GENERAL REILLY-TYPE INEQUALITIES FOR SUBMANIFOLDS OF WEIGHTED EUCLIDEAN SPACES

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ABSTRACT. We prove new upper bounds for the first positive eigenvalue of a family of second order operators, including the Bakry-Émery Laplacian, for submanifolds of weighted Euclidean spaces.

1. INTRODUCTION

A weighted manifold $(\bar{M}, \bar{g}, \bar{\mu}_f)$ is a Riemannian manifold $(\bar{M}, \bar{g})$ endowed with a weighted volume form $\bar{\mu}_f = e^{-f} dv_{\bar{g}}$, where f is a real-valued smooth function on $\bar{M}$ and $dv_{\bar{g}}$ is the Riemannian volume form associated with the metric $\bar{g}$. In the present note, we will focus on the case where $(\bar{M}, \bar{g})$ is the Euclidean space $(\mathbb{R}^N, can)$ with its canonical flat metric and we will consider isometric immersions of Riemannian manifolds (M^n, g) into $(\mathbb{R}^N, can)$. For such an immersion, we define the weighted mean curvature vector $\mathbf{H}_f = \mathbf{H} - (\bar{\nabla} f)^{\perp}$, where $\mathbf{H}$ is the mean curvature vector of the immersion and $(\bar{\nabla} f)^{\perp}$ is the projection of $\bar{\nabla} f$ on the normal bundle $T^{\perp}M$.
We can define on M a divergence and a Laplace operator associated with the volume form $\mu_f = e^{-f} dv_g$ by

$$\mathrm{div}_f Y = \mathrm{div} Y - \langle \nabla f, Y \rangle \quad \text{and} \quad \Delta_f u = -\mathrm{div}_f(\nabla u) = \Delta u + \langle \nabla f, \nabla u \rangle,$$

where ∇ is the gradient on M, that is the projection on TM of $\bar{\nabla}$. We call them the f-divergence and the f-Laplacian which is often called Bakry-Émery Laplacian, Witten Laplacian or drifting Laplacian in the litterature. It is a classical fact that Δ_f has a discrete spectrum composed of an infinite sequence of nonnegative real numbers

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \longrightarrow +\infty.$$

The eigenvalue $\lambda_0 = 0$ has multiplicity one and corresponds to constant functions.
In [4], Batista, Cavalcante and Pyo proved the following upper bound for the first positive eigenvalue of Δ_f:

$$\lambda_1(\Delta_f) \leqslant \frac{\int_M ||\mathbf{H}_f - \bar{\nabla} f||^2 \mu_f}{n\mathrm{Vol}_f(M)} = \frac{\int_M (||\mathbf{H}||^2 + ||\nabla f||^2) \mu_f}{n\mathrm{Vol}_f(M)},$$

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where $\mathrm{Vol}_f(M) = \int_M \mu_f$ is the f-volume of M. This inequality is a weighted version of the classical Reilly inequality (see [9])

$$\lambda_1(\Delta) \leqslant \frac{1}{n\mathrm{Vol}(M)} \int_M ||\mathbf{H}||^2 dv_g.$$

Very recently, Domingo-Juan and Miquel [6] obtained the same inequality with a more complete characterization of the equality case by the use of mean curvature flow.

The aim of this note is to give a general inequality, which contains the above one, for a larger class of f-divergence-type operators. Precisely, for a positive symmetric divergence-free $(1,1)$-tensor T, we define the operator $L_{T,f}$ by

$$L_{T,f}u = -\mathrm{div}_f(T\nabla u),$$

for any $\mathcal{C}^2$ fonction u on M. We prove the following theorem.

Theorem 1.1. *Let (M^n, g) be a connected and oriented closed Riemannian manifold isometrically immersed into the Euclidean space $\mathbb{R}^N$ endowed with a density e^{-f}. Let S and T be two symmetric divergence-free $(1,1)$-tensor over M. Assume moreover that T is positive. Then, the first positive eigenvalue of the operator $L_{T,f}$ satisfies the following inequality*

$$\lambda_1(L_{T,f}) \left(\int_M \mathrm{tr}(S)\mu_f \right)^2 \leqslant \left(\int_M \mathrm{tr}(T)\mu_f \right) \int_M \left(||H_S||^2 + ||S\nabla f||^2 \right)\mu_f.$$

Moreover, if equality holds in the case $S = \mathrm{Id}$ then M is a self-shrinker for the mean curvature flow and $f_{|M} = a - \frac{c}{2}r_p^2$, where r_p is the Euclidean distance to the center of mass p of M. In particular, if $n = N-1$ and $H > 0$ or $n = 2$, $N = 3$ and M is embedded and has genus 0, then M a geodesic hypersphere.

