A DDVV INEQUALITY FOR SUBMANIFOLDS OF WARPED PRODUCTS

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ABSTRACT. We prove a DDVV inequality for submanifolds of warped products of the form $I \times_a \mathbb{M}^n(c)$ where I is an interval and $\mathbb{M}^n(c)$ a real space form of curvature c . As an application, we give a rigidity result for submanifolds of $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$.

RÉSUMÉ. **Une inégalité de type DDVV pour les sous-variétés des produits tordus.** Nous donnons une inégalité de type DDVV pour les sous-variétés des produits tordus de la forme $I \times_a \mathbb{M}^n(c)$ où I est un interval et $\mathbb{M}^n(c)$ un espace modèle réel de courbure constante c. Nous en déduisons un résultat de rigidité pour les sous-variétés de $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$.

Version francaise abrégée.

Soit (M^m, g) une variété riemannienne immergée isométriquement dans une variété riemannienne ambiante $(N^{m+p}, \bar{g})$, de dimension $m + p$. Lorsque N est un espace modèle simplement connexe à courbure sectionnelle constante c, l'inégalité suivante est vérifiée :

$$||H||^2 \geqslant \rho + \rho^\perp - c, \tag{1}$$

où $\rho = \dfrac{2}{n(n-1)} \sum_{i<j} \langle R(e_i, e_j)e_j, e_i \rangle$ est la courbure scalaire normalisée de (M, g) et $\rho^\perp = \dfrac{2}{n(n-1)} \sum_{i<j} \sum_{\alpha<\beta} \langle R^\perp(e_i, e_j)\xi_\alpha, \xi_\beta \rangle$ la courbure scalaire normale (également normalisée), $\{e_1, \cdots, e_n\}$ et $\{\xi_1, \cdots \xi_p\}$ étant respectivement des bases orthonormées locales de TM et $T^\perp M$. Cette inégalité est connue sous le nom de conjecture de DDVV car conjecturée par De Smets-Dillen-Verstraelen-Vrancken [2]. La conjecture a été démontrée récemment par Lu [6] et par Ge-Tang [4] indépendamment. Plus récemment, Chen et Cui [1] ont obtenu une inégalité comparable dans le cas des espaces produits $\mathbb{S}^n \times \mathbb{R}$ et $\mathbb{H}^n \times \mathbb{R}$. Le but de cette note est d'étendre le résultat de Chen-Cui pour les sous-variétés des produits tordus $I \times_a \mathbb{M}^n(c)$, c'est-à-dire $I \times \mathbb{M}^n(c)$ muni de la métrique $\widetilde{g} = dt^2 + a(t)^2 g_{\mathbb{M}^n(c)}$, I étant un intervalle de $\mathbb{R}$ et $a : I \longrightarrow \mathbb{R}$ une fonction lisse ne s'annulant pas. Nous démontrons le résultat suivant.

Théorème 1. *Soient $n > m \geqslant 2$ deux entiers. Soit M^m une sous-variété du produit tordu $I \times_a \mathbb{M}^n(c)$ avec courbure scalaire et courbure scalaire normale normalisées ρ et $\rho^\perp$ et courbure moyenne H. L'inégalité suivante est vérifiée :*

$$||H||^2 \geqslant \rho + \rho^\perp + \left(\frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left(1 - \frac{2}{n} ||T||^2 \right) - \frac{2a''}{na} ||T||^2.$$

Nous en déduisons un résultat pour les surfaces des produits tordus $\mathbb{R} \times_{e^{4\lambda t}} \mathbb{H}^n(c)$.

Corollaire 1. *Soit M^m une sous-variété complète sans bord du produit tordu $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$ avec courbure scalaire et courbure scalaire normale normalisées ρ et $\rho^\perp$ et courbure*

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moyenne H. *Si* $||H||^2 \leqslant \rho+\lambda^2$, *alors*

$$||H||^2 = \rho+\lambda^2, \quad \rho^\perp = 0, \quad n=2 \quad et \quad ||T||=1.$$

En particulier, M *est une surface du type* $\mathbb{R}\times_{e^{\lambda t}}\gamma$, *où* γ *est une courbe dans* $\mathbb{H}^n(c)$.

