# A NEW RESULT ABOUT ALMOST UMBILICAL HYPERSURFACES OF REAL SPACE FORMS

JULIEN ROTH

ABSTRACT. In this short note, we prove that an almost umbilical compact hypersurface of a real space form with almost Codazzi umbilicity tensor is embedded, diffeomorphic and quasi-isometric to a round sphere. Then, we derive a new characterization of geodesic spheres in space forms.

CONTENTS



## 1. Introduction

Let $(M^n, g)$ be a connected and oriented compact Riemannian manifold isometrically immersed into the simply-connected real space form $\mathbb{M}^{n+1}(\delta)$ of constant curvature $\delta$. Let $B$ be the second fundamental form of the hypersurface and $H$ its mean curvature. Since we consider only hypersurfaces, we take $B$ as the real-valued second fundamental form. We denote by $\tau = B - Hg$ the traceless part of the second fundamental form, also called umbilicity tensor. We say that $M$ is totally umbilical if $\tau = 0$.

It is a well-known fact that a compact (without boundary) totally umbilical hypersurface of a simply connected real space form is a geodesic sphere. In the present note, we will investigate the natural question of the stability of this rigidity result. In other words, if a compact hypersurface of a real space form is almost umbilical, is this hypersurface close to a sphere? In what sense?

Shiohama and Xu proved in [18, 19] that if $||\tau||_n$ is small enough, then $M^n$ is homeomorphic to the sphere $\mathbb{S}^n$. Later on, we obtain quantitative results about the closeness of almost hypersurfaces to spheres in [6, 16]. For instance, we prove in [16], always for hypersurfaces of

2010 *Mathematics Subject Classification.* 53A42, 53C20, 53C21.
*Key words and phrases.* Almost-umbilical hypersurfaces, space forms, constant scalar curvature.

space forms, that if $||B-kg||_\infty$ is sufficientely small, for a positive constant $k$, then the hypersurface is quasi-isometric to a sphere of radius $\frac{1}{k}$. In [6] and also in [16], we obtain similar results with as less restrictive assumption replacing the $L^\infty$-norm by the $L^r$-norm ($r > n$). In particular, the hypersurface is diffeomorphic to the sphere $\mathbb{S}^n$. The proximity to the sphere is stronger than for the results of Shiohama and Xu but in counterpart, the assumption on the umbilicity is also stronger. Indeed, the hypothesis implies that the umbilicity tensor is close to zero and in addition that the mean curvature is close to a constant. We add that in a very recent paper [17], Scheuer obtain the following nice result. If $M$ is embedded into $\mathbb{R}^{n+1}$ and mean convex, then $||\tau||_\infty$ sufficiently small implies that $M$ is strictly convex and Hausdorff close to a geodesic sphere of appropriate radius $\sqrt{\frac{n}{\lambda_1}}$, where $\lambda_1$ is the first positive eigenvalue of the Laplacian on $M$.

The aim of the present paper is to give a result comparable to those of [6] and [16] with an alternative condition to the assumption that the mean curvature is almost constant. Starting from the remark that a hypersurface of a real space form has constant mean curvature if and only if its umbilicity tensor is Codazzi, we will prove that a hypersurface of a real space form which is almost umbilical with almost Codazzi umbilicity tensor is close to a sphere in the same sense as in [16] (see Theorem 3.1). Then, we prove a new rigidity result for geodesic spheres in real space forms (Corollary 4.4).

## 2. Preliminaries

Before stating the results, we will introduce some useful notations and recalls. First, we recall that the results obtained in [16] and [6] are consequences of pinching results for the first eigenvalue of the Laplacian proved in [5] and [6]. A key tool for these pinching results is the Michael-Simon's extrinsic Sobolev inequality for submanifolds of the Euclidean space [10] and its generalization by Hoffman and Spruck for any ambient manifold [8]. We begin by recalling the conditions under which these Sobolev inequalities are valid. Let $(N^{n+1},\overline{g})$ be a Riemannian manifold with sectional curvature bounded by above, say $K_N \leqslant b^2$, with $b$ real or purely imaginary and $n \geqslant 2$. Let $0<\alpha<1$, we denote by $\mathcal{H}_V(N,\alpha)$ the set of all connected, oriented and compact Riemannian manifolds without boundary $(M^n,g)$ isometrically immersed into $N$ and satisfying the two following conditions