As a corollary, we obtain a similar inequality for submanifolds of the sphere $\mathbb{S}^N$ which generalizes the corresponding inequality of [4] and [6] for the operator $L_{T,f}$ (see Corollary 4.4). We also prove a general non-weighted Reilly-type inequality (Theorem 5.1).

2. Preliminaries

Let (M^n, g) be a connected and oriented closed Riemannian manifold isometrically immersed into $\mathbb{R}^N$. We denote by X its position vector, B its second fundamental form and $\mathbf{H} = \mathrm{tr}(B)$ its mean curvature vector. For the case of hypersurfaces, we will also consider the real-valued mean curvature $H = \langle \mathbf{H}, \nu \rangle$, where ν is a unit normal vector field (H is defined up to a sign depend of the choice of ν). We denote by $\{\partial_1, \cdots, \partial_N\}$ the canonical frame of $\mathbb{R}^N$ and for $k \in \{1, \cdots, N\}$, $X^k = \langle X, \partial_k \rangle$ the coordinate functions. We begin by giving the following elementary lemma.

Lemma 2.1. *If A is a field of endomorphisms on M, we have*

$$\sum_{k=1}^{N}\langle A(\nabla X^k), \nabla X^k\rangle = \mathrm{tr}(A).$$

Proof: Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of TM. It is a classical fact that $\nabla X^k = \partial_k^\top = \sum_{i=1}^n \langle \partial_k, e_i\rangle e_i$. Hence, we have

$$\begin{aligned}\sum_{k=1}^{N}\langle A(\nabla X^k), \nabla X^k\rangle &= \sum_{k=1}^{N}\sum_{i,j=1}^{n}\langle \partial_k, e_j\rangle\langle \partial_k, e_j\rangle\langle Ae_i, e_j\rangle\\ &= \sum_{i,j=1}^{n}\left(\sum_{k=1}^{N}\langle \partial_k, e_j\rangle\langle \partial_k, e_j\rangle\right)\langle Ae_i, e_j\rangle\\ &= \sum_{i,j=1}^{n}\langle e_i, e_j\rangle\langle Ae_i, e_j\rangle = \mathrm{tr}(A).\end{aligned}$$

□

Note that, in particular, for $A = \mathrm{Id}$, we recover the well known identity $\sum_{k=1}^N ||\nabla X^k||^2 = n$.

Then, we recall briefly by some basic facts about the f-divergence. We first have the weighted version of the divergence theorem:

$$\int_M \mathrm{div}_f Y \mu_f = 0, \tag{1}$$

for any vector field Y on M. From this, we deduce easily the integration by parts formula

$$\int_M u\mathrm{div}_f Y \mu_f = -\int_M \langle \nabla u, X\rangle \mu_f, \tag{2}$$

for any smooth fonction u and any vector field Y on M.

Now, let T be a divergence-free symmetric $(1,1)$-tensor. We associate with T the second order differential operator L_T defined by $L_T u := -\mathrm{div}(T\nabla u)$, for any $\mathcal{C}^2$ function u on M. We also associate with T the following normal vector field:

$$H_T = \sum_{i,j=1}^{n} T(e_i, e_j)B(e_i, e_j), \tag{3}$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame of TM. We also defined a corresponding weighted operator by $L_{T,f}u = -\mathrm{div}_f(T\nabla u)$ for any $\mathcal{C}^2$ function u . We have the following weighted Hsiung-Minkowski formula.

Lemma 2.2. *We have*

$$\int_M \big(\langle X, H_T - T\nabla f\rangle + \mathrm{tr}(T)\big)\mu_f = 0.$$

