UPPER BOUNDS FOR THE FIRST EIGENVALUE OF THE LAPLACIAN OF HYPERSURFACES IN TERMS OF ANISOTROPIC MEAN CURVATURES

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Abstract. We prove upper bounds for the first eigenvalue of the Laplacian of hypersurfaces of Euclidean space involving anisotropic mean curvatures. Then, we study the equality case and its stability.

1. Introduction

Let (M^n, g) be a n-dimensional $(n \geqslant 2)$ compact, connected, oriented manifold without boundary, isometrically immersed by X into the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. The spectrum of Laplacian of (M, g) is a increasing sequence of real numbers

$$0 = \lambda_0(M) < \lambda_1(M) \leqslant \lambda_2(M) \leqslant \cdots \leqslant \lambda_k(M) \leqslant \cdots \longrightarrow +\infty.$$

The eigenvalue 0 (corresponding to constant functions) is simple and $\lambda_1(M)$ is the first positive eigenvalue. In [15], Reilly proved the following well-known upper bound for $\lambda_1(M)$

$$\lambda_1(M) \leqslant \frac{n}{\mathrm{V}(M)} \int_M H^2 dv_g, \tag{1}$$

where H is the mean curvature of the immersion. He also proved an analogous inequality involving the higher order mean curvatures. Namely, for $r \in \{1, \cdots, n\}$

$$\lambda_1(M) \left(\int_M H_{r-1} dv_g \right)^2 \leqslant \mathrm{V}(M) \int_M H_r^2 dv_g, \tag{2}$$

where H_k is the k-th mean curvature, defined by the k-th symmetric polynomial of the principal curvatures. Moreover, Reilly studied the equality cases and proved that equality in (1) as in (2) is attained if and only if $X(M)$ is a geodesic sphere.

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On the other hand, over the past years, many authors considered geometric problems involving anisotropic mean curvature and higher order mean curvatures (see [4, 5, 10, 13] for instance). The setting is the following. Let $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+$ be a smooth function satisfying the following convexity assumption

$$A_F = (\nabla dF + F\mathrm{Id}_{|T_x\mathbb{S}^n})_x > 0, \tag{3}$$

for all $x \in \mathbb{S}^n$, where ∇dF is the Hessian of F. Here, > 0 mean positive definite in the sense of quadratic forms. Now, we consider the following map

$$\begin{array}{rccl} \phi: & \mathbb{S}^n & \longrightarrow & \mathbb{R}^{n+1} \\ & x & \longmapsto & F(x)x + (\mathrm{grad}_{|S^n}F)_x \end{array}$$

The image $W_F = \phi(\mathbb{S}^n)$ is called the Wulff shape of F and is a smooth hypersurface of $\mathbb{R}^{n+1}$. Moreover, from the so-called convexity condition (3), W_F is convex. Note that if F is a positive constant c, the Wulff shape is nothing else but the sphere of radius c.

Now, let (M^n, g) be a n-dimensional compact, connected, oriented manifold without boundary, isometrically immersed by X into $\mathbb{R}^{n+1}$. We denote by ν a normal unit vector field globally defined on M, that is, we have $\nu : M \longrightarrow \mathbb{S}^n$. We set $S_F = A_F \circ \mathrm{d}\nu$, where A_F is defined by (3). The operator S_F is called the F-Weingarten operator and its eigenvalues are the anisotropic principal curvature that we will denote $\kappa_1, \kappa_2, \cdots, \kappa_n$. Finally, for $r \in \{1, \cdots, n\}$, the r-th anisotropic mean curvature is defined by

$$H_r^F = \frac{1}{C_n^r} \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \cdots \kappa_{i_r}.$$

We also set $H_0^F = 1$ for convenience. Note that if $F = 1$ then this is the definition of the classical r-th mean curvature.

During the past years, lots of classical results have by been generalized to the anisotropic case. For instance, in [6], He, Li, Ma and Ge proved an anisotropic version of the well-known Alexandrov theorem [1, 12]. Namely, they proved that a compact manifold without boundary M^n embedded into $\mathbb{R}^{n+1}$ with constant anisotropic r-th mean curvature is the Wulff shape. Many other characterizations of the Wulff shape have been proved (see [4, 5, 13] for instance.

The aim of the present paper is to give an analogue to Reilly's Inequalities (1) and (2) in this anisotropic setting. We prove the following anisotropic version of these upper bounds:

Theorem 1. *Let (M^n, g) be a n-dimensional ($n \geqslant 2$) compact, connected, oriented manifold without boundary, isometrically immersed by X into $\mathbb{R}^{n+1}$, $r \in \{1, \cdots, n\}$, and $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+$ a function satisfying (3). Then*

$$\lambda_1(M) \left(\int_M F(\nu) H_{r-1}^F dv_g \right)^2 \leqslant n\mathrm{V}(M) \int_M (H_r^F)^2 dv_g.$$

Moreover, equality occurs if and only if $X(M)$ is a geodesic hypersphere and F is constant.

Remark 1. *One could expect that the equlity case was attained by the Wulff shape but, as we will see in the proof, the use of coordinates as test functions forces the limitting manifolds to be geodesic spheres. Nevertheless one could expect to have a spectral characterization of the Wulff shape by studying an other second order operator naturally associated with the functional J where $J(X) = \int_M F(\nu)dv_g$ (see [14] for instance). But this is not the subject of the present paper.*

Since we know that if equality holds, then $X(M)$ is a geodesic hypersphere, a natural question is to study the stability of this equality case, that is, if the equality almost holds, is the hypersurface close to a geodesic hypersphere? And what do we understand by *close* ? The following theorems, generalizing results of [2] for $r = 1$, and [16] for any r, give an answer to this question.

Theorem 2. *Let (M^n, g) be a n-dimensional ($n \geqslant 2$) compact, connected, oriented Riemannian manifold without boundary isometrically immersed by X into $\mathbb{R}^{n+1}$ and $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+$ a function satisfying (3). Let $r \in \{1, \cdots, n\}$ and assume that $H_r^F > 0$ if $r > 1$. Then, for any $p \geqslant 2$ and for any $\varepsilon > 0$, there exists a constant C_ε depending on ε, n, r, $||F||_\infty$, $||H||_\infty$, $||H^F||_\infty$, $||H_r^F||_{2p}$ and $\mathrm{V}(M)$ such that*

$$(P_{C_\varepsilon}) \qquad \lambda_1(M)\left(\int_M F(\nu)H_{r-1}^F\right)^2 - n\mathrm{V}(M)^2||H_r^F||_{2p}^2 > -C_\varepsilon \mathrm{V}(M)^2$$

is satisfied, then $d_H\left(X(M), S(p_0, \sqrt{\frac{n}{\lambda_1}})\right) \leqslant \varepsilon$.

Remark 2. *We will see in the proof that $C_\varepsilon \longrightarrow 0$ when $||H||_\infty \longrightarrow \infty$ or $\varepsilon \longrightarrow 0$.*

Theorem 3. *Let (M^n, g) be a n-dimensional ($n \geqslant 2$) compact, connected, oriented Riemannian manifold without boundary isometrically immersed by X into $\mathbb{R}^{n+1}$ and p_0 the center of mass of M. Let $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+$ a function satisfying (3) and $r \in \{1, \cdots, n\}$ and assume that $H_r > 0$ if $r > 1$. Then for any $p \geqslant 2$, there exists a constant K depending on n, r, $||F||_\infty$, $||H^F||_\infty$, $||B||_\infty$, $||H_r^F||_{2p}$ and $\mathrm{V}(M)$ such that if the pinching condition*

$$(P_K) \qquad \lambda_1(M)\left(\int_M F(\nu)H_{r-1}^F\right)^2 - n\mathrm{V}(M)^2||H_r^F||_{2p}^2 > -K\mathrm{V}(M)^2$$

is satisfied, then M is diffeomorphic to $\mathbb{S}^n$.

