Rigidity Results for Geodesic Spheres in Space Forms

Julien Roth

Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris-Est
5, boulevard Descartes, Cité Descartes, 77454 Marne-la-Vallée cedex 2, France

We prove that a hypersurface of a space form with almost constant mean curvature and almost constant scalar curvature is close to a geodesic sphere. In the case of Euclidean space, we deduce new characterizations of geodesic spheres.

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1. Introduction

The well-Known Alexandrov theorem ([1]) claims that any compact without boundary hypersurface embedded into the Euclidean space $\mathbb{R}^{n+1}$ with constant mean curvature (CMC) is a geodesic sphere. Later, A. Ros ([2]) proved that the result also holds for hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$ and the open half sphere $\mathbb{S}^{n+1}_+$.

In these results, the assumption that the hypersurface is embedded is crucial. Indeed, the results are false for immersed hypersurfaces. For instance, the so-called Wente's torus (see[3]) is an example of (non-embedded) immersed surface in $\mathbb{R}^3$ with constant mean curvature which is not a geodesic sphere. Other examples of higher genus are known ([4]).

For surfaces in $\mathbb{R}^3$, Hopf ([5]) proved that CMC immersed spheres are geodesic spheres. Here again, the result is not true in general since, we know examples of CMC spheres in higher dimension which are not geodesic spheres (see[6]).

The goal of this article is to find an alternative assumption to the embedding such that under this assumption, CMC hypersurfaces are geodesic spheres. It is a well-known fact that if the mean curvature H and the scalar curvature Scal are both constant, then the hypersurface is a geodesic sphere. We will show a stability result associated with this assertion. Namely, we have the following result which was proved for $\delta = 0$ in ([7]).

The cases $\delta > 0$ and $\delta < 0$ are new.

Theorem 1.1. *Let (M^n, g) be a compact without boundary, connected and oriented Riemannian manifold, isometrically immersed into the simply connected space form $\mathbb{M}_\delta^{n+1}$ of constant sectional curvature δ. Let R be the extrinsic radius of M. If $\delta > 0$, we assume that $R < \frac{\pi}{4\sqrt{\delta}}$. Let $h > 0$ and $\theta \in]0,1[$. Then, there exists $\varepsilon(n,h,R,\theta) > 0$ such that if*

- $|H - h| < \varepsilon$ *and*
- $|\mathrm{Scal} - s| < \varepsilon$ *for a constant s,*

then $\left|h^2 - \frac{s}{n(n-1)} + \delta\right| \leqslant A\varepsilon$, for a positive constant A depending on n, h, δ and R, and M is diffeomorphic and θ-almost isometric to a geodesic sphere in $\mathbb{M}_\delta^{n+1}$ of radius $t_\delta^{-1}\left(\frac{1}{h}\right)$, where t_δ is the function defined in Section 2.

Remark 1.1. By θ-almost isometric, we mean that there exists a diffeomorphism F from M into a geodesic sphere of appropriate radius such that

$$\left||dF_x(u)|^2 - 1\right| \leq \theta,$$

for any $x \in M$ and any unit vector $u \in T_xM$.

Remark 1.2. The extrinsic radius of M is the radius of the smallest closed ball in $\mathbb{M}_\delta^{n+1}$ containing M.

Then, from this stability result, we will deduce a new characterization of geodesic spheres in $\mathbb{R}^{n+1}$ with a weaker assumption on the scalar curvature (see Section 4).

2. Preliminaries

Let (M^n, g) be a n-dimensional compact, connected, oriented Riemannian manifold without boundary, isometrically immersed into the $(n+1)$-dimensional Euclidean space $(\mathbb{R}^{n+1}, can)$, where can is the canonical metric of $\mathbb{R}^{n+1}$. The (real-valued) second fundamental form B of the immersion is the bilinear symmetric form on $\Gamma(TM)$ defined for two vector fields X, Y by

$$B(X,Y) = -g\left(\overline{\nabla}_X \nu, Y\right),$$

where $\overline{\nabla}$ is the Riemannian connection on $\mathbb{R}^{n+1}$ and ν a normal unit vector field on M. When M is embedded, we choose ν as the inner normal field.

