section $\varphi$ of $\Gamma(U\Sigma)$ solution of the Killing equation (24). Note that $Q_E\simeq M$ and the double covering $\tilde{Q}_E\to Q_E$ is trivial, since $M$ is assumed to be simply connected. Fixing a section $\tilde{s}_E$ of $\tilde{Q}_E$ we get an injective map

$$\begin{array}{ccc}\tilde{Q}_M & \to & \tilde{Q}_M\times_M\tilde{Q}_E=:\tilde{Q}\\ \tilde{s}_M & \mapsto & (\tilde{s}_M,\tilde{s}_E).\end{array}$$

Using

$$Cl_p\simeq Cl^0_{p+1}\subset Cl_{p+1}$$

(induced by the Clifford map $\mathbb{R}^p\to Cl_{p+1}$, $X\mapsto X\cdot e_{p+1}$), we deduce a bundle isomorphism

$$\begin{array}{ccc}\tilde{Q}_M\times_\rho Cl_p & \to & \tilde{Q}\times_\rho Cl^0_{p+1}\quad\subset\Sigma\\ \psi & \mapsto & \psi^*.\end{array}\tag{49}$$

It satisfies the following properties: for all $X\in TM$ and $\psi\in\tilde{Q}_M\times_\rho Cl_p$,

$$(X\cdot\psi)^*=X\cdot\nu\cdot\psi^*\quad\text{ and }\quad\nabla_X(\psi^*)=(\nabla_X\psi)^*.\tag{50}$$

To write down the Killing equation (24) in the bundle $\tilde{Q}_M\times_\rho Cl_p$, we need to decompose the Clifford action of $\Gamma(X)$ into its tangent and its normal parts:

**Lemma 5.1.** *Recall the notation introduced in Remark 2. Then, for all $X\in TM$,*

$$\Gamma(X)=\sum_{i=1}^n\langle X,T_i\rangle\sum_{1\leq j<k\leq n}\Gamma_{ij}^k\left(\frac{1}{2}(T_j\cdot T_k-T_k\cdot T_j)+(f_kT_j-f_jT_k)\cdot\nu\right).\tag{51}$$

*Proof.* We have

$$X=\sum_{i=1}^n\langle X,\underline{e}_i\rangle\underline{e}_i=\sum_{i=1}^n\langle X,T_i\rangle\underline{e}_i,$$

$$\begin{aligned}\Gamma(X)(\underline{e}_j) &= \sum_{i=1}^n \langle X, T_i\rangle \Gamma(\underline{e}_i)(\underline{e}_j)\\ &= \sum_{i=1}^n \langle X, T_i\rangle \sum_{k=1}^n \Gamma_{ij}^k \underline{e}_k\\ &= \sum_{1\le i,k\le n} \Gamma_{ij}^k \langle X, T_i\rangle (T_k + f_k\nu),\end{aligned}$$

and thus

$$\begin{aligned}\Gamma(X) &= \frac{1}{2}\sum_{j=1}^n \underline{e}_j \cdot \Gamma(X)(\underline{e}_j)\\ &= \frac{1}{2}\sum_{j=1}^n (T_j + f_j\nu)\cdot \sum_{1\le i,k\le n} \Gamma_{ij}^k \langle X, T_i\rangle (T_k + f_k\nu)\\ &= \frac{1}{2}\sum_{1\le i,j,k\le n} \Gamma_{ij}^k \langle X, T_i\rangle (T_j + f_j\nu)\cdot(T_k + f_k\nu).\end{aligned}$$

Now

$$(T_j + f_j\nu)\cdot(T_k + f_k\nu) = T_j\cdot T_k + f_k T_j\cdot\nu - f_j T_k\cdot\nu - f_j f_k,$$

and the result follows since $\Gamma_{ij}^k = -\Gamma_{ik}^j$. □

The section $\varphi \in \Gamma(U\Sigma)$ solution of (24) thus identifies to a section $\psi$ of $\tilde{Q}_M \times_\rho Cl_p$ solution of

$$\begin{aligned}\nabla_X\psi &= -\frac{1}{2}\sum_{j=1}^p h(X, e_j) e_j\cdot\psi + \frac{1}{2}\tilde{\Gamma}(X)\cdot\psi\\ &= -\frac{1}{2}S(X)\cdot\psi + \frac{1}{2}\tilde{\Gamma}(X)\cdot\psi\end{aligned}$$

for all $X \in TM$, where

$$\tilde{\Gamma}(X) = \sum_{i=1}^n \langle X, T_i\rangle \sum_{1\le j<k\le n} \Gamma_{ij}^k \left(\frac{1}{2}(T_j\cdot T_k - T_k\cdot T_j) + (f_k T_j - f_j T_k)\right) \tag{52}$$

and $S : TM \to TM$ is the symmetric operator associated to $h$. We deduce the following result:

**Theorem 4.** *Let $S : TM \to TM$ be a symmetric operator. The following two statements are equivalent:*

*(1) there exists an isometric immersion of $M$ into $G$ with shape operator $S$;*
*(2) there exists a normalized spinor field $\psi \in \Gamma(\tilde{Q}_M \times_\rho Cl_p)$ solution of*

$$\nabla_X\psi = -\frac{1}{2}S(X)\cdot\psi + \frac{1}{2}\tilde{\Gamma}(X)\cdot\psi \tag{53}$$

*for all $X \in TM$, where $\tilde{\Gamma}$ is defined in (52).*

*Here, a spinor field $\psi \in \Gamma(\tilde{Q}_M \times_\rho Cl_p)$ is said to be normalized if it is represented in some frame $\tilde{s} \in \tilde{Q}_M$ by an element $[\psi] \in Cl_p \simeq Cl_{p+1}^0$ belonging to $Spin(p+1)$.*

We will see below explicit representation formulas in the cases of the dimensions 3 and 4.

5.4. **Surfaces in a 3-dimensional metric Lie group.** Since $Cl_2 \simeq \Sigma_2$ we have

$$\tilde{Q}_M \times_\rho Cl_2 \simeq \Sigma M,$$

and $\varphi$ is equivalent to a spinor field $\psi \in \Gamma(\Sigma M)$ solution of (53) and such that $|\psi| = 1$. Moreover, the explicit representation formula $F = \int \xi$ may be written in terms of $\psi$ : it may be proved by a computation that

$$\langle\langle X \cdot \varphi, \varphi\rangle\rangle = i2\mathcal{R}e\langle X \cdot \psi^+, \psi^-\rangle + j\left(\langle X \cdot \psi^+, \alpha(\psi^+)\rangle - \langle X \cdot \psi^-, \alpha(\psi^-)\rangle\right) \tag{54}$$

where the brackets $\langle .,.\rangle$ stand here for the natural hermitian product on $\Sigma_2$ and $\alpha : \Sigma_2 \to \Sigma_2$ is the natural quaternionic structure. If $G = \mathbb{R}^3$, this is the explicit representation formula given in [8] (see also [3]).