More precisely, there exists a diffeomorphism G from M into the sphere $\mathbb{S}^n\left(\sqrt{\frac{n}{\lambda_1(M)}}\right)$ of radius $\sqrt{\frac{n}{\lambda_1(M)}}$ which is a quasi-isometry. Namely, for any $\theta \in]0, 1[$, there exists a constant K_θ depending only on θ, n, r, $||F||_\infty$,

$||H^F||_\infty$, $||B||_\infty$, $||H_r^F||_{2p}$ and $\mathrm{V}(M)$ so that the pinching condition with K_θ implies

$$\Big||dG_x(u)|^2-1\Big|\leq\theta,$$

for any unitary vector $u\in T_xM$.

Remark 3. *We use the followimg convention for the L^p-norms. For any function continuous f on M,*

$$||F||_p=\frac{\left(\int_M|f|^pdv_g\right)^{1/p}}{\mathrm{V}(M)^{1/p}}.$$

Remark 4. *As, we will see in the proof (see Lemma 10) the dependence on $||B||_\infty$ can be replaced by a dependence weaker dependence, namely, on q, $||H||_\infty$ and $||B||_q$.*

2. Preliminaries

In this section, we will give some basic recalls about anisotropic higher order mean curvatures.

Let (M^n,g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed into $\mathbb{R}^{n+1}$ by X and denote by ν a normal unit vector field. Let $p_0\in\mathbb{R}^{n+1}$. We denote by $r(x)=d(p_0,x)$ the geodesic distance between p_0 to x in $\mathbb{R}^{n+1}$. We denote by ∇ (*resp.* $\overline{\nabla}$) the gradient associated with (M,g) (*resp.* $\mathbb{R}^{n+1}$).

Then, up to a possible translation, $X=r\overline{\nabla}r$ is the position vector field and $X^T=X-<X,\nu>\nu$ its projection on the tangent bundle of $X(M)$. Obviously, we have $X^T=r\nabla r$.

The second fundamental form B of the immersion is defined by

$$B(X,Y)=\left\langle\overline{\nabla}_X\nu,Y\right\rangle,$$

where $<\cdot,\cdot>$ and $\overline{\nabla}$ are respectively the Riemannian metric and the Riemannian connection of $\mathbb{R}^{n+1}$. We also denote by S the Weingarten operator, which is the associated $(1,1)$-tensor. The mean curvature of the immersion is given by

$$H=\frac{1}{n}\mathrm{tr}\,(B).$$

Let $S_F=A_F\circ S$, where $A_F=\nabla dF+F\mathrm{Id}_{|T_x\mathbb{S}^n}$. The operator S_F is called the F-Weingarten operator and its eigenvalues $\kappa_1,\cdots,\kappa_n$ are the anisotropic principal curvatures Now let us recall the definition of the anisotropic high order mean curvature H_r^F. First, we consider an orthonormal frame $\{e_1,\cdots,e_n\}$ of T_xM. For all $k\in\{1,\cdots,n\}$, the r-th anisotropic mean curvature of the immersion is

$$H_r^F=\begin{pmatrix}n\\r\end{pmatrix}^{-1}\sum_{\substack{1\leqslant i_1,\cdots,i_r\leqslant n\\1\leqslant j_1,\cdots,j_r\leqslant n}}\epsilon\begin{pmatrix}i_1\cdots i_r\\j_1\cdots j_r\end{pmatrix}S^F_{i_1j_1}\cdots S^F_{i_rj_r},$$

where the S^F_{ij} are the coefficients of the F-Weingarten operator. The symbols $\epsilon\begin{pmatrix} i_1\cdots i_r \\ j_1\cdots j_r\end{pmatrix}$ are the usual permutation symbols which are zero if the sets $\{i_1,\cdots,i_r\}$ and $\{j_1,\cdots,j_r\}$ are different or if there exist distinct p and q with $i_p=i_q$. For all other cases, $\epsilon\begin{pmatrix} i_1\cdots i_r \\ j_1\cdots j_r\end{pmatrix}$ is the signature of the permutation $\begin{pmatrix} i_1\cdots i_r \\ j_1\cdots j_r\end{pmatrix}$. By convention, we set $H^F_0=1$ et $H^F_{n+1}=0$.

For $r\in\{1,\cdots,n\}$, the symmetric $(1,1)$-tensor associated to H^F_r is

$$T^F_r=\frac{1}{r!}\sum_{\substack{1\leqslant i,i_1,\cdots,i_r\leqslant n\\ 1\leqslant j,j_1,\cdots,j_r\leqslant n}}\epsilon\begin{pmatrix} i_1\cdots i_r \\ j_1\cdots j_r\end{pmatrix}S^F_{i_1j_1}\cdots S^F_{i_rj_r}e^*_i\otimes e^*_j.$$

This tensor is divergence free, symmetric $(1,1)$-tensor (see [4] for a proof). For any symmetric $(1,1)$-tensor, we define the following function

$$H_T(x)=\sum_{i=1}^n < S^F_x e_i, Te_i >, \tag{4}$$

where $\{e_1,\ldots,e_n\}$ is an orthonormal frame of T_xM. Then, we have the following relations

Lemma 1. *[4] For $r\in\{1,\cdots,n\}$, we have:*

(1) $\mathrm{tr}\,(T^F_r)=m(r)H^F_r$,
(2) $H_{T^F_r}=m(r)H^F_{r+1}$,

where $m(r)=(n-r)\begin{pmatrix} n \\ r\end{pmatrix}$ *and* H_{T_r} *is given by (4).*

We give these integral forumlas proved by He and Li in [4] and which generalized the classical Hsiung-Minkowski formula [9]

$$\int_M \left(F(\nu)H^F_{r-1}+H^F_r < X,\nu >\right)dv_g=0, \tag{5}$$

for any $r\in\{1\cdots,n\}$ with the convention that $H^F_0=1$. These powerful formulas play a crucial role in all the rigidity results involving anisotropic r-th mean curvatures (see [4, 5, 13] for instance) and we will also make an important use of them in this work.

An other classical but important fact about the anisotropic r-th mean curvatures is the following fact. If for $r\in\{1\cdots,n\}$, H^F_r is positive everywhere, then for each $k\in\{1\cdots,r-1\}$, T^F_{k-1} is postive definite and H^F_k is also a positive function. Moreover, we have

$$(H^F_k)^{\frac{1}{k}}\leq(H^F_{k-1})^{\frac{1}{k-1}}\leq\cdots\leq(H^F_2)^{\frac{1}{2}}\leq H^F_1. \tag{6}$$

This can also be found in [4].

3. Upper bounds for the first eigenvalue of the Laplacian

Now, we have all the ingredients to prove Theorem 1.

