A FUNDAMENTAL THEOREM FOR SUBMANIFOLDS OF MULTIPRODUCTS OF REAL SPACE FORMS

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Abstract. We prove a Bonnet theorem for isometric immersions of submanifolds into the products of an arbitrary number of simply connected real space forms. Then, we prove the existence of associate families of minimal surfaces in such products. Finally, in the case of $\mathbb{S}^2 \times \mathbb{S}^2$, we give a *complex* version of the main theorem in terms of the two canonical complex structures of $\mathbb{S}^2 \times \mathbb{S}^2$.

1. Introduction

It is a classical problem of submanifold theory to determine when a Riemannian manifold (M^n, g) can be isometrically immersed into a fixed Riemannian manifold $(\bar{M}^{n+p}, \bar{g})$. The well-known Gauss, Ricci and Codazzi equations relate the intrinsic and extrinsic curvatures, and any submanifold of any Riemannian manifold must satisfy them. Conversely, the classical Bonnet theorem [2] states that on a surface, given first and second fundamental forms satisfying the Gauss and Codazzi equations, this surface is locally isometrically embeddable into the Euclidean 3-space $\mathbb{R}^3$. This result can be generalized to higher dimension and codimension [13], and the classical Fundamental Theorem of Submanifolds states that, in fact, the Gauss, Codazzi and Ricci equations are necessary and sufficient conditions for a Riemannian n-dimensional manifold to admit a (local) immersion into a space of constant sectional curvature of dimension $n + d$.

If the ambient space is not of constant sectional curvature, proving fundamental theorems is technically difficult and there are few results known. Moreover, the Gauss, Codazzi and Ricci equations are in general not sufficient anymore and other conditions are required in order to produce the immersion. In [5], Daniel gave such a characterization for surfaces in the three-dimensional Thurston geometries with four-dimensional isometry groups, by computing the Christoffel symbols explicitly and using the technique of Cartan moving frames. Using this fundamental theorem, Daniel obtained a generalized Lawson correspondence for surfaces in homogeneous 3-space, which appears to be a powerful tool to study surfaces in these spaces and construct new interesting examples. In higher dimensions, he also stated in [4] necessary and sufficient conditions for an n-dimensional Riemannian manifold to be isometrically immersible into the products $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, also using the moving frame technique. This allowed him to study the existence of associate families in the case of minimal surfaces. This result was later generalized by the second author [12] in the case where the ambient space is a Lorentzian product. Very recently, Ortega and the first author [10] proved fundamental theorems characterizing immersions of hypersurfaces into (quasi-)Einstein manifolds, specifically Robertson-Walker warped products. These spaces play an important role in standard models

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of cosmology, arising as solutions of the non-vacuum Einstein equations, and have therefore a great importance in Lorentzian geometry. As an application, conditions were obtained for 3-dimensional hypersurfaces in Robertson-Walker spacetimes to be foliated by surfaces whose mean curvature vector is either lightlike or zero (including maximal surfaces, marginally outer trapped surfaces (MOTS), and mixed cases), hence providing an helpful tool for the study of horizons on Robertson-Walker spacetimes with spacelike or timelike causal character, including marginally outer trapped tubes.
Extending the result of [4], Kowalczyk and Lira-Tojeiro-Vitório proved independently in [9] and [11] the existence and uniqueness of isometric immersions in a product of two spaces forms of constant sectional curvature. In this paper we generalize their result to immersions into multiproducts $\widetilde{P} = M_1 \times \cdots \times M_m$ of real space forms of arbitrary dimension and arbitrary sectional curvature. We follow the technique used by Kowalczyk with the additional key idea to use the projections π_i, $i \in 1, \ldots, m$ into each factor of the product instead of the so called *product structure* in the case of a product of two space forms. Each projection induces then two operators on the tangent bundle and two operators on the normal bundle of the submanifold satisfying some properties and some compatibility equations which can be deduced from the Gauss and Weingarten formulas of the immersion. We prove that, conversely, these conditions together with the Gauss, Codazzi and Ricci equations are necessary and sufficient conditions to immerse a Riemannian manifold isometrically into such an ambient space. The new class of ambient manifolds we study in this paper is of particular interest for general relativity: in fact they are particular examples of generalized Kasner spacetime, which are exact solutions to the vacuum Einstein equations (see for example [8]).
As an application we then prove the existence of a one-parameter associate family of isometric immersions for minimal surfaces in multiproducts. Finally we consider the special case where the ambient space is $\mathbb{S}^2 \times \mathbb{S}^2$ and give a complex version of our fundamental theorem in terms of the induced complex structures. The examples of complex and Lagrangian surfaces are detailed.

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2. Multiproducts of space forms and their submanifolds

We consider the product space $(\widetilde{P} = M_1 \times \cdots \times M_m, \widetilde{g} = g_1 \oplus \cdots \oplus g_m)$ where (M_i, g_i) is the simply connected real space form of dimension n_i and constant sectional curvature c_i. Moreover, without loss of generality, we assume that $c_i \neq 0$ for $i \in \{1, \cdots, m-1\}$ and that c_m may possibly be zero. We denote by π_i the projection of any tangent vector X on TM_k. These projections satisfy the following relations

$$\begin{cases} \widetilde{g}(\pi_i X, Y) = \widetilde{g}(X, \pi_i Y) \text{ for any } X, Y \in \Gamma(TP), \\ \pi_i \circ \pi_i = \pi_i, \\ \pi_i \circ \pi_j = 0 \text{ if } i \neq j, \\ \widetilde{\nabla} \pi_i = 0, \\ rank(\pi_i) = n_i, \\ \sum_{i=1}^m \pi_i = \mathrm{Id}_{T\widetilde{P}}. \end{cases}$$

Moreover, the curvature tensor of $\widetilde{P}$ is given by

$$\widetilde{R}(X,Y)Z = \sum_{i=1}^m c_i \big[\langle \pi_i Y, \pi_i Z \rangle \pi_i X - \langle \pi_i X, \pi_i Z \rangle \pi_i Y \big]. \tag{1}$$

Now, we consider a Riemannian manifold (M^n, g) isometrically immersed by φ into $\widetilde{P}$. We denote by $T^\perp\varphi(M)$ the normal bundle, by $\nabla^\perp$ the normal connection and by $B : TM \times TM \longrightarrow T^\perp\varphi(M)$ the second fundamental form. For any $\nu \in T^\perp\varphi(M)$, A_ν is the Weingarten operator associated to ν and defined by $\widetilde{g}(A_\nu X, Y) = \widetilde{g}(B(X,Y), \nu)$, with X, Y some vectors tangent to M.

For any $i \in \{1, \cdots, m\}$, the projection π_i induces the existence of the following four operators $f_i : TM \to TM$, $h_i : TM \to T^\perp\varphi(M)$, $s_i : T^\perp\varphi(M) \to TM$ and $t_i : T^\perp\varphi(M) \to T^\perp\varphi(M)$, such that

$$\pi_i X = f_i X + h_i X \quad \text{and} \quad \pi_i \nu = s_i \nu + t_i \nu. \tag{2}$$

From the symmetry of the π_i, we obtain that, for any $i \in \{1, \cdots, m\}$, f_i and t_i are symmetric and for any $X \in \Gamma(TM)$ and $\nu \in \Gamma(T^\perp\varphi(M))$

$$\widetilde{g}(h_i X, \nu) = \widetilde{g}(X, s_i \nu). \tag{3}$$

In addition, from the fact that $\sum_{i=1}^{m} \pi_i = Id_{T\widetilde{P}}$, we get the following identities

$$\sum_{i=1}^{m} f_i = Id_{TM}, \quad \sum_{i=1}^{m} t_i = Id_{T^\perp\varphi(M)}, \quad \sum_{i=1}^{m} s_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} h_i = 0. \tag{4}$$

Moreover, we have the following relations between these operators coming from the fact that $\pi_i \circ \pi_j = \delta_i^j \pi_i$

$$f_i \circ f_j + s_i \circ h_j = \delta_i^j f_i, \tag{5}$$

$$t_i \circ t_j + h_i \circ s_j = \delta_i^j t_i, \tag{6}$$

$$f_i \circ s_j + s_i \circ t_j = \delta_i^j s_i, \tag{7}$$

$$h_i \circ f_j + t_i \circ h_j = \delta_i^j h_i, \tag{8}$$

where δ_i^j is the classical Kronecker symbol, that is, 1 if $i = j$ and 0 if $i \neq j$. Moreover, from the fact that π_i is parallel, we deduce easily that for any $X, Y \in TM$ and $\nu \in T^\perp\varphi(M)$, we have

$$\nabla_X(f_i Y) - f_i(\nabla_X Y) = A_{h_i Y} X + s_i(B(X,Y)), \tag{9}$$

$$\nabla^\perp_X(h_i Y) - h_i(\nabla_X Y) = t_i(B(X,Y)) - B(X, f_i Y), \tag{10}$$

$$\nabla^\perp_X(t_i \nu) - t_i(\nabla^\perp_X \nu) = -B(s_i \nu, X) - h_i(A_\nu X), \tag{11}$$

$$\nabla_X(s_i \nu) - s_i(\nabla^\perp_X \nu) = -f_i(A_\nu X) + A_{t_i \nu} X. \tag{12}$$