We also note that the expression (52) of $\tilde{\Gamma}$ simplifies if the Lie group is 3-dimensional:

**Lemma 5.2.** *If $j, k,$ $j \neq k$, belong to $\{1,2,3\}$, let us denote by $l_{jk} \in \{1,2,3\}$ the number such that $(j,k,l_{jk})$ is a permutation of $\{1,2,3\}$ and by $\epsilon_{jk} = \pm 1$ the sign of this permutation. Then, for all $X \in TM$,*

$$\tilde{\Gamma}(X) = \sum_{i=1}^{3} \langle X, T_i\rangle \sum_{1\leq j<k\leq 3} \Gamma_{ij}^k \epsilon_{jk}(f_{l_{jk}} - T_{l_{jk}}) \cdot \omega$$

*where $\omega \in Cl(TM)$ is the area element of $M$.*

*Proof.* Keeping the notation introduced above, we note that

$$\underline{e}_j \cdot \underline{e}_k \cdot \underline{e}_{l_{jk}} = \epsilon_{jk}\ \omega \cdot \nu,$$

which yields

$$\underline{e}_j \cdot \underline{e}_k = -\epsilon_{jk}\ \omega \cdot \nu \cdot \underline{e}_{l_{jk}}.$$

Thus

$$\begin{aligned} T_j \cdot T_k + (f_k T_j - f_j T_k) \cdot \nu - f_j f_k &= -\epsilon_{jk}\ \omega \cdot \nu \cdot (T_{l_{jk}} + f_{l_{jk}} \nu) \\ &= \epsilon_{jk}(f_{l_{jk}} - T_{l_{jk}} \cdot \nu) \cdot \omega \end{aligned}$$

since $T_{l_{jk}} \cdot \nu = -\nu \cdot T_{l_{jk}}$, $T_{l_{jk}} \cdot \omega = -\omega \cdot T_{l_{jk}}$ and $\omega \cdot \nu = \nu \cdot \omega$. Switching the indices $j$ and $k$ we also get

$$\begin{aligned} T_k \cdot T_j + (f_j T_k - f_k T_j) \cdot \nu - f_k f_j &= \epsilon_{kj}(f_{l_{kj}} - T_{l_{kj}} \cdot \nu) \cdot \omega \\ &= -\epsilon_{jk}(f_{l_{jk}} - T_{l_{jk}} \cdot \nu) \cdot \omega \end{aligned}$$

since $\epsilon_{kj} = -\epsilon_{jk}$ and $l_{kj} = l_{jk}$. We deduce that

$$\frac{1}{2}\left(T_j \cdot T_k - T_k \cdot T_j\right) + (f_k T_j - f_j T_k) \cdot \nu = \epsilon_{jk}(f_{l_{jk}} - T_{l_{jk}} \cdot \nu) \cdot \omega.$$

The result is then a consequence of Lemma 5.1 together with the relation

$$\left(\tilde{\Gamma}(X) \cdot \psi\right)^* = \Gamma(X) \cdot \psi^*$$

and the first property in (50). □

5.4.1. *The metric Lie group $S^3$.* A spinor representation of a surface immersed in $S^3$ was already given in [16] (see also [3, 4]). We give here a spinor representation relying on the group structure; it appears that it coincides with the result in [16].

We regard the sphere $S^3$ as the set of the unit quaternions, with its natural group structure. The Lie algebra of $S^3$ identifies to $\mathbb{R}^3$, with the bracket $[X,Y] = 2X \times Y$ for all $X, Y \in \mathbb{R}^3$ ($\times$ is the usual cross product). By the Koszul formula (4), for all $X, Y \in \mathbb{R}^3$,

$$\Gamma(X)(Y) = X \times Y.$$

As a bivector, for all $X = X_1 e_1^o + X_2 e_2^o + X_3 e_3^o \in \mathbb{R}^3$,

$$\begin{aligned}\Gamma(X) &= \frac{1}{2}\left(e_1^o \cdot \Gamma(X)(e_1^o) + e_2^o \cdot \Gamma(X)(e_2^o) + e_3^o \cdot \Gamma(X)(e_3^o)\right)\\ &= X_1 e_2^o \cdot e_3^o + X_2 e_3^o \cdot e_1^o + X_3 e_1^o \cdot e_2^o\\ &= -X \cdot (e_1^o \cdot e_2^o \cdot e_3^o).\end{aligned}$$

Thus, if $\varphi \in \tilde{Q} \times_\rho Cl_3^0$ represents an immersion of an oriented surface $M$ in $S^3$ and if $\psi \in \Gamma(\Sigma M)$ is such that $\varphi = \psi^*$, then, for all $X \in TM$,

$$\begin{aligned}\Gamma(X) \cdot \varphi &= -X \cdot (e_1^o \cdot e_2^o \cdot e_3^o) \cdot \varphi\\ &= -X \cdot \omega \cdot \nu \cdot \varphi\\ &= (X \cdot \nu) \cdot \omega \cdot \varphi\\ &= (X \cdot \omega \cdot \psi)^*\end{aligned}$$

where $\omega$ is the area form of $M$, and $\nu$ is the vector normal to $M$ in $S^3$. Since $\varphi \in \Gamma(U\Sigma)$ is a solution of (24), $\psi \in \Gamma(\Sigma M)$ is a solution of

$$\nabla_X \psi = -\frac{1}{2} S(X) \cdot \psi + \frac{1}{2} X \cdot \omega \cdot \psi$$

and satisfies $|\psi| = 1$. Taking the trace, we get

$$\begin{aligned}D\psi &= e_1 \cdot \nabla_{e_1}\psi + e_2 \cdot \nabla_{e_2}\psi\\ &= H\psi - \omega \cdot \psi\end{aligned}$$

where $(e_1, e_2)$ is a positively oriented and orthonormal basis of $TM$. Now, setting $\overline{\psi} = \psi^+ - \psi^-$ and since $\omega \cdot \psi = -i\overline{\psi}$ (recall that $i\omega$ acts as the identity on $\Sigma^+ M$ and as -identity on $\Sigma^- M$), we get

$$D\psi = H\psi - i\overline{\psi},$$

which is also the spinor characterization given by Morel in [16].

5.4.2. *Surfaces in the 3-dimensional metric Lie groups $E(\kappa, \tau)$, $\tau \neq 0$.* We recover here a spinor characterization of immersions in the 3-dimensional homogeneous spaces $E(\kappa, \tau)$; this result was obtained by the second author in [19], using a characterization of immersions in these spaces by Daniel [7]. We give here an independent proof, and rather obtain the result of Daniel as a corollary.

The metric Lie group $E(\kappa, \tau)$, $\tau \neq 0$, is defined as follows: its Lie algebra is $\mathcal{G} = \mathbb{R}^3$, with the bracket defined on the vectors $e_1^o, e_2^o, e_3^o$ of the canonical basis by

$$[e_1^o, e_2^o] = 2\tau e_3^o, \qquad [e_2^o, e_3^o] = \sigma e_1^o, \qquad [e_3^o, e_1^o] = \sigma e_2^o$$

where $\sigma = \frac{\kappa}{2\tau}$. The metric on $\mathcal{G}$ is the canonical metric, ie the metric such that the basis $(e_1^o, e_2^o, e_3^o)$ is orthonormal. The Levi-Civita connection is then given by

$$\Gamma(X)(Y) = \{\tau(X - \langle X, e_3^o\rangle e_3^o) + (\sigma - \tau)\langle X, e_3^o\rangle e_3^o\} \times Y \tag{55}$$

for $X, Y \in \mathcal{G}$; see e.g. [7].

Let $S : TM \to TM$ be a symmetric operator. We assume that a vector field $T \in \Gamma(TM)$ and a function $f \in C^\infty(M, \mathbb{R})$ are given such that

$$|T|^2 + f^2 = 1, \tag{56}$$

$$\nabla_X T = f(S(X) - \tau JX) \tag{57}$$

and

$$df(X) = -\langle S(X) - \tau JX, T\rangle \tag{58}$$

for all $X \in TM$, where $J : TM \to TM$ denotes the rotation of angle $+\pi/2$ in the tangent planes.