Proof of Theorem 1. First, if necessary, we make a translation so that $X(M)$ is centered at the origin of $\mathbb{R}^{n+1}$, that is $\int_M X^i dv_g = 0$ for all $i \in \{1, \cdots, n+1\}$, where X_i are the functions defined by $X = \sum_{i=1}^{n+1} X_i \partial_i$, where $\{\partial_1, \cdots, \partial_{n+1}\}$ is the canonical frame of $\mathbb{R}^{n+1}$. In other words, the functions X_i are the coordinates functions and can be used as test functions in the variational characterization of $\lambda_1(M)$. So for any $i \in \{1, \cdots, n+1\}$, we have

$$\lambda_1(M) \int_M (X^i)^2 dv_g \leqslant \int_M |\mathrm{d}X^i|^2 dv_g,$$

Form this and using (5) and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\lambda_1(M) \left(\int_M F(\nu) H_{r-1}^F dv_g \right)^2 &\leqslant \lambda_1(M) \left(\int_M H_r^F < X, \nu > dv_g \right)^2 \\
&\leqslant \lambda_1(M) \left(\int_M (H_r^F)^2 dv_g \right) \left(\int_M |X|^2 dv_g \right) \\
&\leqslant \left(\int_M (H_r^F)^2 dv_g \right) \left(\sum_{i=1}^{n} \int_M |\mathrm{d}X^i|^2 dv_g \right) \\
&\leqslant n\mathrm{V}(M) \int_M (H_r^F)^2 dv_g.
\end{aligned}$$

This concludes the proof of the inequality. Now, if equlity occurs, then the coordinate functions are eigenfunctions of the Laplacian and by the classical result of Takahasi [18], $X(M)$ is a geodesic hypersphere $\left(\text{of radius } \sqrt{\frac{n}{\lambda_1}}\right)$ and in particular, we have that $X = -\sqrt{\frac{n}{\lambda_1}}\nu$. We also have equality in the Cauchy-Schwarz inequality, which implies that H_r^F and $< X, \nu >$ are colinear, that is, H_r^F is constant. We conclude by a result of He and Li ([4], Theorem 1.2) that since H_r^F is constant and $< X, \nu >$ has fixed sign, then $X(M)$ is the Wulff shape. But $X(M)$ is a geodesic sphere, the only possibility is that F a positive constant. Precisely, F is the radius of this sphere, that is, $\sqrt{\frac{n}{\lambda_1}}$.

Conversly, if $X(M)$ is a geodesic sphere and F is constant, then, the inequality of Theorem 1 is just the classical Reilly inequality which is an equality since $X(M)$ is a geodesic sphere. $\square$

Corollary 1. *Let (M^n, g) be a n-dimensional $(n \geqslant 2)$ compact, connected, oriented manifold without boundary, isometrically immersed by X into $\mathbb{R}^{n+1}$, $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+$ a function satisfying* (3) *and $r \in \{2, \cdots, n\}$. If H_r^F is a* *positive constante, then*

$$\lambda_1(M) \leqslant \frac{n\mathrm{V}(M)^2}{\left(\int_M F(\nu)dv_g\right)^2}(H_r^F)^{\frac{2}{r}}.$$

Moreover, equality occurs if and only if $X(M)$ is a geodesic hypersphere and F is constant.

Proof: Since H_r^F is positive, we have $(H_r^F)^{\frac{r-1}{r}} \geqslant H_{r-1}^F$ and since F is a positive function, we have

$$\begin{aligned}\int_M F(\nu)H_{r-1}^F dv_g &\geqslant \int_M F(\nu)(H_r^F)^{\frac{r-1}{r}}dv_g \\ &\geqslant (H_r^F)^{\frac{r-1}{r}}\int_M F(\nu)dv_g.\end{aligned}$$

We conclude by using the estimate of Theorem 1, and again, the fact that H_r^F is constant. Equality holds, if and only if, we have equality in Theorem 1, that is, if and only if $X(M)$ is a geodesic hypersphere and F is constant.□

Corollary 2. *Let (M^n, g) be a n-dimensional ($n \geqslant 2$) compact, connected, oriented manifold without boundary, isometrically immersed by X into $\mathbb{R}^{n+1}$, $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+$ a function satisfying* (3) *and $r \in \{2, \cdots, n\}$. If H_r^F is a positive constante, then for any $k \in \{2, \cdots, r-1\}$*

$$\lambda_1(M) \leqslant \frac{n\mathrm{V}(M)^2}{\left(\int_M F(\nu)dv_g\right)^2}\inf_M(H_k^F)^{\frac{2}{k}}.$$

Moreover, equality occurs if and only if $X(M)$ is a geodesic hypersphere and $F = 1$.

Proof: The proof is immediate from Corollary 1. Since H_r^F is a positive constant, we have

$$\begin{aligned}\lambda_1(M) &\leqslant \frac{n\mathrm{V}(M)^2}{\left(\int_M F(\nu)dv_g\right)^2}(H_r^F)^{\frac{2}{r}} \\ &= \frac{n\mathrm{V}(M)^2}{\left(\int_M F(\nu)dv_g\right)^2}\inf_M(H_r^F)^{\frac{2}{r}} \\ &\leqslant \frac{n\mathrm{V}(M)^2}{\left(\int_M F(\nu)dv_g\right)^2}\inf_M(H_k^F)^{\frac{2}{k}},\end{aligned}$$

since for $k \in \{2, \cdots, r-1\}$, we have $(H_r^F)^{\frac{1}{r}} \leqslant (H_k^F)^{\frac{1}{k}}$. Equality case is the same as for Corollary 1, namely if and only if $X(M)$ is a geodesic hypersphere and F is constant. □

4. Stability of equality case

Now, we will study the stability of the equality case of Theorem 1, and hence proved Theorem 2 and 3.

4.1. An L^2-approach to the problem.

First, we will show that the pinching condition (P_C) implies a proximity between $X(M)$ and a geodesic hypersphere in an L^2-sense. We have this first lemma.

Lemma 2. *If the pinching condition (P_C) is satisfied for $C < \frac{n}{2}||H_r^F||_{2p}^2$, then*

$$\frac{n\lambda_1(M)\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^4}{\left[C\mathrm{V}(M)^2+\lambda_1(M)\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^2\right]^2} \leqslant ||X||_2^2 \leqslant \frac{n}{\lambda_1(M)} \leqslant A_1,$$

where A_1 is a positive constant depending only on n, r, $||F||_\infty$, $||H^F||_\infty$ and $||H_r^F||_{2p}$.