1. Introduction

Let (M^n,g) be a n-dimensional Riemannian manifold isometrically immersed into a $(n+p)$- dimensional Riemannian manifold $(N^{n+p},\overline{g})$. When the ambient space is a real space form of constant sectional curvature c, we have the following pointwise inequality

$$||H||^2 \geqslant \rho+\rho^\perp - c, \tag{2}$$

where $\rho = \dfrac{2}{n(n-1)}\displaystyle\sum_{i<j}\langle R(e_i,e_j)e_j,e_i\rangle$ is the normalized scalar curvature of (M,g) and $\rho^\perp = \dfrac{2}{n(n-1)}\displaystyle\sum_{i<j}\sum_{\alpha<\beta}\langle R^\perp(e_i,e_j)\xi_\alpha,\xi_\beta\rangle$ is the normalized normal curvature of the immersion, where $\{e_1,\cdots,e_n\}$ and $\{\xi_1,\cdots\xi_p\}$ are respectively orthonormal frames of TM and $T^\perp M$. This inequality, known as DDVV conjecture, was conjectured by De Smets-Dillen-Verstrealen-Vrancken in [2] and proved recently by Lu [6] and by Ge-Tang [4] independently. More recently, Chen and Cui generalized this inequality in the setting of product spaces $\mathbb{S}^n\times\mathbb{R}$ and $\mathbb{H}^n\times\mathbb{R}$.

In this note, we extend Chen-Cui result by proving a DDVV inequality for submanifolds o warped products $I\times_a\mathbb{M}^n(c)$ where $I\subset\mathbb{R}$ is an interval and $a: I\longrightarrow\mathbb{R}$ is a nowhere vanishing smooth function. We denote by $\partial_t=\frac{\partial}{\partial t}$ the unit vector field tangent to the factor I. We prove the following result.

Theorem 1. *Let* M^m *be a submanifold of the warped product* $I\times_a\mathbb{M}^n(c)$ *with normalized scalar and normal scalar curvatures* ρ *and* $\rho^\perp$ *and mean curvature* H. *Then, we have*

$$||H||^2 \geqslant \rho+\rho^\perp + \left(\frac{(a')^2}{a^2}-\frac{c}{a^2}\right)\left(1-\frac{2}{n}||T||^2\right) - \frac{2a''}{na}||T||^2,$$

where T *is the part of* ∂t *tangent to* M.

Remark 1. *Note that, of course, we recover the DDVV inequality of* [1] *for product spaces* $\mathbb{S}^n\times\mathbb{R}$ *and* $\mathbb{H}^n\times\mathbb{R}$ *as well as for* $\mathbb{R}^{n+1}$ *by taking* $a=1$, *but we also recover the inequality for space forms. Indeed,* $\mathbb{S}^n$ *and* $\mathbb{H}^n$ *can be expressed in term of warped products. Namely, we have*

(1) $\mathbb{S}^n = [0,2\pi]\times_a\mathbb{S}^{n-1}$ *with* $a(t)=\sin(t)$. *Hence the inequality of Theorem 1 becomes* $||H||^2\geqslant\rho+\rho^\perp-1$.

(2) $\mathbb{H}^n=[0,+\infty[\times_a\mathbb{S}^{n-1}$ *with* $a(t)=\sinh(t)$ *or* $\mathbb{H}^n=\mathbb{R}\times_a\mathbb{R}^{n-1}$ *with* $a(t)=e^{-t}$. *For both cases, the inequality of Theorem 1 becomes* $||H||^2\geqslant\rho+\rho^\perp+1$.

2. Preliminaries

Let $\mathbb{M}^n(c)$ be the simply connected real space form of dimension n and constant curvature c. Let $I\subset\mathbb{R}$ an interval and $a: I\longrightarrow\mathbb{R}$ be a nowhere vanishing smooth function. We consider the warped $\widetilde{P}^{n+1}=I\times_a\mathbb{M}^n(c)$, that the product $I\times\mathbb{M}^n(c)$ endowed the metric $\widetilde{g}=dt^2+a(t)^2g_{\mathbb{M}^n(c)}$. We denote by $\partial_t=\frac{\partial}{\partial t}$ the unit vector field tangent to the factor I. We recall (see [5] for instance) that the curvature tensor of $(\widetilde{P}^{n+1},\widetilde{g})$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)Z \;=\;& \left(\frac{(a')^2}{a^2}-\frac{c}{a^2}\right)\Big(\langle X,Z\rangle Y-\langle Y,Z\rangle X\Big)\\ &+\left(\frac{a''}{a}-\frac{(a')^2}{a^2}+\frac{c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,\partial_t\rangle\partial_t-\langle Y,Z\rangle\langle X,\partial_t\rangle\partial_t\\ &-\langle Y,\partial_t\rangle\langle Z,\partial_t\rangle X+\langle X,\partial_t\rangle\langle Z,\partial_t\rangle Y\Big).\end{aligned} \tag{3}$$