Proof: First, it is well known that $L_T X = -H_T$. The proof of this fact is standard and completely analogue to the case $T = \mathrm{Id}$, that is, $\Delta X = -n\mathbf{H}$ and uses the fact that $\mathrm{div}(T) = 0$. From this, we deduce

$$\begin{aligned} L_T||X||^2 &= \sum_{k=1}^{N} L_T((X^k)^2) \\ &= -2\sum_{k=1}^{N} \mathrm{div}(X^k T(\nabla X^k)) \\ &= 2\sum_{k=1}^{N} \big(X^k L_T X^k - \langle \nabla X^k, T(\nabla X^k)\rangle\big) \\ &= -2\langle X, H_T\rangle - 2\mathrm{tr}(T), \end{aligned}$$

where we have used $L_T X = -H_T$ and Lemma 2.1 for the last line. Therefore, we get

$$\begin{aligned} \frac{1}{2}L_{T,f}||X||^2 &= \frac{1}{2}L_T||X||^2 + \frac{1}{2}\langle T(\nabla||X||^2), \nabla f\rangle \\ &= -\langle X, H_T\rangle - \mathrm{tr}(T) + \frac{1}{2}\langle \nabla||X||^2, T\nabla f\rangle, \\ &= -\langle H_T - T\nabla f, X\rangle - \mathrm{tr}(T) \end{aligned}$$

where we have used (4), the symmetry of T and the fact that $\nabla||X||^2 = 2X^\top$. We conclude by integrating over M for the measure μ_f and using the fact that $\int_M L_{T,f}||X||^2\mu_f = 0$ by (1). □

We can obtain a weighted Hsiung-Minkowski inequality by the use of the operator $L_{T,f}$. Namely, we prove the following lemma.

3. PROOF OF THEOREM 1.1

Now, we have all the ingredients to prove the main theorem of this note. First, since we assume that the tensor T is positive, the operator $L_{T,f}$ has a discrete nonnegative spectrum. The first eigenvalue is $\lambda_0 = 0$ is of mulitplicity one and the associated eigenfunctions are the constants. Thus, we denote by $\lambda_1(L_{T,f})$ its first positive eigenvalue. From the definition of $L_{T,f}$ and (2) we have the following the variational characterization of $\lambda_1(L_{T,f})$

$$\lambda_1(L_{T,f}) = \inf\left\{ \frac{\int_M \langle T\nabla u, \nabla u\rangle\mu_f}{\int_M u^2\mu_f} \;\Big|\; u \in \mathcal{C}^\infty(M),\ \int_M u\mu_f = 0 \right\}.$$

Up to a translation if needed, we may assume that the μ_f-center of mass of M is zero, that is, $\int_M X\mu_f = \vec{0}$. Hence, the coordinates can be used as test functions in the Rayleigh quotient and we have

$$\lambda_1(L_{T,f})\int_M ||X||^2\mu_f \quad\leqslant\quad \int_M\sum_{i=1}^N \langle T\nabla X^k, X^k\rangle\mu_f,$$

which gives, by Lemma 2.1,

$$\lambda_1(L_{T,f})\int_M ||X||^2\mu_f \quad\leqslant\quad \int_M \mathrm{tr}(T)\mu_f. \tag{4}$$

Now, we have

$$\begin{aligned}\lambda_1(L_{T,f})\left(\int_M \mathrm{tr}(S)\mu_f\right)^2 &\leqslant \lambda_1(L_{T,f})\left(\int_M (\langle X, H_S - S\nabla f\rangle)\mu_f\right)^2\\ &\leqslant \lambda_1(L_{T,f})\left(\int_M ||X||^2\mu_f\right)\left(\int_M ||H_S - S\nabla f||^2\mu_f\right)\\ &\leqslant \left(\int_M \mathrm{tr}(T)\mu_f\right)\left(\int_M ||H_S - S\nabla f||^2\mu_f\right),\end{aligned}$$

where we have used succesively the weighted Hsiung-Minkowski formula, the Cauchy-Schwarz inequality and (4). Since H_S is normal and $S\nabla f$ is tangent to M, we get the wanted upper bound

$$\lambda_1(L_{T,f})\left(\int_M \mathrm{tr}(S)\mu_f\right)^2 \leqslant \left(\int_M \mathrm{tr}(T)\mu_f\right)\int_M \left(||H_S||^2 + ||S\nabla f||^2\right)\mu_f.$$

Equality case. Now, we assume that $S = \mathrm{Id}$. Then, the inequality becomes

$$\lambda_1(L_{T,f}) \leqslant \left(\int_M \mathrm{tr}(T)\mu_f\right)\int_M \left(||\mathbf{H}||^2 + ||\nabla f||^2\right)\mu_f.$$

If, equality occurs then all the above inequalities are equalities. In particular, equality occurs in the Cauchy-Schwarz inequality and we have $\mathbf{H} - \nabla f = cX$ for some constant c. Identifying tangential and normal parts, we get $\nabla f = -cX^\top$ and $\mathbf{H} = cX^\perp$.