Proof: If (P_C) is satisfied, we have:

$$\lambda_1(M)\left(\int_M F(\nu)H_{r-1}dv_g\right)^2 \geqslant n||H_r^F||_{2p}^2\mathrm{V}(M)^2 - C\mathrm{V}(M)^2.$$

If, in addition, we assume that $C < \frac{n}{2}||H_r^F||_{2p}^2$, we get

$$\lambda_1(M)\left(\int_M F(\nu)H_{r-1}dv_g\right)^2 \geqslant \frac{n}{2}||H_r^F||_{2p}^2\mathrm{V}(M)^2,$$

and so

$$\frac{n}{\lambda_1(M)} \leqslant \frac{2\left(\int_M F(\nu)H_{r-1}dv_g\right)^2}{||H_r^F||_{2p}^2\mathrm{V}(M)^2} \leqslant \frac{2||H^F||_\infty^{2(r-1)}}{||H_r^F||_{2p}^2}||F||_\infty. \tag{7}$$

Moreover, by the variational characterization of $\lambda_1(M)$, we have

$$\lambda_1(M)\int_M |X|^2 \leqslant \int_M\left(\sum_{i=1}^{n+1}|dX_i|^2\right) = n\mathrm{V}(M).$$

So we have $||X||_2^2 \leqslant \dfrac{n}{\lambda_1(M)}$, and by (7),

$$||X||_2^2 \leqslant A_1,$$

where A_1 depends on $n, r, ||F||_\infty, ||H^F||_\infty$ and $||H_r^F||_{2p}$. For the left hand side, we have

$$\begin{aligned}
\lambda_1(M)\left(\int_M |X|^2dv_g\right)\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^4 &\leqslant n\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^4 \\
&\leqslant n\left(\int_M H_r^F\langle X,\nu\rangle dv_g\right)^4 \\
&\leqslant n\left(\int_M (H_r^F)^2dv_g\right)^2\left(\int_M |X|^2dv_g\right)^2.
\end{aligned}$$

Then, by the Hölder inequality, we deduce

$$\lambda_1(M)\left(\int_M F(\nu)H_{r-1}dv_g\right)^4 \leqslant n||H_r^F||_{2p}^2\left(\int_M |X|^2 dv_g\right),$$

and with the pinching condition,

$$||X||_2^2 \geqslant \frac{n\lambda_1(M)\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^4}{\left[C\mathrm{V}(M)^2+\lambda_1(M)\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^2\right]^2}.$$

□

Then, we have the following lemma for the L^2-norm of X^T.

Lemma 3. *The pinching condition* (P_C) *with* $C < \frac{n}{2}||H_r^F||_{2p}^2$ *implies*

$$||X^T||_2^2 \leqslant A_2 C,$$

where A_2 *is a positive constant depending only on* n, r, $||F||_\infty$, $||H^F||_\infty$ *and* $||H_r^F||_{2p}$.

Proof: We saw that

$$\lambda_1(M)\int_M |X|^2 dv_g \quad\leqslant\quad n\mathrm{V}(M),$$

so by the Hsiung-Minkowski formula (5) and the Cauchy-Schwarz inequality

$$\begin{aligned}
\lambda_1(M)\int_M |X|^2 dv_g\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^2 &\leqslant n\mathrm{V}(M)\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^2\\
&\leqslant n\mathrm{V}(M)\left(\int_M H_r^F\left\langle X,\nu\right\rangle dv_g\right)^2\\
&\leqslant ||H_r^F||_2^2\mathrm{V}(M)^2\int_M \left\langle X,\nu\right\rangle^2 dv_g\\
&\leqslant ||H_r^F||_{2p}^2\mathrm{V}(M)^2\int_M \left\langle X,\nu\right\rangle^2 dv_g.
\end{aligned}$$

Then we deduce

$$\begin{aligned}
n||H_r^F||_{2p}^2||X^T||_2^2\mathrm{V}(M) &= n||H_r^F||_{2p}^2\int_M\left(|X|^2-\left\langle X,\nu\right\rangle^2\right)dv_g\\
&\leqslant n||H_r^F||_{2p}^2\left[\int_M |X|^2-\frac{\lambda_1(M)}{\mathrm{V}(M)^2}\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^2\int_M |X|^2 dv_g\right]\\
&\leqslant \left[n||H_r^F||_{2p}^2\mathrm{V}(M)-\frac{\lambda_1(M)}{\mathrm{V}(M)}\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^2\right]||X||_2^2\\
&\leqslant C||X||_2^2\mathrm{V}(M)\leqslant A_1 C\mathrm{V}(M).
\end{aligned}$$

Finally, we get

$$||X^T||_2^2 \leqslant \frac{A_1 C}{n||H_r^F||_{2p}^2} = A_2 C,$$

where A_2 is a positive constant depending only on n, r, $||F||_\infty$, $||H^F||_\infty$ and $||H_r^F||_{2p}$. □

To prove Theorem 2, we need to show that the norm of the position vector $||X||$ is close to $\sqrt{\frac{n}{\lambda_1(M)}}$. For this, we will consider the following function

$$\varphi := |X| \left(|X| - \sqrt{\frac{n}{\lambda_1(M)}} \right)^2.$$

We will show that under the pinchng condition, this function is close to 0. Before getting such an estimate, we introduce the two following vector fields:

$$\begin{cases} Y = nH_r^F \nu + \lambda_1(M) \left(\int_M F(\nu) H_{r-1}^F dv_g \right) X, \\ \\ Z = \sqrt{\dfrac{n}{\lambda_1}} \dfrac{|X|^{1/2} H_r^F \mathrm{V}(M)}{\left(\int_M F(\nu) H_{r-1}^F dv_g \right)} \nu + \dfrac{X}{|X|^{1/2}}. \end{cases}$$

Remark 5. *Note that if F is a positive constant c and $X(M)$ a sphere of radius c, then these two vector fields vanish identically on M.*

We will prove that under the pinching condition, they are close to zero. First, we have the following:

Lemma 4. *The pinching condition (P_C) implies*

$$||Y||_2^2 \leqslant nC.$$

Proof: We have

$$\begin{aligned} ||Y||_2^2 \mathrm{V}(M) &= n^2 \int_M (H_r^F)^2 dv_g + \frac{\lambda_1(M)}{\mathrm{V}(M)^2} \left(\int_M F(\nu) H_{r-1}^F dv_g \right)^2 \int_M |X|^2 dv_g \\ &\quad + 2n \frac{\lambda_1(M)}{\mathrm{V}(M)} \left(\int_M F(\nu) H_{r-1}^F dv_g \right) \int_M H_r^F \langle X, \nu \rangle \, dv_g \\ &\leqslant n^2 ||H_r^F||_{2p}^2 \mathrm{V}(M) + n \frac{\lambda_1(M)}{\mathrm{V}(M)} \left(\int_M F(\nu) H_{r-1}^F dv_g \right)^2 \\ &\quad - 2n \frac{\lambda_1(M)}{\mathrm{V}(M)} \left(\int_M F(\nu) H_{r-1}^F dv_g \right)^2 \\ &\leqslant n \left[n ||H_r^F||_{2p}^2 \mathrm{V}(M) - \frac{\lambda_1(M)}{\mathrm{V}(M)} \left(\int_M F(\nu) H_{r-1}^F \right)^2 dv_g \right] \\ &\leqslant nC\mathrm{V}(M), \end{aligned}$$

where we used the Hsiung-Minkowski formula (5), and the fact that

$$||X||_2^2 \leqslant \frac{n}{\lambda_1(M)}.$$

□

For the vector field Z, we have the following lemma.

Lemma 5. *Assume that $p \geqslant 2$. If the pinching condition (P_C) is satisfied, with $C < \frac{n}{2}||H_r^F||_{2p}^2$, then*

$$||Z||_2^2 \leqslant A_3 C,$$

where A_3 is a positive constant depending only on n, r, $||F||_\infty$, $||H^F||_\infty$ and $||H_r^F||_{2p}$.