**Theorem 5.** [19] *If $M$ is simply connected, the following two statements are equivalent:*

1. *There exists $\psi \in \Gamma(\Sigma M)$ such that $|\psi| = 1$ and*

$$\nabla_X \psi = -\frac{1}{2} S(X) \cdot \psi + \frac{1}{2}\left\{(2\tau - \sigma)\langle X, T\rangle\,(T - f) - \tau X\right\} \cdot \omega \cdot \psi \tag{59}$$

*for all $X \in TM$.*

2. *There exists an isometric immersion of $M$ into $E(\kappa, \tau)$, with shape operator $S$.*

*Proof.* We consider the trivial line bundle $E = \mathbb{R}\nu$, where $\nu$ is a unit section. The bundle $TM \oplus E$ is of rank 3, and is assumed to be oriented by the orientation of $TM$ and by $\nu$. We suppose that it is endowed with the natural product metric. Let us denote by $\times$ the natural cross product in the fibers. We set

$$\underline{e}_3 = T + f\nu,$$

and, for all $X, Y \in TM \oplus E$,

$$\Gamma(X)(Y) = \{\tau(X - \langle X, \underline{e}_3\rangle \underline{e}_3) + (\sigma - \tau)\langle X, \underline{e}_3\rangle \underline{e}_3\} \times Y. \tag{60}$$

Defining $B : TM \times TM \to E$ and its adjoint $B^* : TM \times E \to TM$ by

$$B(X, Y) = \langle S(X), Y\rangle\nu \qquad \text{and} \qquad B^*(X, \nu) = S(X) \tag{61}$$

for all $X, Y \in TM$, the equations (57) and (58) are equivalent to the single equation

$$\nabla_X \underline{e}_3 = \Gamma(X)(\underline{e}_3) - B(X, \underline{e}_3^T) + B^*(X, \underline{e}_3^N) \tag{62}$$

for all $X \in TM$, where $\nabla$ is the sum of the Levi-Civita connection on $TM$ and the trivial connection on $E$. This is (20) for $Z = \underline{e}_3$. We will need the following expression for $\Gamma$ :

**Lemma 5.3.** *For all $X \in TM$, the linear map $\Gamma(X) : TM \oplus E \to TM \oplus E$ defined by (60) is represented by the bivector*

$$\Gamma(X) = \{(2\tau - \sigma)\,\langle X, T\rangle\,(T \cdot \nu - f) - \tau X \cdot \nu\} \cdot \omega.$$

*Proof.* The linear map $\Gamma(X)$ is represented by the bivector

$$\Gamma(X)=\frac{1}{2}\left(\underline{e}_1\cdot\Gamma(X)(\underline{e}_1)+\underline{e}_2\cdot\Gamma(X)(\underline{e}_2)+\underline{e}_3\cdot\Gamma(X)(\underline{e}_3)\right)$$

where $\underline{e}_1,\underline{e}_2$ are such that $\underline{e}_1,\underline{e}_2,\underline{e}_3$ is a positively oriented and orthonormal basis of $TM\oplus E$ (see Lemma A.1); thus, a straightforward computation shows that $\Gamma(X)$ is represented by the bivector

$$\Gamma(X)=-\tau(X\times\underline{e}_3)\cdot\underline{e}_3+(\sigma-\tau)\langle X,\underline{e}_3\rangle\ \underline{e}_1\cdot\underline{e}_2. \tag{63}$$

The following formula may be checked by a direct computation: for $X,Y\in TM\oplus E$,

$$X\times Y=-\left(X\cdot Y+\langle X,Y\rangle\right)\underline{e}_1\cdot\underline{e}_2\cdot\underline{e}_3;$$

this gives

$$\begin{array}{rcl}(X\times\underline{e}_3)\cdot\underline{e}_3&=&-\left(X\cdot\underline{e}_3+\langle X,\underline{e}_3\rangle\right)\underline{e}_1\cdot\underline{e}_2\cdot\underline{e}_3\cdot\underline{e}_3\\&=&\left(X-\langle X,\underline{e}_3\rangle\underline{e}_3\right)\underline{e}_1\cdot\underline{e}_2\cdot\underline{e}_3\\&=&\left(X-\langle X,T\rangle\left(T+f\nu\right)\right)\cdot\omega\cdot\nu\\&=&\left(X\cdot\nu-\langle X,T\rangle\left(T\cdot\nu-f\right)\right)\cdot\omega.\end{array}$$

Moreover,

$$\begin{array}{rcl}\langle X,\underline{e}_3\rangle\ \underline{e}_1\cdot\underline{e}_2&=&\langle X,T\rangle\left(-\underline{e}_1\cdot\underline{e}_2\cdot\underline{e}_3\cdot\underline{e}_3\right)\\&=&\langle X,T\rangle\left(-\omega\cdot\nu\cdot(T+f\nu)\right)\\&=&-\langle X,T\rangle\left(T\cdot\nu-f\right)\cdot\omega.\end{array}$$

Plugging these two formulas in (63) we get the result. □

We deduce the following key lemma:

**Lemma 5.4.** *A spinor field $\varphi\in\Gamma(U\Sigma)$ solution of (24) is equivalent to a spinor field $\psi\in\Gamma(\Sigma M)$ solution of (59).*

*Proof.* We use the identification $\psi\in\Gamma(\Sigma M)\mapsto\psi^*\in\Gamma(\Sigma)$ described at the beginning of the section; we recall that, for all $X\in TM$,

$$\left(\nabla_X\psi\right)^*=\nabla_X(\psi^*)\qquad\text{and}\qquad(X\cdot\psi)^*=X\cdot\nu\cdot(\psi^*). \tag{64}$$

Thus, if $\varphi\in\Gamma(U\Sigma)$ is a solution of (24) and if $\psi\in\Gamma(\Sigma M)$ is such that $\psi^*=\varphi$, using (64) together with the formula

$$\sum_{j=1}^p e_j\cdot B(X,e_j)=\sum_{j=1}^p e_j\cdot\langle S(X),e_j\rangle\nu=S(X)\cdot\nu$$

and Lemma 5.3, we get:

$$\begin{array}{rcl}(\nabla_X\psi)^*&=&\nabla_X\varphi\\&=&-\frac{1}{2}S(X)\cdot\nu\cdot\varphi+\frac{1}{2}\left\{(2\tau-\sigma)\langle X,T\rangle\left(T\cdot\nu-f\right)-\tau X\cdot\nu\right\}\cdot\omega\cdot\varphi\\&=&\left(-\frac{1}{2}S(X)\cdot\psi+\frac{1}{2}\left\{(2\tau-\sigma)\langle X,T\rangle\left(T-f\right)-\tau X\right\}\cdot\omega\cdot\psi\right)^*.\end{array}$$

This gives (59). Reciprocally, if $\psi$ is a solution of (59), the spinor field $\varphi=\psi^*$ satisfies (24). This proves the lemma. □

