FILLING RADIUS AND SHORT CLOSED GEODESICS OF THE TWO-SPHERE

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1. Introduction

Let M be a closed connected smooth Riemannian manifold of dimension n. The Riemannian metric g on M induces a distance d_g on M.
The map $i : (M, d_g) \hookrightarrow (L^\infty(M), ||.||)$ defined by $i(x)(.) = \mathrm{dist}_M(x, .)$ is an embedding from the metric space (M, d_g) into the Banach space $L^\infty(M)$ of all bounded functions on M with the sup-norm $||.||$. This natural embedding is a (strong) isometry between metric spaces, i.e., it preserves the distances. Note that Riemannian embeddings of closed manifolds into Euclidean spaces are not isometric in this sense. Considering M isometrically embedded in the Banach space $L^\infty(M)$, we define $U_\delta(M)$ as the δ-tubular neighborhood of M in $L^\infty(M)$. The homology coefficients will be in $\mathbb{Z}$, if M is orientable, and in $\mathbb{Z}_2$, otherwise.

Definition. The filling radius of M, denoted FillRad(M), is the infimum of positive reals δ such that $(i_\delta)_*([M]) = 0 \in H_n(U_\delta(M))$, where $i_\delta : M \hookrightarrow U_\delta(M)$ is the inclusion and $[M] \in H_n(M)$ is the fundamental class of M.

In this paper, we show the following curvature free estimate.

Theorem 1.1. *Let M be a Riemannian two-sphere, then*

$$FillRad(M) \geq \frac{1}{12} scg(M) \tag{1.1}$$

where $scg(M)$ denotes the length of the shortest nontrivial closed geodesic on M.

This statement admits a stronger version in which occur the Morse index and the number of self-intersection points of closed geodesics. Note, however, that the constant involved is not as good as in the first version.

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Main Theorem 1.2. *Let M be a Riemannian two-sphere, then*

$$FillRad(M) \geq \frac{1}{20} \bar{L}(M) \tag{1.2}$$

where $\bar{L}(M)$ is the length of the shortest nontrivial curve among the simple closed geodesics of index zero or one and the figure eight geodesics of null index.

For metrics all of whose geodesics are non-degenerate (bumpy metrics) the same inequality holds if we replace $\bar{L}(M)$ by $L(M)$, where $L(M)$ represents the length of the shortest nontrivial curve among the simple closed geodesics of index one and the figure eight geodesics of null index.

Some examples illustrating the different cases of the Main Theorem are presented in this paper (see Remark 4.10).

Before going further, let us review some known results to which these inequalities are related.

These two theorems extend to the simply connected case some filling radius estimates related to the one-dimensional systole.
The one-dimensional systole of a non-simply connected closed Riemannian manifold (M, g) is defined as the infinimum of the lengths of noncontractible closed curves. This lower bound, denoted $\mathrm{sys}_1(M, g)$, is attained by the length of a closed geodesic.

In [17] (see also [9], [18] and [19]), M. Gromov showed that every essential manifold of dimension n satisfies the isosystolic inequality

$$\mathrm{Vol}(M, g) \geq C_n \mathrm{sys}_1(M, g)^n \tag{1.3}$$

where C_n is a positive constant depending only on n.

In the above statement, whose converse was established in [3], a closed manifold M is said to be essential if there is a map f from M to a $K(\pi, 1)$ space such that $f_*([M]) \neq 0$. In particular, $T^n, \mathbb{R}P^n$ and all closed aspherical manifolds are essential.

Isosystolic inequalities on surfaces were previously established in [33], [1], [7], [8, p. 43] and [21]. An inequality similar to (1.3) also holds for the stable systole under suitable topological conditions (see [19], [17], [22], [4], [5] and [24]).

Examples of non-essential manifolds with "long" systole and "small" volume can easily be constructed. The product metric on $S^1 \times S^2$ where the length of S^1 is long and the area of S^2 is small provides such an example. However, they may still have a short contractible closed geodesic whose length is bounded from above in terms of the volume.

For Riemannian two-spheres, C. Croke showed in [12] that

$$\mathrm{Area}(M) \geq \frac{1}{(31)^2}\mathrm{scg}(M)^2$$

$$\mathrm{Diam}(M) \geq \frac{1}{9}\mathrm{scg}(M)$$

It is unknown whether or not the length of the shortest nontrivial closed geodesic provides a lower bound on the volume of any non-essential manifold of dimension greater than two. Under some curvature assumptions, upper bounds on the length of the shortest nontrivial closed geodesic exist (see [38], [35] and [30] for general results).

The proof of M. Gromov's isosystolic inequality (1.3) rests on the two following filling radius inequalities.

Theorem 1.3 (M. Gromov)**.** *Let M be a complete Riemannian n-manifold.*

$$\mathit{FillRad}(M) \geq \frac{1}{6}\mathit{sys}_1(M) \text{ if } M \text{ is essential} \tag{1.4}$$

$$\mathit{FillRad}(M) \leq c_n\,\mathit{Vol}(M)^{\frac{1}{n}} \text{ for some } c_n > 0. \tag{1.5}$$

In particular, the first inequality holds for all the closed surfaces except the sphere. The second inequality, more difficult to establish (though in the case of the sphere S^2 it may be obtained in a more elementary way), takes the form $\mathrm{FillRad}(S^2) \leq \mathrm{Area}(S^2)^{\frac{1}{2}}$ for the two-sphere S^2 (see [17, p. 128]).

Thus, the inequalities (1.1) and (1.2) lead to the following corollary which improves C. Croke's result providing an alternative proof.

Corollary 1.4. *Let M be a Riemannian two-sphere, then*

$$\mathit{Area}(M) \geq \frac{1}{(12)^2}\mathit{scg}(M)^2 \tag{1.6}$$

$$\mathit{Area}(M) \geq \frac{1}{(20)^2}\bar{L}(M)^2 \tag{1.7}$$

For bumpy metrics, we can replace $\bar{L}(M)$ by $L(M)$ in the above inequalities. Note that the length of a simple closed geodesic around the waist of an hourglass figure does not provide a "good" lower bound on the area as it can be made arbitrarily small while the area remains constant. These closed geodesics, which still are simple and have a null index after slight perturbations of the metric into a bumpy one, can actually be ignored to give better bound on the area.

Note that the inequality (1.6) is not optimal (C. Croke conjectures that the extremal sphere is composed of two copies of flat equilateral triangles glued together along their boundaries) and that sharp isosystolic inequalities are known only for the two-torus, the projective plane and the Klein bottle (see [9], [33], [6] and [36]).

The proof of the Main Theorem rests on a minimax principle derived from Morse Theory on the space of one-cycles $\mathcal{Z}_1(M,\mathbb{Z})$ on M. This principle, based on F. Almgren's isomorphism $\pi_1(\mathcal{Z}_1(S^2,\mathbb{Z}),\{0\}) \simeq H_2(S^2,\mathbb{Z}) \simeq \mathbb{Z}$ (see [2] and Theorem 2.4 for a more general version), has been established by F. Almgren and J. Pitts using geometric measure theory and has been used by E. Calabi and J. Cao in [10]. The use of the space of one-cycles rather than the ordinary free loop space allows us to cut and paste closed curves using several component loops. This minimax principle proceeds as follows.

Let us consider the one-parameter families $(z_t)_{0\leq t\leq 1}$ of one-cycles on M which satisfy the following conditions

(C.1) z_t starts and ends at null-currents
(C.2) z_t induces a nontrivial class $[z]$ in $\pi_1(\mathcal{Z}_1(M,\mathbb{Z}),\{0\})$.

We define the minimax value

$$L_1(M) := \inf_{[z]\neq 0} \sup_{0\leq t\leq 1} \mathrm{mass}(z_t)$$

For bumpy metrics, we introduce other constructions as follows.
The previous global minimax principle extends to the nontrivial groups $\pi_1(\mathcal{Z}_1^{\leq\kappa_1}(M),\mathcal{Z}_1^{\leq\kappa_0}(M))$, where $\mathcal{Z}_1^{\leq\kappa}(M)=\{z\in\mathcal{Z}_1(M,\mathbb{Z}) \mid \mathrm{mass}(z) \leq \kappa\}$ and $0\leq\kappa_0<\kappa_1$. We refer to Section 4.1 for further details. The lowest positive minimax value of these local minimax processes is noted $L_1'(M)$. We show that $L_1'(M)$ agrees with the mass $L_1''(M)$ of the shortest one-cycle of index one. Here, the index of a one-cycle of mass κ is defined by $\mathrm{ind}_{\mathcal{Z}_1}(z)=\min\{i\in\mathbb{N} \mid \pi_i(\mathcal{Z}_1^{<\kappa}(M)\cup\{z\},\mathcal{Z}_1^{<\kappa}(M)) \text{ is nontrivial}\}$. Further, we show that $\mathrm{scg}(M)\leq L_1'(M)=L_1''(M)\leq L_1(M)$.

We also introduce a new curve-shortening process which permits us to prove the following

Theorem 1.5. *Let M be a bumpy Riemannian two-sphere, then*

$$\mathit{FillRad}(M) \geq \frac{1}{20} L_1''(M)$$

where $L_1''(M)$ is the length of the shortest one-cycle of index one.

We show then that the shortest one-cycle of index one for bumpy metrics is either a simple closed geodesic of index one or a figure eight geodesic of null index. This immediately leads to the Main Theorem.

Contrary to $L_1''(M)$, the invariant $L_1(M)$ provides no universal lower bound on the filling radius of the two-sphere. More precisely, we have

Theorem 1.6. *There exists a sequence g_n of Riemannian metrics on S^2 which satisfies*

$$\lim_{n\to\infty} \frac{FillRad(S^2, g_n)}{L_1(S^2, g_n)} = 0$$

Using techniques involved in the proof of Theorem 1.1, we also prove

Theorem 1.7. *Let M be a Riemannian two-sphere of diameter $Diam(M)$, then*

$$scg(M) \leq 4Diam(M)$$

For other simply connected manifolds, it is still unknown whether or not a similar inequality holds. Note that C. Croke already showed in [12] that $\mathrm{scg}(M) \leq 9\mathrm{Diam}(M)$ for two-spheres. This inequality was then improved with the constant 5 by M. Maeda in [26].
Theorem 1.7 may also be derived from Theorem 1.1 and the sharp general filling inequality $\mathrm{FillRad}(M) \leq \frac{1}{3}\mathrm{Diam}(M)$ established by M. Katz in [23]. However, we present its short proof because it illustrates in a simple way some techniques used in this paper.

After having written the final version of this paper, the author learned that A. Nabutovsky and R. Rotman have independently established similar results. Specifically, on two-spheres, they have obtained in [28] the same improvement for the diameter lower bound as us (cf. Theorem 1.7) and a better one for the area lower bound ($\frac{1}{64}$ instead of $\frac{1}{144}$ in (1.6)). They have also obtained in [29] a lower bound on the filling radius of any closed Riemannian manifold in terms of the mass of the shortest stationary one-cycle.

We refer the reader to the recent survey [13] and the references therein for an account of further curvature-free geometric inequalities.

In Section 2, we study a minimax principle on the space of one-cycles which yields a nontrivial closed geodesic on the two-sphere. Then, we introduce a new curve-shortening process. In Section 3, we illustrate the general use of this minimax principle and show that the length of the shortest closed geodesic on the two-sphere provides a lower bound on the filling radius. Section 4 is devoted to the proof of the filling radius estimates of the Main Theorem. The geometrical structure of the shortest one-cycle of index one is described here. In Section 5, we construct a (counter)-example showing that the minimax value of the global minimax principle does not provide any lower bound on the filling radius.