Finally, from the expression for the curvature tensor $\widetilde{R}$, we get the following Gauss, Codazzi and Ricci equations

$$R(X,Y)Z = \sum_{i=1}^{m} c_i \Big[\langle f_i Y, Z \rangle\, f_i X - \langle f_i X, Z \rangle\, f_i Y \Big] + A_{B(Y,Z)} X - A_{B(X,Z)} Y, \tag{G}$$

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \sum_{i=1}^{m} c_i \Big[\langle f_i Y, Z \rangle\, h_i X - \langle f_i X, Z \rangle\, h_i Y \Big], \tag{C}$$

$$R^\perp(X,Y)\nu = \sum_{i=1}^{m} c_i \Big[\langle h_i Y, \nu \rangle\, h_i X - \langle h_i X, \nu \rangle\, h_i Y \Big] + B(A_\nu Y, X) - B(A_\nu X, Y). \tag{R}$$

3. Main result

Now, conversely, given (M^n, g) a Riemannian manifold, we want to give some sufficient conditions to have an isometric immersion of (M^n, g) into the multiproduct $\widetilde{P}$. In order to get such conditions, we need to introduce E, a d-dimensional vector bundle over M endowed with a metric $\overline{g}$ and a compatible connection $\overline{\nabla}$, which will play the role of the normal bundle. We also consider $B : TM \times TM \longrightarrow E$ a symmetric $(2,1)$-tensor and $f_i : TM \longrightarrow TM$, $h_i : TM \longrightarrow E$ and $t_i : E \longrightarrow E$ be some $(1,1)$-tensors for $i \in \{1, \cdots, m\}$. We define s_i as the dual of h_i with respect to the metric $\widetilde{g} := g \oplus \overline{g}$ on $TM \oplus E$, that is, for any $X \in T_xM$ and $\nu \in E_x$,

$$\overline{g}_x(h_i X, \nu) = g_x(X, s_i \nu).$$

Finally, for any $\nu \in \Gamma(E)$, we define A_ν by

$$\langle A_\nu X, Y\rangle = \langle B(X,Y), \nu\rangle,$$

for any $X, Y \in \Gamma(TM)$. Note that B is the candidate to be the second fundamental form and that the tensors f_j, h_j, s_j, t_j will coincide with the maps π_i over M.
From the discussions of Section 2, we introduce now the following natural definition.

Definition 3.1. *We say that $(M, g, E, \overline{g}, \overline{\nabla}, B, f_i, h_i, t_i)$ satisfies the compatibility equations for the multiproduct $\widetilde{P} = M_1 \times \cdots \times M_m$, where M_i is the simply connected Riemannian space form of dimension n_i and sectional curvature c_i, if*

i) The maps f_i and t_i are symmetric for any $i \in \{1, \cdots, m\}$,
ii) For any $i \in \{1, \cdots, m\}$, Equations (4)-(11) are satisfied, that is,

$$\sum_{i=1}^m f_i = Id_{TM}, \quad \sum_{i=1}^m t_i = Id_E, \quad \sum_{i=1}^m s_i = 0 \quad \text{and} \quad \sum_{i=1}^m h_i = 0$$
$$f_i \circ f_j + s_i \circ h_j = \delta_i^j f_i,$$
$$t_i \circ t_j + h_i \circ s_j = \delta_i^j t_i,$$
$$f_i \circ s_j + s_i \circ t_j = \delta_i^j s_i,$$
$$h_i \circ f_j + t_i \circ h_j = \delta_i^j h_i,$$
$$\nabla_X(f_i Y) - f_i(\nabla_X Y) = A_{h_i Y} X + s_i(B(X,Y)),$$
$$\overline{\nabla}_X(h_i Y) - h_i(\nabla_X Y) = t_i(B(X,Y)) - B(X, f_i Y),$$
$$\overline{\nabla}_X(t_i \nu) - t_i(\overline{\nabla}_X \nu) = -B(s_i \nu, X) - h_i(A_\nu X),$$

The maps $\pi_i : TM \otimes E \to TM \otimes E$ given by $\pi_i X = f_i X + h_i X$ if $X \in TM$ and $\pi_i \nu = s_i \nu + t_i \nu$ if $\nu \in E$ satisfy that $\operatorname{rank}(\pi_i) = n_i$ *and $\sum_{i=1}^m n_i = n + p$ where n is the dimension of M and d the dimension of E.*

iii) The Gauss, Ricci and Codazzi equations (G)*,* (C) *and* (R) *are satisfied. Namely for any* $X, Y, Z \in \Gamma(TM)$ *and any* $\nu \in \Gamma(E)$*,*

$$R(X,Y)Z = \sum_{i=1}^{m} c_i \Big[\langle f_i Y, Z\rangle\, f_i X - \langle f_i X, Z\rangle\, f_i Y \Big] + A_{B(Y,Z)}X - A_{B(X,Z)}Y,$$

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \sum_{i=1}^{m} c_i \Big[\langle f_i Y, Z\rangle\, h_i X - \langle f_i X, Z\rangle\, h_i Y \Big],$$

$$\overline{R}(X,Y)\nu = \sum_{i=1}^{m} c_i \Big[\langle h_i Y, \nu\rangle\, h_i X - \langle h_i X, \nu\rangle\, h_i Y \Big] + B(A_\nu Y, X) - B(A_\nu X, Y),$$

where $\overline{R}$ *is the curvature associated with the connection* $\overline{\nabla}$*.*

We can now state the main result of the paper.

Theorem 3.2. *Let* (M^n, g) *be a simply connected Riemannian manifold and* E *a* d*-dimensional vector bundle over* M *endowed with a metric* $\overline{g}$ *and a compatible connection* $\overline{\nabla}$*. Moreover, let* $B : TM \times TM \longrightarrow E$ *be a symmetric* $(2,1)$*-tensor and* $f_i : TM \longrightarrow TM$*,* $h_i : TM \longrightarrow E$ *and* $t_i : E \longrightarrow E$ *be some* $(1,1)$*-tensors for* $i \in \{1, \cdots, m\}$*. If* $(M, g, E, \overline{g}, \overline{\nabla}, B, f_i, h_i, t_i)$ *satisfies the compatibility equations for the multiproduct* $\widetilde{P} = M_1 \times \cdots \times M_m$ *then, there exists an isometric immersion* $\varphi : M \longrightarrow \widetilde{P}$ *such that the normal bundle of* M *for this immersion is isomorphic to* E *and such that the second fundamental form* II *and the normal connection* $\nabla^\perp$ *are given by* B *and* $\overline{\nabla}$*. Precisely, there exists a vector bundle isometry* $\widetilde{\varphi} : E \longrightarrow T^\perp\varphi(M)$ *so that*

$$II = \widetilde{\varphi} \circ B,$$
$$\nabla^\perp \widetilde{\varphi} = \widetilde{\varphi}\overline{\nabla}.$$

Moreover, we have

$$\pi_i(\varphi_* X) = \varphi_*(f_i X) + \widetilde{\varphi}(h_i X),$$
$$\pi_i(\widetilde{\varphi}\nu) = \varphi_*(s_i \nu) + \widetilde{\varphi}(t_i \nu),$$

and this isometric immersion is unique up to an isometry of $\widetilde{P}$*.*

Our approach to prove this theorem is not based on the moving frame technique, but is in the spirit of [9] and uses techniques introduced in [6] and [7]. The general strategy is to construct an isometric immersion of (M, g) into an appropriate pseudo-Euclidean space using the classical fact that a simply connected n-dimensional real space form of constant positive (*resp. negative*) constant curvature has a canonical isometrical embedding into the $(n+1)$-dimensional (pseudo)-Euclidean space of signature $(n+1, 0)$ (*resp.* $(n, 1)$). After that, we will show that this immersion lies in fact into $\widetilde{P}$ which is a subset of the above pseudo-Euclidean space.

Proof: We give the proof for the case $c_m \neq 0$, the case $c_m = 0$ can be proved analogously with minor changes. First, for any $i \in \{1, \ldots, m\}$, let us denote by E_i a trivial line bundle over M equipped with the Euclidean metric, if $c_i > 0$, and minus the Euclidean metric, if $c_i < 0$. We consider the vector bundle F over M,

$$F = TM \oplus E \overset{m}{\underset{i=1}{\oplus}} E_i,$$

defined by the orthogonal Whitney sum of Riemannian vector bundles. We denote by $\widetilde{g}$ the metric over F obtained from g, $\overline{g}$ and the metrics on each E_i. For any $i \in \{1, \ldots, m\}$, we consider a section ξ_i of E_i such that $\widetilde{g}(\xi_i,\xi_i)=\frac{1}{c_i}$. We introduce now the following connection on F, denoted by D

$$\begin{aligned}
&D_XY=\nabla_XY+B(X,Y)-\sum_{i=1}^m c_i g(f_iX,Y)\xi_i,\\
&D_X\nu=\overline{\nabla}_X\nu-A_\nu X-\sum_{i=1}^m c_i\overline{g}(h_iX,\nu)\xi_i,\\
&D_X\xi_i=f_iX+h_iX,
\end{aligned}$$

for any vector fields X,Y tangent to M and any section ν of E.