$$b^2(1-\alpha)^{-2/n}(\omega_n^{-1}Vol(M))^{2/n} \leqslant 1, \tag{1}$$

$$2\rho_0 \leqslant \mathrm{inj}_M(N), \tag{2}$$

where $\mathrm{inj}_M(N)$ is the injectivity radius of $N$ restricted to $M$ and $\rho_0$ is given by

$$\rho_0 = \begin{cases} b^{-1}\sin^{-1}\left(b(1-\alpha)^{-1/n}(\omega_n^{-1}Vol(M))^{1/n}\right) & \text{if } b \text{ is real,} \\ (1-\alpha)^{-1/n}(\omega_n^{-1}Vol(M))^{1/n} & \text{if } b \text{ is imaginary.} \end{cases}$$

Under these hypotheses, Hoffman and Spruck showed that for any $\mathcal{C}^1$ function $f$ on $M$, the following extrinsic Sobolev inequality holds

$$\left(\int_M f^{\frac{n}{n-1}}dv_g\right)^{\frac{n-1}{n}} \leqslant K(n,\alpha)\int_M \left(|\nabla f| + |Hf|\right)dv_g, \tag{3}$$

where $K(n,\alpha)$ depends only on $n$ and $\alpha$ (not on $b$). We remark that for Euclidean and hyperbolic spaces, that is $\delta \leqslant 0$, (1) and (2) are trivially satisfied, whereas for spheres ($\delta > 0$) (1) and (2) resume to $Vol(M) \leqslant \frac{(1-\alpha)\omega_n}{\delta^{n/2}}$, where $\omega_n$ is the volume of $n$-dimensional unit sphere. An immediate consequence of this inequality is that $1 \leqslant K(n,\alpha)||H||_\infty Vol(M)^{1/n}$ by taking $f=1$. This extrinsic Sobolev inequality is of crucial importance to obtain pinching results for the first eigenvalue of the Laplacian (see [5, 6]). An other important fact under these assumptions is that the diameter of the hypersurface is bounded from above in terms of the mean curvature. Namely, Topping (for the Euclidean space [20]) and Wu and Zheng (for any ambient manifold [21]) proved that if Equations (1) and (2) hold, then there exists a constant $C(n,\alpha)$ depending only on $n$ and $\alpha$ so that

$$\mathrm{diam}(M) \leqslant C(n,\alpha)\int_M |H|^{n-1}dv_g. \tag{4}$$

For the statements of our results, we introduce the following subset of $\mathcal{H}_V(\mathbb{M}^{n+1}(\delta),\alpha)$. Let $q>n$ and $A>0$, we denote by $\mathcal{H}_V(n,\delta,\alpha,q,A)$ the subset of all manifolds in $\mathcal{H}_V(\mathbb{M}^{n+1}(\delta),\alpha)$ satisfying $\max_M\left\{||H||_\infty Vol(M)^{1/n}, ||B||_q Vol(M)^{1/n}\right\} \leqslant A$ if $\delta \geqslant 0$ and $\max_M\left\{||H||_\infty Vol(M)^{1/n}, ||B||_q Vol(M)^{1/n}, \frac{||H||_\infty}{\sqrt{||H||_\infty^2+\delta}}\right\} \leqslant A$ if $\delta<0$. Note that the quantity $||H||_\infty^2+\delta$ is positive, even if $\delta$ is negative and hence, its square root has a sense. For instance, we can see this fact from the upper bound for the first positive eigenvalue of the Laplacian $0<\lambda_1(\Delta)\leqslant n(||H||_\infty^2+\delta)$, obtained by Heinze [7]. Note also that this last condition is invariant by any dilatation of the metric of the ambiant space.

We also introduce the following useful functions. Let $r(x)=d(p,x)$ is the distance function to a base point $p$ (in the sequel, $p$ will be the center of mass of $M$). We denote by $Z$ the position vector defined by $Z=s_\delta(r)\overline{\nabla}r$, where $\overline{\nabla}$ is the connection of $\mathbb{M}^{n+1}(\delta)$. Moreover the functions $c_\delta$ and $s_\delta$ are defined by

$$c_\delta(t)=\begin{cases}\cos(\sqrt{\delta}t) & \text{if } \delta>0\\ 1 & \text{if } \delta=0\\ \cosh(\sqrt{|\delta|}t) & \text{if } \delta<0\end{cases} \quad \text{and} \quad s_\delta(t)=\begin{cases}\frac{1}{\sqrt{\delta}}\sin(\sqrt{\delta}t) & \text{if } \delta>0\\ t & \text{if } \delta=0\\ \frac{1}{\sqrt{|\delta|}}\sinh(\sqrt{|\delta|}t) & \text{if } \delta<0\end{cases}$$