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2. Generalities and preliminaries

In this part, we introduce the space of one-cycles and present a minimax principle providing a nontrivial closed geodesic on the two-sphere. Then, we define a curve-shortening process and state its main properties.

2.1. The minimax principle on the space of one-cycles.

Originally, G. D. Birkhoff established the existence of a nontrivial closed geodesic on the two-sphere by using a minimax argument on the free loop space. This argument was extended in higher dimension by Fet and Lyusternik (see [25]). Since we shall need a modified version of this minimax principle, we briefly recall it.

Let M be a compact Riemannian manifold such that $\pi_{k+1}(M) \neq 0$.
The free loop space ΛM, formed of piecewise smooth curves $\gamma : S^1 \longrightarrow M$ parametrized proportionally to arclength, is endowed with the compact-open topology and the length functional L. The subspace of point curves is noted $\Lambda^0 M$.
Recall that the closed geodesics of M agree with the critical points of the energy functional E on the free loop space of M. The (analytical) Morse index of a closed geodesic c is then defined as the number (counted with multiplicity) of negative eigenvalues of $D^2E(c)$. It is noted $\text{ind}_\Lambda(c)$.

A homotopically nontrivial smooth map $\varphi : S^{k+1} \longrightarrow M$ induces a continuous map $\psi : (B^k, \partial B^k) \longrightarrow (\Lambda M, \Lambda^0 M)$ representing a nontrivial class in $\pi_k(\Lambda M, \Lambda^0 M)$ (see [25]). Define $\ell := \inf_{\varphi \not\simeq 0} \sup_{t \in B^k} L(\psi_t)$.

Theorem 2.1. *There exists a nontrivial closed geodesic of length ℓ.*

In particular, let $(\sigma_t)_{0 \leq t \leq 1}$ be a one-parameter family of closed curves starting and ending at point curves such that the induced map $\sigma : S^2 \longrightarrow S^2$ has nonzero degree. From Theorem 2.1, there exists a closed geodesic on S^2 of length ℓ where $\ell := \inf_{deg\, \sigma \neq 0} \sup_{0 \leq t \leq 1} L(\sigma_t)$.

Remark 2.2. This method sometimes yields a “short” geodesic. Actually, for two-spheres with nonnegative curvature, ℓ represents the length of the shortest nontrivial closed geodesic (see [10]). Such a result is no longer true for an arbitrary metric. In fact, the minimax value ℓ may be quite “long”, for instance for a sphere with three long spikes (see Remark 4.10 for a more precise description of this example).

Although C. Croke established a lower bound on the area in terms of the length of the shortest closed geodesic on the sphere using the Birkhoff minimax principle on ΛS^2, the free loop space is not very well adapted to this problem. We will rather use the space of one-cycles.

In the remainder of this section, we introduce the space of integral one-cycles and generalize on it the minimax principle (see [27],[32] and [10]).

We define the space $\mathcal{Z}_1(S^2,\mathbb{Z})$ of one-cycles with integral coefficients over S^2 as follows

$$\mathcal{Z}_1(S^2,\mathbb{Z}) = \left\{\sum a_i T_i \mid a_i \in \mathbb{Z} \text{ and } T_i \text{ is a closed rectifiable one-current}\right\}$$

It is endowed with the weak topology. Note that the weak and flat norm topologies coincide on the space of one-cycles (see [27, Sect. 4.3]).
The space $\mathcal{Z}_k(M,\mathbb{Z})$ of k-dimensional integral cycles is defined in higher dimension in a similar manner.

The mass of a one-current T is defined by

$$\text{mass}(T) = \sup\{T(\omega) \mid \omega \text{ one-form on } M \text{ with } \sup_x |\omega(x)| \leq 1\}$$

Note that the mass functional is only semicontinuous with respect to the weak topology on the space of one-currents.

Note that a piecewise smooth closed curve γ with only finitely many intersection points induces by integration a one-cycle with $\text{mass}(\gamma) = L(\gamma)$.

The support of a one-form ω, noted $\text{Supp}(\omega)$, is defined as the closure of $\{x \in M \mid \omega(x) \text{ is not null in } T_xM\}$ in M.
By duality, the support of a one-current T is the smallest closed set C such that

$$\text{Supp}(\omega) \cap C = \emptyset \text{ implies that } T(\omega) = 0 \text{ for every one-form } \omega \text{ on } M.$$

Let T be a one-current whose support decomposes into connected components C_i. The connected components of T are the one-currents T_i defined by

$$T_i(\omega) = T(\varphi_i.\omega) \text{ for every one-form } \omega \text{ on } M$$

where φ_i is a smooth function which equals 1 on C_i and vanishes on $\bigcup_{j\neq i} C_j$. This definition does not depend on the choice of φ_i.

The structure of the one-cycles is the following. A one-cycle is indecomposable if it is induced by an oriented simple closed curve. Every one-cycle T decomposes (not necessarily in a unique way) into a sum of indecomposable one-cycles T_i, i.e., $T = \sum_{i\in\mathbb{N}} T_i$, such that $\text{mass}(T) = \sum_{i\in\mathbb{N}} \text{mass}(T_i)$ (see [14, p. 420]).

We describe now this structure for the local minima of the mass functional. A local minimum of the mass functional is a locally minimizing current, that is, a disjoint union of simple closed geodesics (see [27, Sect. 8]), which are of null index. The converse, given by the following lemma, is an application of Theorem 2 in [37].

Lemma 2.3. *Let M be a bumpy Riemannian two-sphere. The local minima of the mass functional are strict local minima. They agree with the finite sums of disjoint simple closed geodesics of null index.*

The following theorem, established by F. Almgren in [2], is a central tool in this paper. It determines the homotopy groups of the space of integral cycles.

Theorem 2.4. *Let M be a complete manifold. There exists a natural isomorphism*

$$\pi_m(\mathcal{Z}_k(M,\mathbb{Z}),\{0\}) \simeq H_{m+k}(M,\mathbb{Z}) \text{ for every } m,k \in \mathbb{N}.$$

In particular, $\pi_1(\mathcal{Z}_1(S^2,\mathbb{Z}),\{0\}) \simeq H_2(S^2,\mathbb{Z}) \simeq \mathbb{Z}$.

For the rest of this paper, M designates a Riemannian two-sphere. The latter isomorphism of Theorem 2.4 permits us to apply the F. Almgren-J. Pitts minimax principle to the one-cycle space of the two-sphere.

Let us consider the one-parameter families $(z_t)_{0 \leq t \leq 1}$ of one-cycles on M which satisfy the following conditions

(C.1) z_t starts and ends at null-currents

(C.2) z_t induces a nontrivial class $[z]$ in $\pi_1(\mathcal{Z}_1(M,\mathbb{Z}),\{0\})$.

We define the minimax value

$$L_1(M) := \inf_{[z]\neq 0} \sup_{0\leq t\leq 1} \operatorname{mass}(z_t) \tag{2.1}$$

The F. Almgren-J. Pitts principle leads to the following result (see [31], [32, Theorem 4.10] and [10, Appendix] for a proof). Note that another proof is presented in Section 4.1 (see Remark 4.4).

Theorem 2.5. *Let M be a Riemannian two-sphere. There exists a nontrivial closed geodesic on M of total length $\leq L_1(M)$, i.e., scg$(M) \leq L_1(M)$.*

Remark 2.6. This minimax process can be extended in higher dimension but it is no longer clear that the obtained critical points correspond to closed geodesics (see [31] for a study of critical one-cycles on any manifold and [32] for a generalization in term of minimal hypersurfaces).

2.2. **Curve-shortening process.** First, we state the main properties of the curve-shortening process. The construction of the process will be presented afterwards.

For the rest of this paper, we will assume that the metric on the sphere M is bumpy, that is, the closed geodesics of M are non-degenerate. It is a generic condition: the space of bumpy metrics contains an open dense set of the space of metrics (see [25, p. 163]). Note that non-degenerate closed geodesics are isolated in ΛM. Therefore, there are finitely many closed geodesics of a length uniformly bounded.

The bumpy assumption is not restrictive in order to prove the inequalities of Main Theorem 1.2 and Corollary 1.4. Indeed, suppose we have established these inequalities for bumpy metrics with $L(M)$ instead of $\bar{L}(M)$ (see the introduction for the definitions). Let g be an arbitrary Riemannian metric on M and g_n be a sequence of bumpy metrics which converges to g. The area and diameter of (M, g_n) converge to the ones of (M, g). Let c_n be a sequence of closed geodesics on (M, g_n) of lengths uniformly bounded. From the Palais-Smale condition (see [25, p. 26]), a subsequence of c_n, still noted c_n, converges to a closed geodesic c of (M, g). From the lower semi-continuity of the index, we have $\mathrm{ind}_\Lambda(c) \leq \mathrm{ind}_\Lambda(c_n)$ for n large enough. Furthermore, if the curves c_n are simple closed geodesics (resp. figure eight geodesics), then the same holds for c. Therefore, $\liminf_{n\to\infty} L(M, g_n) \geq \bar{L}(M, g)$.

Let $\mathcal{C}$ be a finite collection of piecewise smooth curves formed either of closed curves in ΛM or of paths and closed curves lying in a convex domain Ω homeomorphic to a disk such that the endpoints of the paths of $\mathcal{C}$ lie in the boundary of Ω. A closed domain Ω is said to be convex if there is an $\varepsilon > 0$ such that for all $x, y \in \Omega$ with $d(x, y) < \varepsilon$, the minimizing geodesic from x to y lies in Ω.
The curve-shortening process depends on the collection $\mathcal{C}$ and deforms simultaneously the curves of $\mathcal{C}$. The flow of $\gamma \in \mathcal{C}$ which it induces is noted γ_t.

Theorem 2.7. *Let $\mathcal{C}$ be a finite collection of curves as above. There exists a curve-shortening process defined on $\mathcal{C}$, which deforms simultaneously the curves γ of $\mathcal{C}$ through γ_t and satisfies the following properties:*

i) it leaves the endpoints of the paths fixed
ii) it does not increase the lengths: $\forall \gamma \in \mathcal{C}$, $\forall 0 \leq s \leq t$, $L(\gamma_s) \geq L(\gamma_t)$
iii) simple curves remain simple through the process
iv) non-intersecting simple curves do not intersect through the process
v) for every $\gamma \in \mathcal{C}$, the family γ_t converges to a geodesic of null index
vi) the geodesics of null indices are the only fixed points of this process.

Remark 2.8. The curve-shortening process depends on the whole finite collection $\mathcal{C}$ of curves. It is not defined on the whole loop space ΛM but only for the curves of $\mathcal{C}$.
- The simple closed curves of $\mathcal{C}$ converge to local minima of the mass functional from Lemma 2.3. The closed geodesics of $\mathcal{C}$ of positive indices converge to closed geodesics of null indices like any other closed curve of $\mathcal{C}$.
- We will use all the properties of the curve-shortening process listed in Theorem 2.7, in particular in the proof of Theorem 4.8
- Other curve-evolution processes exist that satisfy some of these properties.

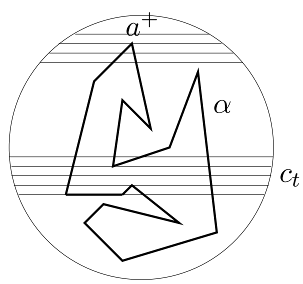
For instance, the Birkhoff process (see [12, p. 4]) shortens curves but simple curves do not necessarily remain simple. The disk flow defined in [20] satisfies most of these properties but it may increase lengths at non-integral time, which does not yield a precise enough control on the lengths to apply minimax processes. The curvature flow defined on ΛM satisfies these properties for embedded curves (see [16]). But the proof of this result, involving analytic techniques, has been established only for simple closed curves and not for arcs with fixed endpoints. Moreover, it is still unknown if the flow converges for non-embedded curves.