Lemma 3.3. *The connection D is compatible with the metric $\widetilde{g}$.*

Proof: This comes easily from the definition. Let $X,Y,Z\in\Gamma(TM)$, $\nu,\eta\in\Gamma(E)$. We have

$$\begin{aligned}
X\widetilde{g}(Y,Z) &= Xg(Y,Z)\\
&= g(\nabla_XY,Z)+g(Y,\nabla_XZ)\\
&= \widetilde{g}(D_XY,Z)+\widetilde{g}(Y,D_XZ),
\end{aligned}$$

since the tangential parts of D_XY and D_XZ are ∇_XY and ∇_XZ respectively. Similarly, we have

$$\begin{aligned}
X\widetilde{g}(\nu,\eta) &= X\overline{g}(\nu,\eta)\\
&= \overline{g}(\overline{\nabla}_X\nu,\eta)+\overline{g}(\nu,\overline{\nabla}_X\eta)\\
&= \widetilde{g}(D_X\nu,\eta)+\widetilde{g}(\nu,D_X\eta),
\end{aligned}$$

since the normal parts of $D_X\nu$ and $D_X\eta$ are $\overline{\nabla}_X\nu$ and $\overline{\nabla}_X\eta$ respectively. Moreover, we have

$$\begin{aligned}
X\widetilde{g}(\xi_i,\xi_j) &= 0\\
&= \widetilde{g}(f_iX+h_iX,\xi_j)+\widetilde{g}(\xi_i,f_jX+h_jX)\\
&= \widetilde{g}(D_X\xi_i,\xi_j)+\widetilde{g}(\xi_i,D_X\xi_j).
\end{aligned}$$

Finally for mixed terms, we have

$$\begin{aligned}
X\widetilde{g}(\xi_i,Y) &= 0\\
&= g(f_iX,Y)-c_i\widetilde{g}(\xi_i,\xi_i)g(f_iX,Y)\\
&= \widetilde{g}(f_iX+h_iX,Y)+\widetilde{g}\left(\xi_i,\nabla_XY+B(X,Y)-\sum_{i=1}^m c_ig(f_iX,Y)\xi_i\right)\\
&= \widetilde{g}(D_X\xi_i,Y)+\widetilde{g}(\xi_i,D_XY),
\end{aligned}$$

and

$$\begin{aligned}
X\widetilde{g}(\xi_i,\nu) &= 0\\
&= g(h_iX,\nu)-c_i\widetilde{g}(\xi_i,\xi_i)g(h_iX,\nu)\\
&= \widetilde{g}(f_iX+h_iX,\nu)+\widetilde{g}\left(\xi_i,\overline{\nabla}_X\nu-A_\nu X-\sum_{i=1}^m c_ig(h_iX,\nu)\xi_i\right)\\
&= \widetilde{g}(D_X\xi_i,\nu)+\widetilde{g}(\xi_i,D_X\nu).
\end{aligned}$$

By bilinearity, we get the property for any sections of F. $\square$

Now, we consider the curvature tensor associated with the connection D, denoted by $\mathcal{R}^D$ and defined by $\mathcal{R}^D(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$. We can prove the following

Lemma 3.4. *The connection D is flat, that is, $\mathcal{R}^D = 0$.*

Proof: Let $X, Y, Z \in \Gamma(TM)$ and $\nu \in \Gamma(E)$. We will prove that $\mathcal{R}^D(X,Y)Z = 0$, $\mathcal{R}^D(X,Y)\nu = 0$ and $\mathcal{R}^D(X,Y)\xi_i = 0$ for any $i \in \{1, \cdots, m\}$. Then by linearity of the curvature $\mathcal{R}^D$ in its third argument, we will get that $\mathcal{R}^D = 0$. First, we have

$$\begin{aligned} D_X D_Y Z &= \nabla_X \nabla_Y Z + B(X, \nabla_Y Z) - \sum_{i=1}^m c_i g(f_i X, \nabla_Y Z)\xi_i + \overline{\nabla}_X B(Y,Z) - A_{B(Y,Z)}X \\ &\quad - \sum_{i=1}^m c_i \Big[\overline{g}(B(Y,Z), h_i X)\xi_i + (f_i X + h_i X) + g(\nabla_X f_i Y, Z)\xi_i + g(f_i Y, \nabla_X Z)\xi_i\Big]. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \mathcal{R}^D(X,Y)Z &= R(X,Y)Z - \sum_{i=1}^m c_i \Big[g(f_i Y, Z) f_i X - g(f_i X, Z) f_i Y\Big] - A_{B(Y,Z)}X + A_{B(X,Z)}Y \\ &\quad + (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) - \sum_{i=1}^m c_i \Big[g(f_i Y, Z) h_i X - g(f_i X, Z) h_i Y\Big] \\ &\quad - \sum_{i=1}^m c_i \Big[g\big((\nabla_X f_i)Y - A_{h_i Y}X + s_i B(X,Y), Z\big)\Big] \\ &\quad - \sum_{i=1}^m c_i \Big[g\big((\nabla_Y f_i)X - A_{h_i X}Y + s_i B(X,Y), Z\big)\Big] \\ &= 0 \end{aligned}$$

by using the Gauss equation (first line), the Codazzi equation (second line) and equation (9) (third and fourth lines). Similarly, we have

$$\begin{aligned} D_X D_Y Z\nu &= \overline{\nabla}_X \overline{\nabla}_Y \nu - \nabla_X A_\nu Y - B(X, A_\nu Y) - A_{\overline{\nabla}_X \nu} Y \\ &\quad + \sum_{i=1}^m c_i \Big[g(f_i X, A_\nu Y)\xi_i + \overline{g}(\nu, h_i Y)(f_i X + h_i X)\Big] \\ &\quad - \sum_{i=1}^m c_i \Big[\overline{g}(\overline{\nabla}_Y \nu, h_i X) - \overline{g}(\overline{\nabla}_X \nu, h_i Y)\overline{g}(\nu, \overline{\nabla}_X h_i Y)\Big]. \end{aligned}$$

And hence,

$$\begin{aligned} \mathcal{R}^D(X,Y)\nu &= \overline{R}(X,Y)\nu - B(X, A_\nu Y) + B(Y, A_\nu X) - \sum_{i=1}^m c_i \Big[\overline{g}(h_i Y, \nu) h_i X - \overline{g}(h_i X, \nu) h_i Y\Big] \\ &\quad + \nabla_Y A_\nu X + A_{\overline{\nabla}_Y \nu} X - \nabla_X A_\nu Y - A_{\overline{\nabla}_X \nu} Y - \sum_{i=1}^m c_i \Big[\overline{g}(h_i Y, \nu) f_i X - \overline{g}(h_i X, \nu) f_i Y\Big] \\ &\quad + \sum_{i=1}^m c_i \Big[g(f_i X, A_\nu Y) - \overline{g}(\nu, (\nabla_Y h_i)X) + \overline{g}(t_i B(X,Y), \nu)\Big] \\ &\quad + \sum_{i=1}^m c_i \Big[g(f_i Y, A_\nu X) - \overline{g}(\nu, (\nabla_X h_i)Y) + \overline{g}(t_i B(X,Y), \nu)\Big]. \end{aligned}$$

The first line in the right hand side vanishes due to Ricci equation. The second line vanishes by Codazzi equation and the third and fourth lines vanish by using equation (10). Finally, for any $i \in \{1, \cdots, m\}$, we have

$$\begin{aligned} D_X D_Y \xi_j &= \nabla_X f_j Y + B(X, f_j Y) - \sum_{i=1}^{m} c_i g(f_i X, f_i Y)\xi_i \\ &\quad +\overline{\nabla}_X h_j Y - A_{h_j Y} X - \sum_{i=1}^{m} c_i g(h_i X, h_i Y)\xi_i. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{R}^D(X,Y)\xi_j &= (\nabla_X f_j)Y - A_{h_j Y}X - s_j B(X,Y) - (\nabla_Y f_j)X + A_{h_j X}Y + s_j B(X,Y) \\ &\quad \overline{\nabla}_X h_j Y + B(X, f_j Y) - t_j B(X,Y) - \overline{\nabla}_Y h_j X - B(Y, f_j X) - t_j B(X,Y) \\ &= 0 \end{aligned}$$

by equations (9) and (10). Thus, we get that the connection D is flat. □

We define now for any $i \in \{1, \cdots, m\}$ the map $\pi_i : F \longrightarrow F$ by

$$\pi_i X = f_i X + h_i X,$$

$$\pi_i \nu = s_i \nu + t_i \nu,$$

$$\pi_i \xi_j = \delta_i^j \xi_j,$$

for any $X \in \Gamma(TM)$ and any $\nu \in \Gamma(E)$. We have the following properties

Lemma 3.5. *For any $i \in \{1, \cdots, m\}$, the map π_i is symmetric with respect to $\widetilde{g}$, parallel with respect to D and we have*

(1) $\pi_i \circ \pi_j = \delta_i^j \pi_j$, for any $i, j \in \{1, \cdots, m\}$
(2) $\sum_{i=1}^m \pi_i = Id_F$.