The normal equation $\mathbf{H} = cX^\perp$ is exactly the definition of a self-similar solution of the mean curvature flow. Since M is a compact submanifold of $\mathbb{R}^N$, c cannot be zero. The case $c > 0$ is no more possible. Indeed, if $c > 0$, then M is a self-expander, but it is well known that there exists no compact self-expander. Hence, the only possibility is $c < 0$, that is M is a self-shrkiner.

In addition, since $X^\top = \frac{1}{2}\nabla||X||^2$, the tangential equation becomes $\nabla(f + \frac{c}{2}||X||^2) = 0$. Since M is connected, there exists a constant a such that $f_{|M} = a - \frac{c}{2}||X||^2$.
In the particular cases $N = n-1$ and $H > 0$ or $n = 2$, $N = 3$ and M is embedded

and has genus 0, then we know from [8] and [5] respectively that M has to be a geodesic hypersphere. This finishes the proof of the equality case.

4. Some corollaries

In this secction, we state some corollaries obtained from Theorem 1.1. The first corollary is just a particular case of Theorem 1.1 invovling higher order mean curvatures. We before stating it, we recall briefely the definition of higher order mean curvatures and their associated tensors. For $r \in \{1, \cdots, n\}$, we set

$$T_r = \frac{1}{r!} \sum_{\substack{i, i_1, \cdots, i_r \\ j, j_1, \cdots, j_r}} \epsilon \begin{pmatrix} i, i_1, \cdots, i_r \\ j, j_1, \cdots, j_r \end{pmatrix} \langle B_{i_1 j_1} B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}} B_{i_r j_r} \rangle e_i^* \otimes e_j^*,$$

if r is even and

$$T_r = \frac{1}{r!} \sum_{\substack{i, i_1, \cdots, i_r \\ j, j_1, \cdots, j_r}} \epsilon \begin{pmatrix} i, i_1, \cdots, i_r \\ j, j_1, \cdots, j_r \end{pmatrix} \langle B_{i_1 j_1} B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}} B_{i_r j_r} \rangle B_{i_r, j_r} \otimes e_i^* \otimes e_j^*,$$

where the B_{ij}'s are the coefficients of the second fundamental form B in a local orthonormal frame $\{e_1, \cdots, e_n\}$ and ϵ is the standard signature for permutations. Here, $\{e_1^*, \cdots, e_n^*\}$ is the dual coframe of $\{e_1, \cdots, e_n\}$. By definition, the r-th mean curvature is $H_r = \frac{1}{c(r)} \mathrm{tr}(T_r)$, where $c(r) = (n-r)\binom{r}{n}$. Note that H_r is a real function if r is even and a normal vector field if r is odd. By convention, we set $H_0 = 1$. Moreover, always if r is even, we show easily that $H_{T_r} = c(r) H_{r+1}$, where H_{T_r} is given by the relation (3).

In the case of hypersurfaces, we can consider the higher order mean curvatures as scalar functions also for odd indices by taking B as the real-valued second fundamental form.

By the symmetry of B, these tensors are clearly symmetric. Morover, we have the following well-known lemma (the proof of this lemma can be found in [7] for instance).

Lemma 4.1. (1) *If* $n = N-1$, *then for any* $r \in \{0, \cdots, n-1\}$, *we have* $\mathrm{div}(T_r) = 0$.
(2) *If* $n \leqslant N-2$, *then for any even* $r \in \{0, \cdots, n-1\}$, *we have* $\mathrm{div}(T_r) = 0$.

The tensor T_r is the linearized operator associated with the r-th mean curvature and plays a crucial role in the study of the r-stability of hypersurfaces with constant r-th mean curvature (see [1] for instance).

We can state the following corollary obtained immediately form Theorem 1.1, since the tensors T_r are divergence-free. Note that this corollary is a weighted version of an inequality of Alias and Malacarne [2].