Proof of Theorem 2.7. Without loss of generality, we can assume that the curves of $\mathcal{C}$ are polygonal (made of finitely many segments) and that none of them is a geodesic of positive index. Indeed, it is possible to deform the curves of $\mathcal{C}$ into such curves through homotopies satisfying i-iv).
The curve-shortening process of $\mathcal{C}$ rests on two constructions.

Construction of the straightening process:
Let D be a disk of M of radius less than $\frac{1}{2}\mathrm{inj}(M)$. Given a geodesic foliation c_t of D, we define a real function h on D by $h^{-1}(t) = c_t$. Let α be a polygonal arc and c be the segment joining the endpoints of α. Assume that α and c form a simple closed curve and that c lies in some leaf of c_t.
The goal of this straightening process is to define a homotopy α_t in D between α and c which satisfies i-iii). We can slightly deform α through a homotopy of simple polygonal arcs, which satisfies i-iii), so that the restriction h_α of h to α satisfies:

(H.1) for all t, the set $h_\alpha^{-1}(t)$ is finite or empty
(H.2) the values of h_α, at its local extrema, are distinct

Here, by definition, the local extrema are different from the endpoints of α.



We will argue by induction on the number of local extrema of h_α. The arc α and the segment c bound a topological disk Δ in D. Changing the sign of h and adding a constant if necessary, we can assume that h_α admits a global maximum a^+ (different from the endpoints of α) and that c lies in c_0. Let α^+ be the subarc of α, starting and ending at local minima of h, such that a^+ is the only extremum of h in the interior of α^+. The pair of

the endpoints of α^+ is noted $\partial\alpha^+$.
Now, we define a deformation of α^+ which will permit us to construct a homotopy in Δ between α and c. We deform α^+ by replacing the subarc of α^+ above c_s with endpoints in c_s by the segment with the same endpoints, where s decreases from $h(a^+)$ to $\max h(\partial\alpha^+)$. This deformation of α^+, with fixed endpoints, is noted α_t^+ with $0 \leq t \leq 1$. Note that if a^+ is the only extremum of h_α then $\alpha^+ = \alpha$ and α_t^+ is a homotopy in Δ between α and c which satisfies i-iii).

Let us now define a homotopy α_t in Δ between α and c, which satisfies i-iii). We argue by induction on the number of local extrema of h_α.

Suppose that the homotopy α_t^+ does not pass by a local maximum of h_α different from a^+. This homotopy lies in Δ and stops for $t = 1$ when α_t^+ agrees with a segment one of whose endpoints x_0 is a local minimum of h_α. Note $[x_0, x_1]$ this segment. We slightly extend the homotopy α_t^+ by deforming the segment $[x_0, x_1]$ into $[x_0, x_t]$ where x_t moves down along α for $t \geq 1$. Note that it is possible from (H.2). The homotopy α_t^+ so extended defines a deformation α_t in Δ from α to a new simple polygonal arc β. The restriction h_β of h to β satisfies the properties (H.1-2) and has fewer local extrema than h. We can then conclude by induction.

Suppose that the homotopy α_t^+ passes through a local maximum of h. Let t_0 be the greatest t at which α_t^+ passes through a local maximum x_0 of h, different from a^+. The homotopy α_t^+ lies in Δ for $0 \leq t \leq t_0$. The set $\Delta \setminus \cup_{0 \leq t \leq t_0} \alpha_t^+$ is formed of two disjoint topological disks. One of them, noted Δ^+, does not meet the segment c. Let α' be the simple polygonal arc $\partial\Delta^+ \setminus \alpha_{t_0}^+$. Let c' be the segment of Δ joining the endpoints of α'. By construction, the segment c' lies in some leaf of c_t and the endpoints of α' are the only intersection points with c'. Moreover, the restriction $h_{\alpha'}$ of h to α' satisfies the properties (H.1-2) and has fewer local extrema than h_α. By induction, there exists a homotopy α'_t in Δ between α' and $c' = [x_0, x_1]$. We slightly extend the homotopy α'_t as previously, by deforming the segment $[x_0, x_1]$ into $[x_0, x_t]$ where x_t moves up along α. The homotopy α'_t so extended defines a deformation α_t in Δ from α to a new arc β such that h_β has fewer local extrema than h_α. By induction, the arc β straightens into c. This achieves the construction of the straightening process.

Given a polygonal arc α and a segment c with the same endpoints which form a simple closed curve, it is always possible to define a geodesic foliation c_t such that c lies in c_0 in order to apply the straightening process to α.

We can now present the basic step of the curve-shortening process.
Let D be a disk of M of radius less than $\frac{1}{2}\text{inj}(M)$, whose boundary ∂D meets every curve of $\mathcal{C}$ transversely. If $\mathcal{C}$ is formed of arcs and closed curves lying in a convex domain Ω, we replace D by $D \cap \Omega$. The intersections of the curves of $\mathcal{C}$ with D form a collection of subarcs α_i, noted Γ. The following

construction straightens every arc α_i into the segment of the same endpoints.

Construction of the curve-shortening process with respect to D:
Using the straightening process, we can deform in D, one at a time, the closed curves of Γ and the loops formed of the subarcs of the curves of Γ, into point curves. Thus, we can assume that no closed curve of $\mathcal{C}$ lies in D and that the arcs α_i are simple. Let c_i be the segment joining the endpoints of α_i.
Let us define a homotopy in D between c_1 and α_1. We can slightly deform the arcs α_i through a homotopy of polygonal arcs, which satisfies i-iv), so that the segment c_1 cuts the new arcs α_i so deformed, finitely many times, at transverse intersection points. We still denote by Γ the new collection of arcs α_i so deformed. The arcs α_i of Γ such that c_i has no transverse intersection point with c_1 form a subcollection Γ_1 of Γ. We will argue by induction on the number $N(c_1, \Gamma) = \sum_{\alpha_i \in \Gamma_1} \mathrm{card}(c_1 \cap \alpha_i)$ of intersection points between c_1 and the α_i's, with $\alpha_i \in \Gamma_1$. Let α be a subarc of some arc $\overline{\alpha}$ of Γ_1 such that c_1 and α bound a topological disk Δ of D. We can choose α and $\overline{\alpha}$ so that Δ is minimal for the inclusion. We deform in Δ the arc α into a segment c of c_1 by applying the straightening process. This yields a homotopy between $\overline{\alpha}$ and a new arc $\overline{\alpha}'$ such that c lies in $\overline{\alpha}'$. The new collection of arcs so obtained is noted Γ'. Note that if no arc α_i different from α_1 cuts the segment c_1, we get a homotopy between α_1 and c_1. Now, we perturb $\overline{\alpha}'$ through a homotopy of polygonal arcs by slightly deforming the segment c, so that $N(c_1, \Gamma')$ is less than $N(c_1, \Gamma)$. Such a perturbation of Γ' satisfying i-iv) always exists. We conclude then by induction and get a homotopy between α_1 and c_1. The original collection Γ is thus deformed into a new one, noted $\overline{\Gamma}$.

We can now apply this construction to $\overline{\Gamma}$ to straighten another arc. After a finite number of iterations, all the arcs of the original collection are straightened into segments with the same endpoints. This defines the curve-shortening process with respect to D.

The construction of the global curve-shortening process rests on an iteration principle similar to the one of the disk flow defined in [20].
Let D_i be a periodic sequence of disks of radius less than $\frac{1}{2}\mathrm{inj}(M)$. We choose the disks D_i well-positioned relative to $\mathcal{C}$ (see [20, p. 26]) such that the union covers M. The flow γ_t of $\gamma \in \mathcal{C}$ is defined for $i-1 \leq t \leq i$ as the result of performing the curve-shortening process with respect to D. Note that it agrees with the disk flow at integral times. The convergence of this flow to closed geodesics follows now from the proof of [20, Theorem 1.8]. We recall that the geodesics of M are isolated and non-degenerate. Consider the curves γ_t, where $\gamma \in \mathcal{C}$, which converge to geodesics γ_∞ of positive index. These curves can be deformed into γ_t', for t large enough through homotopies of curves satisfying i-iv), so that $L(\gamma_t') < L(\gamma_\infty)$. Perturbing the flow

in this way, we can assume that the curves of $\mathcal{C}$ converge to geodesics of null index. This defines the curve-shortening process. By construction, the properties i-vi) are satisfied. □

Remark 2.9. For a non-generic metric, the curve-shortening process may not converge to closed geodesics but oscillate between them.

The curve-shortening process satisfies the following result whose proof follows [12, Lemma 2.2] and [10, Lemma 1.1].

Lemma 2.10. *Let γ be a simple closed curve bounding a convex domain Ω of M with $L(\gamma) < scg(M)$. There exists a homotopy $\gamma_t \in \Lambda M$ obtained by the curve-shortening process which satisfies*

– *$\gamma_0 = \gamma$, γ_1 is a point curve, γ_t is a simple curve of Ω bounding*
– *the convex domain $\Omega_t := \{x \in \gamma_s \mid t \leq s \leq 1\}$ $L(\gamma_t)$ is non-increasing*
– *$\{\gamma_t \mid 0 \leq t \leq 1\}$ gives rise in a natural way to a map of degree ± 1 from the two-disk D^2 onto Ω.*

3. Diameter, filling radius and the shortest closed geodesic

In this part, we first establish an upper bound on the length of the shortest closed geodesic in terms of the diameter of the two-sphere. Then, we show that the filling radius of the two-sphere is bounded from below in terms of the length of the shortest closed geodesic. Some constructions and arguments used in this part will be developed further afterwards.

3.1. **Diameter and the shortest closed geodesic.**

Following the ideas used in [12], we can prove

Theorem 3.1. *Let M be a Riemannian two-sphere , then*

$$scg(M) \leq 4Diam(M)$$

where $scg(M)$ is the length of the shortest nontrivial closed geodesic on M.

Remark 3.2. The example of the round sphere and of the sphere composed of two copies of flat disks glued together along their boundaries shows that the optimal constant is not less than 2.

Proof. Without loss of generality, we will assume the metric generic.
Let $x, y \in M$ such that $d(x, y) = \text{Diam}(M)$. We can assume that $\text{Diam}(M) < \frac{1}{2}\text{scg}(M)$. Berger's lemma (see [11, p. 106]) asserts that for any non-zero tangent vector $v \in T_xM$ there exists a minimizing geodesic γ from x to y such that the angle $\angle(v, \dot\gamma(0))$ between v and $\dot\gamma(0)$ is not greater than $\frac{\pi}{2}$. In the case of the sphere, it can be stated in the following more precise way (see [12, p. 13]).

Lemma 3.3. *Let x, $y \in M$ such that $d(x, y) = Diam(M)$. There exist $(\gamma_i)_{1 \leq i \leq m}$ minimizing geodesics from x to y such that the closed curves $\gamma_i \cup (-\gamma_{i+1})$ bound convex domains Ω_i in M with $i = 1, \ldots, m$.*

Thus, M decomposes into convex domains Ω_i with $L(\partial\Omega_i) = 2\mathrm{Diam}(M) < \mathrm{scg}(M)$. Lemma 2.10 asserts that there is a homotopy $c_t^i \in \Lambda M$ lying in Ω_i between $\alpha_i := \gamma_i \cup (-\gamma_{i+1}) = \partial\Omega_i$ and a point curve p_i which satisfies $L(c_t^i) \leq L(\partial\Omega_i)$.