Proof: The symmetry is clear because of the symmetry of f_i, t_i and the fact that s_i is the dual of h_i. The fact that π_i is D-parallel comes from the definition of D and equations (5) to (12). Indeed, we have for $X, Y \in \Gamma(TM)$,

$$\begin{aligned} (D_X \pi_i)Y &= D_X(\pi_i Y) - \pi_i(D_X Y) \\ &= D_X(f_i Y + h_i Y) - \pi_i \left(\nabla_X Y + B(X,Y) - \sum_{k=1}^{m} c_k g(f_k X, Y)\xi_k \right) \\ &= \nabla_X (f_i Y) + B(X, f_i Y) - \sum_{k=1}^{m} c_k g(f_k X, f_i Y)\xi_k \\ &\quad +\nabla_X (h_i Y) - A_{h_i Y} X - \sum_{k=1}^{m} c_k g(h_k X, h_i Y)\xi_k \\ &\quad -f_i(\nabla_X Y) + h_i(\nabla_X Y) - s_i B(X,Y) - t_i B(X,Y) + c_i g(f_i X, Y)\xi_i. \end{aligned}$$

By the use of equations (9) and (10), we get

$$\begin{aligned}(D_X\pi_i)Y &= c_i g(f_iX,Y)\xi_i - \sum_{k=1}^m c_k\Big[g(f_kX,f_iY)+g(h_kX,h_iY)\xi_k\Big]\\ &= c_i g(f_iX,Y)\xi_i - \sum_{k=1}^m c_k\Big[g(f_i\circ f_kX+s_i\circ h_kX,Y)\Big]\\ &= 0,\end{aligned}$$

since, by (5), we have $f_i\circ f_k + s_i\circ h_k = \delta_i^k f_i$. The computations are analogous for a section ν of E or for one of the ξ_i.
The relation $\pi_i\circ\pi_j = \delta_i^j\pi_j$ is obvious from the definition of π_i and relations (5) to (8). Finally, we get immediately that $\sum_{i=1}^m \pi_i = Id_F$ from the definition and assumption (4). □

We consider the subsets F^i of F defined by

$$F^i = \Big\{\alpha\in F \mid \pi_j\alpha = \delta_i^j\alpha \text{ for any } j\in\{1,\cdots,m\}\Big\}.$$

Note that, since the π_i are symmetric, then the subbundles F^i are orthogonal with respect to $\widetilde{g}$. We finally need a last lemma

Lemma 3.6. *For any $i\in\{1,\cdots,m\}$, there exists orthonormal n_i+1 parallel sections $\sigma_1^i,\cdots,\sigma_{n_i+1}^i$ of F^i.*

Proof: Let p be a point of M. For any $i\in\{1,\cdots,m\}$, let $\{v_1^i,\cdots,v_{n_1+1}^i\}$ be an orthonormal basis of F_i^p, which is of dimension n_i+1 by the assumption on the rank of π_i. Moreover, since ξ_i clearly belongs to F_i, we can choose $v_1^i=\sqrt{|c_i|}\xi_1(p)$. Thus, we have $\widetilde{g}(v_1^i,v_1^i)=\operatorname{sign}(c_i)$ and $\widetilde{g}(v_k^i,v_k^i)=1$ for $k\in\{2,\cdots,n_i+1\}$. Since the F^i are orthogonal, the set of all v_i^k forms an orthogonal basis of F_p. Now, since the connection D is flat and M is simply connected, then for any $i\in\{1,\cdots,m\}$ there exists a family of parallel sections $\sigma_1^i,\cdots,\sigma_{n_i+1}^i$, such that $\sigma_k^i(p)=v_k^i$. Moreover, since D is compatible with the metric $\widetilde{g}$, then the sections are orthonormal. Finally, since the maps π_i are D-parallel, then, for any $i\in\{1,\cdots,m\}$ and any $k\in\{1,\cdots,n_i+1\}$, $\pi_i(\sigma_k^i)=\sigma_k^i$, that is σ_k^i is a section of F^i. This concludes the proof of the lemma. □

We will construct now the isometric immersion from M into $\widetilde{P}$. For this, we consider the following functions. For $i\in\{1,\cdots,m\}$ and $k\in\{1,\cdots,n_i+1\}$, let φ_k^i be defined by

$$\varphi_k^i = \widetilde{g}(\sigma_k^i,\xi_i).$$

The candidate for the isometric immersion is

$$\varphi : M \longrightarrow \mathbb{E}_1\times\cdots\times\mathbb{E}_m,$$

where $\mathbb{E}_i$ is the Euclidean space $\mathbb{R}^{n_i+1}$ if $c_i>0$ and the Minkowski space $\mathbb{L}^{n_i+1}$ if $c_i<0$. We will show that the map φ goes into $\widetilde{P}\subset\mathbb{E}_1\times\cdots\times\mathbb{E}_m$ and satisfies all the properties stated in Theorem 3.2.

First, we have

$$\begin{aligned}\frac{1}{c_i} = \widetilde{g}(\xi_i.\xi_i) &= \sum_{k=1}^{n_i+1} \widetilde{g}(\xi_i,\sigma_k^i)^2\, \widetilde{g}(\sigma_k^i,\sigma_k^i) \\ &= \mathrm{sign}(c_i)\widetilde{g}(\xi_i,\sigma_1^i)^2 + \sum_{k=2}^{n_i+1}\widetilde{g}(\xi_i,\sigma_k^i)^2 \\ &= \mathrm{sign}(c_i)(\varphi_1^i)^2 + \sum_{k=2}^{n_i+1}(\varphi_k^i)^2.\end{aligned}$$

Thus, we get that $(\varphi_1^i,\cdots,\varphi_{n_i+1}^i)\in M_i$, the n_i-dimensional simply connected space form of curvature c_i, and so, $\varphi(M)$ lies in $\widetilde{P}$.
Moreover, from the definition of φ, we have clearly

$$\varphi_* X = \sum_{i=1}^{m}\sum_{k=1}^{n_1+1}\widetilde{g}(\sigma_k^i,\pi_i X)e_k^i.$$

Now, we will show that φ is an immersion. For this, let $p\in M$ and $v\in T_pM$ so that $\varphi_*(v)=0$. From the definition of φ, the fact that $\varphi_*(v)=0$ implies

$$\widetilde{g}(\sigma_k^i(p),\pi_i v) = \widetilde{g}(v_k^i,\pi_i v) = 0,$$

for any $i\in\{1,\cdots,m\}$ and $k\in\{1,\cdots,n_i+1\}$. Since, for any i, $\{v_k^i\}$ is an orthonormal basis of F_p^i, we get that $\pi_i v = 0$. Moreover, from Lemma 3.5, $\sum_{i=1}^m \pi_i = Id_F$, then $v=0$. This holds for any p and any v, so we get that φ is an immersion.
Moreover, for $v,w\in T_pM$, we have

$$\begin{aligned}\langle\varphi_*(v),\varphi_*(w)\rangle &= \sum_{i=1}^{m}\left(\mathrm{sign}(c_i)\widetilde{g}(\sigma_1^i,\pi_i v)\widetilde{g}(\sigma_1^i,\pi_i w) + \sum_{k=2}^{n_i+1}\widetilde{g}(\sigma_k^i,\pi_i v)\widetilde{g}(\sigma_k^i,\pi_i w)\right) \\ &= \sum_{i=1}^{m}\widetilde{g}(\pi_i v,\pi_i w) \\ &= \widetilde{g}(v,w),\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the (pseudo)-Euclidean metric on $\mathbb{E}_1\times\cdots\times\mathbb{E}_m$. Hence, φ is an isometric immersion from M into $\widetilde{P}$.
Now, we define the following bundle isomorphism

$$\Phi : F \longrightarrow T(\mathbb{E}_1\times\cdots\times\mathbb{E}_m)_{|\varphi(M)},$$

by $\Phi(\sigma_k^i) = e_k^i$, where $\{e_1^i,\cdots,e_{n_i+1}^i\}$ is the canonical frame of $T\mathbb{E}_i$ restricted to $\varphi(M)$. For $X\in\Gamma(TM)$, we have

$$\begin{aligned}\Phi(X) &= \sum_{i=1}^{m}\sum_{k=1}^{n_i+1}\widetilde{g}(X,\sigma_k^i)e_k^i \\ &= \sum_{i=1}^{m}\sum_{k=1}^{n_i+1}\widetilde{g}(\pi_i X,\sigma_k^i)e_k^i \\ &= \varphi_*(X).\end{aligned}$$

Moreover, for any $i \in \{1, \cdots, m\}$, we have

$$\Phi(\xi_i) = \sum_{k=1}^{n_i+1} \widetilde{g}(\xi_i, \sigma_k^i) e_k^i. = \sum_{k=1}^{n_i+1} \varphi_k^i e_k^i.$$

Hence, $\Phi(\xi_i)$ is the normal direction of M_i in $\mathbb{E}_i$. Since Φ is an isometry of the fibers, we deduce that $\Phi(E)$ is the normal bundle $T^\perp\varphi(M)$ of $\varphi(M)$ in $\widetilde{P}$. We denote by $\widetilde{\varphi}$ the restriction of Φ to E. It is clear that $\widetilde{\varphi}$ is an isomorphism of vector bundles between E and $T^\perp\varphi(M)$.