A last notion will be useful for the sequel, the extrinsic radius. We recall that the extrinsic radius $R(M)$ of $M$ is defined by

$$R(M)=\inf\{\rho>0|\ \exists x\in\mathbb{M}^{n+1}(\delta) \text{ s.t. } \phi(M)\subset B(x,r)\},$$

where $\phi$ is the immersion of $M$ into $\mathbb{M}^{n+1}(\delta)$. By a slight abuse of notation, we denote it $R(M)$ but, this radius depends not only on $M$ but also on the immersion $\phi$. Since in this paper, the considered immersion will be fixed, this notation does not lead to any ambiguity.
Finally, even if, this is not optimal, we remark that $R(M)\leqslant \mathrm{diam}(M)$.

## 3. Almost umbilical hypersurfaces

Now, we have all the ingredients to state the main result of this note, which gives a new result about the closeness to spheres for almost umbilical hypersurfaces..

**Theorem 3.1.** *Let $(M^n,g)\in\mathcal{H}_V(n,\delta,\alpha,q,A)$. There exists $\varepsilon_0\in]0,1[$ depending on $n$, $q$ and $A$ so that if $\varepsilon\leqslant\varepsilon_0$ and*

$$||\tau||_\infty\leqslant\varepsilon||H||_\infty \quad \textit{and} \quad ||d^\nabla\tau||_\infty\leqslant\frac{\varepsilon||H||_\infty^2}{2nC(n,\alpha)A^n},$$

*then $M$ is embedded and $(M,g)$ is $\varepsilon$-quasi-isometric to the round sphere $S\left(p, s_\delta^{-1}\left(\frac{1}{\sqrt{||H||_\infty^2+\delta}}\right)\right)$, where $p$ is the center of mass of $M$. In particular, $M$ is diffeomorphic to the sphere $\mathbb{S}^n$.*

Before giving the proof of this theorem, we prove the following lemma

**Lemma 3.2.** *Let $(M^n,g)$ be a hypersurface of $\mathbb{M}^{n+1}(\delta)$. For any tangent vector fields $X, Y$, we have*

$$d^\nabla \tau(X,Y) = Y(H)X - X(H)Y.$$

**Proof:** We compute the curvature $\overline{R}(X,Y)\nu$. We have

$$\begin{aligned}\overline{R}(X,Y)\nu &= \overline{\nabla}_X\overline{\nabla}_Y\nu - \overline{\nabla}_Y\overline{\nabla}_X\nu - \overline{\nabla}_{[X,Y]}\nu \\ &= -\overline{\nabla}_X A(Y) + \overline{\nabla}_Y A(X) + A([X,Y]) \\ &= -\overline{\nabla}_X\tau(Y) + \overline{\nabla}_Y\tau(X) + \tau([X,Y]) - X(H)Y + Y(H)X \\ &\quad +H\overline{\nabla}_Y X - H\overline{\nabla}_X Y + H[X,Y] \\ &= -d^\nabla\tau(X,Y) - X(H)Y + Y(H)X,\end{aligned}$$

where we have used the fact $\overline{\nabla}_Y X - \overline{\nabla}_X Y + [X,Y] = 0$ since $\overline{\nabla}$ is torsion-free. Moreover, since $M$ lies into a space of constant curvature, $\overline{R}(X,Y)\nu = 0$, which concludes the proof of the lemma. □

**Remark 3.3.** *We deduce immediately from this lemma that $M$ have constant mean curvature if and only if $\tau$ is Codazzi.*