We define a one-parameter family of one-cycles on M as follows.
We start from the sum of the points $p_1, \ldots, p_{m-1}$. We deform p_1 into α_1 via the homotopy c_t^1, then we deform p_2 into α_2 via the homotopy c_t^2. At this stage, the one-cycle obtained is the sum $\alpha_1 + \alpha_2 = \gamma_1 - \gamma_3$ (as one-cycles). Then, we deform p_3 to α_3 through the homotopy c_t^3. We get the one-cycle $\alpha_1 + \alpha_2 + \alpha_3 = \gamma_1 - \gamma_4$. We repeat this process so on, deforming p_i to α_i through the homotopy c_t^i one at a time. We obtain a homotopy from the null-current to $\gamma_1 - \gamma_m$. Using the homotopy c_t^m, we contract this latter curve into the null-current.

Since the homotopies c_t^i give rise to maps of degree one onto each domain Ω_i of the decomposition of M, the one-parameter family of one-cycles z_t defined above satisfies (C.1-2). Moreover, the mass of each of these one-cycles is bounded from above by $4\mathrm{Diam}(M)$. Therefore, from the minimax principle of Theorem 2.5, there exists a nontrivial closed geodesic of length $\leq 4\mathrm{Diam}(M)$, i.e., $\mathrm{scg}(M) \leq 4\mathrm{Diam}(M)$. □

3.2. **Filling radius and the shortest closed geodesic.**
In what follows, the sphere M is considered isometrically embedded, as metric space, in $L^\infty(M)$ (see the definition of the filling radius in the introduction).

Let $L_0(M)$ be the length of the shortest nontrivial simple closed geodesic on M of null index. Equivalently, $L_0(M)$ is the smallest length of the nontrivial local minima of the mass functional (see Lemma 2.3). In particular, every simple closed curve of length less than $L_0(M)$ converges to a point curve through the curve-shortening process. We refer to (2.1) for the definition of $L_1(M)$.

Theorem 3.4. *Let M be a Riemannian two-sphere, then*

$$\mathit{FillRad}(M) \geq \min\{\frac{1}{6}L_0(M), \frac{1}{12}L_1(M)\}$$

In particular, $\mathit{FillRad}(M) \geq \frac{1}{12}\mathit{scg}(M)$ where $\mathit{scg}(M)$ is the length of the shortest nontrivial closed geodesic on M.

Remark 3.5. The constant $\frac{1}{12}$ in the last inequality of Theorem 3.4 should be compared with the constant $\frac{1}{6}$ in the sharp filling inequality (1.4), which is reached for instance by the flat tori.

Proof. We follow the arguments of [17, Lemma 1.2.B]. By definition, the fundamental class $[M]$ of M vanishes in $U_\delta(M) \subset L^\infty(M)$ where $\delta >$

FillRad(M). Therefore, there exists a 3-dimensional chain c in $U_\delta(M)$, with coefficients in $\mathbb{Z}$, which fills M, i.e., whose boundary ∂c contained in M represents $[M]$. Using a piecewise linear approximation of c, we construct a 3-dimensional polyhedron P in $U_\delta(M)$ representing c and containing a subcomplex $Q \subset M$ which represents ∂c, that is, $[M]$. In this case, the induced natural map $i_* : H_2(M) \longrightarrow H_2(U_\delta(M))$ is null.

Let P be a 3-dimensional polyhedron in $U_\delta(M)$ containing M as a subpolyhedron, i.e., $M \subset P \subset U_\delta(M) \subset L^\infty(M)$. Fix $\delta > 0$. We show that if $\delta < \min\{\frac{1}{6}L_0(M), \frac{1}{12}L_1(M)\}$, the inclusion $M \hookrightarrow P$ admits a retraction. Thus M does not bound in $U_\delta(M)$, hence the inequality.

Subdividing P if necessary, we can assume that the diameter of all simplices in P is less than $\varepsilon > 0$ with $\varepsilon < \min\{\frac{1}{3}L_0(M), \frac{1}{6}L_1(M)\} - 2\delta$ and $\varepsilon < \mathrm{conv}(M)$, where $\mathrm{conv}(M)$ is the convexity radius of M. Using the Birkhoff process with fixed endpoints if necessary, we can also assume that the edges of P lying in M are minimizing segments. We are going to construct successively on the skeleton of P a retraction $r : P \longrightarrow M$.

We define $r : P^0 \longrightarrow M$ on the 0-skeleton of P by sending each vertex p_i of P to a nearest point v_i of M, as we wish. The vertices of M are thus fixed.
Let us extend this map to the 1-skeleton P^1 of P. Since $i : M \hookrightarrow U_\delta(M)$ preserves the distances, we have for every pair p_i, p_j of adjacent vertices of P

$$\begin{aligned} d_M(v_i, v_j) = d_{L^\infty(M)}(v_i, v_j) &\leq d_{L^\infty(M)}(v_i, p_i) + d_{L^\infty(M)}(p_i, p_j) + d_{L^\infty(M)}(p_j, v_j) \\ &\leq 2\delta + \varepsilon =: \rho < \min\{\frac{1}{3}L_0(M), \frac{1}{6}L_1(M)\} \end{aligned}$$

where v_i and v_j are the images by r of p_i and p_j. We map $[p_i, p_j]$ to a minimizing segment $[v_i, v_j] \subset M$ joining v_i to v_j. This defines an extension $P^1 \longrightarrow M$ which leaves the edges of M fixed.
Let us now extend this map to P^2. Let Δ^2 be a 2-simplex of P which lies in M. We extend r by the identity on Δ^2. Since the edges of Δ^2 are minimizing segments and $\mathrm{Diam}(\Delta^2) \leq \mathrm{conv}(M)$, Δ^2 is convex and the curve-shortening process contracts $\partial\Delta^2$ into a point through a homotopy of free loops lying in Δ^2 (see Lemma 2.10). Let Δ^2 be a 2-simplex of P which does not lie in M. Since the boundary $\partial\Delta^2$ maps onto a geodesic triangle T of perimeter less than $3\rho < L_0(M)$, the curve-shortening process defines a map from the disk $D \simeq \Delta^2$ to M which sends $\partial D \simeq \partial\Delta^2$ onto T. This construction yields a map $r : P^2 \longrightarrow M$ whose restriction to M is the identity.
We now want to extend it to P^3. This is possible if the restriction $\varphi : \partial\Delta \longrightarrow M$ of $r : P^2 \longrightarrow M$ to the boundary $\partial\Delta$ of each 3-simplex Δ of P is of null degree. We recall that $\partial\Delta$ is homeomorphic to a sphere. The curve-shortening process used in the construction of $r : P^2 \longrightarrow M$ provides homotopies c_t^i

defined on each face Δ_i of Δ between $\partial\Delta_i$ and point curves p_i such that $L(\varphi(c_t^i)) \leq L(\partial\Delta_i)$.
We define a one-parameter family of one-cycles on M putting together the homotopies $c_t^1 + c_t^2$ and $c_t^3 + c_t^4$. The construction is analogous to the one carried out in the proof of Theorem 3.1. We start from the sum of p_1 and p_2. We deform it through the sum of the homotopies c_t^1 and c_t^2 into $\partial\Delta_1 + \partial\Delta_2$ with $\partial\Delta_1 + \partial\Delta_2 = -\partial\Delta_3 - \partial\Delta_4$ as one-cycles. Then, we deform $-\partial\Delta_3 - \partial\Delta_4$ into $-p_3 - p_4$ using the homotopies c_t^3 and c_t^4.
The one-parameter family z_t of one-cycles on $\partial\Delta$ so defined starts and ends at null-currents. Since the homotopies c_t^i give rise to maps of degree one on each face Δ_i of $\partial\Delta$, it represents a generator of $\pi_1(\mathcal{Z}_1(\partial\Delta), \{0\}) \simeq \pi_2(\partial\Delta) \simeq \mathbb{Z}$. Moreover, $\text{mass}(\varphi(z_t)) \leq 6\rho < L_1(M)$. Therefore, by definition of $L_1(M)$, the family $\varphi(z_t)$ represents a null-class in $\pi_1(\mathcal{Z}_1(M), \{0\})$.

$$\text{So,} \quad \varphi_* : \begin{array}{ccc} \pi_1(\mathcal{Z}_1(\partial\Delta), \{0\}) & \longrightarrow & \pi_1(\mathcal{Z}_1(M), \{0\}) \\ | \wr & & | \wr \\ \pi_2(\partial\Delta) & & \pi_2(M) \end{array} \quad \text{is trivial.}$$

That is the degree of φ is null.

In conclusion, $\varphi : \partial\Delta \longrightarrow M$ can be extended to Δ for each 3-simplex Δ of P^3. Therefore, the inclusion $M \hookrightarrow P$ admits a retraction. □

4. Filling radius and the minimal one-cycle of index one

We will first introduce a local minima process on the one-cycle space and define the index of the one-cycles. Then, we will describe the structure of the shortest one-cycle of index one on the two-sphere. Finally, we will show that the length of the latter provides a lower bound on the filling radius, extending the results of the previous section.

4.1. **Indices of the one-cycles.** In this section, we will usethe language of the varifolds to state some results due to F. Almgren and J. Pitts. However, in order to be consistant with the use of the one-cycles, we will restate these results in the language of one-cycles.
The varifolds considered in this section correspond to unions of closed curves counted with multiplicity. Since we will not really make use of the varifolds in the proofs of the results which follow, the latter will not be defined here. Instead, we refer to [32, p. 60-62] for the definition of the one-dimensional integral varifolds and to [32, p. 74] for the one of stationary varifolds.

The global minimax process on the one-cycle space described by (C.1-2) in Section 2.1 admits a local version presented below:

Given $\kappa > 0$, let $\mathcal{Z}_1^{\leq\kappa}(M) = \{z \in \mathcal{Z}_1(M, \mathbb{Z}) \mid \text{mass}(z) \leq \kappa\}$.
Similarly, we define $\mathcal{Z}_1^{<\kappa}(M)$, $\Lambda^{\leq\kappa}(M)$ and $\Lambda^{<\kappa}(M)$. Fix $0 \leq \kappa_0 < \kappa_1$.

Let us consider the one-parameter families $(z_t)_{0 \leq t \leq 1}$ of one-cycles on M which satisfy the following conditions

(C'.1) z_0, z_1 lie in $\mathcal{Z}_1^{\leq \kappa_0}(M)$

(C'.2) z_t induces a nontrivial class $[z]$ in $\pi_1(\mathcal{Z}_1^{\leq \kappa_1}(M), \mathcal{Z}_1^{\leq \kappa_0}(M))$.

We define the minimax value $L(\kappa_0, \kappa_1) := \inf\limits_{[z] \neq 0} \sup\limits_{0 \leq t \leq 1} \text{mass}(z_t)$.

This local minimax process gives rise to a stationary one-dimensional integral varifold of total length $L(\kappa_0, \kappa_1)$ corresponding to a union of closed curves counted with multiplicity (see [31, p. 468] and [32, Theorem 4.10]). Strictly speaking, this result is stated for the global minimax process (C.1-2), i.e., when $\kappa = 0$, but it also holds with the local minimax process (C'.1-2). Thus, there exists a one-cycle z of mass $L(\kappa_0, \kappa_1)$ which induces a stationary varifold. Note that since the varifold is stationary, it decomposes M into convex domains. Therefore, $L(\kappa_0, \kappa_1) \geq \text{conv}(M)$ where $\text{conv}(M)$ is the convexity radius of M.