Since Φ sends the orthonormal parallel sections $\{\sigma_k^i\}$ of F onto the orthonormal parallel sections $\{e_k^i\}$ of $\mathbb{E}_1 \times \cdots \times \mathbb{E}_m$, we have

$$\Phi(D_X Y) = \nabla^0_{\varphi_*(X)} \varphi_*(Y), \tag{13}$$

$$\Phi(D_X \nu) = \nabla^0_{\varphi_*(X)} \widetilde{\varphi}(\nu), \tag{14}$$

$$\Phi(D_X \xi_i) = \nabla^0_{\varphi_*(X)} \Phi(\xi_i), \tag{15}$$

where ∇^0 is the Levi-Civita connection of $\mathbb{E}_1 \times \cdots \times \mathbb{E}_m$.

For any $i \in \{1, \cdots, m\}$, we define the map $\widetilde{\pi}_i = \Phi \circ \pi_i \circ \Phi^{-1}$. From this definition, it is clear that $\widetilde{\pi}_i(e_k^j) = \delta_i^j e_k^j$. Then, it follows that the maps $\widetilde{\pi}_i$ are symmetric, parallel along $\varphi(M)$ and satisfy $\widetilde{\pi}_i \circ \widetilde{\pi}_j = \delta_i^j \widetilde{\pi}_i$ and $\sum_{i=1}^m \widetilde{\pi}_i = \mathrm{Id}_{T(\mathbb{E}_1 \times \cdots \times \mathbb{E}_m)}$. Thus, it is clear that these maps are the restrictions on $\varphi(M)$ of the projections on each factor $T\mathbb{E}_i$ of $T(\mathbb{E}_1 \times \cdots \times \mathbb{E}_m)$.

Moreover, from the definition of $\widetilde{\pi}_i$, we deduce immediately that

$$\widetilde{\pi}_i(\varphi_* X) = \varphi_*(f_i X) + \Phi(h_i X),$$

and

$$\widetilde{\pi}_i(\Phi\xi) = \varphi_*(s_i X) + \Phi(t_i \xi).$$

Indeed, we have

$$\widetilde{\pi}_i(\varphi_* X) = \Phi(\pi_i(X)) = \Phi(f_i X + h_i X) = \varphi_*(f_i X) + \widetilde{\varphi}(h_i X),$$

and

$$\widetilde{\pi}_i(\widetilde{\varphi}(\nu)) = \Phi(\pi_i(\nu)) = \Phi(s_i \nu + h_i \nu) = \varphi_*(s_i \nu) + \widetilde{\varphi}(t_i \nu).$$

Finally, we will prove that the second fundamental form is given by B and the normal connection is given by $\overline{\nabla}$. From Equation (13), we have

$$\begin{aligned}
\nabla^0_{\varphi_*(X)} \varphi_*(Y) &= \Phi(D_X Y) \\
&= \Phi(\nabla_X Y + B(X,Y) - \sum_{i=1}^m c_i g(f_i X, Y)\xi_i) \\
&= \varphi_*(\nabla_X Y) + \widetilde{\varphi}(B(X,Y)) - \sum_{i=1}^m c_i g(f_i X, Y)\Phi(\xi_i).
\end{aligned}$$

Then the normal part in $T\widetilde{P}$ is $\widetilde{\varphi}(B(X,Y))$, which implies that the second fundamental form of the immersion φ is $\widetilde{\varphi} \circ B$. Moreover, from Equation (14), we have

$$\begin{aligned}
\nabla^0_{\varphi_*(X)} \widetilde{\varphi}(\nu) &= \Phi(D_X \nu) \\
&= \Phi(\overline{\nabla}_X \nu - A_\nu X - \sum_{i=1}^m c_i g(h_i X, \nu)\xi_i) \\
&= \widetilde{\varphi}(\overline{\nabla}_X \nu) + \varphi_*(A_\nu X) - \sum_{i=1}^m c_i g(h_i X, \nu)\Phi(\xi_i).
\end{aligned}$$

Thus, the normal part in $T\widetilde{P}$ is $\widetilde{\varphi}(\overline{\nabla}_X\nu)$ and we deduce that $\nabla^{\perp}_{\varphi_*X}\widetilde{\varphi}(\nu)=\widetilde{\varphi}(\overline{\nabla}_X\nu)$. Then, we get $\nabla^{\perp}\widetilde{\varphi}=\widetilde{\varphi}\overline{\nabla}$. This concludes the proof of the existence in Theorem 3.2.

Now, we will prove the uniqueness of this isometric immersion up to an isometry of $\widetilde{P}$. This follows directly from the following proposition.

Proposition 3.7. *Let $\varphi,\varphi':M\longrightarrow\widetilde{P}$ be two isometric immersions with respective normal bundles E,E' and second fundamental forms B,B'. Let f_i, h_i and f'_i, h'_i be the $(1,1)$-tensors defined by (2) for φ and φ' respectively. Assume that*

i) $f_iX=f'_iX$ for any $i\in\{1,\cdots,m\}$ and $X\in\Gamma(TM)$,
ii) there exists an isometry of vector bundles $\phi:E\longrightarrow E'$ so that

$$\phi(h_iX)=h'_iX,$$

$$\phi(B(X,Y))=B'(X,Y),$$

$$\phi(\overline{\nabla}_X\nu)=\overline{\nabla}'_X\phi(\nu),$$

for any $i\in\{1,\cdots,m\}$, $X,Y\in\Gamma(TM)$ and $\nu\in E$.
Then, there exists an isometry α of $\widetilde{P}$ such that $\varphi'=\alpha\circ\varphi$ and $\alpha_{*|E}=\phi$.

Proof: We give the complete proof for $c_m\neq 0$, the case c_m can be proven with a minor modification.
As previously, we denote by $\{e^i_k\}$, for $i\in\{1,\cdots,m\}$ and $k\in\{1,\cdots,n_i+1\}$ the canonical frame of $\mathbb{E}_1\times\cdots\times\mathbb{E}_m$. Hence, we denote by φ^i_k and $(\varphi')^i_k$ the components of φ and φ' respectively in the frame $\{e^i_k\}$. We consider the map $G:M\longrightarrow GL(\mathbb{E}_1\times\cdots\times\mathbb{E}_m)$ defined by

$$\begin{cases} G_p(\varphi_*(X))=\varphi'_*(X),\\ G_p(\nu)=\phi(\nu),\\ G_p(\xi_i)=\xi'_i, \end{cases} \tag{16}$$

for any $i\in\{1,\cdots,m\}$, $X,Y\in\Gamma(TM)$ and $\nu\in E$, and where ξ_i and ξ'_i are defined by

$$\xi_i=\sum_{k=1}^{n_i+1}\varphi^i_ke^i_k$$

and

$$\xi'_i=\sum_{k=1}^{n_i+1}(\varphi')^i_ke^i_k.$$

We will show that the map G is constant, that is, that it does not depend on the point p. First of all, we remark that for any i and any $X\in\Gamma(TM)$, we have $\nabla^0_X\xi_i=\pi_i(\varphi_*(X))$, where π_i is the projection on $T\mathbb{E}_i$ and ∇^0 is the Levi-Civita connection of $\mathbb{E}_1\times\cdots\times\mathbb{E}_m$. Now, we will show that $\nabla^0G=0$, or equivalently that $\nabla^0_X(G(V))-G(\nabla^0_XV)=0$ for any $X\in\Gamma(TM)$ and $V\in\Gamma(T(\mathbb{E}_1\times\cdots\times\mathbb{E}_m)_{|\varphi(M)})$.