Proof: We have

$$\begin{aligned} ||Z||_2^2 \mathrm{V}(M) &= \frac{n\mathrm{V}(M)^2}{\lambda_1(M)\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^2}\int_M |X|(H_r^F)^2 dv_g \\ &\quad + \int_M |X| dv_g + 2\frac{\sqrt{\frac{n}{\lambda_1(M)}}\mathrm{V}(M)}{\int_M F(\nu)H_{r-1}^F dv_g}\int_M H_r^F \langle X, \nu\rangle dv_g \\ &\leqslant \frac{n}{\lambda_1(M)\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^2}\int_M |X| H_r^2 dv_g + \int_M |X| dv_g - 2\sqrt{\frac{n}{\lambda_1(M)}}\mathrm{V}(M), \end{aligned}$$

where we used (5). Now, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned} ||Z||_2^2 \mathrm{V}(M) &\leqslant \frac{n\mathrm{V}(M)^2}{\lambda_1(M)\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^2}\left(\int_M (H_r^F)^4 dv_g\right)^{1/2}\left(\int_M |X|^2 dv_g\right)^{1/2} \\ &\quad + \left(\int_M |X|^2 dv_g\right)^{1/2}\mathrm{V}(M)^{1/2} - 2\sqrt{\frac{n}{\lambda_1(M)}}\mathrm{V}(M). \end{aligned}$$

From Lemma 2, we have $||X||_2 \leqslant \sqrt{\frac{n}{\lambda_1(M)}}$, so using this fact and the Hölder inequality, we get

$$\begin{aligned} ||Z||_2^2 \mathrm{V}(M) &\leqslant \sqrt{\frac{n}{\lambda_1(M)}}\left[\frac{n}{\lambda_1(M)}\frac{||H_r^F||_{2p}^2 \mathrm{V}(M)^2}{\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^2} - 1\right]\mathrm{V}(M) \\ &\leqslant \left(\frac{n}{\lambda_1(M)}\right)^{3/2}\frac{\mathrm{V}(M)}{n\left(\int_M F(\nu)H_{r-1}^F dv_g\right)^2}\left[n||H_r^F||_{2p}^2\mathrm{V}(M)^2 - \lambda_1(M)\left(\int_M F(\nu)H_{r-1}^F\right)^2\right] \\ &\leqslant A_3 C\mathrm{V}(M), \end{aligned}$$

where A_3 depends only on n, r, $||F||_\infty$, $||H^F||_\infty$ and $||H_r^F||_{2p}$. Note that we have used the fact that $\frac{n}{\lambda_1(M)} \leqslant \frac{2||H^F||_\infty^{2(r-1)}}{||H_r^F||_{2p}^2}||F||_\infty$ and the pinching condition (P_C). Finally, note that at this point, we have assumed that $p \geqslant 2$ when we used the Hölder inequality. □

Lemma 6. *The pinching condition (P_C) with $C < \frac{n}{2}||H_r^F||_{2p}^2$ implies*

$$||\varphi||_2 \leqslant A_4 ||\varphi||_\infty^{3/4} C^{1/4},$$

where A_4 is a positive constant depending only on n, r, $||F||_\infty$, $||H^F||_\infty$ and $||H_r^F||_{2p}$.

Proof: We have

$$||\varphi||_2 = \frac{1}{\mathrm{V}(M)^{1/2}}\left(\int_M \varphi^{3/2}\varphi^{1/2}dv_g\right)^{1/2} \leqslant ||\varphi||_\infty^{3/4}||\varphi^{1/2}||_1^{1/2}.$$

Moreover,

$$\begin{aligned}\frac{1}{\mathrm{V}(M)}\int_M \varphi^{1/2}dv_g &= \left|\left||X|^{1/2}X - \sqrt{\frac{n}{\lambda_1(M)}}\frac{X}{|X|^{1/2}}\right|\right|_1\\ &= \left|\left| - \frac{|X|^{1/2}\mathrm{V}(M)}{\lambda_1(M)\int_M F(\nu)H_{r-1}^F dv_g}Y - \sqrt{\frac{n}{\lambda_1(M)}}Z\right|\right|_1\\ &\leqslant \left|\left|\frac{|X|^{1/2}\mathrm{V}(M)}{\lambda_1(M)\int_M F(\nu)H_{r-1}^F dv_g}Y\right|\right|_1 + \sqrt{\frac{n}{\lambda_1(M)}}||Z||_1.\end{aligned}$$

By the Hölder inequality, we get

$$\left|\left|\frac{|X|^{1/2}\mathrm{V}(M)}{\lambda_1(M)\int_M F(\nu)H_{r-1}^F dv_g}Y\right|\right|_1 \leqslant \frac{1}{\lambda_1(M)\int_M F(\nu)H_{r-1}^F dv_g}||X||_2^{1/2}||Y||_2$$

Since we assume that the pinching condition (P_C) holds for $C < \frac{n}{2}||H_r^F||_{2p}^2$, we have

$$\lambda_1(M)\left(\int_M F(\nu)H_{r-1}dv_g\right)^2 \geqslant \frac{n}{2}||H_r^F||_{2p}^2\mathrm{V}(M)^2,$$

and then

$$\begin{aligned}\left|\left|\frac{|X|^{1/2}\mathrm{V}(M)}{\lambda_1(M)\int_M F(\nu)H_{r-1}^F dv_g}Y\right|\right|_1 &\leqslant \frac{2||X||_2^{1/2}||Y||_2}{n||H_r^F||_{2p}^2}\\ &\leqslant \frac{A_1^{1/4}}{n^{1/2}||H_r^F||_{2p}^2}C^{1/2},\end{aligned}$$

where we used Lemmas 2 and 4. Finally, from this and Lemma 5, we obtain

$$||\varphi^{1/2}||_1^{1/2} \leqslant A_4C^{1/4},$$

where A_4 is a positive constant depending only on n, r, $||F||_\infty$, $||H^F||_\infty$ and $||H_r^F||_{2p}$. □

4.2. **Proof of Theorem 2.** Now, we can use these L^2 estimates to get L^∞ estimates by iterating processes. First, we have this lemma for the norm of the position vector X.

Lemma 7. *We have* $||X||_\infty^2 \leqslant \Gamma(n)\mathrm{V}(M)||H||_\infty^n||X||_2^2$, *where* $\Gamma(n)$ *is a constant depending only on* n.

The proof of the lemma 7 uses a Nirenberg-Moser type of proof (see [2, 3]) based on a Sobolev inequality due to Michael-Simon and Hoffman-Spruck (see [7], [8] and [11]).

Proof: Let us put $\phi = |X|$. An easy computation shows that $|d\phi^{2\alpha}| \leqslant 2\alpha\phi^{2\alpha-1}$. Hence, using the Sobolev inequality (see [7], [8] and [11])

$$\|f\|_{\frac{n}{n-1}} \leqslant K(n)\mathrm{V}(M)^{\frac{1}{n}}\left(\|df\|_1 + \|Hf\|_1\right) \tag{8}$$

we get for any $\alpha \geqslant 1$ and $f = \phi^{2\alpha}$

$$\|\phi\|^{2\alpha}_{\frac{2\alpha n}{n-1}} \leqslant K(n)\mathrm{V}(M)^{1/n}2\alpha||H||_\infty\|\phi\|_\infty\|\phi\|^{2\alpha-1}_{2\alpha-1}$$