Instead of $\psi \in \Gamma(\Sigma M)$ solution of (59) we may thus consider $\varphi \in \Gamma(U\Sigma)$ solution of (24). Theorem 5 will thus be a consequence of Theorem 1 if we can define a bundle isomorphism $f : TM \oplus E \to M \times \mathcal{G}$ such that (19) and (20) hold. Let us set

$$f(Z) = \langle\langle Z \cdot \varphi, \varphi \rangle\rangle.$$

We first observe that $f(\underline{e}_3)$ is constant: indeed, for all $X \in TM$,

$$\partial_X(f(\underline{e}_3)) = \langle\langle \nabla_X \underline{e}_3 \cdot \varphi, \varphi \rangle\rangle + \langle\langle \underline{e}_3 \cdot \nabla_X \varphi, \varphi \rangle\rangle + \langle\langle \underline{e}_3 \cdot \varphi, \nabla_X \varphi \rangle\rangle = 0$$

in view of (62), (24) and the computation in (29). Moreover, since $f$ preserves the norm of the vectors, $f(\underline{e}_3)$ is a unit vector. Replacing $\varphi$ by $\varphi \cdot a$ for some $a \in Spin(\mathcal{G})$ if necessary, we may thus assume that $f(\underline{e}_3) = e_3^o$. We now check (19): since the map $f$ is an orientation preserving isometry and using $f(\underline{e}_3) = e_3^o$, we have, for all $X, Y \in TM$,

$$\begin{aligned} f(\Gamma(X)(Y)) &= f\left(\{\tau(X - \langle X, \underline{e}_3 \rangle \underline{e}_3) + (\sigma - \tau)\langle X, \underline{e}_3 \rangle \underline{e}_3\} \times Y\right) \\ &= \{\tau(f(X) - \langle f(X), f(\underline{e}_3) \rangle f(\underline{e}_3)) + (\sigma - \tau)\langle f(X), f(\underline{e}_3) \rangle f(\underline{e}_3)\} \times f(Y) \\ &= \{\tau(f(X) - \langle f(X), e_3^o \rangle e_3^o) + (\sigma - \tau)\langle f(X), e_3^o \rangle e_3^o\} \times f(Y) \\ &= \Gamma(f(X))(f(Y)). \end{aligned}$$

Finally, the proof of (20) is very similar to the proof of this identity made in Section 5.1 for $G = \mathbb{R}^n$ : we only have to add the term involving $\Gamma$ which appears in the expression (24) of the covariant derivative of $\varphi$; we leave the details to the reader. □

**Remark 8.** *We also get an explicit representation formula: the immersion is given by the Darboux integral of $\xi : X \mapsto \langle\langle X \cdot \varphi, \varphi \rangle\rangle$, which may be written in terms of $\psi$ by the formula (54).*

We deduce the following result, first obtained by Daniel in [7] using the moving frame method:

**Corollary 2.** *If $S$, $T$, $f$, $\kappa$ and $\tau$ satisfy (56)-(58), the Gauss equation*

$$K = \det S + \tau^2 + \left(\kappa - 4\tau^2\right) f^2 \tag{65}$$

*and the Codazzi equation*

$$\nabla_X(SY) - \nabla_Y(SX) - S([X,Y]) = (\kappa - 4\tau^2) f(\langle Y, T \rangle X - \langle X, T \rangle Y), \tag{66}$$

*then there exists an isometric immersion of $M$ into $E(\kappa, \tau)$ with shape operator $S$. Moreover the immersion is unique up to a global isometry of $E(\kappa, \tau)$ preserving the orientations.*

*Proof.* The equations (65) and (66) are equivalent to the Gauss and Codazzi equations (33) and (35) where $B$ is defined by (61). They are thus exactly the integrability conditions for (24), and consequently also for (59). □

5.4.3. *Three-dimensional semi-direct products.* We consider here a semi-direct product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

if $(e_1^o, e_2^o, e_3^o)$ stands for the canonical basis of $\mathcal{G} = \mathbb{R}^2 \times \mathbb{R}$, the Lie bracket is given by

$$[e_1^o, e_2^o] = 0, \qquad [e_3^o, e_1^o] = ae_1^o + ce_2^o, \qquad [e_3^o, e_2^o] = be_1^o + de_2^o.$$

We equip $\mathbb{R}^2\rtimes_A\mathbb{R}$ with the left invariant metric such that $(e_1^o,e_2^o,e_3^o)$ is orthonormal. By the Koszul formula, we get

$$\nabla_{e_1^o}e_1^o=a\ e_3^o,\qquad \nabla_{e_1^o}e_2^o=\frac{b+c}{2}\ e_3^o,\qquad \nabla_{e_1^o}e_3^o=-a\ e_1^o-\frac{b+c}{2}\ e_2^o, \tag{67}$$

$$\nabla_{e_2^o}e_1^o=\frac{b+c}{2}\ e_3^o,\qquad \nabla_{e_2^o}e_2^o=d\ e_3^o,\qquad \nabla_{e_2^o}e_3^o=-\frac{b+c}{2}\ e_1^o-d\ e_2^o \tag{68}$$

and

$$\nabla_{e_3^o}e_1^o=\frac{c-b}{2}\ e_2^o,\qquad \nabla_{e_3^o}e_2^o=\frac{b-c}{2}\ e_1^o,\qquad \nabla_{e_3^o}e_3^o=0, \tag{69}$$

and deduce

$$\Gamma(X)=\left(aX_1+\frac{b+c}{2}X_2\right)e_1^o\cdot e_3^o+\left(\frac{b+c}{2}X_1+dX_2\right)e_2^o\cdot e_3^o+\frac{c-b}{2}X_3e_1^o\cdot e_2^o$$

for all $X\in\mathcal{G}$. We first assume that $M$ is an oriented surface in $G=\mathbb{R}^2\rtimes_A\mathbb{R}$. Recalling that

$$(T_j+f_j\nu)\cdot(T_k+f_k\nu)=\epsilon_{jk}(f_{l_{jk}}-T_{l_{jk}}\cdot\nu)\cdot\omega$$

(see the proof of Lemma 5.2), we obtain

$$\begin{aligned}\Gamma(X)\quad &=\quad -\left(aX_1+\frac{b+c}{2}X_2\right)(f_2-T_2\cdot\nu)\cdot\omega\\ &\qquad+\left(\frac{b+c}{2}X_1+dX_2\right)(f_1-T_1\cdot\nu)\cdot\omega+\frac{c-b}{2}X_3(f_3-T_3\cdot\nu)\cdot\omega\end{aligned}$$

and

$$\begin{aligned}\tilde{\Gamma}(X)\quad &=\quad -\left(aX_1+\frac{b+c}{2}X_2\right)(f_2-T_2)\cdot\omega\\ &\qquad+\left(\frac{b+c}{2}X_1+dX_2\right)(f_1-T_1)\cdot\omega+\frac{c-b}{2}X_3(f_3-T_3)\cdot\omega.\end{aligned} \tag{70}$$

Conversely, we consider an oriented Riemannian surface $M$, and a symmetric operator $S:TM\to TM$. We suppose that there exist tangent vectors fields $T_i\in\Gamma(TM)$ and functions $f_i\in C^\infty(M)$ for $1\leqslant i\leqslant 3$ satisfying

$$\langle T_i,T_j\rangle+f_if_j=\delta_i^j \tag{71}$$

for all $1\leq i,j\leq 3$, and the equations (22) and (23) in Remark 2, with the coefficients $\Gamma_{ij}^k$ given by (67)-(69). Theorem 4 then yields the following result:

**Theorem 6.** *If $M$ is simply connected, the following two statements are equivalent:*

1. *there exists an isometric immersion of $M$ into $\mathbb{R}^2\rtimes_A\mathbb{R}$ with shape operator $S$;*
2. *there exists $\psi\in\Gamma(\Sigma M)$ such that $|\psi|=1$ and*

$$\nabla_X\psi=-\frac{1}{2}S(X)\cdot\psi+\frac{1}{2}\tilde{\Gamma}(X)\cdot\psi \tag{72}$$

*for all $X\in TM$.*

**The metric Lie group $Sol_3$.** Now, we describe the special case of a surface in $Sol_3$ : this achieves the spinor representation of immersions of surfaces into 3-dimensional Riemannian homogeneous spaces [19].