We also define

$$L'_1(M) = \inf\{L(\kappa_0, \kappa_1) \mid 0 \leq \kappa_0 < \kappa_1 \text{ such that the conditions (C'.1-2) apply}\}$$

Note that $L'_1(M)$ is positive. From the definition of $L_1(M)$ (see (2.1)), we have $0 < L'_1(M) \leq L_1(M)$.

Remark 4.1. From the structure, previously described, of the critical points of the local minimax processes and the compactness theorem for stationary one-dimensional integral varifolds (see [32, p. 77]), there exists a one-cycle z of mass $L'_1(M)$ which induces a stationary varifold.

We define the index of the one-cycles by analogy with the index of the closed geodesics. Recall that the metric is supposed bumpy. Using classical Morse theory on a finite dimensional approximation of ΛM (see also [25, Section 2.4]), we see that the index of a closed geodesic γ of length κ agrees with $\text{ind}_\Lambda(\gamma) = \min\{i \in \mathbb{N} \mid \pi_i(\Lambda^{<\kappa}(M) \cup \{\gamma\}, \Lambda^{<\kappa}(M)) \text{ is nontrivial}\}$.
The index of a one-cycle of mass κ is similarly defined by
$\text{ind}_{\mathcal{Z}_1}(z) = \min\{i \in \mathbb{N} \mid \pi_i(\mathcal{Z}_1^{<\kappa}(M) \cup \{z\}, \mathcal{Z}_1^{<\kappa}(M)) \text{ is nontrivial}\}$.
In particular, the one-cycles of null indices agree with the local minima of the mass functional described in Lemma 2.3. Note that the varifolds induced by one-cycles of finite indices are stationary.

We define

$$L''_1(M) = \inf\{\text{mass}(z) \mid z \text{ one-cycle of index one}\}$$

The following useful claim, which is a version of the mountain pass lemma, comes from the definition of $L'_1(M)$.

Claim 4.2. *Let $(z_t)_{0 \leq t \leq 1}$ be a homotopy of one-cycles joining two local minima of the mass functional. If $\sup_{0 \leq t \leq 1} \text{mass}(z_t) < L'_1(M)$, then the two local minima of the mass functional agree.*

Proof. Define $\kappa_0 = \max\{\text{mass}(z_0), \text{mass}(z_1)\}$ and $\kappa_1 = \sup_{0 \leq t \leq 1} \text{mass}(z_t)$. Since $\kappa_1 < L_1'(M)$, the group $\pi_1(\mathcal{Z}_1^{\kappa_1}(M), \mathcal{Z}_1^{\kappa_0}(M))$ is trivial. Therefore, the family z_t is homotopic to another family of one-cycles z_t' with the same endpoints such that $\text{mass}(z_t') \leq \kappa_0$. Since the endpoints z_0 and z_1 of z_t' are strict local minima of the mass functional (see Lemma 2.3), the family z_t' is constant. In particular, z_0 and z_1 agree. □

The structure of the one-cycles critical for the mass functional is described by the following

Theorem 4.3. *Let M be a bumpy Riemannian two-sphere.*
We have $L_1'(M) = L_1''(M)$. Moreover, there exists a (shortest) one-cycle of index one of mass $L_1''(M)$.
The shortest one-cycle of index one agrees with the shortest curve among
- *the simple closed geodesics of indices one as geodesic*
- *the figure eight geodesics of null indices as geodesic.*

Remark 4.4. - Both simple closed geodesics of index one and figure eight geodesics of null index can occur as the shortest one-cycles of index one. Examples are presented in Remark 4.10.
- Since $L_1'(M) \leq L_1(M)$, Theorem 4.3 admits Theorem 2.5 as corollary.

We will need the following result.

Lemma 4.5.

i) Every simple closed geodesic of positive index γ satisfies $L(\gamma) \geq L_1'(M)$. Furthermore, the inequality is strict if $\text{ind}_\Lambda(\gamma) > 1$.
ii) Every figure eight geodesic γ satisfies $L(\gamma) \geq L_1'(M)$. Furthermore, the inequality is strict if $\text{ind}_\Lambda(\gamma) > 0$.

Proof. Let us show first the point i). Let γ be a simple closed geodesic of positive index. The curve γ decomposes M into two disks D^+ and D^-.

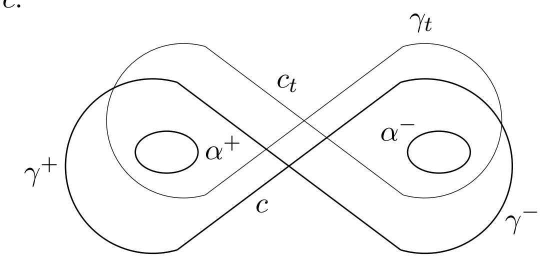
We are going to define a homotopy $\gamma_t \in \Lambda M$ with $-1 \leq t \leq 1$, joining two simple closed geodesics of null indices, such that $\gamma_0 = \gamma$, γ_t lies in D^+ for $t \geq 0$, γ_t lies in D^- for $t \leq 0$ and $L(\gamma_t) < L(\gamma)$ for $t \neq 0$. But first, let us show how this yields the point i) of the lemma.
If γ_1 and γ_{-1} are not both trivial, the endpoints of the family of one-cycles z_t induced by γ_t are different. Therefore, $L(\gamma) = \sup_{0 \leq t \leq 1} L(\gamma_t) \geq L_1'(M)$ from Claim 4.2. If γ_1 and γ_{-1} are not both trivial, the homotopies $(\gamma_t)_{0 \leq t \leq 1}$ and $(\gamma_t)_{-1 \leq t \leq 0}$ give rise to maps of degree one onto D^+ and D^-. Thus, the family z_t satisfies the conditions (C.1-2). Therefore, $L(\gamma) = \sup_{0 \leq t \leq 1} L(\gamma_t) \geq L_1(M) \geq L_1'(M)$.
In both cases, the homotopy γ_t starting and ending at local minima of the mass functional satisfies (C'.1-2) for some $0 \leq \kappa_0 < \kappa_1$. If further $\text{ind}_\Lambda(\gamma) > 1$, then the family γ_t is homotopic in $\Lambda^{\kappa_1} M$ to another family γ_t' with the same endpoints such that $\sup_{0 \leq t \leq 1} L(\gamma_t') < L(\gamma)$. Therefore,

$L(\gamma) > L_1'(M)$.

Now, let us define the homotopy γ_t. The subspace of the space of piecewise smooth normal vectorfields on γ spanned by the eigenvectors of $D^2E(\gamma)$ corresponding to negative eigenvalues is nontrivial. Let V be an eigenvector corresponding to the smallest eigenvalue such that $\sup_{x\in\gamma}|V(x)| = 1$. Let $\gamma_t \in \Lambda M$ be a homotopy of simple closed curves with $-1 \leq t \leq 1$ such that $\gamma_0 = \gamma$, $\frac{d}{dt}(\gamma_t)_{|t=0} = V$ and $L(\gamma_t) < L(\gamma)$ for $t \neq 0$. We can also assume that $\gamma_t(x)$ is fixed at the points x where $V(x)$ vanishes.
If the vectors $V(x)$ point toward D^+ for every $x \in \gamma$, then γ_t lies in D^+ for $t \geq 0$ and γ_t lies in D^- for $t \leq 0$. We deform γ_1 in D^+ and γ_{-1} in D^- into simple closed geodesics of null indices through the curve-shortening process. This yields the desired homotopy. The same holds true if the vectors $V(x)$ point toward D^- for every $x \in \gamma$.
Thus, we can suppose that there are two subarcs γ_t^+ and γ_t^- of γ_t with fixed endpoints such that γ_t^+ lies in D^+ and γ_t^- lies in D^-.
We have $\frac{d}{dt}L(\gamma_t^+)_{|t=0} \leq 0$. Otherwise, the normal vectorfield on γ defined by $\bar{V} = V$ off γ_0^+ and $\bar{V} = -V$ on γ_0^+ would satisfy $\sup_{x\in\gamma}|\bar{V}(x)| = 1$ and $D^2E_{|\gamma}(\bar{V},\bar{V}) < D^2E_{|\gamma}(V,V)$. Hence a contradiction with the definition of V.
We can now define a homotopy $\bar{\gamma}_t$ starting from γ, such that $\bar{\gamma}_t$ lies in D^+ for $t \geq 0$ and $L(\bar{\gamma}_t) < L(\gamma)$ for $t \neq 0$, by deforming γ_0^+ through γ_t^+ and smoothing out at the endpoints of γ_t^+. Then, we deform $\bar{\gamma}_1$ in D^+ into a simple closed geodesic of null index through the curve-shortening process. We perform the same construction in D^- with γ_t^- and put together the homotopies in order to obtain the desired homotopy.

Let us now show the point ii). We will argue as previously.
Let γ be a figure eight geodesic. The curve γ decomposes into two geodesic loops γ^+ and γ^-, which bound convex domains Ω^+ and Ω^-. These loops form a closed curve $c = \gamma^+ \cup (-\gamma^-)$, which bounds the convex domain $\Omega := M \setminus (\Omega^+ \cup \Omega^-)$. The curves c, γ^+ and γ^-, bounding the convex domains Ω, Ω^+ and Ω^-, converge respectively in Ω, Ω^+ and Ω^- to local minima α, α^+ and α^- of the mass functional through homotopies h_t, h_t^+ and h_t^- given by the curve-shortening process. If the curve α is trivial, the homotopy h_t gives rise to a map of degree one onto Ω (see Lemma 2.10). The same goes for γ^+ and γ^-. Put together, the homotopies of one-cycles h_t and $h_t^+ - h_t^-$ form a one-parameter family of one-cycles z_t with endpoints α and $\alpha^+ - \alpha^-$. As previously, this family z_t, starting and ending at local minima of the mass functional, satisfies (C'.1-2) for some $0 \leq \kappa_0 < \kappa_1$. Therefore, $L(\gamma) = \sup_{0\leq t\leq 1} \mathrm{mass}(z_t) \geq L_1'(M)$.

Suppose now that the index of γ is positive. The geodesic γ can be slightly deformed through a path $(\gamma_t)_{0\leq t\leq 1} \in \Lambda M$ of figure eight curves with $\gamma_0 = \gamma$ and $L(\gamma_t) < L(\gamma)$ for $t \neq 0$. The curves of the path γ_t decompose into loops γ_t^+ and γ_t^-. These latter give rise to a path $c_t = \gamma_t^+ \cup (-\gamma_t^-) \in \Lambda M$ starting from c.



Let us now define another family of one-cycles homotopic to z_t with the same endpoints. Put together, the homotopies c_t and h_t define a homotopy in ΛM between c_1 and α. The length of the curves of this homotopy is $< L(\gamma)$, except for c. Since c is not a geodesic, this homotopy deforms in ΛM into a path $c_{1,t}$ with the same endpoints c_1 and α such that $L(c_{1,t}) < L(\gamma)$ for every t. Similarly, we define a homotopy $\gamma_{1,t}^+$ (resp. $\gamma_{1,t}^-$) between γ_1^+ and α^+ (resp. γ_1^- and α^-) using γ_t^+ and h_t^+ (resp. γ_t^- and h_t^-). Put together, the homotopies of one-cycles $c_{1,t}$ and $\gamma_{1,t}^+ - \gamma_{1,t}^-$ form a one-parameter family of one-cycles $z_{1,t}$ homotopic to z_t with the same endpoints α and $\alpha^+ - \alpha^-$. Thus, the family $z_{1,t}$ also satisfies (C'.1-2) for some $0 \leq \kappa_0 < \kappa_1$. By construction, $\sup_{0 \leq t \leq 1} \operatorname{mass}(z_{1,t}) < L(\gamma)$. Hence the conclusion. □

We will also need the following result.