**Proof of Theorem 3.1:** Let $\varepsilon > 0$. We set $\eta = \frac{\varepsilon||H||_\infty^2}{2nC(n,\alpha)A^n}$. From the previous Lemma and the assumption $||d^\nabla\tau||_\infty \leqslant \eta$, we deduce that $|X(H)| \leqslant \eta$ for any unitary vector $X$. Thus, we have $||\nabla H|| \leqslant n\eta$. Now, let $p \in M$ be a point where the maximum of $|H|$ is achieved. Then, for any $x \in M$, by the mean value inequality, we have

$$|H(p) - H(x)| \leqslant n\eta d(p,x) \leqslant n\eta \mathrm{diam}(M).$$

Since by assumption $M \in \mathcal{H}_V(n,\delta,\alpha,A)$, the diameter is bounded in terms the mean curvature by (4). Namely, we have

$$\mathrm{diam}(M) \leqslant C(n,\alpha)\int_M |H|^{n-1}dv_g.$$

Hence, we get

$$|H(p) - H(x)| \leqslant n\eta C(n,\alpha)||H||_\infty^{n-1}Vol(M).$$

Thus, we have

$$|H^2 - ||H||_\infty^2| \leqslant 2n\eta C(n,\alpha)||H||_\infty^n Vol(M).$$

Since $\eta = \frac{\varepsilon||H||_\infty^2}{2nC(n,\alpha)A^n}$ and the assumption $||H||_\infty^n Vol(M) \leqslant A^n$, we get

$$|H^2 - ||H||_\infty^2| \leqslant \varepsilon||H||_\infty^2.$$

This, together with the other assuption $||\tau||_\infty \leqslant \varepsilon||H||_\infty$ leads to the conclusion that if $\varepsilon \leqslant \varepsilon_0$, where $\varepsilon_0$ is a constant depending on $n$, $q$ and $A$ given by [6, Theorem 1.3], then $M$ is $\varepsilon$-quasi-isometric to the the sphere $S\left(p, s_\delta^{-1}\left(\frac{1}{\sqrt{||H||_\infty^2+\delta}}\right)\right)$, where, $p$ is the center of mass of

$M$. In particular, $M$ is diffeomorphic to $\mathbb{S}^n$. Moreover, in the proof of [6, Theorem 1.3], the diffeomorphism is explicitely given. Namely, this diffeomorphism is the map

$$\begin{array}{rcl} F : M & \longrightarrow & S(p,\rho) \\ x & \longmapsto & \exp_p\left(\rho\frac{X}{|X|}\right), \end{array}$$

where $\rho = s_\delta^{-1}\left(\frac{1}{\sqrt{||H||_\infty^2+\delta}}\right)$ is the radius of the sphere, $\phi$ is the immersion of $M$ into $\mathbb{M}^{n+1}(\delta)$ and $X = \exp_p^{-1}(\phi(x))$ is the position vector. Since $F$ which is of the form $F = G\circ\phi$ is a diffeomorphism, then $\phi$ is necessarely injective. Thus, the immersion $\phi$ is an embedding. This concludes the proof. □

**Remark 3.4.** *The first condition $||\tau||_\infty \leqslant \varepsilon||H||_\infty$ is invariant by homothety. But, the second condition $||d^\nabla\tau||_\infty \leqslant \frac{\varepsilon||H||_\infty^2}{2nC(n,\alpha)A^n}$ is not invariant by homothety because of the square on $||H||_\infty$. This square is required since the diameter of $M$ appears by the use of the mean value inequality.*

## 4. A new characterization of geodesic spheres

From Theorem 3.1, we can obtain a new characterization of geodesic spheres in real space forms. We are motivated by the well-known Alexandrov theorem and the Yau conjecture [22]. Indeed, the Alexandrov theorem [2] states that a compact CMC hypersurface embedded into the Euclidean space, the hyperbolic space or the half-sphere is a geodesic sphere. Many generalizations of this result have been proved after. For instance, Ros proved that the same holds for higher order mean curvatures [12, 13]. In particular, in the Euclidean space, the second mean curvature $H_2 = \sigma_2(B)$ defined as the elementary symmetric homogeneous polynomial of the second fundamental form is (up to a multiplicative constant) the scalar curvature. Precisely, we have $scal = n(n-1)H_2$. More generally, in $\mathbb{M}^{n+1}(\delta)$, we have $scal = n(n-1)(H_2+\delta)$.
In the famous *Problem section* of [22], Yau conjectured that the embedding is not necessary for the Alexandrov theorem for the scalar curvature. This conjecture is still open, even if several partial answers have been given. We can cite for instance, the cases where the hypersurface is convex [4], stable [1], of cohomogeneity 2 [11], locally conformally flat [3] or with pinched second fundamental form [9]. In [14] and [15], we prove it with the assumption that the mean curvature is almost constant. In this section, we give another partial answer to this conjecture with a different additional assumption. Namely, we will show that hypersurfaces with constant scalar curvature and almost Codazzi umbilicity tensor are geodesic spheres. This characterization is valid in the three ambient space forms (Euclidean space, hyperbolic space and half-sphere).