Let us recall that $Sol_3$ is the only metric Lie group whose isometry group is 3-dimensional. It is defined as follows: its Lie algebra is $\mathcal{G}=\mathbb{R}^3$, with the bracket defined on the canonical basis $(e_1^o,e_2^o,e_3^o)$ by

$$[e_1^o,e_2^o]=0,\qquad [e_2^o,e_3^o]=-e_2^o,\qquad [e_3^o,e_1^o]=-e_1^o.$$

This is the semi-direct product $\mathbb{R}^2\rtimes_A\mathbb{R}$ with $a=-1,b=c=0,d=1$. The metric on $\mathcal{G}$ is the canonical metric, i.e., the metric such that the basis $(e_1^o,e_2^o,e_3^o)$ is orthonormal. By the formulas (67)-(69), the Levi-Civita connection is then such that

$$\Gamma_{11}^3=-\Gamma_{13}^1=-1,\qquad \Gamma_{22}^3=-\Gamma_{23}^2=1 \tag{73}$$

and $\Gamma_{ij}^k=0$ for the other indices.

Let us consider an oriented Riemannian surface $M$, and a symmetric operator $S:TM\to TM$. We suppose that there exist tangent vectors fields $T_i\in\Gamma(TM)$ and functions $f_i\in C^\infty(M)$ for $1\leqslant i\leqslant 3$ satisfying

$$\langle T_i,T_j\rangle+f_if_j=\delta_i^j \tag{74}$$

for all $1\le i,j\le 3$, and, for all $X\in TM$,

$$\begin{aligned}\nabla_XT_i&=(-1)^i\langle X,T_i\rangle T_3+f_iS(X),\\ df_i(X)&=(-1)^i\langle X,T_i\rangle f_3-\langle SX,T_i\rangle\end{aligned} \tag{75}$$

for $1\leqslant i\leqslant 2$,

$$\begin{aligned}\nabla_XT_3&=\sum_{j=1}^2(-1)^{j+1}\langle X,T_j\rangle T_j+f_3S(X),\\ df_3(X)&=\sum_{j=1}^2(-1)^{j+1}\langle X,T_j\rangle f_j-\langle S(X),T_3\rangle.\end{aligned} \tag{76}$$

The equations (75) and (76) are the equations (22) and (23) in Remark 2, with the coefficients $\Gamma_{ij}^k$ given by (73). According to (70) with $a=-1,b=c=0$ and $d=1$ we set

$$\tilde{\Gamma}(X)=\{\langle X,T_1\rangle(f_2-T_2)+\langle X,T_2\rangle(f_1-T_1)\}\cdot\omega \tag{77}$$

for all $X\in TM$. Theorem 6 then gives a spinor characterization of an immersion in $Sol_3$.

As a corollary, we obtain a new proof of a result by Lodovici [11] concerning existence and uniqueness of isometric immersions in $Sol_3$, since equation (72) is solvable if and only if the equations of Gauss and Codazzi hold (see Section 4).

**$\mathbb{H}^2\times\mathbb{R}$ as a metric Lie group.** Finally, viewing $\mathbb{H}^2\times\mathbb{R}$ as a metric Lie group, we obtain a new spinor characterization of an immersion in $\mathbb{H}^2\times\mathbb{R}$ which differs from [19] where the product point of view was used.
We recall that $\mathbb{H}^2\times\mathbb{R}$ is the semi-direct product $\mathbb{R}^2\rtimes_A\mathbb{R}$ with $a=1,b=c=d=0$.

The metric on $\mathcal{G}$ is the canonical metric, i.e., the metric such that the basis $(e_1^o, e_2^o, e_3^o)$ is orthonormal. Lie bracket is given by

$$[e_1^o, e_2^o] = 0, \qquad [e_3^o, e_1^o] = e_1^o, \qquad [e_3^o, e_2^o] = 0.$$

By the formulas (67)-(69), the Levi-Civita connection is then such that

$$\Gamma_{11}^3 = -\Gamma_{13}^1 = 1 \tag{78}$$

and $\Gamma_{ij}^k = 0$ for the other indices.
Let us consider an oriented Riemannian surface $M$, and a symmetric operator $S : TM \to TM$. We suppose that there exist tangent vectors fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$ for $1 \leqslant i \leqslant 3$ satisfying

$$\langle T_i, T_j\rangle + f_i f_j = \delta_i^j \tag{79}$$

for all $1 \leq i, j \leq 3$, and, for all $X \in TM$,

$$\begin{aligned} \nabla_X T_1 &= \langle X, T_1\rangle T_3 + f_1 S(X), \\ df_1(X) &= \langle X, T_1\rangle f_3 - \langle SX, T_1\rangle, \end{aligned} \tag{80}$$

$$\begin{aligned} \nabla_X T_2 &= f_2 S(X), \\ df_2(X) &= -\langle SX, T_2\rangle, \end{aligned} \tag{81}$$

$$\begin{aligned} \nabla_X T_3 &= -\langle X, T_3\rangle T_1 + f_3 S(X), \\ df_1(X) &= -\langle X, T_3\rangle f_1 - \langle SX, T_3\rangle. \end{aligned} \tag{82}$$

With these identities and according to (70) with $a = 1$,$b = c = d = 0$, we set

$$\tilde{\Gamma}(X) = -\langle X, T_1\rangle (f_2 - T_2)\cdot\omega \tag{83}$$

for all $X \in TM$. Theorem 6 then gives a spinor characterization of an immersion in $\mathbb{H}^2 \times \mathbb{R}$.

5.5. **CMC-surfaces in a 3-dimensional metric Lie group.** The aim here is to show that the representation formula for CMC-surfaces in a 3-dimensional metric Lie group by Meeks, Mira, Perez and Ros [13, Theorem 3.12] may be obtained as a consequence of the general representation formula in Theorem 1. For sake of brevity we assume that the group $G$ is unimodular and only give the principal arguments, without details. Under this hypothesis, there exists an orthonormal basis $(e_1^o, e_2^o, e_3^o)$ of the Lie algebra $\mathcal{G}$ and constants $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ such that the Levi-Civita connection of $G$ is given by

$$\begin{aligned} \Gamma(X)(e_1^o) &:= \nabla_X e_1^o = X_3\mu_3 e_2^o - X_2\mu_2 e_3^o, \\ \Gamma(X)(e_2^o) &:= \nabla_X e_2^o = -X_3\mu_3 e_1^o + X_1\mu_1 e_3^o, \\ \Gamma(X)(e_3^o) &:= \nabla_X e_3^o = X_2\mu_2 e_1^o - X_1\mu_1 e_2^o \end{aligned}$$

(see e.g. [13, Section 2.6]), i.e.

$$\begin{aligned} \Gamma(X) &= \frac{1}{2}\left(e_1^o\cdot\Gamma(X)(e_1^o) + e_2^o\cdot\Gamma(X)(e_2^o) + e_3^o\cdot\Gamma(X)(e_3^o)\right) \\ &= X_1\mu_1 e_2^o\cdot e_3^o + X_2\mu_2 e_3^o\cdot e_1^o + X_3\mu_3 e_1^o\cdot e_2^o \end{aligned} \tag{84}$$

for all $X \in \mathcal{G}$. Following [13] we introduce the $H$-potential of the group $G$

$$R(g) = H\left(1+|g|^2\right)^2 - \frac{i}{2}\left(\mu_1|1-g^2|^2 + \mu_2|1+g^2|^2 + 4\mu_3|g|^4\right) \tag{85}$$

for all $g \in \overline{\mathbb{C}}$. The importance of this quantity appears in the following lemma, which will permit to express the right-hand side of the Dirac equation (27):