From B, we can define the mean curvature,

$$H = \frac{1}{n}\mathrm{tr}\,(B).$$

Now, we recall the Gauss formula. For $X, Y, Z, W \in \Gamma(TM)$,

$$R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + \langle AX, Z\rangle \langle AY, W\rangle - \langle AY, Z\rangle \langle AX, W\rangle \quad (1)$$

where R and $\overline{R}$ are respectively the curvature tensor of M and $\mathbb{M}_\delta^{n+1}$, and A is the Weingarten operator defined by $AX = -\overline{\nabla}_X \nu$.

By taking the trace and for $W = Y$, we get

$$\mathrm{Ric}(Y) = \overline{\mathrm{Ric}}(Y) - \overline{R}(\nu, Y, \nu, Y) + nH \langle AY, Y\rangle - \left\langle A^2 Y, Y\right\rangle \quad (2)$$

Since, the ambient space is of constant sectional curvature δ, by taking the trace a seconde time, we have

$$\mathrm{Scal} = n(n-1)\delta + n^2 H^2 - |A|^2, \quad (3)$$

or equivalently

$$\mathrm{Scal} = n(n-1)\left(H^2 + \delta\right) - |\tau|^2, \quad (4)$$

where $\tau = B - H\mathrm{Id}$ is the umbilicity tensor.

Now, we define the higher order mean curvatures, for $k \in \{1, \cdots, n\}$, by

$$H_k = \frac{1}{\binom{n}{k}} \sigma_k(\kappa_1, \cdots, \kappa_n),$$

where σ_k is the k-th elementary symmetric polynomial and $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of the immersion.

From the definition, it is obvious that H_1 is the mean curvature H. We also remark from the Gauss formula (1) that

$$H_2 = \frac{1}{n(n-1)} \mathrm{Scal} - \delta. \quad (5)$$

On the other hand, we have the well-known Hsiung-Minkowski formula

$$\int_M \Big(H_{k+1} \langle Z, \nu\rangle + c_\delta(r) H_k \Big) = 0, \quad (6)$$

where $r(x) = d(p_0, x)$ is the distance function to a base point p_0, Z is the position vector defined by $Z = s_\delta(r)\overline{\nabla} r$, and the functions c_δ and s_δ are defined by

$$c_\delta(t) = \begin{cases} \cos(\sqrt{\delta}t) & \text{if } \delta > 0 \\ 1 & \text{if } \delta = 0 \\ \cosh(\sqrt{-\delta}t) & \text{if } \delta < 0 \end{cases} \quad \text{and} \quad s_\delta(t) = \begin{cases} \frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta}t) & \text{if } \delta > 0 \\ t & \text{if } \delta = 0 \\ \frac{1}{\sqrt{-\delta}} \sinh(\sqrt{-\delta}t) & \text{if } \delta < 0 \end{cases}$$

Finally, we define the function $t_\delta = \frac{s_\delta}{c_\delta}$.

We finish this section of preliminaries by the following recollection about the first eigenvalue of the Laplacian. In ([8]), Heintze proved the following upper bound for $\lambda_1(\Delta)$

$$\lambda_1(\Delta) \leqslant n(||H||_\infty^2 + \delta), \tag{7}$$

with equality for geodesic spheres. Grosjean ([9]) proved a stability result associated with Heintze's estimate. Precisely, he proved the following

Theorem 2.1 (Grosjean, 2007). *Let M be compact without boundary, connected and oriented hypersurface of $\mathbb{M}_\delta^{n+1}$. If $\delta > 0$, we assume that M is contained in an open ball of $\mathbb{M}_\delta^{n+1}$ of radius $\frac{\pi}{4\sqrt{\delta}}$. Let $\theta \in]0,1[$, then there exitst a constant $C_\theta(n, ||B||_\infty, V(M), \delta) > 0$ such that if*

$$n(||H||_\infty^2 + \delta) - C_\theta < \lambda_1(\Delta),$$

then M is diffeomorphic and θ-almost isometric to a geodesic sphere of radius $\sqrt{\frac{n}{\lambda_1}}$.

Now, we have all the ingredients to prove Theorem 1.1.

3. Proof of Theorem 1.1

We begin the proof of Theorem 1.1 by the following lemma.

Lemma 3.1. *Let h and s be two positive constants. If the mean curvature and the scalar curvature satisfy*

- $|H - h| < \varepsilon$ *and*
- $|\mathrm{Scal} - s| < \varepsilon$,

for some positive ε, then

$$\left| h^2 - \frac{s}{n(n-1)} + \delta \right| \leqslant A\varepsilon,$$

where A is a positive constante depending on h, R, n and δ.