First, for $V \in \varphi_*(TM)$, that is, $V = \varphi_*(Y)$ with Y tangent to M, we have

$$\begin{aligned}
\nabla^0_X(G(\varphi_*(Y))) - G(\nabla^0_X\varphi_*(Y)) &= \nabla^0_X\varphi'_*(Y) - G(\varphi_*(\nabla_X Y) + B(X,Y)) \\
&\quad + \sum_{i=1}^m c_i g(f_i(X), Y)\xi'_i \\
&= \varphi'_*(\nabla_X Y) + B'(X,Y) - \sum_{i=1}^m c_i g(f'_i(X), Y)\xi'_i \\
&\quad - G(\varphi_*(\nabla_X Y) + B(X,Y)) + \sum_{i=1}^m c_i g(f_i(X), Y)\xi'_i \\
&= 0,
\end{aligned}$$

since, by assumption, $f_i X = f'_i X$ and $\phi(B(X,Y)) = B'(X,Y)$. Now, if $\nu \in E$, we have

$$\begin{aligned}
\nabla^0_X(G(\nu)) - G(\nabla^0_X \nu) &= -A'_{\phi(\nu)}X + \overline{\nabla}'_X\phi(\nu) - \sum_{i=1}^m c_i\widetilde{g}(\phi(\nu), h'_i(X))\xi' \\
&\quad - G\left(-A_\nu X + \overline{\nabla}_X\nu - \sum_{i=1}^m c_i\widetilde{g}(\phi(\nu), h_i(X))\xi\right) \\
&= 0,
\end{aligned}$$

since by assumption, $\phi(B(X,Y)) = B'(X,Y)$, $\phi(h_i X) = h'_i X$ and $\phi(\overline{\nabla}_X\nu) = \overline{\nabla}'_X\phi(\nu)$. Finally, we have

$$\begin{aligned}
\nabla^0_X(G(\xi)) - G(\nabla^0_X\xi) &= \pi_i(\varphi'_*(X)) - G(\pi_i(\varphi_*(X))) \\
&= \varphi'_*(f'_i X) + h'_i X - G(\varphi_*(f_i X) + h_i X) \\
&= 0,
\end{aligned}$$

since $f_i X = f'_i X$ and $\phi(h_i X) = h'_i X$.
These last three identities prove that the linear map G is the same at any point of M and so we can identify G with an element $\alpha \in GL(\mathbb{E}_1 \times \cdots \times \mathbb{E}_m)$. First, by (16) and since φ_* and ϕ are isometries, it is obvious that α belongs in fact to $O(\mathbb{E}_1 \times \cdots \times \mathbb{E}_m)$. Moreover, by definition again, we have for any $i \in \{1, \cdots, m\}$, $\alpha(\xi_i) = \xi'_i$, which implies that α is an isometry of $\mathbb{E}_1 \times \cdots \times \mathbb{E}_m$ for which $\widetilde{P}$ is stable. Hence, by a slight abuse of notation, we can consider α as an isometry of $\widetilde{P}$. Finally, since we have defined G by $G_p(\varphi_*(X)) = \varphi'_*(X)$ and $G_p(\nu) = \phi(\nu)$, for any X tangent to M and $\nu \in E$, we get immediately that $\varphi' = \alpha \circ \varphi$ and $\alpha_{*|E} = \phi$. This concludes the proof. □

By this proposition, the uniqueness of the isometric immersion up to rigid motion is proved, which concludes the proof of Theorem 3.2. □

Remark 3.8. *We want to point out that Theorem 3.2 can be easily extended for isometric immersions of pseudo-Riemannian manifolds into mutliproducts of pseudo-Riemannian space forms just by changing signs appropriately in the compatibility equations.*

4. Associate families of minimal surfaces

In this section, we use Theorem 3.2 to prove the existence of associate families of minimal surfaces into the multiproduct $\widetilde{P}$.
Let (Σ, g) be an oriented Riemannian surface. We denote by J its complex structure, that is,

the rotation of angle $\frac{\pi}{2}$ on TM. For any $\theta \in \mathbb{R}$, we set $\mathcal{R}_\theta = \cos(\theta)I + \sin(\theta)J$. Remark, that $\mathcal{R}_\theta$ is parallel. First, we have the following proposition.

Proposition 4.1. *Assume that $(\Sigma, g, E, \overline{g}, \overline{\nabla}, B, f_i, h_i, t_i)$ satisfies the compatibility equation for $\widetilde{P}$ and that B is trace-free for any $\nu \in E$, then $(\Sigma, g, E, \overline{g}, \overline{\nabla}, B_\theta, f_{i,\theta}, h_{i,\theta}, t_{i,\theta})$ also satisfies the compatibility equations for $\widetilde{P}$, where*

$$\begin{aligned}
&B_\theta(X,Y) = B(X, \mathcal{R}_\theta^{-1}Y),\\
&f_{i,\theta} = \mathcal{R}_\theta \circ f_i \circ \mathcal{R}_\theta^{-1},\\
&h_{i,\theta} = h_i \circ \mathcal{R}_\theta^{-1},\\
&t_{i,\theta} = t_i.
\end{aligned}$$

Moreover, B_θ is also trace-free for any $\nu \in E$.

Proof: We want to point out that since the immersion is minimal, the complex structure J anti-commutes with the second fundamental form and hence $R_\theta A = AR_\theta^{-1}$. Consequently we have that $B(R_\theta X, Y) = B(X, R_\theta^{-1}Y)$.
First, from the definition of $f_{i,\theta}$, $h_{i,\theta}$ and $t_{i,\theta}$ and the fact that

$$\begin{aligned}
&f_i \circ f_j + s_i \circ h_j = \delta_i^j f_i,\\
&t_i \circ t_j + h_i \circ s_j = \delta_i^j t_i,\\
&f_i \circ s_j + s_i \circ t_j = \delta_i^j s_i,\\
&h_i \circ f_j + t_i \circ h_j = \delta_i^j h_i,
\end{aligned}$$

we get immediately that $f_{i,\theta}$ and $t_{i,\theta}$ are symmetric and

$$\begin{aligned}
&f_{i,\theta} \circ f_{j,\theta} + s_{i,\theta} \circ h_{j,\theta} = \delta_i^j f_{i,\theta},\\
&t_{i,\theta} \circ t_{j,\theta} + h_{i,\theta} \circ s_{j,\theta} = \delta_i^j t_{i,\theta},\\
&f_{i,\theta} \circ s_{j,\theta} + s_{i,\theta} \circ t_{j,\theta} = \delta_i^j s_{i,\theta},\\
&h_{i,\theta} \circ f_{j,\theta} + t_{i,\theta} \circ h_{j,\theta} = \delta_i^j h_{i,\theta}.
\end{aligned}$$

It is also clear that with this definition, the rank of $\pi_{i,\theta}$ is the same that the rank of π_i and that

$$\sum_{i=1}^m f_{i,\theta} = Id_{TM}, \quad \sum_{i=1}^m t_{i,\theta} = Id_E \quad \text{and} \quad \sum_{i=1}^m h_{i,\theta} = 0.$$

Now, we will show that analogues of Equations (5)-(7) are satisfied for $(\Sigma, g, E, \overline{g}, \overline{\nabla}, B_\theta, f_{i,\theta}, h_{i,\theta}, t_{,\theta})$. First, we have for X, Y tangent to Σ

$$\begin{aligned}
\nabla_X f_{i,\theta} Y - f_{i,\theta}(\nabla_X Y) &= \nabla_X (\mathcal{R}_\theta f_i \mathcal{R}_\theta^{-1} Y) - \mathcal{R}_\theta f_i \mathcal{R}_\theta^{-1}(\nabla_X Y)\\
&= \mathcal{R}_\theta \nabla_X (f_i \mathcal{R}_\theta^{-1} Y) - \mathcal{R}_\theta f_i \mathcal{R}_\theta^{-1}(\nabla_X Y),
\end{aligned}$$

since $\mathcal{R}_\theta$ is parallel. Moreover, using (5), we get

$$\begin{aligned}
\nabla_X f_{i,\theta} Y - f_{i,\theta}(\nabla_X Y) &= \mathcal{R}_\theta A_{h_i(\mathcal{R}_\theta^{-1}Y)} X + \mathcal{R}_\theta s_i \left(B(X, \mathcal{R}_\theta^{-1} Y) \right)\\
&\quad + \mathcal{R}_\theta f_i \mathcal{R}_\theta^{-1}(\nabla_X Y) - \mathcal{R}_\theta f_i (\nabla_X \mathcal{R}_\theta^{-1} Y)\\
&= A^\theta_{h_{i,\theta}Y} X + s_{i,\theta}\left(B_\theta(X,Y) \right),
\end{aligned}$$

which is the desired equation. The two other equations can be shown in a similar way.
Finally, we prove that Gauss, Codazzi and Ricci equations are also fulfilled.
First we consider the Gauss equation. We notice that, for a surface, we have

$$\begin{aligned}
&\sum_{i=1}^{m} c_i \Big[\langle f_{i,\theta}Y, Z\rangle f_{i,\theta}X - \langle f_{i,\theta}X, Z\rangle f_{i,\theta}Y \Big] + A_{\theta B_\theta(Y,Z)}X - A_{\theta B_\theta(X,Z)}Y \\
&= \sum_{i=1}^{m} c_i \Big[\mathcal{R}_\theta f_i \mathcal{R}_\theta^{-1} X \wedge \mathcal{R}_\theta f_i \mathcal{R}_\theta^{-1} Y \Big] Z + \det A_\theta \\
&= \sum_{i=1}^{m} c_i \det \mathcal{R}_\theta f_i \mathcal{R}_\theta^{-1} + \det A_\theta = \sum_{i=1}^{m} c_i \det f_i + \det A = R(X,Y)Z
\end{aligned}$$

since determinants are invariant under rotations. Hence Gauss equation is satisfied.
Let $\widetilde{\nabla}_X A_\theta^\nu = \nabla_X A_\theta^\nu Y - A_\theta^\nu \nabla_X Y - A_{\theta\nabla^\perp_X \nu}^\nu Y$. Considering now Codazzi equation, we have, using the property of h_i,