**Lemma 5.5.** *Let us consider a positively oriented and orthonormal basis* $e_1, e_2, \nu$ *of* $\mathcal{G}$ *and set, for* $\nu = \nu_1 e_1^o + \nu_2 e_2^o + \nu_3 e_3^o$,

$$T(\nu) = \mu_1 \nu_1 e_1^o + \mu_2 \nu_2 e_2^o + \mu_3 \nu_3 e_3^o, \tag{86}$$

$A = \frac{1}{2}\langle e_2, T(\nu)\rangle$ *and* $B = -\frac{1}{2}\langle e_1, T(\nu)\rangle$. *Then, if*

$$g = \frac{\nu_1 + i\nu_2}{1 + \nu_3}$$

*is the stereographic projection of* $\nu \in S^2$ *with respect to the south pole* $-e_3^o$ *of* $S^2$, *we have*

$$\begin{aligned} H\nu + \frac{1}{2}\left(e_1 \cdot \Gamma(e_1) + e_2 \cdot \Gamma(e_2)\right) &= \frac{1}{(1+|g|^2)^2}\left(\Re e\ R(g) - \Im m\ R(g)\ e_1 \cdot e_2\right) \cdot \nu \\ &\quad + Ae_1 + Be_2. \end{aligned}$$

*Proof.* For $i \in \{1, 2, 3\}$, let us denote by

$$p(e_i^o) := \langle e_i^o, e_1\rangle e_1 + \langle e_i^o, e_2\rangle e_2$$

the orthogonal projection of the vector $e_i^o$ onto the plane generated by $e_1$ and $e_2$. By (84) we have

$$e_1 \cdot \Gamma(e_1) + e_2 \cdot \Gamma(e_2) = \mu_1\ p(e_1^o) \cdot e_2^o \cdot e_3^o + \mu_2\ p(e_2^o) \cdot e_3^o \cdot e_1^o + \mu_3\ p(e_3^o) \cdot e_1^o \cdot e_2^o.$$

The proof is then a direct and long computation using that $p(e_i^o) = e_i^o - \langle e_i^o, \nu\rangle\nu$ together with the formulas

$$\nu_1 = \frac{2\ \Re e\ g}{1 + |g|^2}, \qquad \nu_2 = \frac{2\ \Im m\ g}{1 + |g|^2}, \qquad \nu_3 = \frac{1 - |g|^2}{1 + |g|^2}. \tag{87}$$

□

We consider the Clifford map

$$\begin{aligned} \mathcal{G} &\rightarrow \mathbb{H}(2) \\ x_1 e_1^o + x_2 e_2^o + x_3 e_3^o &\mapsto \begin{pmatrix} ix_3 + j(x_1 - ix_2) & 0 \\ 0 & -ix_3 - j(x_1 - ix_2) \end{pmatrix} \end{aligned} \tag{88}$$

which identifies $\mathcal{G}$ to the imaginary quaternions so that

(89)

$$e_1^o \simeq \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} \simeq j, \qquad e_2^o \simeq \begin{pmatrix} -ji & 0 \\ 0 & ji \end{pmatrix} \simeq -ji, \qquad e_3^o \simeq \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \simeq i.$$

It identifies $Cl(\mathcal{G})$ to the set of matrices

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{H} \right\} \tag{90}$$

and $Spin(\mathcal{G})$ to the group of unit quaternions

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad a \in \mathbb{H},\ |a| = 1 \right\} \simeq \{a \in \mathbb{H},\ |a| = 1\}.$$

We choose a conformal parameter $z = x + iy$ of the surface, and denote by $\mu$ the real function such that the metric is $\mu^2(dx^2 + dy^2)$. In a spinorial frame above the orthonormal frame $e_1 = \frac{1}{\mu}\partial_x$, $e_2 = \frac{1}{\mu}\partial_y$, the spinor field $\varphi$ is represented by $[\varphi] = z_1 + jz_2$ where $z_1, z_2 \in \mathbb{C}$ are such that $|z_1|^2 + |z_2|^2 = 1$.

**Lemma 5.6.** *The Dirac equation (27) is equivalent to the system*

$$\frac{1}{\sqrt{\mu}}\partial_{\overline{z}}\left(\sqrt{\mu}\ \overline{z_1}\right) = i\frac{\mu}{2}\frac{\overline{R(g)}}{(1+|g|^2)^2}\overline{z_2}+\frac{\mu}{2}(A+iB)\overline{z_1} \tag{91}$$

$$\frac{1}{\sqrt{\mu}}\partial_{\overline{z}}\left(\sqrt{\mu}\ z_2\right) = -i\frac{\mu}{2}\frac{\overline{R(g)}}{(1+|g|^2)^2}z_1+\frac{\mu}{2}(A+iB)z_2. \tag{92}$$

*Moreover, the $\mathcal{G}-$valued 1-form $\xi$ in Theorem 1 is*

$$\begin{aligned}\xi(X) &= i\left\{2x_1\ \Im m(z_1\overline{z_2})-2x_2\ \Re e(z_1\overline{z_2})+x_3\left(|z_1|^2-|z_2|^2\right)\right\}\\ &\quad +j\left\{x_1(z_1^2+z_2^2)-ix_2(z_1^2-z_2^2)-2ix_3z_1z_2\right\}\end{aligned} \tag{93}$$

*for all $X=x_1e_1+x_2e_2+x_3\nu\in TM\oplus E$.*

*Proof.* We use here the identification $\psi\in\Gamma(\Sigma M)\mapsto\psi^*\in\Gamma(\Sigma)$ satisfying the properties (64): according to Lemma 5.5, the spinor field $\psi\in\Gamma(\Sigma M)$ such that $\psi^*=\varphi$ is solution of

$$D\psi=\frac{1}{(1+|g|^2)^2}\left(\Re e\ R(g)-\Im m\ R(g)\ e_1\cdot e_2\right)\cdot\psi+(Ae_1+Be_2)\cdot\psi. \tag{94}$$

We identify $Cl_2$ to $\mathbb{H}$ using the Clifford map

$$\begin{array}{ccc}\mathbb{R}^2 & \to & \mathbb{H}\\ (x_1,x_2) & \mapsto & j(x_1-ix_2)\end{array} \tag{95}$$

so that, in the fixed spinorial frame above $e_1=\frac{1}{\mu}\partial_x$, $e_2=\frac{1}{\mu}\partial_y$,

$$[e_1]=j,\qquad [e_2]=-ji,\qquad [e_1\cdot e_2]=i.$$

Using moreover that

$$[\nabla_{\partial x}\psi]=\partial_x[\psi]-\frac{i}{2\mu}\partial_y\mu\ [\psi]\qquad [\nabla_{\partial y}\psi]=\partial_y[\psi]+\frac{i}{2\mu}\partial_x\mu\ [\psi]$$

(by (13), and the computation of the Christoffel symbols), the left-hand side of (94) is

$$[D\psi]=\frac{1}{\mu}j\left\{\partial_x[\psi]-\frac{i}{2\mu}\partial_y\mu\ [\psi]\right\}-\frac{1}{\mu}ji\left\{\partial_y[\psi]+\frac{i}{2\mu}\partial_x\mu\ [\psi]\right\}$$

whereas the right-hand side is

$$\left(\frac{\overline{R(g)}}{(1+|g|^2)^2}+j(A-iB)\right)[\psi].$$

We finally need to precise the identification $\psi\mapsto\psi^*$ : in spinorial frames above $e_1,e_2$ and $e_1,e_2,\nu$, since the second property in (64) is required and using the Clifford maps (88) and (95), it is not difficult to see that the map $\psi\mapsto\psi^*$ corresponds to the map

$$u+jv\mapsto\left(\begin{array}{cc}u+jiv & 0\\ 0 & u+jiv\end{array}\right);$$

$\psi$ is thus represented by the quaternion $[\psi]=z_1-jiz_2$. Direct computations then give the system (91)-(92).