Lemma 4.6. *The connected one-cycles of mass $\leq L_1'(M)$ which induce stationary varifolds are either simple closed geodesics or figure eight geodesics.*

Proof. Let γ be a connected one-cycle of mass $\leq L_1'(M)$ which induces a stationary varifold, in particular $\operatorname{mass}(\gamma) \leq L_1'(M)$. Assume that γ is neither a simple closed curve nor a figure eight curve. In which case, γ would be geodesic.
There exist k domains D_i such that $M \setminus \gamma = \amalg_{i \in I} D_i$ where $I = \{0, \ldots, k-1\}$. Since the varifold induced by γ is stationary and is not a simple closed geodesic, the domains D_i are convex and their boundaries γ_i are not geodesics. Thus, the boundaries γ_i of the convex domains D_i converge in D_i to local minima α_i of the mass functional through homotopies $\gamma_{i,t}$ given by the curve-shortening process. Given $x_0 \in D_0$, the index of D_i is defined as the winding number of γ around any point of D_i in the plane $M \setminus x_0$. The set I decomposes into the disjoint union of I^+ and I^- where the index of D_i is even if $i \in I^+$ and odd if $i \in I^-$. We have $\sum_{i \in I^+} \gamma_i = -\sum_{i \in I^-} \gamma_i$, where the orientation of γ_i is given by the one on D_i. Put together, the homotopies of one-cycles $\sum_{i \in I^+} \gamma_{i,t}$ and $-\sum_{i \in I^-} \gamma_{i,t}$ form a one-parameter family of one-cycles z_t between $\sum_{i \in I^+} \alpha_i$ and $-\sum_{i \in I^-} \alpha_i$ with $\operatorname{mass}(z_t) \leq \operatorname{mass}(\gamma)$. Since the α_i's are disjoint, the family z_t starts and ends at local minima of the mass functional (see Lemma 2.3). Moreover, z_0 and z_1 are either different or both trivial. As previously, the family z_t satisfies (C'.1-2) for some

$0 \leq \kappa_0 < \kappa_1$. Therefore, $\mathrm{mass}(\gamma) \geq \sup_{0 \leq t \leq 1} \mathrm{mass}(z_t) \geq L'_1(M)$.

We are going to modify z_t into a homotopic family z'_t with the same endpoints which satisfies $\sup_{0 \leq t \leq 1} \mathrm{mass}(z'_t) < \mathrm{mass}(\gamma)$. This implies that $\mathrm{mass}(\gamma) > L'_1(M)$ and leads to a contradiction since $\mathrm{mass}(\gamma) \leq L'_1(M)$.

Suppose that the indices of two adjacent domains D_i have the same parity, even for instance. Renumbering the D_i's if necessary, we can assume that these two domains are D_0 and D_2.
Let c be a subarc of γ_0 formed of two segments one of whose agrees with a common side of D_0 and D_2. The arc c is a piecewise geodesic broken at some point p. We shorten c in D_0 at the neighborhood of p while keeping its endpoints fixed by smoothing it out at p. This yields a length-decreasing homotopy c_t from c to some arc c'.
Let us consider the homotopy of one-cycles $\gamma_{0,t} + \sum_{i \in I^+ \setminus \{0\}} \alpha_i$ starting from $\sum_{i \in I^+} \alpha_i$ and joining $\gamma_0 + \sum_{i \in I^+ \setminus \{0\}} \alpha_i$. We extend it by deforming first the curve α_2 into γ_2 through $\gamma_{2,t}$. We get a one-cycle which agrees with $\gamma'_0 + \gamma_2 + \sum_{i \in I^+ \setminus \{0,2\}} \alpha_i$ where $\gamma'_0 = (\gamma_0 \setminus c) \cup c'$. Note that γ'_0 and γ_2 have a nontrivial segment in common with opposite orientations. This segment vanishes out in the sum of one-cycles $\gamma'_0 + \gamma_2$. Then, we deform c' into c through c_t and α_i into γ_i through $\gamma_{i,t}$ to obtain $\sum_{i \in I^+} \gamma_i = -\sum_{i \in I^-} \gamma_i$. Finally, we deform the one-cycle so obtained into $-\sum_{i \in I^-} \alpha_i$ through $-\sum_{i \in I^-} \gamma_{i,t}$. As a result, we get a one-parameter family of one-cycles z'_t homotopic to z with the same endpoints. By construction, there exists $\varepsilon > 0$ such that $\mathrm{mass}(z_t) + \varepsilon \leq \mathrm{mass}(\gamma) \leq L'_1(M)$. Hence the result.

We can now assume that the indices of adjacent domains D_i have different parities. From this assumption and since γ is not simple, there exists $p \in \gamma$ such that the (finite) set $\mathcal{V} \subset U_pM$ of unit vectors based at p tangent to γ has at least four vectors. Every pair of adjacent vectors of $\mathcal{V}$ spans a domain D_i for some $i \in I$. There exist two different domains in $\{D_i \mid i \in I\}$ spanned by two pairs, $\{u, v\}$ and $\{u', v'\}$, of vectors of $\mathcal{V}$ such that u, v, u' and v' are consecutive for the cyclic order on U_pM. Renumbering the D_i's if necessary, we can assume that the pairs $\{u, v\}$, $\{v, u'\}$ and $\{u', v'\}$ respectively span D_0, D_1 and D_2, where the indices of D_0 and D_2 are both even. The arcs $\gamma_0 \cup \gamma_2$ and γ_1 agree along a piecewise geodesic subarc c_1 broken at p and pointing to the directions of v and u'. We shorten c_1 in D_1 into c'_1 while keeping its endpoints fixed by smoothing it out at p. The simple closed curve $\gamma'_1 = (\gamma_1 \setminus c_1) \cup c'_1$ converges to α_1 along a path $\gamma'_{1,t}$ of one-cycles with $L(\gamma'_{1,t}) \leq L(\gamma'_1)$.
Suppose that there are at least three domains of even index. Let us consider the homotopy of one-cycles $\gamma_{0,t} + \gamma_{2,t} + \sum_{i \in I^+ \setminus \{0,2\}} \alpha_i$ starting from $\sum_{i \in I^+} \alpha_i$ and joining $\gamma_0 + \gamma_2 + \sum_{i \in I^+ \setminus \{0,2\}} \alpha_i$. We extend it by deforming first the subarc c_1 of $\gamma_0 \cup \gamma_2$ into c'_1, then the curves α_i into γ_i through $\gamma_{i,t}$

for $i \in I^+ \setminus \{0,2\}$. We get a one-cycle which agrees with $-\gamma_1' - \sum_{i\in I^- \setminus \{1\}} \gamma_i$. Using the homotopies $\gamma_{1,t}'$ and $\gamma_{i,t}$, we deform the one-cycle so obtained into $-\sum_{i\in I^-} \alpha_i$.
The above construction defines a one-parameter family of one-cycles z_t' which starts at $\sum_{i\in I^+} \alpha_i$ and ends at $-\sum_{i\in I^-} \alpha_i$. By construction, the family z_t' is homotopic to z_t and satisfies $\sup_{0\leq t\leq 1} \mathrm{mass}(z_t') < L(\gamma)$.

Suppose that D_0 and D_2 are the only domains of even index. Since γ is not a figure eight curve, γ_0 and γ_2 cannot be both geodesic loops based at p. We shall assume that γ_0 is not a geodesic loop. Let c_0 be a piecewise geodesic subarc of γ_0 broken at some point $q \neq p$. We shorten c_0 in D_0 into c_0' by smoothing it out at q. We extend the homotopy of one-cycles $\gamma_{0,t} + \alpha_2$ starting from $\alpha_0 \cup \alpha_2$ and joining $\gamma_0 \cup \alpha_2$ by deforming the subarc c_0 of γ_0 into c_0'. We perturb the one-cycle so obtained by deforming first the subarc c_1 into c_1', then the subarc c_0' into c_0. At this stage, we get a one-cycle which agrees with $-\gamma_1' - \sum_{i\in I^- \setminus \{1\}} \gamma_i$. The latter deforms into $-\sum_{i\in I^-} \alpha_i$, as previously. The one-parameter family of one-cycles z_t' so defined is homotopic to z_t with the same endpoints and satisfies $\sup_{0\leq t\leq 1} \mathrm{mass}(z_t') < L(\gamma)$. □

We can now established Theorem 4.3.

Proof of Theorem 4.3. Let γ be a one-cycle of mass $\kappa = L_1'(M)$ which induces a stationary varifold as in Remark 4.1. At least one connected component of γ is not a local minimum of the mass functional. From Lemma 4.6 and Lemma 2.3, this component is either a simple closed geodesic of positive index or a figure eight geodesic. Thus, from Lemma 4.5, γ is connected and agrees with either a simple closed geodesic of index one or a figure eight geodesic of null index. From the proof of Lemma 4.5, there exists a homotopy γ_t in $\mathcal{Z}_1^{<\kappa}(M) \cup \{\gamma\}$ which induces a nontrivial class in $\pi_1(\mathcal{Z}_1^{\leq\kappa}(M), \mathcal{Z}_1^{\leq\kappa_0}(M))$ for some $\kappa_0 < \kappa$. In particular, the group $\pi_1(\mathcal{Z}_1^{<\kappa}(M) \cup \{\gamma\}, \mathcal{Z}_1^{<\kappa}(M))$ is nontrivial. Therefore, $\mathrm{ind}_{\mathcal{Z}_1}(\gamma) = 1$ and $L_1''(M) \leq L_1'(M)$. Moreover, $L_1''(M)$ agrees with the length of the shortest curve among the simple closed geodesics of index one and the figure eight geodesic of null index.

Suppose that $L_1''(M) < L_1'(M)$. Let γ be a one-cycle of index one such that $\mathrm{mass}(\gamma) < L_1'(M)$. As previously, at least one connected component of γ is not a local minimum of the mass functional. Therefore, from Lemma 4.6, Lemma 2.3 and Lemma 4.5, the length of this connected component is $\geq L_1'(M)$. Hence a contradiction. □

4.2. **Filling radius and the minimal one-cycle of index one.**

An admissible geodesic on the sphere M is by definition a nontrivial simple closed geodesic of length $\leq \frac{1}{2} L_1'(M)$ and of null index as geodesic. From Lemma 2.3, an admissible geodesic is also a local minimum of the mass functional. Note that admissible geodesics exist on M if and only if $L_0(M) \leq \frac{1}{2} L_1'(M)$ (we refer to Section 3.2 for the definition of $L_0(M)$).
An admissible geodesic, which divides M into two disks D and D', is said to be extremal with respect to D if no other admissible geodesic lies in D.

Lemma 4.7. *Let γ be an admissible geodesic of M. Every simple geodesic arc of length $\leq \frac{1}{2}L'_1(M)$ with endpoints in γ is a subarc of γ.*

Proof. Let us suppose that there exists a simple geodesic arc c of length $\leq \frac{1}{2}L'_1(M)$ with endpoints in γ which lies in the interior of D. The arc c divides D into two convex domains D_1 and D_2 with $L(\partial D_1) + L(\partial D_2) \leq L'_1(M)$. The simple closed curves ∂D_1 and ∂D_2 converge in the convex domains D_1 and D_2 to simple closed geodesics disjoint from γ and from each other by the curve-shortening process. The sum of the homotopies so defined is a one-parameter family of one-cycles z_t. This family joins two different local minima of the mass functional and satisfies $\text{mass}(z_t) < L'_1(M)$. We get therefore a contradiction with Claim 4.2. □

Now, we can prove

Theorem 4.8. *Let M be a bumpy Riemannian two-sphere, then*

$$\mathit{FillRad}(M) \geq \frac{1}{20}L'_1(M)$$

where $L'_1(M)$ is the mass of the shortest one-cycle of index one on M.