Proof. The proof of this lemma comes directly from the Hisung-Minkowski formula (6). Indeed, the Hisung-Minkowski formula for $k = 1$ is the following

$$\int_M \Big(H_2 \langle Z, \nu \rangle + c_\delta(r) H \Big) = 0. \tag{8}$$

Since we assume that $|\mathrm{Scal} - s| < \varepsilon$, we get easily from (5) that

$$\left| H_2 - \left(\frac{s}{n(n-1)} - \delta \right) \right| < \frac{1}{n(n-1)} \varepsilon. \tag{9}$$

For more convenience, we will denote $h_2 = \frac{s}{n(n-1)} - \delta$. Then, from (8)

$$\begin{aligned}
0 &= \int_M \Big(H_2 \langle Z, \nu \rangle + c_\delta(r) H \Big) \\
&= \int_M \Big(h_2 \langle Z, \nu \rangle + c_\delta(r) H \Big) + \int_M (H_2 - h_2) \langle Z, \nu \rangle \\
&= \frac{h_2}{h} \int_M h \langle Z, \nu \rangle + \int_M c_\delta(r) H + \int_M (H_2 - h_2) \langle Z, \nu \rangle \\
&= \frac{h_2}{h} \int_M H \langle Z, \nu \rangle + \frac{h_2}{h} \int_M (h - H) \langle Z, \nu \rangle + \int_M c_\delta(r) h + \int_M c_\delta(r)(H - h) \\
&\quad + \int_M (H_2 - h_2) \langle Z, \nu \rangle
\end{aligned}$$

Now, we use the Hsiung-Minkowski formula for $k = 0$, that is

$$\int_M \Big(H \langle Z, \nu \rangle + c_\delta(r) \Big) = 0, \tag{10}$$

to get

$$\begin{aligned}
0 &= -\frac{h_2}{h} \int_M c_\delta(r) + \frac{h_2}{h} \int_M (h - H) \langle Z, \nu \rangle + \int_M c_\delta(r) h + \int_M c_\delta(r)(H - h) \\
&\quad + \int_M (H_2 - h_2) \langle Z, \nu \rangle \\
&= \left(h - \frac{h_2}{h} \right) \int_M c_\delta(r) + \frac{h_2}{h} \int_M (h - H) \langle Z, \nu \rangle + \int_M c_\delta(r)(H - h) \\
&\quad + \int_M (H_2 - h_2) \langle Z, \nu \rangle
\end{aligned}$$

Then, since s_δ is an increasing function, we deduce

$$\left| h - \frac{h_2}{h} \right| \int_M c_\delta(r) \leqslant \frac{h_2}{h} \varepsilon \int_M s_\delta(R) + \varepsilon \int_M c_\delta(r) + \frac{\varepsilon}{n(n-1)} \int_M s_\delta(R)$$

Using the fact that $|H_2| \leqslant H^2$, we deduce from the assumptions on h and h_2 that

$$|h_2| \leqslant h^2 + A_1(n, h, \delta) \varepsilon,$$

and then we have

$$\Big|h-\frac{h_2}{h}\Big|\int_M c_\delta(r)\leqslant\varepsilon\int_M c_\delta(r)+A_2(n,h,\delta)\int_M s_\delta(R)$$

We conclude using the fact that $\mathrm{V(M)}\leqslant\int_M c_\delta(r)\leqslant A_3(n,\delta,R)\mathrm{V(M)}$, which yields

$$|h^2-h_2|\leqslant A_4(n,h,R,\delta)\varepsilon,$$

which gives the wanted assertion. □

Let $0<\varepsilon<1$. We will choose a particular ε later. We assume $|H-h|<\varepsilon$ and $|\mathrm{Scal}-s|<\varepsilon$, then from (3) and Lemma 3.1, we have

$$\begin{aligned}|\tau|^2&=n(n-1)H^2-\mathrm{Scal}\\&\leqslant A'(n,h,R,\delta)\varepsilon.\end{aligned}\tag{11}$$

