

VOLUME OF MINIMAL HYPERSURFACES IN MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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ABSTRACT. We establish a min-max estimate on the volume width of a closed Riemannian manifold with nonnegative Ricci curvature. More precisely, we show that every closed Riemannian manifold with nonnegative Ricci curvature admits a PL Morse function whose level set volume is bounded in terms of the volume of the manifold. As a consequence of this sweep-out estimate, there exists an embedded, closed (possibly singular) minimal hypersurface whose volume is bounded in terms of the volume of the manifold.

1. INTRODUCTION

In this article, we show the following result.

Theorem 1.1. *Let M be a closed Riemannian n -manifold with nonnegative Ricci curvature. There exists an embedded closed minimal hypersurface S in M with a singular set of Hausdorff dimension at most $n - 8$ such that*

$$\text{vol}_{n-1}(S) \leq C_n \text{vol}_n(M)^{\frac{n-1}{n}}$$

where C_n is an explicit positive constant depending only on n .

Observe that both the Ricci curvature condition and the inequality are scale invariant in this theorem. We do not know whether the curvature condition can be dropped in dimension greater than two. In dimension two, it is the case according to [BZ88, Theorem 5.3.1], [He81] and [Cr88]: every closed Riemannian surface has a closed geodesic of length bounded from above in terms of the area of the surface. Thus, Theorem 1.1 can be viewed as a partial generalization of this result on closed geodesics on surfaces in terms of minimal hypersurfaces.

Another way to generalize this result would be to find an upper bound on the length of the shortest closed geodesic in a closed Riemannian manifold in terms of the volume. Except for essential manifolds, where systolic inequalities hold [Gr83], and the two-sphere [Cr88], this question is completely open, even on manifolds with nonnegative Ricci curvature.

The condition on the size of the singular set is customary in this context. To our knowledge, it is still unknown if there exists a closed *smooth* minimal hypersurface in every closed Riemannian manifold even with nonnegative Ricci curvature. The existence of a minimal hypersurface in Theorem 1.1 actually derives from a min-max principle due to Almgren-Pitts in geometric measure theory (see [Pi81, SS81, CD03, DT13] for $n \geq 3$ and [CC92] for

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$n = 2$). When the Ricci curvature is positive, the geometry of the min-max minimal hypersurface S has recently been described in [Zh].

In fact, Theorem 1.1 is a consequence of the following sweep-out estimate.

Theorem 1.2. *Let M be a closed Riemannian n -manifold with nonnegative Ricci curvature. There exists a PL Morse function $f : M \rightarrow \mathbb{R}$ such that*

$$\sup_t \text{vol}_{n-1}(f^{-1}(t)) \leq C_n \text{vol}_n(M)^{\frac{n-1}{n}}$$

where C_n is an explicit positive constant depending only on n .

Recall that a function $f : M \rightarrow \mathbb{R}$ is a PL Morse function if there exists a simplicial complex structure on M such that f is linear on every simplex of M and takes pairwise distinct values at the vertices of M .

As already mentioned, the sweep-out estimate of Theorem 1.2 is stronger than Theorem 1.1. A slightly more general version is given by Theorem 6.1.

Observe again that both the Ricci curvature condition and the inequality are scale invariant. Contrary to Theorem 1.1 where the question is open, this result fails without any curvature condition for every closed n -manifold with $n \geq 3$ (see Proposition 7.1). However, the curvature assumption can be dropped for closed Riemannian surfaces [BS10], see also [Ba, Li14]. In this case, the multiplicative constant depends on the genus of the surface (and it has to).

None of the multiplicative constants in Theorems 1.1 or 1.2 is optimal, even on two-dimensional convex spheres despite a local extremality result [Ba10, Sa10]. For positively curved 3-manifolds, related sharp upper bounds have been obtained in [MN12].

When M is a smooth convex hypersurface in a Euclidean n -space, the result of Theorem 1.2 follows from [Tr85]. A similar result also holds true for domains of the Euclidean n -space (see [Fa80, Gu07]). Thus, our sweep-out estimate can be seen as a partial generalization of these results to a non-Euclidean setting.

Actually, Theorem 1.2 can be formulated in terms of a min-max estimate by taking the infimum over all PL Morse functions of the supremum of the volume of their fibers. A related estimate holds for manifolds with nonnegative sectional curvature by replacing the volume of the fibers by the diameter (see [Pe95]). Further consideration of min-max principles can be found in [Gr83, Gr88, Gr03, Gu09]. Note that min-max processes are also involved in concentration phenomena.

As a consequence of our sweep-out estimate, we also derive the following isoperimetric inequality.

Corollary 1.3. *There exists a positive constant C_n such that every closed Riemannian n -manifold M with nonnegative Ricci curvature decomposes into two connected domains with the same volume whose common boundary S satisfies*

$$\text{vol}_{n-1}(S) \leq C_n \text{vol}_n(M)^{\frac{n-1}{n}}.$$

Before giving an outline of the arguments, we mention some recent applications of the min-max process on three-dimensional manifolds to emphasize its importance in current research. (The most recent results mentioned below even appeared after this paper was submitted for publication, which

illustrates the rapid emergence of new ideas related to this theme in the past few years.) In [CM05], T. Colding and W. Minicozzi established min-max estimates through the Ricci flow which allowed them to simplify some arguments in Perelman's proof of the Poincaré conjecture through a more conceptual approach. In [DP10] and [Ke], the authors established genus bound for minimal surfaces constructed via min-max arguments on every closed Riemannian 3-manifold. Recently, F. Marques and A. Neves proved the Willmore conjecture [MN14] and a conjecture of Yau about the existence of infinitely many minimal hypersurfaces in closed Riemannian manifolds with positive Ricci curvature [MN] by using a min-max process à la Almgren-Pitts.