First, we give this technical lemma.

**Lemma 4.1.** *Let $(M^n,g)\in\mathcal{H}_V(n,\delta,\alpha,q,A)$ and $s$ a positive constants. Let $\varepsilon>0$ and assume that*

$$\Big|H-||H||_\infty\Big| \leqslant \varepsilon||H||_\infty \quad et \quad ||scal-s||_\infty \leqslant \varepsilon||H||_\infty^2,$$

*then*

$$||\tau||_\infty \leqslant D||H||_\infty\varepsilon,$$

*where $D>1$ is an explicit constant depending on $n, ||H||_\infty, \delta$ and $A$.*

**Remark 4.2.** *(1) The constant $D$ does not depend on $s$. Moreover, we will see in the proof that $s$ is then close to $n(n-1)(||H||_\infty^2+\delta)$.*
*(2) If $\delta\geqslant 0$, $D$ does not depend on $||H||_\infty$.*

**Proof:** The proof of this lemma comes directly from the Hisung-Minkowski formulae. We recall that the Hsiung-Minkowksi formulae are integral formulae involving two consecutive higher order mean curvatures. In particular, we have the first two ones

$$\int_M \Big(H\left\langle Z,\nu\right\rangle + c_\delta(r)\Big)dv_g = 0, \tag{5}$$

$$\int_M \Big(H_2\left\langle Z,\nu\right\rangle + c_\delta(r)H\Big)dv_g = 0. \tag{6}$$

Since we assume that $|scal - s| < \varepsilon$, and $scal = n(n-1)(H_2+\delta)$, we get easily

$$\left|H_2 - \left(\frac{s}{n(n-1)} - \delta\right)\right| < \frac{1}{n(n-1)}\varepsilon||H||_\infty^2. \tag{7}$$

For more convenience, we will denote $h_2 = \frac{s}{n(n-1)} - \delta$ and $||H||_\infty = h$. Then, from (6)

$$\begin{aligned}
0 &= \int_M \Big(H_2\left\langle Z,\nu\right\rangle + c_\delta(r)H\Big)dv_g \\
&= \int_M \Big(h_2\left\langle Z,\nu\right\rangle + c_\delta(r)H\Big)dv_g + \int_M (H_2 - h_2)\left\langle Z,\nu\right\rangle dv_g \\
&= \frac{h_2}{h}\int_M h\left\langle Z,\nu\right\rangle dv_g + \int_M c_\delta(r)Hdv_g + \int_M (H_2 - h_2)\left\langle Z,\nu\right\rangle dv_g \\
&= \frac{h_2}{h}\int_M H\left\langle Z,\nu\right\rangle dv_g + \frac{h_2}{h}\int_M (h-H)\left\langle Z,\nu\right\rangle dv_g + \int_M c_\delta(r)hdv_g \\
&\quad + \int_M c_\delta(r)(H-h)dv_g + \int_M (H_2 - h_2)\left\langle Z,\nu\right\rangle dv_g.
\end{aligned}$$

Now, we use the other Hsiung-Minkowski formula (5) to get

$$\begin{aligned}
0 &= -\frac{h_2}{h}\int_M c_\delta(r)dv_g + \frac{h_2}{h}\int_M (h-H)\left\langle Z,\nu\right\rangle dv_g \\
&\quad + \int_M c_\delta(r)hdv_g + \int_M c_\delta(r)(H-h)dv_g + \int_M (H_2 - h_2)\left\langle Z,\nu\right\rangle dv_g \\
&= \left(h - \frac{h_2}{h}\right)\int_M c_\delta(r)dv_g + \frac{h_2}{h}\int_M (h-H)\left\langle Z,\nu\right\rangle dv_g + \int_M c_\delta(r)(H-h)dv_g \\
&\quad + \int_M (H_2 - h_2)\left\langle Z,\nu\right\rangle dv_g.
\end{aligned}$$

Then, since $|Z| = s_\delta(r)$ and using the assumption $|H-h| \leqslant h\varepsilon$ and (7), we deduce

$$\left|h - \frac{h_2}{h}\right|\int_M c_\delta(r)dv_g \leqslant h_2\varepsilon\int_M s_\delta(r)dv_g + \varepsilon h\int_M c_\delta(r)dv_g + \frac{\varepsilon h^2}{n(n-1)}\int_M s_\delta(r)dv_g.$$