Then putting $\beta = \frac{n}{n-1}$ and $\alpha = \frac{a_p+1}{2}$ where $a_{p+1} = (a_p+1)\beta$ and $a_0 = 2$ we have

$$\begin{aligned}\|\phi\|^{\frac{a_{p+1}}{\beta}}_{a_{p+1}} &\leqslant K(n)\mathrm{V}(M)^{\frac{1}{n}}(a_p+1)||H||_\infty\|\phi\|_\infty\|\phi\|^{a_p}_{a_p} \\ &\leqslant K(n)\mathrm{V}(M)^{\frac{1}{n}}a_{p+1}||H||_\infty\|\phi\|_\infty\|\phi\|^{a_p}_{a_p}\end{aligned}$$

Then by iterating we find

$$\begin{aligned}\|\phi\|^{\frac{a_{p+1}}{\beta^{p+1}}}_{a_{p+1}} &\leqslant \left(K(n)\mathrm{V}(M)^{\frac{1}{n}}a_{p+1}||H||_\infty\|\phi\|_\infty\right)^{1/\beta^p}\|\phi\|^{\frac{a_p}{\beta^p}}_{a_p} \\ &\leqslant \left(\prod_{i=0}^p a_{i+1}^{\frac{1}{\beta^i}}\right)\left(K(n)\mathrm{V}(M)^{\frac{1}{n}}||H||_\infty\|\phi\|_\infty\right)^{\sum_{i=1}^p\frac{1}{\beta^i}}\|\phi\|^{a_0}_{a_0}\end{aligned}$$

Now since $\frac{a_p}{\beta^p}$ converges to $a_0 + n = n+2$ and $\sum_{i=1}^\infty \frac{1}{\beta^i} = \frac{1}{1-1/\beta} = n$, we get

$$\|\phi\|^2_\infty \leqslant \Gamma(n)\mathrm{V}(M)||H||^n_\infty\|\phi\|^2_2,$$

where $\Gamma(n) = \prod_{i=0}^\infty a_{i+1}^{\frac{1}{\beta^i}}$ is a constant depending only on β, that is, only on n. □

Now, we can prove in the same way that under the pinching condition, the L^∞-norm of the function φ is controlled.

Lemma 8. *For $p \geqslant 2$ and any $\eta > 0$, there exists K_η depending on n, r, $||F||_\infty$, $||H||_\infty$, $||H_r^F||_{2p}$ and $\mathrm{V}(M)$ so that if (P_{K_η}) is true, then $||\varphi||_\infty \leqslant \eta$. Moreover, $K_\eta \longrightarrow 0$ when $||H||_\infty \longrightarrow \infty$ or $\eta \longrightarrow 0$.*

Proof: Let $\alpha \geqslant 1$ then

$$\begin{aligned}|d\varphi^{2\alpha}| &= \alpha\varphi^{2\alpha-2}|d\varphi^2| \\ &= \alpha\varphi^{2\alpha-2}\left||X| - \sqrt{\frac{n}{\lambda_1(M)}}\right|\left|3|X| - \sqrt{\frac{n}{\lambda_1(M)}}\right||d|X|| \\ &\leqslant 3\alpha\varphi^{2\alpha-2}\left(||H||_\infty + \sqrt{\frac{n}{\lambda_1(M)}}\right)^2\end{aligned}$$

Proceeding as in the proof of Lemma 7 we find that $|d\varphi^{2\alpha}| \leqslant \alpha E\varphi^{2\alpha-2}$ where $E = 3\left(||X||_\infty + \sqrt{\frac{n}{\lambda_1(M)}}\right)^2$. It follows that

$$\|\varphi\|^{2\alpha}_{\frac{2\alpha n}{n-1}} \leqslant K(n)\mathrm{V}(M)^{1/n}(\alpha E + \|\varphi\|^2_\infty||H||_\infty)\|\varphi\|^{2\alpha-2}_{2\alpha-2}$$

From this we deduce that

$$\|\varphi\|_{\frac{2\alpha n}{n-1}}^{2\alpha} \leqslant K(n)\mathrm{V}(M)^{1/n}\alpha E'\|\varphi\|_{2\alpha-2}^{2\alpha-2}$$

where $E' = E + ||X||_\infty \left(||X||_\infty ||H||_\infty + 1\right)^2$. Now we put $a_{p+1} = (a_p + 2)\nu$ with $\nu = \frac{n}{n-1}$, $a_0 = 1$ and $\alpha = \frac{a_p+2}{2}$. Then noting that $\frac{a_p}{\nu^p}$ converges to $a_0 + 2n$, the end of the proof is similar to that Lemma 7 and we find

$$\|\varphi\|_\infty^{1+2n} \leqslant K(n)\mathrm{V}(M)(E')^n||\varphi||_1 \leqslant K(n)\mathrm{V}(M)(E')^n||\varphi||_2.$$

Now from Lemma 7 we have that E' is a constant depending on n, r, $||F||_\infty$, $||H||_\infty$, $||H^F||_\infty$ and $||H_r^F||_{2p}$. Thus, this last inequality combined with Lemma 6 allows us to conclude that under the pinching condition (P_C), we have

$$\|\varphi\|_\infty \leqslant \left(K(n)\mathrm{V}(M)(E')^n A_4\right)^{\frac{4}{8n+1}} C^{\frac{1}{8n+1}}.$$

Now, we choose $C = K_\eta = \left(\dfrac{1}{K(n)\mathrm{V}(M)(E')^n A_4}\right)^4 \eta^{8n+1}$ and we get $||\varphi||_\infty \leqslant \eta$. Moreover, from the expression of E', we know that E' tends to $+\infty$ when $||H||_\infty$ tends to $+\infty$. Since the other factors of K_η do not depend on $||H||_\infty$, we get that K_η tends to 0 when $||H||_\infty$ tends to $+\infty$. Obvioulsy, we also have that K_η tends to 0 when η tends to 0. □

Now, we recall this geometric lemma proved by Colbois-Grosjean.

Lemma 9 ([2])**.** *Let x_0 be a point of the sphere $S(0, R)$ in $\mathbb{R}^{n+1}$ with the center at the origin and of radius R. Assume that $x_0 = Re$ with $e \in \mathbb{S}^n$. Now let (M^n, g) be a compact, connected, oriented n-diemsnional Riemannian manifold without boundary isometrically mmersed in $\mathbb{R}^{n+1}$. If the image of M is contained in $\Big(B(0, R+\eta) \setminus B(0, R-\eta)\Big) \setminus B(x_0, \rho)$ with $\rho = 4(2n-1)\eta$, then there exists a point $y_0 \in M$ so that the mean curvature of M in y_0 satisfies $|H(y_0)| \geqslant \frac{1}{4n\eta}$.*

With these last two lemmas, we are able to prove Theorem 1 using the last two lemmas. Let $\varepsilon > 0$ and consider the function

$$f(t) := t\left(t - \sqrt{\frac{n}{\lambda_1(M)}}\right)^2.$$

We set

$$\eta(\varepsilon) := \inf\left\{f\left(\sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon\right), f\left(\sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon\right), \frac{1}{27||H||_\infty^3}\right\}.$$

By definition, $\eta(\varepsilon) > 0$, and by Lemma 8, there exists $K_{\eta(\varepsilon)}$ such that for all $x \in M$,

$$f(|X|(x)) \leqslant \eta(\varepsilon). \tag{9}$$

Now to prove the theorem, it is sufficient to assume $\varepsilon < \frac{2}{3||H||_\infty}$. We will show that either

$$\sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon \leqslant |X| \leqslant \sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon \quad \text{or} \quad |X| < \frac{1}{3}\sqrt{\frac{n}{\lambda_1(M)}} \tag{10}$$

By examining the function f, it is easy to see that f has a unique local maximum at $\frac{1}{3}\sqrt{\frac{n}{\lambda_1(M)}}$. Moreover, from the definition of $\eta(\varepsilon)$, we have

$$\eta(\varepsilon) < \frac{4}{27||H||_\infty^3} \leqslant \frac{4}{27}\left(\frac{n}{\lambda_1(M)}\right)^{3/2} = f\left(\frac{1}{3}\sqrt{\frac{n}{\lambda_1(M)}}\right).$$

Since we assume $\varepsilon < \frac{2}{3||H||_\infty} \leqslant \frac{2}{3}\sqrt{\frac{n}{\lambda_1(M)}}$, we have

$$\frac{1}{3}\sqrt{\frac{n}{\lambda_1(M)}} < \sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon,$$

which with (9) yields (10).