Corollary 4.2. *Let (M^n, g) be a connected and oriented closed Riemannian manifold isometrically immersed into the Euclidean space $\mathbb{R}^N$ endowed with a density e^{-f}. Let $r, s \in \{1, \cdots, n-1\}$. Assume that r and s are even if $N > n-1$ and assume moreover that T_r is positive. Then, the first positive eigenvalue of the operator $L_{r,f} = L_{T_r,f}$ satisfies the following inequality*

$$\lambda_1(L_{r,f}) \left(\int_M H_s \mu_f \right)^2 \leqslant \frac{c(r)}{c(s)} \left(\int_M H_r \mu_f \right) \int_M \Big(c(s)^2 ||H_{s+1}||^2 + ||T_s \nabla f||^2 \Big) \mu_f.$$

Remark 4.3. *In the case of hypersurfaces, it is sufficient to have $H_{r+1} > 0$ to ensure that T_r is positive (see* [3] *for instance).*

Now, using the embedding of the sphere $\mathbb{S}^N$ into the Euclidean space $\mathbb{R}^{N+1}$, we can prove this second corollary for submanifolds of the sphere $\mathbb{S}^N$. Precisely, we have the following result.

Corollary 4.4. *Let (M^n, g) be a connected and oriented closed Riemannian manifold isometrically immersed into the sphere $\mathbb{S}^N$ endowed with a density e^{-f}. Let S and T be two symmetric divergence-free $(1,1)$-tensor over M. Assume moreover that T is positive. Then, the first positive eigenvalue of the operator $L_{T,f}$ satisfies the following inequality*

$$\lambda_1(L_{T,f}) \left(\int_M \mathrm{tr}(S) \mu_f \right)^2 \leqslant \left(\int_M \mathrm{tr}(T) \mu_f \right) \int_M \big(||H_S||^2 + \mathrm{tr}(S)^2 + ||S \nabla f||^2 \big) \mu_f.$$

Proof: The proof comes easily form Theorem 1.1. We denote by ϕ the immersion of M into $\mathbb{S}^N$ and we consider the canonical immersion i of $\mathbb{S}^N$ into $\mathbb{R}^{N+1}$ and we extend the weight f defined on $\mathbb{S}^N$ to a weight $\widetilde{f}$ on $\mathbb{R}^{N+1}$, for instance by taking $\widetilde{f}(x) = |x| f\left(\frac{x}{|x|}\right)$ for any $x \in \mathbb{S}^N$ and $\widetilde{f}(0) = 0$. From Theorem 1.1 we have

$$\lambda_1(L_{T,f}) \left(\int_M \mathrm{tr}(S) \mu_f \right)^2 \leqslant \left(\int_M \mathrm{tr}(T) \mu_f \right) \int_M \big(|H'_S|^2 + |S \nabla \widetilde{f}|^2 \big) \mu_f, \tag{5}$$

where H'_S is defined by $H_S = \sum_{i,j=1}^n S(e_i, e_j) B'(e_i, e_j)$ with B' the second fundamental form of the immersion of M into $\mathbb{R}^{N+1}$. Obviously, the second fundamental forms B of ϕ and B' of $i \circ \phi$ are linked by the relation $B' = B - g\phi$. Hence, we get immediately $H'_S = H_S - \mathrm{tr}(S)\phi$. Therefore, we deduce that $||H'_S||^2 = ||H_S||^2 + \mathrm{tr}(S)^2$, since H_S and ϕ are orthogonal and $||\phi|| = 1$ since M is contained in the sphere $\mathbb{S}^N$. Reporting this in (5), and since f coincides with $\widetilde{f}$ on M, we have $\nabla \widetilde{f} = \nabla f$ and so

$$\lambda_1(L_{T,f}) \left(\int_M \mathrm{tr}(S) \mu_f \right)^2 \leqslant \left(\int_M \mathrm{tr}(T) \mu_f \right) \int_M \big(||H_S||^2 + \mathrm{tr}(S)^2 + ||S \nabla f||^2 \big) \mu_f.$$

This concludes the proof. □

For submanifolds of spheres, we have immediately the following corollary involving higher order mean curvatures.

Corollary 4.5. *Let (M^n, g) be a connected, oriented closed Riemannian manifold isometrically immersed into the sphere $\mathbb{S}^N$ endowed with a density e^{-f}. Let $r, s \in \{1, \cdots, n-1\}$. Assume that r and s are even if $N > n-1$ and assume moreover that T_r is positive. Then, the first eigenvalue of the operator $L_{r,f}$ satisfies the following inequality*

$$\lambda_1(L_{r,f})\left(\int_M H_s\mu_f\right)^2 \leqslant \frac{c(r)}{c(s)}\left(\int_M H_r\mu_f\right)\int_M\left(c(s)^2||H_{s+1}||^2+c(s)^2H_s^2+||T_s\nabla f||^2\right)\mu_f.$$