Proof. We can assume that there exists an admissible geodesic, otherwise Theorem 3.4 immediately yields the result. Since there are finitely many closed geodesics of length uniformly bounded, there exists an admissible geodesic γ_0 which divides M into two connected components D and D' extremal with respect to D.

We argue as in the proof of Theorem 3.4. Fix $\delta > 0$. Let P be a 3-dimensional polyhedron in $U_\delta(M)$ containing M as a subpolyhedron. We show that if $\delta < \frac{1}{20}L'_1(M)$, the inclusion $M \hookrightarrow P$ admits a retraction. Therefore, M does not bound in $U_\delta(M)$.

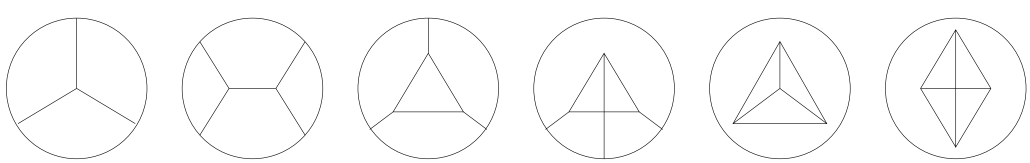
We subdivide P and define a map $r : P^1 \longrightarrow M$ on the 1-skeleton of P which leaves fixed the edges of M as in the proof of Theorem 3.4. The map r takes the edges of P to segments of M. Slightly deforming P and r if necessary, we can further assume that no vertex of P is sent to γ_0. Since $i : M \hookrightarrow U_\delta(M)$ preserves the distances, the length of the images of the edges of P are bounded from above by ρ with $\rho < \frac{1}{10}L'_1(M)$. Therefore, the images of the edges of P cut γ_0 at most once from Lemma 4.7. In particular, the images of the boundaries of the 2-simplices of P cut γ_0 at most twice. On every edge of P whose image cuts γ_0, we introduce a new vertex given by the pre-image of the intersection with γ_0. A 3-simplex Δ^3 of P having new vertices has exactly three or four. Every new vertex divides the edges where it lies into two new edges. Every pair of new vertices lying in a same face defines a new edge. These three or four new edges of Δ^3 bound a new face in Δ^3. This defines a new simplicial structure on P modeled on three kinds of building blocks and not only on standard simplices anymore. These building blocks are given by truncating th! e standard simplex along a plane which does not pass through a vertex.

The intersections of the images of $\partial\Delta^2$ with D, where Δ^2 is a face of the original simplicial structure of P, form a finite collection of curves, noted $\mathcal{C}$. The curves of $\mathcal{C}$ are either simple arcs of D with endpoints in $\partial\Delta^2$ or simple closed curves of D.

Let p and q be two new vertices of P lying in the same face Δ^2 of the original simplicial structure of P. The intersection of the image of $\partial\Delta^2$ with D form a simple arc γ_{pq} with endpoints in γ_0 The endpoints, $\overline{p}$ and $\overline{q}$, of'γ_{pq} are the images by r of p and q. We apply the curve-shortening process to the collection $\mathcal{C}$. The arc γ_{pq} converges to a simple geodesic arc $\overline{pq}$ of D with endpoints $\overline{p}$ and $\overline{q}$. Since $L(\partial\Delta^2) < \frac{1}{2}L'_1(M)$, this arc lies in γ_0 from Lemma 4.7. We extend the map r on the 1-skeleton P^1 of the new simplicial structure of P sending the new edges $[p,q]$ onto the arcs $\overline{pq}$ of γ_0.

We now want to extend r to P^2. Let Δ^2 be a face of a building block of P. When Δ^2 lies in M, we fill the boundary $\partial\Delta^2$ as in the proof of Theorem 3.4. If the image of $\partial\Delta^2$ lies in D', we fill it by the disk it bounds in D'. Otherwise, it lies in D. In this case, if $\partial\Delta^2$ has an edge which does not lie in the original simplicial structure of P, the image of $\partial\Delta^2$ retracts onto the image of this edge by construction. If it does not, its image, of length $< \frac{1}{2}L'_1(M)$, contracts either into an admissible geodesic of D, that is, γ_0, which is homotopic to a point in D', or into a point in D.
In conclusion, we get a map of degree one from Δ^2 onto its image in M which agrees with r on the boundary. This yields an extension $r : P^2 \longrightarrow M$ whose restriction to M is the identity.

Let us extend it to P^3. The restriction of $r : P^2 \longrightarrow M$ to the boundary Σ of a building block Δ^3 of P is noted $\varphi : \Sigma \longrightarrow M$. When the image of Σ lies in D or D', it extends to a map defined on Δ^3 whose image lies in D or D'. In the general case, the image of Σ lies in M and induces a class $[\Sigma]$ in $H_2(M, D') \simeq H_2(D, \partial D)$. The map φ extends to $\Delta^3 \longrightarrow M$ if and only if $[\Sigma]$ vanishes in $H_2(D, \partial D)$. Since the image of the edges of Σ are minimizing segments, the image of the 1-skeleton of Σ is isotopic to one of the following graphs when it is not degenerated. Note that we can always assume it is the case by slightly perturbing the map r.



Graphs 1-6

In the three following cases, Σ is identified with its image by φ.

Case 1. Suppose that only one vertex of Σ maps in the interior of D (see graph 1). Let c_1, c_2 and c_3 be the three edges of Σ based at this vertex and α_1, α_2 and α_3 be the other edges of Σ, lying in ∂D, with the same endpoints as $c_2 \cup c_3$, $c_3 \cup c_1$ and $c_1 \cup c_2$. By construction, these three latter arcs are simple and respectively converge to α_1, α_2 and α_3 through homotopies given by the curve-shortening process. The sum of these homotopies defines a one-parameter family z_t of one-cycles of D which represents $[\Sigma]$. From the property iv) of the curve-shortening process (see Theorem 2.7), the family z_t, which starts from a null-current, ends at a one-cycle lying in ∂D representing either a null-current or ∂D. Since $\text{mass}(z_t) \leq 6\rho < L_1'(M)$, this one-cycle represents a null-current, otherwise γ_0 would not be a local minimum of the mass functional from Claim 4.2. Therefore, $[\Sigma] = 0$ in $H_2(D, \partial D)$.
Suppose now that only two vertices of Σ map in the interior of D (see graph 2). We argue in the same way. We construct as previously a one-parameter family of one-cycles z_t in D representing $[\Sigma]$ with $\text{mass}(z_t) \leq 10\rho < L_1'(M)$. This homotopy, starting from a null-current, ends also at a null-current. Therefore, it induces a trivial class in homology, that is, $[\Sigma] = 0$ in $H_2(D, \partial D)$.

Case 2. Suppose that exactly three vertices of Σ map in the interior of D (see graphs 3 and 4). They define a face Δ^2 of Σ. The edges of the boundary $\partial\Delta^2$ are noted e_1, e_2 and e_3. Let c_1, c_2 and c_3 be the three edges of Σ which join the vertices of Δ^2 to ∂D (we recall that Σ is identified with its image by φ). By construction, the arcs $c_2 \cup e_1 \cup c_3$, $c_3 \cup e_2 \cup c_1$ and $c_1 \cup e_3 \cup c_2$ are simple and converge to the edges of Σ lying in ∂D through homotopies given by the curve-shortening process. The sum of these homotopies defines a one-parameter family of one-cycles z_t in D. This family starts from $\partial\Delta^2$ and ends at a one-cycle lying in ∂D representing either a null-current or ∂D from Theorem 2.7.iv).
Suppose that it ends at a null-current. Since simple closed curves remain simple through the curve-shortening process, $\partial\Delta^2$ converges either into a point or to ∂D. This homotopy extends the one-parameter family z_t. Since $\text{mass}(z_t) \leq 9\rho < L_1'(M)$, we conclude as previously that $[\Sigma] = 0$.
Suppose that the family z_t ends at ∂D. If $\partial\Delta^2$ contracts into a point, we extend z_t and conclude similarly. Otherwise, $\partial\Delta^2$ is homotopic to ∂D by the curve-shortening process. The sum of this homotopy with z_t, endowed with the orientations induced by the one on Σ, defines a new one-parameter family of one-cycles. Since the orientations of the homotopies are induced by the one on Σ, this new one-parameter family starts at a null-current. Furthermore, since simple curves remain simple through the curve-shortening process, it also ends at a null-current. Thus, the class $[\Sigma]$ it represents vanishes in $H_2(D, \partial D)$.

Case 3. Suppose that the four vertices of Σ map in the interior of D (see graphs 5 and 6). The boundaries of the faces of Σ lie in the interior of D and converge either to points or to ∂D through homotopies z_i^t given by the curve-shortening process. The faces of Σ decompose into two pairs such that the edges of the faces of each pair do not intersect transversely (see graphs 5 and 6) . Since the images by the curve-shortening process of simple curves which do not cut each other do not cross (see Theorem 2.7.iv)), at most two homotopies z_i^t converge to ∂D.
Suppose that z_1^t, z_2^t and z_3^t converge to points. The one-parameter family of one-cycles z_t formed of $z_1^t + z_2^t + z_3^t$ and z_4^t starts from a null-current and ends either at a null-current or at ∂D. Since $\text{mass}(z_t) \leq 9\rho < L_1'(M)$, we conclude as previously that the class $[\Sigma]$ it represents vanishes in $H_2(D, \partial D)$.
Suppose that z_1^t and z_2^t converge to ∂D. Since the orientations of the homotopies are induced by the one on Σ, their sum ends at a null-current. Thus, the one-parameter family of one-cycles $z_1^t + z_2^t + z_3^t + z_4^t$, which represents $[\Sigma]$, starts and ends at null-currents. This implies that $[\Sigma] = 0$.

In conclusion, we have extended r on each building block of P. This defines a retraction $r : P \longrightarrow M$. Hence the result. □

Theorems 4.3 and 4.8 immediately lead to the main theorem.

Remark 4.9. Let $\mathcal{D}$ be the set of disks of M with geodesic boundary containing no nontrivial simple closed geodesic of null index in their interior. The shortest one-cycles of index one of D form a collection of curves when D runs in $\mathcal{D}$. The proof of Theorem 4.8 shows that the mass of the longest of these one-cycles provides a lower bound on the filling radius. This statement yields a better estimate on the filling radius. Indeed, some small one-cycles of index one can be ignored in the case, for instance, of a sphere with small bumps looking like mushrooms.

Remark 4.10. We show in the following examples that it is not possible to restrict the set of curves considered in Theorem 4.8 to a single type of geometry.

The round sphere has no non-simple closed geodesic. Therefore, the length of the shortest simple closed geodesic of index one provides in this example the lower bound on the filling radius.

Three long tubes capped with half-hemispheres glued along the boundary components of a hyperbolic pant forms a sphere. This sphere can be pertubed into a bumpy three-leg sphere with arbitrarily small area such that the simple closed geodesics and the closed geodesics of index one are arbitrarily long. In this example of a sphere with three long spikes, the length of the shortest figure eight geodesic of null index provides the lower bound on the filling radius.