Now, from Lemma 9, we deduce that there exists a point $y_0 \in M$ such that

$$|X(y_0)|^2 \geqslant \frac{n\lambda_1(M)\left(\int_M H_{k-1}\right)^4}{\left(K_{\eta(\varepsilon)} + \lambda_1(M)\left(\int_M H_{k-1}\right)^2\right)^2}.$$

Since $K_{\eta(\varepsilon)} < \frac{n}{2}||H_k||_{2p}^2$, the condition (P_C) implies

$$K_{\eta(\varepsilon)} < \frac{n}{2}||H_k||_{2p}^2 \leqslant \lambda_1(M)\left(\int_M H_{k-1}\right)^2 \leqslant 2\lambda_1(M)\left(\int_M H_{k-1}\right)^2.$$

We deduce that

$$|X(y_0)| \geqslant \frac{1}{3}\sqrt{\frac{n}{\lambda_1(M)}}.$$

Since M is connected, for any $x \in M$,

$$\sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon \leqslant |X|(x) \leqslant \sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon. \tag{11}$$

Now, we assume that the pinching condition (P_{C_ε}) holds with $C_\varepsilon = K_{\eta\left(\frac{\varepsilon}{4(2n-1)}\right)}$. Then (11) is still valid.

Let $x = \sqrt{\frac{n}{\lambda_1(M)}}\, e \in S\left(0, \sqrt{\frac{n}{\lambda_1(M)}}\right)$, with $e \in \mathbb{S}^n$ and assume that $B(x,\varepsilon) \cap M = \emptyset$. We can apply Lemma 9. So, there exists a point $y_0 \in M$ such that $|H(y_0)| \geqslant \frac{2n-1}{n\varepsilon} > ||H||_\infty$ since we assumed $\varepsilon < \frac{2}{3||H||_\infty} \leqslant \frac{2n-1}{n||H||_\infty}$. This is a contradiction and so $B(x,\varepsilon) \cap M \neq \emptyset$. This, in addition with (11) implies that the Hausdorff distance between $X(M)$ and the sphere $S\left(0, \sqrt{\frac{n}{\lambda_1}}\right)$ is smaller that ε. Moreover, $C_\varepsilon \longrightarrow 0$ when $||H||_\infty \longrightarrow \infty$ or $\varepsilon \longrightarrow 0$. $\square$

4.3. **Proof of Theorem 3.** We will prove Theorem 3. For this, we will use a part of Theorem 2, namely the fact the $X(M)$ is close to the sphere for the Hausdorff distance and in particular (11). Before that, we prove the following lemma.

Lemma 10. *Let $p \geqslant 2$ and $q > n$.*
We set $\Lambda = \max\Big(||H||_\infty \mathrm{V}(M)^{1/n}, ||B||_q \mathrm{V}(M)^{1/n}\Big)$. Then for any $\eta > 0$, there exists K_η depending on η, n, r, q, Λ, $||F||_\infty$, $||H^F||_\infty$ and $||H_r^F||_{2p}$ so that if (P_{K_η}) is true, then $||X^T||_\infty \leqslant \eta$.

Proof: We set $\xi = |X^T|$ and we have $\left|d\xi^{2\alpha}\right| = 2\alpha\xi^{2\alpha-2}\left(|\nabla r|^2 + r\,|d|\nabla r||\right)$. Now, let $x \in M$, we will estimate $|d|\nabla r||$ at this point x. For this, let $\{e_1 \cdots, e_n\}$ be an orthonormal frame of T_xM. We have

$$\begin{aligned}
|d|\nabla r|| &= \frac{1}{4}\left|d < \overline{\nabla} r, \nu >^2\right|^2 \\
&= \frac{< \overline{\nabla} r, \nu >^2}{|\nabla r|^2}\sum_{i=1}^n (e_i < \nabla r, \nu >)^2 \\
&= \frac{< \overline{\nabla} r, \nu >^2}{|\nabla r|^2}\left(\sum_{i=1}^n \left(\overline{\nabla} dr(e_i,\nu) + B(e_i,\nabla r)\right)\right)^2 \\
&\leqslant \frac{2}{|\nabla r|^2}\left(\sum_{i=1}^n \overline{\nabla} dr(e_i,\nu)^2 + |B|^2|\nabla r|^2\right)
\end{aligned}$$

Now $\displaystyle\sum_{i=1}^n \overline{\nabla} dr(e_i,\nu)^2 \leqslant |\overline{\nabla} dr|^2 \leqslant \sum_{i=1}^{n+1} \overline{\nabla} dr(u_i,u_i)$ where $(u_i)_{1\leqslant i\leqslant n+1}$ is an orthonormal basis which diagonalizes $\overline{\nabla} dr$. From the comparison theorems (see for instance [17] p. 153) we deduce that

$$\sum_{i=1}^{n+1} \overline{\nabla} dr(u_i,u_i)^2 \leqslant \frac{1}{r^2}\sum_{i=1}^{n+1} |u_i - \langle u_i, \overline{\nabla} r\rangle \overline{\nabla} r|^2 = \frac{n}{r^2}.$$

Then, we have

$$|d|\nabla r|| \leqslant \frac{2n}{r^2|\nabla r|^2} + 2|B|^2.$$

Therefore, we have

$$\begin{aligned}
|d\xi^{2\alpha}| &\leqslant 2\alpha\xi^{2\alpha-1}C(n)\left(1 + \frac{1}{|\nabla r|} + r|B|\right) \\
&\leqslant 2\alpha\xi^{2\alpha-1}C(n)\left(\frac{2}{|\nabla r|} + r|B|\right) \\
&\leqslant 2\alpha\xi^{2\alpha-1}C(n)||X||_\infty\left(\frac{2}{\xi} + |B|\right) \\
&\leqslant 2\alpha\xi^{2\alpha-2}C(n)||X||_\infty\left(2 + ||\xi||_\infty|B|\right) \qquad (12)
\end{aligned}$$