This means that M is almost umbilical. Moreover, since $|H-h|\leqslant\varepsilon$, Inequality (11) is equivalent to

$$|B-h\mathrm{Id}\,|\leqslant A''(n,h,R,\delta)\varepsilon.\tag{12}$$

This last inequality combining with the Gauss formula (2) yields

$$\Big|\mathrm{Ric}(Y)-(n-1)(h^2+\delta)|Y|^2\Big|\leqslant A'''(n,h,R,\delta)\varepsilon.\tag{13}$$

Now, we use the Lichnerowicz formula ([10]) to obtain the following lower bound of the first eigenvalue of the Laplacian on M

$$\lambda_1(\Delta)\geqslant n(h^2+\delta)-\alpha_1(\varepsilon),\tag{14}$$

or equivalently

$$\lambda_1(\Delta)\geqslant n(||H||_\infty^2+\delta)-\alpha_2(\varepsilon),\tag{15}$$

where the positive functions α_1 and α_2 depend on n,h,R and δ and tend to 0 when ε tends to 0. Now, we fix $\theta\in]0,1[$. Since α_2 tends to 0 when ε tends to 0, there exists $\varepsilon_1(n,h,R,\delta)>0$, such that $\alpha_2(\varepsilon_1)\leqslant\frac{\theta}{2}$. Then, we use Theorem 2.1 to conclude that M is diffeomorphic and $\frac{\theta}{2}$-almost isometric to a geodesic sphere of radius $\sqrt{\frac{n}{\lambda_1}}$. Moreover, because of the pinching of $\lambda_1(\Delta)$, the radii $\frac{1}{h}$ and $\sqrt{\frac{n}{\lambda_1}}$ are close. So, there exists $\varepsilon_2(n,h,R,\delta)>0$ such that geodesic spheres of radii $\frac{1}{h}$ and $\sqrt{\frac{n}{\lambda_1}}$ are $\frac{\theta}{2}$-almost isometric. Then, we take $\varepsilon=inf\{\varepsilon_1,\varepsilon_2\}$ and then M is θ-quasi-isometric to a geodesic sphere of radius $\frac{1}{h}$. □

4. Rigidity results in the Euclidean space

For the Euclidean case, that is $\delta = 0$, we can obtain from Theorem 1.1 new characterizations of geodesic spheres. Namely, we have the following

Corollary 4.1. *Let (M^n, g) be a compact, connected and oriented Riemannian manifold without boundary isometrically immersed into $\mathbb{R}^{n+1}$. Let h be a positive constant. Then, there exists $\varepsilon(n, h) > 0$ such that if*

(1) $H = h$ *and*
(2) $|\mathrm{Scal} - s| \leqslant \varepsilon$,

for some constant s, then M is a geodesic sphere of radius $\frac{1}{h}$.

Remark 4.1. Note that in this result, ε does not depend on the extrinsic radius R. Indeed, since the mean curvature is constant, from (10), we have $\int_M \langle Z, \nu\rangle = \frac{1}{h}\mathrm{V(M)}$. So we do not have to control the extrinsic radius.

Proof. This corollary is a direct consequence of Theorem 1.1. Indeed, we know that there exists a diffeomorphism F from M to $\mathbb{S}^n\left(\frac{1}{h}\right)$. But, in the Euclidean space, this diffeomorphism is explicit. Namely,

$$F(x) = \frac{1}{h}\frac{\phi(x)}{|\phi(x)|},$$

where ϕ is the immersion on M into $\mathbb{R}^{n+1}$. See ([11]) to get this expression.

The fact that F is a diffeomorphism implies that the immersion ϕ is necessarily a one-to-one map, that is an embedding. Since M is embedded with constant mean curvature, it is a geodesic sphere by the Alexandrov theorem. □

Now, we state a second characterization of geodesic spheres.

Corollary 4.2. *Let (M^n, g) be a compact, connected and oriented Riemannian manifold without boundary isometrically immersed into $\mathbb{R}^{n+1}$. Let s be a positive constant. Then, there exists $\varepsilon(n, s) > 0$ such that if*

(1) $\mathrm{Scal} = s$
(2) $|H - h| \leqslant \varepsilon$,

for some constant h, then M is a geodesic sphere of radius $\sqrt{\frac{n(n-1)}{s}}$.

Proof. The proof is the same, using an Alexandrov type theorem for the scalar curvature due to Ros ([2]). □

This second corollary gives a partial answer to a conjecture by Yau which states that the only immersed hypersurfaces of the Euclidean space with constant scalar curvature are geodesic spheres.

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