Using the fact that $|H_2| \leqslant H^2$, we deduce from the assumptions on $h$ and $h_2$ that

$$\begin{aligned}
h_2 &\leqslant |H_2| + \frac{1}{n(n-1)}\varepsilon h^2 \\
&\leqslant H^2 + \frac{1}{n(n-1)}\varepsilon h^2 \\
&\leqslant h^2 + 2\varepsilon h^2 + \frac{1}{n(n-1)}\varepsilon h^2 \leqslant h^2(1+3\varepsilon).
\end{aligned}$$

Hence, we get

$$\left|h - \frac{h_2}{h}\right| \int_M c_\delta(r) dv_g \leqslant \varepsilon h \int_M c_\delta(r) dv_g + \left(1 + 3\varepsilon + \frac{1}{n(n-1)}\right) \varepsilon h^2 \int_M s_\delta(r) dv_g.$$

Therefore, unsing the fact that $\varepsilon < 1$, we get

$$\begin{aligned} \left|h^2 - h_2\right| &\leqslant \left[1 + h\left(1 + 3\varepsilon + \frac{1}{n(n-1)}\right) \frac{\int_M s_\delta(r)dv_g}{\int_M c_\delta(r)dv_g}\right] \varepsilon h^2 \\ &\leqslant \left[1 + 5h\frac{\int_M s_\delta(r)dv_g}{\int_M c_\delta(r)dv_g}\right] \varepsilon h^2. \end{aligned}$$

From now on, we will discuss the three cases $\delta = 0$, $\delta > 0$ and $\delta < 0$.

*First case:* $\delta = 0$. In this case, we have $s_\delta(r) = r$ and $c_\delta(r) = 1$, then, $\frac{\int_M s_\delta(r)dv_g}{\int_M c_\delta(r)dv_g} \leqslant R$. Since $R$ is the extrinsic radius, we have this obvious relation with the interior diameter $R \leqslant \mathrm{diam}(M)$ and therefore, by the result of Topping

$$\frac{\int_M s_\delta(r)dv_g}{\int_M c_\delta(r)dv_g} \leqslant R \leqslant \mathrm{diam}(M) \leqslant C(n,\alpha) \int_M |H|^{n-1} dv_g \leqslant C(n,\alpha)\frac{A^n}{h}.$$

*Second case:* $\delta > 0$. In this case, $c_\delta(r) = \cos(\sqrt{\delta} r)$ and $s_\delta(r) = \frac{1}{\sqrt{\delta}}\sin(\sqrt{\delta} r)$ and we have $\frac{\int_M s_\delta(r)dv_g}{\int_M c_\delta(r)dv_g} \leqslant t_\delta(R) \leqslant R$, since we have assumed that $M$ is contained in a sphere a radius $\frac{\pi}{4\sqrt{\delta}}$. Thus, like in the case $\delta = 0$, we have

$$\frac{\int_M s_\delta(r)dv_g}{\int_M c_\delta(r)dv_g} \leqslant C(n,\alpha)\frac{A^n}{h}.$$

*Third case:* $\delta < 0$. In this case, $c_\delta(r) = \cosh(\sqrt{|\delta|})$ and $s_\delta(r) = \frac{1}{\sqrt{|\delta}}\sinh(\sqrt{|\delta|})$. Hence, we get immediately that $\frac{\int_M s_\delta(r)dv_g}{\int_M c_\delta(r)dv_g} \leqslant \frac{1}{\sqrt{|\delta|}}$. Then, in the three cases, we have

$$\left|h^2 - h_2\right| \leqslant E\varepsilon h^2,$$

where

$$E = \begin{cases} 1 + 5C(n,\alpha)A^n & \text{if } \delta \geqslant 0, \\ 1 + \frac{5h}{\sqrt{|\delta|}} & \text{if } \delta < 0, \end{cases}$$

is a constant depending on $n$, $\alpha$, $\delta$, $h$ and $A$.