Expression (93) also follows from a direct computation: we have, in $Cl_3$,

$$\begin{aligned}\xi(X) &= \tau[\varphi][X][\varphi]\\ &\simeq (\overline{z_1}-j\overline{z_2})(ix_3+j(x_1-ix_2))(z_1+jz_2),\end{aligned}$$

which easily gives the result. $\square$

We set

$$g = i\frac{\overline{z_2}}{z_1}, \qquad f = -2\mu z_1^2. \tag{96}$$

The function $g$ is the left invariant Gauss map of the surface, stereografically projected with respect to the south pole of $S^2$, since

$$\nu = i\left(|z_1|^2 - |z_2|^2\right) - 2jiz_1z_2$$

is a unit vector normal to the immersion ($x_1 = x_2 = 0$ and $x_3 = 1$ in (93)) and

$$\frac{\nu_1 + i\nu_2}{1+\nu_3} = \frac{2i\ \overline{z_1}\ \overline{z_2}}{1+|z_1|^2-|z_2|^2} = \frac{2i\ \overline{z_1}\ \overline{z_2}}{2|z_1|^2} = i\frac{\overline{z_2}}{z_1}.$$

Direct computations then show that equations (91)-(92) are equivalent to

$$f = 4\frac{\partial_z g}{R(g)} \tag{97}$$

and

$$\frac{\partial_{\overline{z}} f}{f} = -\frac{2}{1+|g|^2}\partial_{\overline{z}}\overline{g}\ g + \mu(A+iB), \tag{98}$$

and that (93) reads

$$\xi = \Re e\left(\ \frac{1}{2}f(\overline{g}^2-1)dz,\ \frac{i}{2}f(\overline{g}^2+1)dz,\ f\overline{g}dz\right) \tag{99}$$

in $(e_1^o, e_2^o, e_3^o)$ (recall (89)). This last formula is the Weierstrass-type representation given in [13, Theorem 3.15]. Using that

$$A = \left\langle \xi\left(\frac{\partial_y}{\mu}\right), T(\nu)\right\rangle \qquad \text{and} \qquad B = -\left\langle \xi\left(\frac{\partial_x}{\mu}\right), T(\nu)\right\rangle$$

(Lemma 5.5) together with (99) and (86) we get that

$$A+iB = -\frac{i}{4\mu}\overline{f}\left(\mu_1\nu_1(g^2-1) - i\mu_2\nu_2(g^2+1) + 2\mu_3\nu_3 g\right). \tag{100}$$

Differentiating (97) with respect to $\overline{z}$ and using (98) together with (100) and (87) we see by a further computation that $g$ satisfies

$$g_{z\overline{z}} = \frac{R_g}{R}g_z g_{\overline{z}} + \left(\frac{R_{\overline{g}}}{R} - \frac{\overline{R_g}}{\overline{R}}\right)|g_z|^2, \tag{101}$$

which is the structure equation for the left invariant Gauss map in [13, Theorem 3.15].

## Appendix A. Skew-symmetric operators and bivectors

We consider $\mathbb{R}^n$ endowed with its canonical scalar product. A skew-symmetric operator $u : \mathbb{R}^n \to \mathbb{R}^n$ naturally identifies to a bivector $\underline{u} \in \Lambda^2\mathbb{R}^n$, which may in turn be regarded as belonging to the Clifford algebra $Cl_n(\mathbb{R})$. We precise here the relations between the Clifford product in $Cl_n(\mathbb{R})$ and the composition of endomorphisms. If $a$ and $b$ belong to the Clifford algebra $Cl_n(\mathbb{R})$, we set

$$[a,b] = \frac{1}{2}\left(a\cdot b - b\cdot a\right),$$

where the dot $\cdot$ is the Clifford product. We denote by $(e_1,\ldots,e_n)$ the canonical basis of $\mathbb{R}^n$.

**Lemma A.1.** *Let $u:\mathbb{R}^n\to\mathbb{R}^n$ be a skew-symmetric operator. Then the bivector*

$$\underline{u}=\frac{1}{2}\sum_{j=1}^{n}e_j\cdot u(e_j)\quad\in\quad\Lambda^2\mathbb{R}^n\subset Cl_n(\mathbb{R})\tag{102}$$

*represents $u$, and, for all $\xi\in\mathbb{R}^n$,*

$$[\underline{u},\xi]=u(\xi).$$

*In the paper, and for sake of simplicity, we will use the same letter $u$ to denote $\underline{u}$.*

*Proof.* For $i<j$, we consider the linear map

$$u:\qquad e_i\mapsto e_j,\qquad e_j\mapsto -e_i,\qquad e_k\mapsto 0\quad\text{if}\quad k\neq i,j;$$

it is skew-symmetric and corresponds to the bivector $e_i\wedge e_j\in\Lambda^2\mathbb{R}^n$; it is thus naturally represented by $\underline{u}=e_i\cdot e_j=\frac{1}{2}\left(e_i\cdot e_j-e_j\cdot e_i\right)$, which is (102). We then compute, for $k=1,\ldots,n$,

$$[\underline{u},e_k]=\frac{1}{2}\left(e_i\cdot e_j\cdot e_k-e_k\cdot e_i\cdot e_j\right)$$

and easily get

$$[\underline{u},e_k]=e_j\quad\text{if}\quad k=i,\quad -e_i\quad\text{if}\quad k=j,\quad 0\quad\text{if}\quad k\neq i,j.$$

The result follows by linearity. □

**Lemma A.2.** *Let $u:\mathbb{R}^n\to\mathbb{R}^n$ and $v:\mathbb{R}^n\to\mathbb{R}^n$ be two skew-symmetric operators, represented in $Cl_n(\mathbb{R})$ by*

$$u=\frac{1}{2}\sum_{j=1}^{n}e_j\cdot u(e_j)\qquad\textit{and}\qquad v=\frac{1}{2}\sum_{j=1}^{n}e_j\cdot v(e_j)$$

*respectively. Then $[u,v]\in\Lambda^2\mathbb{R}^n\subset Cl_n(\mathbb{R})$ represents $u\circ v-v\circ u$.*

*Proof.* For $\xi\in\mathbb{R}^n$, the Jacobi equation yields

$$[[u,v],\xi]=[u,[v,\xi]]-[v,[u,\xi]].$$

Thus, using Lemma A.1 repeatedly, $[u,v]$ represents the map

$$\begin{array}{rcl}\xi\quad\mapsto\quad[[u,v],\xi]&=&[u,[v,\xi]]-[v,[u,\xi]]\\&=&[u,v(\xi)]-[v,u(\xi)]\\&=&(u\circ v-v\circ u)(\xi),\end{array}$$

and the result follows. □

We now assume that $\mathbb{R}^n=\mathbb{R}^p\oplus\mathbb{R}^q$, $p+q=n$.

**Lemma A.3.** *Let us consider a linear map $u:\mathbb{R}^p\to\mathbb{R}^q$ and its adjoint $u^*:\mathbb{R}^q\to\mathbb{R}^p$. Then the bivector*

$$\underline{u}=\sum_{j=1}^{p}e_j\cdot u(e_j)\quad\in\quad\Lambda^2\mathbb{R}^n\subset Cl_n(\mathbb{R})$$

*represents*

$$\left(\begin{array}{cc}0&-u^*\\u&0\end{array}\right):\qquad\mathbb{R}^p\oplus\mathbb{R}^q\to\mathbb{R}^p\oplus\mathbb{R}^q,$$

*we have*

$$\underline{u}=\frac{1}{2}\left(\sum_{j=1}^{p}e_j\cdot u(e_j)+\sum_{j=p+1}^{n}e_j\cdot(-u^*(e_j))\right) \tag{103}$$

*and, for all* $\xi=\xi_p+\xi_q\in\mathbb{R}^n$,

$$[\underline{u},\xi]=u(\xi_p)-u^*(\xi_q).$$

*As above, we will simply denote* $\underline{u}$ *by* $u$.