Let (M^m, g) be a Riemannian manifold isometrically immersed into $\widetilde{P}$. We denote by B its second fundamental form and A the shape operator defined for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^{\perp}M)$ by $\langle B(X,Y), \xi\rangle = \langle A_{\xi}X, Y\rangle$. Moreover, ∂_t can be written

$$\partial_t = T + \sum_{\alpha=1}^{p} f_{\alpha}\xi_{\alpha},$$

where T is a vector field tangent to M, $\{\xi_1 \cdots, \xi_p\}$ is a local orthonormal frame of $T^{\perp}M$ and $f_1, \cdots, f_p$ are smooth functions over M. We will simply denote $A_{\xi_{\alpha}}$ by A_{α}.
From the expression of the curvature tensor of $\widetilde{P}$, we get immediately the Gauss, Codazzi and Ricci equations for a submanifold of $\widetilde{P}$. Namley, if we denote by R and $R^{\perp}$ the curvature tensor of (M, g) and the normal curvature respectively, we have the following

Proposition 2.1. *The Gauss, Codazzi and Ricci equations of the immersion of M into $\widetilde{P}$ are respectively*

$$\begin{aligned}\langle R(X,Y)Z,W\rangle &= \langle B(Y,Z),B(X,W)\rangle - \langle B(Y,W),B(X,Z)\rangle \\ &\quad + \left(\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\Big) \\ &\quad + \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,T\rangle\langle W,T\rangle - \langle Y,Z\rangle\langle X,T\rangle\langle W,T\rangle \\ &\quad - \langle Y,T\rangle\langle Z,T\rangle\langle X,W\rangle + \langle X,T\rangle\langle Z,T\rangle\langle Y,W\rangle\Big),\end{aligned}$$

$$\langle(\widetilde{\nabla}_X B)(Y,Z),\xi_{\alpha}\rangle = \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2}\right) f_{\alpha}\Big(\langle Y,T\rangle\langle X,Z\rangle - \langle X,T\rangle\langle Y,Z\rangle\Big),$$

$$\langle R^{\perp}(X,Y)\nu,\xi\rangle = \langle [A_{\nu}, A_{\xi}]X, Y\rangle.$$

The proof is straightforward form the expression of $\widetilde{R}$.

Finally, we recall that the DDVV conjecture was reduced to the following algebraic result (see [3]) proved by Lu.

Theorem ([6])**.** *Let $n, p \geqslant 2$ be two integers and M_1, $M_2, \cdots, M_p$ be some $n \times n$ real symmetric and trace-free matrices. Then, we have*

$$\sum_{\alpha,\beta=1}^{p} ||[M_{\alpha}, M_{\beta}]||^2 \leqslant \left(\sum_{\alpha=1}^{p} ||M_{\alpha}||^2\right)^2.$$

Now, we are able to prove Theorem 1.

3. Proof of theorem 1

First, from the definition of ρ and using the Gauss equation, we have

$$\begin{aligned}\rho &= \frac{2}{n(n-1)}\sum_{i<j}\langle R(e_i,e_j)e_j,e_i\rangle\\ &= \frac{1}{n(n-1)}\sum_{i\neq j}\langle R(e_i,e_j)e_j,e_i\rangle\\ &= \frac{1}{n(n-1)}\sum_{i\neq j}\left(\langle B(e_j,e_j),B(e_i,e_i)\rangle-||B(e_i,e_j)||^2+\left(\frac{(a')^2}{a^2}-\frac{c}{a^2}\right)\right.\\ &\quad\left.+\left(\frac{a''}{a}-\frac{(a')^2}{a^2}+\frac{c}{a^2}\right)(\langle T,e_i\rangle^2+\langle T,e_j\rangle^2)\right)\\ &= \left(\frac{(a')^2}{a^2}-\frac{c}{a^2}\right)+\frac{1}{n(n-1)}\left(n^2||H||^2-||B||^2+2(n-1)\left(\frac{a''}{a}-\frac{(a')^2}{a^2}+\frac{c}{a^2}\right)||T||^2\right)\end{aligned}$$

Now, we set $\tau=B-Hg$ the traceless part of the second fundamental form. Clearly, we have $||\tau||^2=||B||^2-n||H||^2$. Hence we get

$$\rho = \left(\frac{(a')^2}{a^2}-\frac{c}{a^2}\right)\left(1-\frac{2}{n}||T||^2\right)-\frac{2a''}{na}||T||^2+||H||^2-\frac{1}{n(n-1)}||\tau||^2. \tag{4}$$

Moreover, for any $\alpha\in\{1,\cdots,p\}$, we set $S_\alpha:TM\to TM$ the operator defined by $\langle S_\alpha X,Y\rangle=\langle\tau(X,Y),\xi_\alpha\rangle$. Obviously, we have $S_\alpha=A_\alpha-\langle H,\xi_\alpha\rangle\mathrm{Id}$ and $[A_\alpha,A_\beta]=[S_\alpha,S_\beta]$. From the Ricci Equation, given in Proposition 2.1, we have

$$\rho^\perp=\frac{1}{n(n-1)}\sqrt{\sum_{\alpha,\beta=1}^p||[A_\alpha,A_\beta]||^2}=\frac{1}{n(n-1)}\sqrt{\sum_{\alpha,\beta=1}^p||[S_\alpha,S_\beta]||^2}.$$