Now, we recall the Gauss formula. For $X, Y, Z, W \in \Gamma(TM)$,

$$R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + \langle SX, Z\rangle \langle SY, W\rangle - \langle SY, Z\rangle \langle SX, W\rangle$$

where $R$ and $\overline{R}$ are respectively the curvature tensor of $M$ and $\mathbb{M}^{n+1}(\delta)$.
By taking the trace in $X$ and $Z$ and for $W = Y$, we get

$$Ric(Y) = \overline{Ric}(Y) - \overline{R}(\nu, Y, \nu, Y) + nH\langle SY, Y\rangle - \left\langle S^2Y, Y\right\rangle,$$

where $S$ is the shape operator. Since, the ambient space is of constant sectional curvature $\delta$, by taking the trace a second time, we have

$$scal = n(n-1)\delta + n^2H^2 - |S|^2,$$

or equivalently

$$scal = n(n-1)\left(H^2+\delta\right) - |\tau|^2.$$

Hence, we have

$$\begin{aligned} ||\tau||^2 &= n(n-1)(H^2 - H_2) \\ &\leqslant n(n-1)\left(|H^2-h^2| + |h^2-h_2|\right) \\ &\leqslant n(n-1)\left(2h^2\varepsilon + Eh^2\varepsilon\right) \\ &\leqslant Dh^2\varepsilon, \end{aligned}$$

where we have set $D = n(n-1)(2+E)$. This concludes the proof of the lemma. □

Now, from this lemma, we can prove the following result.

**Theorem 4.3.** *Let $(M^n, g) \in \mathcal{H}_V(n,\delta,\alpha,q,A)$. There exists $D > 1$ depending on $n$, $\alpha$, $\delta$, $||H||_\infty$ and $A$ and there exists $\eta_0 \in ]0,1[$ depending on $n$,$q$ and $A$ and so that if $\eta \leqslant \eta_0$ and*

$$||scal - s||_\infty \leqslant \frac{\eta ||H||_\infty^2}{D} \quad \text{and} \quad ||d^\nabla \tau||_\infty \leqslant \frac{\eta||H||_\infty^2}{2nC(n,\alpha)AD},$$

*then $M$ is embedded and $(M,g)$ is $\eta$-quasi-isometric to the round sphere $S\left(p, s_\delta^{-1}\left(\sqrt{\frac{s}{n(n-1)}}\right)\right)$, where $p$ is the center of mass of $M$. In particular, $M$ is diffeomorphic to the sphere $\mathbb{S}^n$.*

**Proof:** First, the constant $D$ of the theorem is the one computed in Lemma 4.1. The second assupmtion $||d^\nabla\tau||_\infty \leqslant \frac{\eta||H||_\infty^2}{2nC(n,\alpha)AD}$ implies by the computations of the proof of Theorem 3.1 that $\Big|H - ||H||_\infty\Big| \leqslant \frac{\eta||H||_\infty}{D}$. Thus, we can apply Lemma 4.1 with $\varepsilon = \frac{\eta}{D}$ to get

$$||\tau||_\infty \leqslant D||H||_\infty\varepsilon = \eta||H||_\infty. \tag{8}$$

Moreover, since $D > 1$, we have

$$||d^\nabla\tau||_\infty \leqslant \frac{\eta||H||_\infty^2}{2nC(n,\alpha)AD} \leqslant \frac{\eta||H||_\infty^2}{2nC(n,\alpha)A}. \tag{9}$$

Thus, (8) and (9) are exactely the hypotheses of Theorem 3.1. Then, we conclude that $(M,g)$ is $\eta$-quasi-isometric to the round sphere $S\left(p, s_\delta^{-1}\left(\sqrt{\frac{s}{n(n-1)}}\right)\right)$, where $p$ is the center of mass of $M$. In particular, $M$ is diffeomorphic to $\mathbb{S}^n$. In addition Theorem 3.1 insures that $M$ is embedded into $\mathbb{M}^{n+1}(\delta)$. □

We deduce easily the following corollary which is a new characterization of geodesic spheres and gives a new partial answer to the Yau conjecture.

**Corollary 4.4.** *Let $(M^n,g) \in \mathcal{H}_V(n,\delta,\alpha,q,A)$ and $s$ a positive constant. There exists $D > 1$ depending on $n$, $\alpha$, $\delta$, $||H||_\infty$ and $A$ and there exists $\eta_0 \in ]0,1[$ depending on $n$, $q$ and $A$ so that if $\eta \leqslant \eta_0$ and*

$$scal = s \quad \text{and} \quad ||d^\nabla\tau||_\infty \leqslant \frac{\eta||H||_\infty^2}{2nC(n,\alpha)A},$$

*then $M$ is a geodesic sphere of radius $s_\delta^{-1}\left(\sqrt{\frac{s}{n(n-1)}}\right)$.*