5. Filling radius and the global minimax process

In the previous section, we showed that the lowest minimax value of the local minima processes described in (C'.1-2) provides a lower bound on the filling radius of the two-sphere. However, the minimax value corresponding to the global minimax process described in (C.1-2) of Section 2.1 does not provide any lower bound on the filling radius. The goal of this section is to establish this result by constructing a sequence of metrics on the two-sphere. More precisely, we have

Theorem 5.1. *There exists a sequence g_n of Riemannian metrics on S^2 which satisfies*

$$\lim_{n\to\infty} \frac{FillRad(S^2, g_n)}{L_1(S^2, g_n)} = 0$$

Let us construct a sequence of metrics g_n satisfying the above theorem. Fix $\varepsilon > 0$. Three copies S, S' and S'' of the circle $\mathbb{R}/2\varepsilon\mathbb{Z}$ are said to be glued together if the following identifications hold

– $x \in S$ and $\varepsilon - x \in S'$ are identified for $-\varepsilon \leq x \leq 0$
– $x \in S$ and $x \in S''$ are identified for $0 \leq x \leq \varepsilon$
– $x \in S'$ and $\varepsilon - x \in S''$ are identified for $0 \leq x \leq \varepsilon$

Fix $n \in \mathbb{N}^*$. The quotient of the strip $\{(x, y) \in \mathbb{R}^2 \mid -n \leq y \leq n\}$ by the subgroup of displacements Γ generated by the translation of vector $(2\varepsilon, 0)$ defines a flat rectangle cylinder C. The curves $\sigma_y = \{(x, y) \mid x \in \mathbb{R}\}/\Gamma$ where $-n \leq y \leq n$ are called sections of C. The connected components of ∂C agree with $C^+ = \sigma_n$ and $C^- = \sigma_{-n}$. The cylinder C decomposes into two semi-cylinders $\{\sigma_y \mid -n \leq y < 0\}$ and $\{\sigma_y \mid 0 \leq y \leq n\}$.

We define by induction the cylindrical tree of height n for every $n \in \mathbb{N}^*$. The cylindrical tree of height 1 is the flat rectangle cylinder C. Its basis is C^-. The cylindrical tree of height $n+1$ is defined by gluing together the basis of two copies of cylindrical trees of height n with the boundary C^+ of C. The boundary C^- of C forms the basis of the cylindrical tree of height $n+1$.

The tubes modeled on C which form the cylindrical tree T_n of height n are noted M_i with $i \in I$. The union of M_i and the semi-cylinders of T_n adjacent to M_i is noted $\mathcal{U}M_i$.

Let M be the Riemmanian sphere (S^2, g_n) obtained by gluing round hemispheres of circumference 2ε along the connected components of the boundary of T_n. It is clear that $\mathrm{FillRad}(S^2, g_n) \leq \varepsilon$. Let us show that the sequence $L_1(S^2, g_n)$ is unbounded.

Let $(z_t)_{0\leq t\leq 1}$ be a one-parameter family of one-cycles on M satisfying the conditions (C.1-2) of Section 2.1. The truncated family z_s with $0 \leq s \leq t$

induces a 2-current S_t with $\partial S_t = z_t$ and $\partial \mathrm{Supp}(S_t) \subset \mathrm{Supp}(z_t)$ (see [2] and [34, Sect. III.1.4]). The one-parameter family S_t is continuous.
We define the index of a tube at the time t by N_t^i where

$$N_t^i = \begin{cases} 1 & \text{if there is a simple closed curve of } M_i \text{ which lies in the interior} \\ & \text{of } \mathrm{Supp}(S_t) \text{ and induces a nontrivial class in } H_1(M_i) \\ 0 & \text{otherwise} \end{cases}$$

For n large enough, we can assume that $\mathrm{mass}(z_t) < n$ for every $0 \leq t \leq 1$, otherwise the result is immediate.

At this stage, we need the three following lemmas.

Lemma 5.2. *Let M_i and M_j be two adjacent tubes such that $N_t^i = 1$ and $N_t^j = 0$ with t fixed. Then, there is an irreductible component, i.e., a simple closed curve, of Supp$(z_t) \cap \mathcal{U}M_i$ homotopic in T_n to a section of M_i or M_j.*

Proof. Assume that no irreductible component of $\mathrm{Supp}(z_t) \cap \mathcal{U}M_i$ is homotopic to a section of M_j in T_n. Let M_k be the tube adjacent to both M_i and M_j. Let c be a simple closed curve of M_i lying in the interior of $\mathrm{Supp}(S_t)$ and representing a nontrivial class in $H_1(M_i)$. The connected component of $\mathrm{Supp}(S_t)$ containing c is noted $\mathrm{Supp}_0(S_t)$. Recall that we have supposed $\mathrm{mass}(z_t) < n$. Therefore, from the coarea inequality, there exist c_j and c_k in $\mathcal{U}M_i$, sections of M_j and M_k, which do not intersect $\mathrm{Supp}(z_t)$. In particular, c_j is disjoint from $\partial \mathrm{Supp}(S_t)$. Therefore, either c_j lies in $\mathrm{Supp}(S_t)$ (which is impossible since $N_t^j = 0$) or c_j is disjoint from $\mathrm{Supp}(S_t)$.
Let $\Upsilon \subset \mathcal{U}M_i$ be the connected domain bounded by c, c_j and c_k. Let $\mathrm{Supp}_0(S_t)$ be the connected component of $\mathrm{Supp}(S_t)$ containing c. The curve c is homologuous in $\Upsilon_t := \mathrm{Supp}_0(S_t) \cap \Upsilon$ to $\partial \Upsilon_t \setminus c$. We have $\partial \Upsilon_t \setminus c \subset [\mathrm{Supp}(z_t) \cap \Upsilon] \amalg [(\mathrm{Supp}_0(S_t) \cap \partial \Upsilon) \setminus c]$ where the union is disjoint. Since c_j is disjoint from $\partial \mathrm{Supp}_0(S_t)$ and $N_t^j = 0$, the term $(\mathrm{Supp}_0(S_t) \cap \partial \Upsilon) \setminus c$ is either the empty set or c_k. By assumption, no irreductible component of $\mathrm{Supp}(z_t) \cap \mathcal{U}M_i$ is homotopic to c_j in T_n. Therefore, the irreductible components of the term $\mathrm{Supp}(z_t) \cap \Upsilon$ are either homotopically trivial in T_n or homotopic to c or c_k in T_n. In conclusion, there is an irreductible component of $\mathrm{Supp}(z_t) \cap \mathcal{U}M_i$ homotopic to a section of M_i in T_n, otherwise the curve c would be homologuous to c_k in T_n, which is absurd. □

We define by induction the height of a tube on a cylindrical tree T_n. The tube at the basis of T_n is at the height zero. Every tube adjacent to a tube of height k which is not itself of height $\leq k$ is by definition of height $k+1$. The height of a tube $M_i \subset T_n$, noted $h(M_i)$, satisfies $0 \leq h(M_i) \leq n-1$.

Define N_t as the cardinal of

$$A_t = \{(i,j) \mid M_i \text{ and } M_j \text{ are adjacent with} N_t^i = 1 \text{ and } N_t^j = 0\}$$

Lemma 5.3. *Let $0 \leq t \leq 1$. We have mass$(z_t) \geq \frac{1}{8} N_t \varepsilon$.*

Proof. The set A_t decomposes into the disjoint union of $A'_t = \{(i,j) \in A_t \mid h(M_i) \text{ is odd}\}$ and $A''_t = \{(i,j) \in A_t \mid h(M_i) \text{ is even}\}$. Define the projection π on the first coordinate by $\pi(i,j) = i$. It is possible to divide A'_t into two subsets whose π-projections contain no pair $\{i, i'\}$ such that M_i and $M_{i'}$ are adjacent. The same goes for A''_t. In this way, the set A_t decomposes into four disjoint subsets. One of them, noted B_t, contains at least $\frac{1}{4}N_t$ elements. From Lemma 5.2, there exists an irreductible component of $\mathrm{Supp}(z_t) \cap \mathcal{U}M_i$ of length $\geq 2\varepsilon$ for every $i \in \pi(B_t)$. By construction, the neighborhoods $\mathcal{U}M_i$ where $i \in \pi(B_t)$ are disjoint. Therefore, $\mathrm{mass}(z_t) \geq 2|\pi(B_t)|\varepsilon$. But $|\pi(B_t)| \geq \frac{1}{4}|B_t| \geq \frac{1}{16}N_t$, hence the result. □

Lemma 5.4. *Let $0 \leq t \leq 1$. The number of tubes of M of index one at the time t can be written under the form $\sum_{i \in S} \epsilon_i(2^{n_i} - 1)$ where $S \subset \{0, \ldots, N_t\}$, $\epsilon_i = \pm 1$ and $1 \leq n_i \leq n$.*
In particular, there are at most $(2n)^{N_t+1}$ such numbers.

Proof. We follow the proof of [15, Lemma 6]. Define $B'_t = \{(i,j) \in A_t \mid h(M_i) > h(M_j)\}$ and $B''_t = \{(i,j) \in A_t \mid h(M_i) < h(M_j)\}$. The tube $M_i \subset T_n$ is the basis of a unique cylindrical tree of maximal height $n - h(M_i)$ contained in T_n. Given $k \in \mathbb{N}$, let τ_k^+ be the union of the cylindrical trees of T_n of basis M_i and of maximal height with $h(M_i) = k$ and $(i,j) \in B'_t$ for some j. For $k = 0$, τ_0^+ agrees with the empty set if the index of the basis of T_n is null or with T_n otherwise. Similarly, let τ_k^- be the union of the cylindrical trees of T_n of basis M_j and of maximal height with $h(M_i) = k$ and $(i,j) \in B''_t$ for some j.
The set of the tubes of M of index one at the time t agrees with the disjoint union $(\ldots(((\tau_0^+ \setminus \tau_0^-) \amalg \tau_1^+) \setminus \tau_1^-) \amalg \tau_2^+ \cdots \setminus \tau_{n-2}^-) \amalg \tau_{n-1}^+$. Therefore, the number of the tubes of M of index one at the time t equals
$\sum_{(i,j) \in B'_t} (2^{n-h(M_i)} - 1) - \sum_{(i,j) \in B''_t} (2^{n-h(M_j)} - 1)$ if the index of the basis of T_n is null and
$\sum_{(i,j) \in B'_t} (2^{n-h(M_i)} - 1) - \sum_{(i,j) \in B''_t} (2^{n-h(M_j)} - 1) + 2^n - 1$ otherwise.
In both cases, it is of the desired form. □

Let us show how to conclude from the previous lemmas.
Slightly perturbing the homotopy z_t if necessary, we can suppose that the indices N_t^i change one at a time throughout the homotopy. Note that the small perturbations used do not increase the mass of the one-cycles z_t more than any preliminary given number. When t varies from 0 to 1, the number of tubes of M of index one varies from 0 at $t = 0$ to $2^n - 1$ at $t = 1$. Thus, each integer between 0 and $2^n - 1$ occurs as the number of tubes of M of index one at some time t. Therefore, we have $(2n)^{N_t+1} \geq 2^n$ for some t, from Lemma 5.4. That is, N_t is bigger than $c\frac{n}{\ln n}$ for some t where c is a positive constant independent of n and t. In particular, we get

$\sup_{0\leq t\leq 1} \text{mass}(z_t) \geq c' \frac{n}{\ln n}\varepsilon$ with $c' > 0$ independent of n from Lemma 5.3. Therefore, $L_1(S^2, g_n)$ is unbounded.

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