$$\begin{aligned}
(\widetilde{\nabla}_X A_\theta^\nu)Y - (\widetilde{\nabla}_Y A_\theta^\nu)X &= \mathcal{R}_\theta \Big[(\widetilde{\nabla}_X A^\nu)Y - (\widetilde{\nabla}_Y A^\nu)X \Big] \\
&= \mathcal{R}_\theta \sum_{i=1}^{m} c_i \Big[f_i Y \langle X, s_i\nu\rangle - f_i X \langle Y, s_i \nu\rangle \Big] \\
&= \mathcal{R}_\theta \sum_{i=1}^{m} c_i f_i (X \wedge Y) s_i \nu = \sum_{i=1}^{m} c_i \mathcal{R}_\theta f_i (X\wedge Y) \mathcal{R}_\theta^{-1} \mathcal{R}_\theta s_i \nu \\
&= \sum_{i=1}^{m} c_i (\mathcal{R}_\theta f_i \mathcal{R}_\theta^{-1} X \wedge Y) s_{i,\theta}\nu = \sum_{i=1}^{m} c_i (f_{i,\theta} X \wedge Y) s_{i,\theta}\nu \\
&= \sum_{i=1}^{m} c_i \Big[f_{i,\theta} Y \langle X, s_{i,\theta}\nu\rangle - f_{i,\theta} X \langle Y, s_{i,\theta}\nu\rangle \Big] = \sum_{i=1}^{m} c_i \Big[f_{i,\theta} Y \langle h_{i,\theta} X, \nu\rangle - f_{i,\theta} X \langle h_{i,\theta} Y, \nu\rangle \Big],
\end{aligned}$$

and Codazzi is satisfied.
Similarly we get for the Ricci equation, using the properties of the wedge product

$$\begin{aligned}
R^\perp(X,Y)\xi &= \sum_{i=1}^{m} c_i \Big[h_i \mathcal{R}_\theta^{-1} X \wedge h_i \mathcal{R}_\theta^{-1} Y \Big] \xi + B_\theta(A_{\theta\xi}Y, X) - B_\theta(A_{\theta\xi}X, Y) \\
&= \sum_{i=1}^{m} c_i \Big[h_i X \wedge h_i Y \Big] \xi + B(A_\xi Y, X) - B(A_\xi X, Y)
\end{aligned}$$

Using the fact that the shape operator anti-commutes with J and we have indeed

$$\begin{aligned}
B_\theta^\nu(A_{\theta\xi}Y, X) - B_\theta^\nu(A_{\theta\xi}X, Y) &= \langle [A_{\theta\nu}, A_{\theta\xi}]X, Y\rangle = \langle (\mathcal{R}_\theta A_\nu \mathcal{R}_\theta A_\xi - \mathcal{R}_\theta A_\xi \mathcal{R}_\theta A_\nu)X, Y\rangle \\
&= \langle (A_\nu \mathcal{R}_\theta^{-1} \mathcal{R}_\theta A_\xi - A_\xi \mathcal{R}_\theta^{-1} \mathcal{R}_\theta A_\nu)X, Y\rangle = \langle [A_\nu, A_\xi]X, Y\rangle.
\end{aligned}$$

Finally, let $(e_1, e_2 = Je_1)$ be a local orthonromal frame of Σ. We have

$$\begin{aligned}
\operatorname{tr}(B_\theta) = B_\theta(e_1,e_1) + B_\theta(e_2,e_2) &= B(\mathcal{R}_\theta e_1, e_1) + B(\mathcal{R}_\theta e_2, e_2) \\
&= \cos(\theta)\left[B(e_1,e_1) + B(e_2,e_2)\right] = 0,
\end{aligned}$$

since B is trace-free. □

From this proposition, we can prove easily the following theorem about associate families of minimal surfaces in multiproducts. Namely, we get the following statement.

Theorem 4.2. *Let Σ be a simply connected surface and $\varphi : M \longrightarrow \widetilde{P}$ be a conformal minimal immersion with normal bundle E, second fundamental form B and normal connection $\nabla^{\perp}$. Let f_i, h_i, s_i and t_i be the $(1,1)$-tensors induced by the projections π_i. Let $p_0 \in \Sigma$. Then, there exists a unique family $(\varphi_\theta)_{\theta\in\mathbb{R}}$ of conformal minimal immersions $\varphi_\theta : \Sigma \longrightarrow \widetilde{P}$ so that*

i) $\varphi_\theta(p_0) = \varphi(p_0)$ *and* $d(\varphi_\theta)_{p_0} = (d\varphi)_{p_0}$*,*
ii) the metric induced by φ and φ_θ are the same,
iii) the second fundamental form of $\varphi_\theta(\Sigma)$ in $\widetilde{P}$ is given by $B_\theta(X,Y) = B(R_\theta X, Y)$, for any $X, Y \in \Gamma(T\Sigma)$.
iv) for any $i \in \{1, \cdots, m\}$, $X \in \Gamma(T\Sigma)$ and $\xi \in \Gamma(E)$,

$$\pi_i(d\varphi_\theta X) = d\varphi_\theta(f_{i,\theta}X) + h_{i,\theta}X \quad \text{and} \quad \pi_i(\xi) = d\varphi_\theta(s_{i,\theta}X) + t_{i,\theta}X,$$

Moreover, $\varphi_0 = \varphi$ and the family $(\varphi_\theta)_{\theta\in\mathbb{R}}$ is continuous with respect to θ.

Proof: We just proved that $(\Sigma, g, E, \overline{g}, \overline{\nabla}, B_\theta, f_{i,\theta}, h_{i,\theta}, t_{,\theta})$ satisfies the compatibility equations for each θ. The theorem is then a direct consequence of theorem 3.2. The continuity is ensured by the construction of Theorem 3.2. □

5. Surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$

Let J be the complex structure on $\mathbb{S}^2$. We consider the following complex structures on $\mathbb{S}^2 \times \mathbb{S}^2$

$$J_1 = (J, J), \quad J_2 = (J, -J).$$

Obviously J_1 and J_2 commute with each other and the projections π_1 and π_2 on each of the factors are given by

$$\pi_1 = \frac{\mathrm{Id} - J_1J_2}{2}, \quad \pi_2 = \frac{\mathrm{Id} + J_1J_2}{2}.$$

From equation (1) we get

$$\widetilde{R}(X,Y)Z = \big[\langle Y,Z\rangle X - \langle X,Z\rangle Y\big] + \big[\langle J_1Y, J_2Z\rangle J_1J_2X - \langle J_1X, J_2Z\rangle J_1J_2Y\big]. \tag{17}$$

Let now Σ be a surface isometrically immersed into $\mathbb{S}^2 \times \mathbb{S}^2$. For $i \in \{1,2\}$, we define four operators $j_i : T\Sigma \to T\Sigma$, $k_i : T\Sigma \to N\Sigma$, $l_i : N\Sigma \to T\Sigma$ and $m_i : N\Sigma \to N\Sigma$ such that $J_i = j_i + k_i + l_i + m_i$.
From $J_1J_2 = J_2J_1$ we get the following equations

$$j_1j_2 + l_1k_2 = j_2j_1 + l_2k_1, \tag{18}$$
$$k_1j_2 + m_1k_2 = k_2j_1 + m_2k_1, \tag{19}$$
$$j_1l_2 + l_1m_2 = j_2l_1 + l_2m_1, \tag{20}$$
$$k_1l_2 + m_1m_2 = k_2l_1 + m_2m_1. \tag{21}$$

The property $J_i^2 = -\mathrm{Id}$ yields

$$j_i^2 + l_ik_i = -\mathrm{Id}_{T\Sigma}, \tag{22}$$
$$k_ij_i + m_ik_i = 0, \tag{23}$$
$$j_il_i + l_im_i = 0, \tag{24}$$
$$k_il_i + m_i^2 = -\mathrm{Id}_{N\Sigma}. \tag{25}$$

Moreover the fact that the operators J_i are antisymmetric implies the antisymmetry of the operators j_i and t_i as well as the property $\langle k_iX,\nu\rangle = -\langle X, l_i\nu\rangle$.
The parallelity of J_i gives

$$\nabla_X(j_iY) - j_i(\nabla_X Y) = A_{k_iY}X + l_i(B(X,Y)), \tag{26}$$

$$\nabla^\perp_X(k_iY) - k_i(\nabla_X Y) = m_i(B(X,Y)) - B(X, j_iY), \tag{27}$$

$$\nabla^\perp_X(m_i\xi) - m_i(\nabla^\perp_X\xi) = -B(l_i\xi, X) - k_i(A_\xi X), \tag{28}$$

$$\nabla_X(l_i\xi) - l_i(\nabla^\perp_X\xi) = -j_i(A_\xi X) + A_{m_i\xi}X. \tag{29}$$