Now, assume that $\alpha \geqslant 1$ and apply the Sobolev inequality (8) to the function $\xi^{2\alpha}$. We have

$$\begin{aligned} ||\xi||^{2\alpha}_{\frac{2\alpha n}{n-1}} &\leqslant K(n)\mathrm{V}(M)^{1/n}||d\xi^{2\alpha}||_1 + ||H\xi||_1 \\ &\leqslant K(n)\mathrm{V}(M)^{1/n}\Big[2\alpha C(n)||X||_\infty||\xi||^{2\alpha-2}_{\frac{(2\alpha-2)q}{q-1}}(2+||X||_\infty||B||_q) \\ &\quad +||H||_\infty||X||^2_\infty||\xi||^{2\alpha-2}_{2\alpha-2}\Big], \end{aligned}$$

where we used (12) and the Hölder inequality.
Now, we have that $\max\left(||H||_\infty \mathrm{V}(M)^{1/n}, ||B||_q\mathrm{V}(M)^{1/n}\right) = \Lambda$. Since $\xi = |X^T| \leqslant |X|$, and using the fact that $||X||^2_\infty||H||^2_\infty \geqslant 1$, we get

$$||\xi||^{2\alpha}_{\frac{2\alpha n}{n-1}} \leqslant K'(n)\alpha A\Lambda||X||^2_\infty||\xi||^{2\alpha-2}_{\frac{(2\alpha-2)q}{q-1}},$$

where $K'(n)$ is a constant depending only on n. We use an iteration process comparable to those of Lemmas 7 and 8, by setting $\beta = \frac{n(q-1)}{(n-1)q}$, $a_0 = 2$ and $a_{p+1} = a_p\beta + \frac{2n}{n-1}$ to get

$$||\xi||^{\frac{a_{p+1}}{\beta^{p+1}}}_{a_{p+1}} \leqslant \left(\prod_{i=1}^{p+1} a_i^{\frac{1}{\beta^i}}\right)^{\frac{n}{n-1}} \left(K'(n)\Lambda||X||^2_\infty\right)^{\frac{n}{n-1}\left(\sum_i^{p+1}\frac{1}{\beta^i}\right)} ||\xi||^{a_0}_{a_0}$$

Since $\frac{a_p}{\beta^p} \longrightarrow a_0 + \frac{2nq}{q-n}$ when p goes to infinity, we get

$$||\xi||_\infty \leqslant C(n,q)\left(\Lambda||X||^2_\infty\right)^{\frac{\gamma}{2(1+\gamma)}}||\xi||_2^{\frac{1}{1+\gamma}}, \tag{13}$$

with $\gamma = \frac{nq}{q-n}$. Moreover, from Lemmas 2 and 7, we know that

$$||X||_\infty \leqslant \Gamma(n)\Lambda^{n/2}||X||_2 \leqslant \Gamma(n)A_1\Lambda^{n/2}, \tag{14}$$

where A_1 is a positive constant depending on n, $||F||_\infty$, $||H^F||_\infty$ and $||H^F_r||_{2p}$. Combing (13), (14) and Lemma 3, we get that if (P_C) holds, then

$$||\xi||_\infty = ||X^T||_\infty \leqslant A_5 C,$$

where A_5 is a positive constant depending on n, q, Λ, $||F||_\infty$, $||H^F||_\infty$ and $||H^F_r||_{2p}$. Now, let $\eta > 0$, we set $K_\eta = \left(\frac{\eta}{A_5}\right)^{1+\gamma}$ and if (P_{K_η}) holds, then $||X^T||_\infty \leqslant \eta$. By construction, K_η depends on η, n, q, Λ, $||F||_\infty$, $||H^F||_\infty$ and $||H^F_r||_{2p}$. This achieves the proof. □

Now, we will prove Theorem 3. Let $\varepsilon < \frac{1}{2}\sqrt{\frac{n}{||B||_q}} \leqslant \frac{1}{2}\sqrt{\frac{n}{\lambda_1(M)}}$. This choice of ε implies that if the pinching condition (P_{C_ε}) is true, then $|X|$ never vanishes, and so we can consider the following map

$$\begin{array}{rcl} G\,:\,M & \longrightarrow & S\left(0,\sqrt{\frac{n}{\lambda_1(M)}}\right)\\ x & \longmapsto & \sqrt{\frac{n}{\lambda_1(M)}}\,\frac{X}{|X|}. \end{array}$$

Without any pinching condition, a straightforward computation yields to

$$\left||dG_x(u)|^2-1\right|\leqslant\left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2}-1\right|+\frac{n}{\lambda_1(M)}\frac{1}{|X|^4}\left\langle u,X\right\rangle^2, \tag{15}$$

for any unitary vector $u\in T_xM$. But,

$$\left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2}-1\right|=\frac{1}{|X|^2}\left|\frac{n}{\lambda_1(M)}-|X|^2\right|\leqslant\varepsilon\frac{\sqrt{\frac{n}{\lambda_1(M)}}+|X|}{|X|^2}\leqslant\varepsilon\frac{2\sqrt{\frac{n}{\lambda_1(M)}}+\varepsilon}{\left(\sqrt{\frac{n}{\lambda_1(M)}}-\varepsilon\right)^2}$$

We recall that $\frac{n}{A_1}\leqslant\lambda_1\leqslant||B||_q^2$. Since we assume $\varepsilon<\frac{1}{2}\sqrt{\frac{n}{||B||_q}}$, the right hand side is bounded by a constant σ depending only on n, r, q, Λ, $||F||_\infty$, $||H^F||_\infty$ and $||H_r^F||_{2p}$. So we have

$$\left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2}-1\right|\leqslant\varepsilon\sigma. \tag{16}$$

Moreover, let $\eta>0$, since $C_\varepsilon\longrightarrow0$ when $\varepsilon\longrightarrow0$, there exists ε so that $C_\varepsilon\leqslant K_\eta$ (where K_η is the constant of Lemma 10) and so, $||X^T||_\infty\leqslant\eta$. Note that ε depends on n, r, q, Λ, $||F||_\infty$, $||H^F||_\infty$, $||H_r^F||_{2p}$ and η. As before, there exists a constant δ depending also on n, r, q, Λ, $||F||_\infty$, $||H^F||_\infty$ and $||H_r^F||_{2p}$ such that

$$\frac{n}{\lambda_1(M)}\frac{1}{|X|^4}\left\langle u,X\right\rangle^2\leqslant\frac{n}{\lambda_1(M)}\frac{1}{|X|^4}||X^T||_\infty^2\leqslant\eta^2\delta. \tag{17}$$

Then, from (15), (16) and (17), we deduce that (P_{C_ε}) implies

$$\left||dG_x(u)|^2-1\right|\leqslant\varepsilon\sigma+\eta^2\delta.$$

We fix $\theta\in(0,1)$ and we take $\eta=\sqrt{\frac{\theta}{2\delta}}$. We can assume that ε is small enough to have $\varepsilon\sigma\leqslant\frac{\theta}{2}$. Finally, we have proved that for any $\theta\in(0,1)$, there exists $\varepsilon>0$ depending on n, r, q, Λ, $||F||_\infty$, $||H^F||_\infty$, $||H_r^F||_{2p}$ and θ such that

$$\left||dG_x(u)|^2-1\right|\leqslant\theta.$$

Hence, G is θ-quasi-isometry. In particular, G is a local diffeomorphism from M into $S\left(0,\sqrt{\frac{n}{\lambda_1(M)}}\right)$. Since $S\left(0,\sqrt{\frac{n}{\lambda_1(M)}}\right)$ is simply connected for $n\geqslant2$, the map G is a global diffeomorphism. Theorem 3 is proved, since the dependence on Λ can be replaced by a dependence on $||H||_\infty$, $||B||_q$ and $\mathrm{V}(M)$, or as stated in Theorem 3, by $||B||_\infty$ and $\mathrm{V}(M)$. □

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