Since, the operators S_α are symmetric and trace-free, we can apply the theorem of Lu at any point of M to get

$$\sum_{\alpha,\beta=1}^p||[S_\alpha,S_\beta]||^2\leqslant\left(\sum_{\alpha=1}^p||S_\alpha||^2\right)^2.$$

Thus,

$$\rho^\perp\leqslant\frac{1}{n(n-1)}\sum_{\alpha=1}^n||S_\alpha||^2=\frac{1}{n(n-1)}||\tau||^2.$$

Reporting this in (4), we obtain

$$||H||^2\geqslant\rho+\rho^\perp+\left(\frac{(a')^2}{a^2}-\frac{c}{a^2}\right)\left(1-\frac{2}{n}||T||^2\right)-\frac{2a''}{na}||T||^2,$$

which concludes the proof. □

4. An application to submanifolds of $\mathbb{R}\times_{e^{\lambda t}}\mathbb{H}^n(c)$

We finish this note by the following application of Theorem 1 to submanifolds of warped product of the type $\mathbb{R}\times_a\mathbb{H}^n(c)$ where a is the real function defined by $a(t)=e^{\lambda t}$ with λ a real constant.

Corollary 1. *Let M^m be a submanifold of the warped product $\mathbb{R}\times_{e^{\lambda t}}\mathbb{H}^n(c)$ with normalized scalar and normal scalar curvatures ρ and $\rho^\perp$ and mean curvature H. Then, we have*

$$||H||^2\geqslant\rho+\rho^\perp+\lambda^2-ce^{-2\lambda t}\left(1-\frac{2}{n}||T||^2\right).$$

Proof: This comes directly from Theorem 1, with the fact that $\frac{(a')^2}{a^2} - \frac{c}{a^2} = \lambda^2 - ce^{-2\lambda t}$ and $\frac{a''}{a^2} = \lambda^2$. Hence the term $\left(\frac{(a')^2}{a^2} - \frac{c}{a^2}\right)\left(1 - \frac{2}{n}||T||^2\right) - \frac{2a''}{na}||T||^2$ becomes $\lambda^2 - ce^{-2\lambda t}\left(1 - \frac{2}{n}||T||^2\right)$. □

Comparing $||H||^2$ with ρ is a natural question which leads to rigidity results. Indeed, by the Gauss formula, we know that, for hypersurfaces of space forms, ρ is up to a constant (which is the sectional curvature k of the ambiant space form) the second mean curvature H_2, that is the second elementary symmetric polynomial in the prinicpal curvatures. Moreover, it is a classical fact that $H^2 \geqslant H_2$ with equality at umbilical points. Hence, assuming $H^2 \leqslant \rho - k$ implies that M is a hypersphere. In this spirit, and using the above DDVV inequality, we give the following rigidity result.

Corollary 2. *Let M^m be a complete submanifold without boundary of the warped product $\mathbb{R} \times_{e^{\lambda t}} \mathbb{H}^n(c)$ with normalized scalar and normal scalar curvatures ρ and $\rho^\perp$ and mean curvature H. If $||H||^2 \leqslant \rho + \lambda^2$, then*

$$||H||^2 = \rho + \lambda^2, \quad \rho^\perp = 0, \quad n = 2 \quad \textit{and} \quad ||T|| = 1.$$

Hence, M is a surface of the type $\mathbb{R} \times_{e^{\lambda t}} \gamma$, where γ is a curve in $\mathbb{H}^n(c)$.

Proof: First note that since $n \geqslant 2$ and $||T||^2 \leqslant 1$ and $c < 0$, we have $ce^{-2\lambda t}\left(1 - \frac{2}{n}||T||^2\right) \leqslant 0$. Note also that, by defintion, $\rho^\perp \geqslant 0$. Hence, from Corollary 1, $||H||^2 \leqslant \rho + \lambda^2$ is possible if and only if $||H||^2 = \rho + \lambda^2$, $\rho^\perp = 0$, $n = 2$ and $||T|| = 1$. Since $n = 2$, then M is a surface and the fact that $||T|| = 1$ implies that $T = \partial_t$ and so M is of the type $I \times_{e^{\lambda t}} \gamma$, where γ is a curve in $\mathbb{H}^n(c)$. Since we assume that M is complete and without boundary, $I = \mathbb{R}$. This concludes the proof.

□

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