Finally, from (17), we get the Gauss equation

$$\begin{aligned} K \;=\; & \frac{1}{2}\Big[1 + \Big(\langle j_1e_1, j_2e_2\rangle + \langle k_1e_1, k_2e_2\rangle\Big)\Big(\langle j_1e_2, j_2e_1\rangle + \langle k_1e_2, k_2e_1\rangle\Big) \\ & -\Big(\langle j_1e_1, j_2e_1\rangle + \langle k_1e_1, k_2e_1\rangle\Big)\Big(\langle j_1e_2, j_2e_2\rangle - \langle k_1e_2, k_2e_2\rangle\Big)\Big] \\ & +2|H|^2 - \frac{|B|^2}{2}, \end{aligned} \tag{30}$$

the Codazzi equation

$$\begin{aligned} &(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \\ &\quad \frac{1}{2}\Big[\langle Y, (j_1j_2 + l_1k_2)Z\rangle(k_1j_2 + m_1k_2)X - \langle X, (j_1j_2 + l_1k_2)Z\rangle(k_1j_2 + m_1k_2)Y\Big], \end{aligned} \tag{31}$$

and the Ricci equation

$$\begin{aligned} K^\perp \;=\; & \Big[\langle (k_1j_2 + m_1k_2)e_2, \nu_1\rangle\, \langle (k_1j_2 + m_1k_2)e_1, \nu_2\rangle \\ & \langle k_1j_2 + m_1k_2)e_1, \nu_1\rangle\, \langle (k_1j_2 + m_1k_2)e_2, \nu_2\rangle\Big] \\ & +\langle [A_{\nu_2}, A_{\nu_1}]e_1, e_2]\rangle. \end{aligned} \tag{32}$$

Remark 5.1. *In* [14] *the Gauss, Codazzi and Ricci equations are expressed with the help of the two Kähler functions C_1 and $C_2 : \Sigma \to \mathbb{R}$ defined by $\varphi^*\omega_i = C_i\omega_\Sigma$, $i = 1, 2$, with ω_Σ the area form on Σ. A tidious but straightforward computation shows that those two formulations are equivalent.*

Now, we are able to reformulate the main theorem in the case of $\mathbb{S}^2 \times \mathbb{S}^2$ in terms of complex structures instead of projections on each factor.

Corollary 5.2. *Let (Σ^2, g) be a Riemannian surface and $(E, \langle\cdot,\cdot\rangle_E, \nabla^E)$ a rank 2 vector bundle over Σ endowed with a scalar product and a compatible connection. Suppose that there exists a symmetric $(2,1)$-tensor field $B : T\Sigma\times T\Sigma \to E$ and eight operators $j_i : T\Sigma \to T\Sigma$, $k_i : T\Sigma \to E$, $l_i : E \to T\Sigma$ and $m_i : E \to E$, $i = 1, 2$ such that j_i and t_i are antisymmetric and satisfying conditions* (18) *to* (29) *and the Gauss, Codazzi and Ricci equations* (30), (31) *and* (32)*. Then, there exists a unique (up to isometries of $\mathbb{S}^2 \times \mathbb{S}^2$) isometric immersion from Σ into $\mathbb{S}^2 \times \mathbb{S}^2$ with E as normal bundle, B as second fundamental form and such that the restrictions of the complex structures J_i over Σ are given by j_i, k_i, l_i and m_i.*

Proof: Define the following operators

$$\begin{aligned} f_1 &= \frac{1}{2}\left(\mathrm{Id}_{T\Sigma} - j_1 j_2 - l_1 k_2\right), \quad f_2 = \frac{1}{2}\left(\mathrm{Id}_{T\Sigma} - j_1 j_2 - l_1 k_2\right) \\ h_1 &= -\frac{1}{2}\left(k_1 j_2 + m_1 k_2\right) = -h_2, \\ s_1 &= -\frac{1}{2}\left(j_1 l_2 + l_1 m_2\right) = -s_2 \\ t_1 &= \frac{1}{2}\left(\mathrm{Id}_E - k_1 l_2 - m_1 m_2\right), \quad t_2 = \frac{1}{2}\left(\mathrm{Id}_E + k_1 l_2 + m_1 m_2\right). \end{aligned}$$

We can show easily that equations (18) to (32) imply that these operators satisfy the compatibility equations for $\mathbb{S}^2 \times \mathbb{S}^2$ given by Definition 3.1. The conclusion follows easily from Theorem 3.2. □

Examples 5.1. *We finish this paper by some particular cases of Corollary 5.2, namely complex and Lagrangian surfaces and give some simple examples. The study of such surfaces, especially in the product of spheres is a very active topic (see for example* [1], [14]*).*
We remind (see for example [14]*) that an immersion* $\varphi : \Sigma \longrightarrow \mathbb{S}^2 \times \mathbb{S}^2$ *is called complex if it is complex with respect to* J_1 *or* J_2*. It is called Lagrangian, if it is Lagrangian with respect to* J_1 *or* J_2*.*

First Case: Σ *is a complex surface with respect to one of the complex structures* J_i*. Then it is automatically minimal. Moreover* $k_i = l_i = 0$*,* j_i *and* m_i *are parallel complex structures on* $T\Sigma$ *and* $T^{\perp}\varphi(\Sigma)$ *respectively, and* j_1 *commutes with* j_2*, as well as* m_1 *with* m_2*. Assume without loss of generality that* Σ *is complex with respect to* J_1*, then the Gauss, Codazzi and Ricci equations simplify to*

$$K = \frac{1}{2}\Big[1 + \langle j_1 e_1, j_2 e_2\rangle\langle j_1 e_2, j_2 e_1\rangle - \langle j_1 e_1, j_2 e_1\rangle\langle j_1 e_2, j_2 e_2\rangle\Big] - \frac{|B|^2}{2},$$

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \frac{1}{2}\Big[\langle Y, j_1 j_2 Z\rangle m_1 k_2 X - \langle X, j_1 j_2 Z\rangle m_1 k_2 Y\Big],$$

$$K^{\perp} = \Big[\langle m_1 k_2 e_2, \nu_1\rangle\langle m_1 k_2 e_1, \nu_2\rangle - \langle m_1 k_2 e_1, \nu_1\rangle\langle m_1 k_2 e_2, \nu_2\rangle\Big] + \langle [A_{\nu_2}, A_{\nu_1}] e_1, e_2\rangle.$$

Notice that the only examples of complex surfaces with respect to both complex structures J_1 *and* J_2 *(bi-complex surfaces) are slices* $\mathbb{S}^2 \times p = \{(x,p) \in \mathbb{S}^2 \times \mathbb{S}^2 | x \in \mathbb{S}^2\}$ *and* $p \times \mathbb{S}^2 = \{(p,x) \in \mathbb{S}^2 \times \mathbb{S}^2 | x \in \mathbb{S}^2\}$*.*

Second Case: Σ *is Lagrangian with respect to* J_i*, then* $j_i = m_i = 0$*. Assuming again without loss of generality that* Σ *is Lagrangian with respect to* J_1*, the Gauss, Codazzi and Ricci equations simplify in the following way*

$$K = \frac{1}{2}\Big[1 - \langle k_1 e_1, k_2 e_2\rangle\langle k_1 e_2, k_2 e_1\rangle + \langle k_1 e_1, k_2 e_1\rangle\langle k_1 e_2, k_2 e_2\rangle\Big] + 2|H|^2 - \frac{|B|^2}{2},$$

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \frac{1}{2}\Big[\langle Y, l_1 k_2 Z\rangle k_1 j_2 X - \langle X, l_1 k_2 Z\rangle k_1 j_2 Y\Big],$$

$$K^{\perp} = \Big[\langle k_1 j_2 e_2, \nu_1\rangle\langle k_1 j_2 e_1, \nu_2\rangle - \langle k_1 j_2 e_1, \nu_1\rangle\langle k_1 j_2 e_2, \nu_2\rangle\Big] + \langle [A_{\nu_2}, A_{\nu_1}] e_1, e_2\rangle.$$

Notice that Σ *is Lagrangian for both* J_1 *and* J_2 *(bi-Lagrangian surfaces) if and only if it is the product* $\varphi(s,t) = (\alpha(s), \beta(t))$ *of two curves* α, β *in* $\mathbb{S}^2$*. The Clifford torus is the only example of a minimal such surface.*

Third Case: Σ *is Lagrangian with respect to* J_1 *(hence* $j_1 = m_1 = 0$*) and complex with respect to* J_2 *(hence* $k_2 = l_2 = 0$*).*

$$K = \frac{1}{2} - \frac{|B|^2}{2},$$
$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = 0,$$
$$K^\perp = \Big[\langle k_1 j_2 e_2, \nu_1 \rangle \langle k_1 j_2 e_1, \nu_2 \rangle - \langle k_1 j_2 e_1, \nu_1 \rangle \langle k_1 j_2 e_2, \nu_2 \rangle \Big] + \langle [A_{\nu_2}, A_{\nu_1}] e_1, e_2] \rangle.$$

The only example of such a surface is the diagonal $\mathbf{D} = \{(x,x) \in \mathbb{S}^2 \times \mathbb{S}^2 | x \in \mathbb{S}^2\}$.
We point out that there are no associate families for bi-Lagrangian surfaces, bi-complex surfaces or for the mixed case. In fact those surfaces are the only totally geodesic minimal surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ *(see* [3]*). Hence the family is trivial.*

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