5. A GENERAL NON-WEIGHTED INEQUALITY

In the classical case, that is, without density, the equality case can be characterized in a more rigid way. Namely, we have the following result

Theorem 5.1. *Let (M^n, g) be a connected, oriented closed Riemannian manifold isometrically immersed into $\mathbb{R}^N$. Assume that M is endowed with two symmetric and divergence-free $(1,1)$-tensors S et T. Assume in addition that T is positive definite. Then, the first positive eigenvalue of the operator L_T satisfies*

$$\lambda_1(L_T)\left(\int_M \mathrm{tr}(S)dv_g\right)^2 \leqslant \left(\int_M \mathrm{tr}(T)dv_g\right)\left(\int_M ||H_S||^2dv_g\right). \tag{6}$$

Moreover, if $N > n-1$ and H_S does not vanish identically and equality occurs, then $\mathrm{tr}(S)$ and $||H_S||$ are non-zero constants and M is S-minimally immersed into a geodesic hypersphere of $\mathbb{R}^N$ of radius $\frac{|\mathrm{tr}(S)|}{||H_S||}$.
In particular, if $n = N-1$ and H_S does not vanish identically then if equality holds, then $\mathrm{tr}(S)$ and H_S are non-zero constants and M is a geodesic hypersphere of radius $\frac{|\mathrm{tr}(S)|}{|H_S|}$.

Remarks 5.2.
(1) *Note that for this theorem, contrary to Theorem 1.1, we do not need to assume that M is embedded to characterize the equality case, the embedding is obtained as a consequence.*
(2) *For $T = \mathrm{Id}$, we have*

$$\lambda(\Delta)\left(\int_M \mathrm{tr}(S)dv_g\right)^2 \leqslant n\mathrm{Vol}(M)\left(\int_M ||H_S||^2dv_g\right),$$

which was proved by Grosjean in [7].

Proof: The inequality is immediate form Theorem 1.1 with f identically zero. If equality occurs, then all the above inequalities in the proof of Theorem 1.1 become equalities. In particular, we have $H_S = cX$ from the equality case of Cauchy-Schwarz

inequality, where c is a non-zero constant. This means that the position vector X is everywhere normal to M. But, on the other hand, since $\nabla||X||^2 = 2X^\top$, we get that $\nabla||X||^2 = 0$. Hence, since M is connected, then $||X|| = r$ is constant and M lies in a geodesic hypersphere of radius r. Moreover, since $H_S = cX$, we get that $||H_S||$ is also constant and from Equation (4), we conclude that $\mathrm{tr}(S) = -\langle X, H_S\rangle = -\frac{1}{c}||H_S||^2$. Thus, $\mathrm{tr}(S)$ is also constant. Note that, since we assume that H_S does not vanish identically, $\mathrm{tr}(S)$ and $||H_S||$ are non-zero constants and we have $r = \frac{|\mathrm{tr}(S)|}{||H_S||}$.
Now, we will show that the immersion of M in this hypersphere $\mathbb{S}^{N-1}(r)$ is S-minimal, that is, $\widetilde{H}_S = 0$, where is defined by

$$H_S = \sum_{i,j=1}^{n} S(e_i, e_j)\widetilde{B}(e_i, e_j),$$

with $\widetilde{B}$ the second fundamental form of M in $\mathbb{S}^{N-1}(r)$. Clearly, we have $B = \widetilde{B} + \overline{B}$ where $\overline{B}$ is the second fundamental form of $\mathbb{S}^{N-1}$ into $\mathbb{R}^N$ and is given by $\overline{B}_{ij} = -\frac{1}{r^2}\delta_{ij}X$. From this fact and the definition of H_S and $\widetilde{H}_S$, we get

$$\begin{aligned} H_S &= \widetilde{H}_S - \frac{1}{r^2}\sum_{i,j}^{n} S(e_i, e_j)\delta_{ij}X \\ &= \widetilde{H}_S - \frac{1}{r^2}\mathrm{tr}(S)X \\ &= \widetilde{H}_S - \frac{|H_S|^2}{\mathrm{tr}(S)}X \\ &= \widetilde{H}_S + cX = \widetilde{H}_S + H_S. \end{aligned}$$

We deduce that $\widetilde{H}_S = 0$, that is M is S-minimally immersed into $\mathbb{S}^{N-1}(r)$.
If $n = N-1$, if equality occurs, by the above discussion and since M has no boundary, then M is $\mathbb{S}^{N-1}(r)$. This concludes the proof. □

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