**Proof:** This Corollary is a direct consequence of Theorem 4.3 together with the Alexandrov theorem for the scalar curvature proved by Ros [13]. Indeed, from the assumptions, we can apply Theorem 4.3 and get that $M$ is embedded. Since $M$ is assumed to have constant scalar curvature, by [13], $M$ is a geodesic sphere. Moreover, the radius of this sphere is determined by the scalar curvature and is $s_\delta^{-1}\left(\sqrt{\frac{s}{n(n-1)}}\right)$. □

## References


[1] H. Alencar, M.P. Do Carmo & H. Rosenberg, *On the first eigenvalue of Linearized operator of the r-th mean curvature of a hypersurface*, Ann. Glob. Anal. Geom., **11** (1993), 387-395.

[2] A.D. Alexandrov, *A characteristic property of spheres*, Ann. Mat. Pura Appl., **58** (1962), 303–315.

[3] Q.M. Cheng, *Complete hypersurfaces in a Euclidean space $\mathbb{R}^{n+1}$ with constant scalar curvature*, Indiana Univ. Math. J. **51** (2002), 53-68.

[4] S.Y. Cheng, S.T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann. **225** (1977), 195-204.

[5] B. Colbois and J.F. Grosjean, *A pinching theorem for the first eigenvalue of the Laplacian on hypersurfaces of the Euclidean space*, Comment. Math. Helv.**82** (2007), 175-195.

[6] J.F. Grosjean and J. Roth, *Eigenvalue pinching and application to the stability and the almost umbilicity of hypersurfaces*, Math. Z. **271 (1)** (2012), 469-488.

[7] E. Heinzte, *Extrinsic upper bound for $\lambda_1$*, Math.Ann. **280**, (1988), 389-402.

[8] D. Hoffman and D. Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds*, Comm. Pure Appl. Math. **27** (1974), 715-727.

[9] H. Li, *Hypersurfaces with constant mean curvature in space forms*, Math. Ann. **305** (1996), 665-672.

[10] J. H. Michael, L. M. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of $R^n$*, Comm. Pure Appl. Math. **26** (1973), 361-379.

[11] T. Okayasu, *On compact hypersurfaces in a Euclidean space with constant scalar curvature*, Kodai Math. J., **28** (2005), 577-585.

[12] A. Ros, *Compact hypersurfaces with constant higher order mean curvatures*, Revista Mat. Iberoamer. 3 (1987) 447-453.

[13] A. Ros, *Compact hypersurfaces with constant scalar curvature and a congruence theorem*, J. Diff. Geom. 27 (1988) 215-220.

[14] J. Roth, *Rigidity results for geodesic sphere in space forms*, Differential Geometry, Proceeding of the VIII International Coloquium in Differential Geometry, Santiago de Compostela, pp. 156-163, World Scientific, 2009

[15] J. Roth, *Une nouvelle caractérisation des sphères godésiques dans les espaces modèles*, Comptes-rendus - Mathématique, **347** no.19-20 (2009), 1197-1200.

[16] J. Roth, *A remark on almost umbilical hypersurfaces*, Arch. Math. (Brno), **49 (1)** (2013), pp 1-7.

[17] J. Scheuer, *Quantitative oscillation estimates for almost-umbilical closed hypersurfcaes in Euclidean space*, arXiv:1404.2525 (2014).

[18] K. Shiohama and H. Xu *Rigidity and sphere theorems for submanifolds*, Kyushu J. Math. **48**, (1994), no 2, 291-306.

[19] K. Shiohama and H. Xu *Rigidity and sphere theorems for submanifolds II*, Kyushu J. Math. **54**, (2000), no 1, 103-109.

[20] P. Topping, *Relating diameter and mean curvature for submanifolds of Euclidean space*, Comment. Math. Helv. **83** (2008), 539-546.

[21] J.Y. Wu & Y. Zheng, *Relating diameter and mean curvature for Riemannian submanifolds*, Proc. Amer. Math. Soc., **139 (11)** (2011), 4097-4104.

[22] S. T. Yau, *Seminar on Differential Geometry, Problem Section*, Ann. of Math. Studies **102**, Princeton, 1982, 669-706



Julien Roth, Laboratoire d'Analyse et de Mathématiques Appliquées, UPEM-UPEC, CNRS, F-77454 Marne-la-Vallée

*E-mail address*: julien.roth@u-pem.fr