*Proof.* In view of Lemma A.1, $\underline{u}$ represents the linear map $\xi\mapsto[\underline{u},\xi]$. We compute, for $\xi\in\mathbb{R}^p$,

$$\begin{aligned}
[\underline{u},\xi] &= \frac{1}{2}\left(\sum_{j=1}^{p}e_j\cdot u(e_j)\cdot\xi-\xi\cdot\sum_{j=1}^{p}e_j\cdot u(e_j)\right)\\
&= -\frac{1}{2}\sum_{j=1}^{p}(e_j\cdot\xi+\xi\cdot e_j)\cdot u(e_j)\\
&= \sum_{j=1}^{p}\langle\xi,e_j\rangle\ u(e_j)\\
&= u(\xi),
\end{aligned}$$

and, for $\xi\in\mathbb{R}^q$,

$$\begin{aligned}
[\underline{u},\xi] &= \frac{1}{2}\left(\sum_{j=1}^{p}e_j\cdot u(e_j)\cdot\xi-\xi\cdot\sum_{j=1}^{p}e_j\cdot u(e_j)\right)\\
&= \frac{1}{2}\sum_{j=1}^{p}e_j\cdot(u(e_j)\cdot\xi+\xi\cdot u(e_j))\\
&= -\sum_{j=1}^{p}e_j\ \langle u(e_j),\xi\rangle\\
&= -\sum_{j=1}^{p}e_j\ \langle e_j,u^*(\xi)\rangle\\
&= -u^*(\xi).
\end{aligned}$$

Finally,

$$\underline{u}=\sum_{j=1}^{p}e_j\cdot u(e_j)=\frac{1}{2}\left(\sum_{j=1}^{p}e_j\cdot u(e_j)+\sum_{j=1}^{p}-u(e_j)\cdot e_j\right)$$

with

$$\begin{aligned}\sum_{j=1}^{p} -u(e_j)\cdot e_j &= -\sum_{i=p+1}^{p+q}\sum_{j=1}^{p}\langle u(e_j),e_i\rangle\ e_i\cdot e_j\\ &= \sum_{i=p+1}^{p+q} e_i\cdot\left(-\sum_{j=1}^{p}\langle e_j,u^*(e_i)\rangle\ e_j\right)\\ &= \sum_{i=p+1}^{p+q} e_i\cdot(-u^*(e_i)),\end{aligned}$$

which gives (103). □

**Lemma A.4.** *Let us consider two linear maps $u:\mathbb{R}^p\to\mathbb{R}^q$ and $v:\mathbb{R}^n\to\mathbb{R}^n$, with $v$ skew-symmetric, and the associated bivectors*

$$u=\sum_{j=1}^{p} e_j\cdot u(e_j),\qquad v=\frac{1}{2}\sum_{j=1}^{n} e_j\cdot v(e_j).$$

*Then $[u,v]\in\Lambda^2\mathbb{R}^n$ represents the map*

$$\xi=\xi_p+\xi_q\mapsto -u^*(v(\xi)_q)+v(u^*(\xi_q))+u(v(\xi)_p)-v(u(\xi_p)),$$

*where the sub-indices $p$ and $q$ mean that we take the components of the vectors in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. In view of Lemma A.1, this may also be written in the form*

$$[[u,v],\xi]=-u^*(v(\xi)_q)+v(u^*(\xi_q))+u(v(\xi)_p)-v(u(\xi_p))$$

*for all $\xi\in\mathbb{R}^n$.*

*Proof.* From Lemmas A.2 and A.3, the bivector $[u,v]\in\Lambda^2\mathbb{R}^n$ represents

$$\begin{pmatrix}0 & -u^*\\ u & 0\end{pmatrix}\circ v - v\circ\begin{pmatrix}0 & -u^*\\ u & 0\end{pmatrix},$$

that is the map

$$\begin{aligned}\xi\quad\mapsto&\quad\begin{pmatrix}0 & -u^*\\ u & 0\end{pmatrix}\begin{pmatrix}v(\xi)_p\\ v(\xi)_q\end{pmatrix}-v\begin{pmatrix}0 & -u^*\\ u & 0\end{pmatrix}\begin{pmatrix}\xi_p\\ \xi_q\end{pmatrix}\\ &=\begin{pmatrix}-u^*(v(\xi)_q)+v(u^*(\xi_q))\\ u(v(\xi)_p)-v(u(\xi_p))\end{pmatrix},\end{aligned}$$

which gives the result. □

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## References

[1] C. Bär, *Extrinsic bounds for the eigenvalues of the Dirac operator*, Ann. Glob. Anal. Geom. **16** (1998) 573-596.

[2] P. Bayard, *On the spinorial representation of spacelike surfaces into 4-dimensional Minkowski space*, J. Geom. Phys. **74** (2013) 289-313.

[3] P. Bayard, M.A. Lawn & J. Roth, *Spinorial representation of surfaces in four-dimensional Space Forms*, Ann. Glob. Anal. Geom. **44:4** (2013) 433-453.

[4] P. Bayard, M.A. Lawn & J. Roth, *Spinorial representation of submanifolds in Riemannian space forms*, arXiv:1602.02919 (2016) 1-24.

[5] P. Bayard & V. Patty, *Spinor representation of Lorentzian surfaces in* $\mathbb{R}^{2,2}$, J. Geom. Phys. **95** (2015) 74-75.
[6] D.A. Berdinskii & I.A. Taimanov, *Surfaces in three-dimensional Lie groups*, Sibirsk Mat. Zh. **46:6** (2005) 1248-1264.
[7] B. Daniel, *Isometric immersions into 3-dimensional homogeneous manifolds*, Comment. Math. Helv. **82** (2007) 87-131.
[8] T. Friedrich, *On the spinor representation of surfaces in Euclidean* 3*-space*, J. Geom. Phys. **28** (1998) 143-157.
[9] M.-A. Lawn, *A spinorial representation for Lorentzian surfaces in* $\mathbb{R}^{2,1}$, J. Geom. Phys. **58:6** (2008) 683-700.
[10] M.-A. Lawn & J. Roth, *Spinorial characterization of surfaces in pseudo-Riemannian space forms*, Math. Phys. Anal. and Geom. **14:3** (2011) 185-195.
[11] S.D. Lodovici, *An isometric immersion theorem in* $Sol_3$, Matemática Contemporânea **30** (2006) 109-123.
[12] P. Lounesto, *Clifford Algebras and Spinors*, London Mathematical Society, Lecture Note Series **286**, Cambridge University Press (2001).
[13] W. Meeks & J. Perez, *Constant mean curvature surfaces in metric Lie groups*, in "Geometric Analysis: Partial Differential Equations and Surfaces", Contemp. Math. **570** (2012) 25-110.
[14] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. **21** (1976) 293-329.
[15] J. Milnor, *Remarks concerning spin manifolds*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton Univ. Press (1965) 55-62.
[16] B. Morel, *Surfaces in* $\mathbb{S}^3$ *and* $\mathbb{H}^3$ *via spinors*, Actes du séminaire de théorie spectrale, Institut Fourier, Grenoble, **23** (2005) 9-22.
[17] V. Patty, *A generalized Weierstras representation of Lorentzian surfaces in* $\mathbb{R}^{2,2}$ *and applications*, Int. J. Geometric methods in modern Physics **13:6** (2016) 26 pages.
[18] P. Piccione & D.V. Tausk, *An existence theorem for G-structure preserving affine immersions*, Indiana Univ. Math. J. **57:3** (2008) 1431-1465.
[19] J. Roth, *Spinorial characterizations of surfaces into 3-homogeneous manifolds*, J. Geom. Phys. **60** (2010) 1045-1061.