We also mention that P. Glynn-Adey and Y. Liokumovich recently placed a preprint on arxiv [GAL] in which they establish Theorem 1.1 among other results. The arguments also rely on sweep-out estimates. Another recent preprint considering sweep-outs on surfaces is [Li].

Outline. The proof of the main theorem, namely Theorem 6.1, follows from a divide-and-conquer strategy. The idea is to decompose a given Riemannian manifold with nonnegative Ricci curvature into two (smaller) domains with comparable volume along a hypersurface of volume bounded from above in terms of the volume of the manifold, *cf.* Sections 3 and 4. This decomposition relies on the non-negativity of the Ricci curvature and involves capacity estimates obtained from packing bounds given by the Bishop-Gromov inequality. Each of these (smaller) domains can be swept out by a family of hypersurfaces (more precisely, the level sets of a PL Morse function) of volume bounded from above in terms of the volume of the domain. By merging the sweep-outs, we construct a sweep-out on the original manifold without increasing the volume of the hypersurfaces too much, *cf.* Section 5. Finally, we show that the resulting sweep-out satisfies the desired volume bound, *cf.* Section 6.

Throughout this paper, a domain of a complete n -manifold with piecewise smooth boundary is an n -submanifold with piecewise smooth boundary.

2. CW COMPLEX STRUCTURE WITH QUASI-CONVEX CELLS OF SMALL SIZE

In this section, we introduce some key notions for the rest of the article and establish preliminary results which will be used later.

The definition of a CW complex can be found in [Ha02, Appendix] along with some related topological properties and applications. We simply recall that every k -cell of a CW complex X is attached to the $(k - 1)$ -skeleton $X^{(k-1)}$ of X . The map from this k -cell to X is referred to as the characteristic map of the k -cell.

We will denote by $B^n(\rho)$ the (open) ρ -ball centered at the origin of \mathbb{R}^n . Recall also that a λ -quasi-isometry between two metric space is a homeomorphism which is λ -Lipschitz and whose reciprocal map is also λ -Lipschitz.

The geometry of a CW complex structure can be measured through the following notion.

Definition 2.1. Let $\rho > 0$ and $\lambda > 1$. A compact Riemannian n -manifold D with (possibly empty) piecewise smooth boundary admits a CW complex structure with quasi-convex cells of size (ρ, λ) if

- (1) There exists a λ -quasi-isometry $\varphi : P \rightarrow D$ between a piecewise flat simplicial n -complex P and D .
- (2) The simplicial n -complex P decomposes into n -cells $\Delta_k = \cup_i \sigma_{k,i}$ formed of a union of simplices $\sigma_{k,i}$ such that each n -cell Δ_k is λ -quasi-isometric to a convex Euclidean polyhedron \mathcal{E}_k lying in $B^n(6\rho)$ and containing $B^n(2\rho)$. This quasi-isometry is denoted by $\chi_k : \Delta_k \rightarrow \mathcal{E}_k$.
- (3) For every k , the composite $\varphi \circ \chi_k^{-1} : \mathcal{E}_k \rightarrow \varphi(\Delta_k)$ is λ -quasi-isometric.

The decomposition of P into n -cells Δ_k gives rise to a CW complex structure on D where the characteristic maps are given by the quasi-isometries $\varphi \circ \chi_k^{-1}$ described in (3).

For our purpose, we can think of D as an n -polyhedron whose n -faces are quasi-isometric to convex Euclidean polyhedra. Despite the risk of confusion, we will often identify the n -cells of D with their images by the characteristic maps.

The n -simplices composing the piecewise flat simplicial n -complex P are not necessarily regular as their edges may have different lengths. However the n -cells of P have a uniform size, roughly the size of $B^n(\rho)$, while still being almost convex. (This is the reason why we introduce this definition.) In particular, the volume of the n -cells is between $\lambda^{-n} \omega_n 2^n \rho^n$ and $\lambda^n \omega_n 6^n \rho^n$, where ω_n represents the volume of the unit ball centered at the origin of \mathbb{R}^n . Thus, if N represents the number of n -cells in D , we immediately obtain

$$N \lambda^{-n} \omega_n 2^n \rho^n \leq \text{vol}_n(D) \leq N \lambda^n \omega_n 6^n \rho^n. \quad (2.1)$$

The following result shows the existence of CW complex structures with quasi-convex cells of small size on closed Riemannian manifolds.

Proposition 2.2. *Let D be a bounded domain of a complete Riemannian n -manifold M . For every $\rho > 0$ small enough, the domain D admits a CW complex structure with quasi-convex cells of size (ρ, λ_ρ) with $\lim_{\rho \rightarrow 0} \lambda_\rho = 1$.*

Proof. Without loss of generality, we can assume that the boundary of D is smooth (otherwise we smooth it out). Let $\rho \in (0, \frac{1}{100} \text{inj}(M))$ with ρ smaller than half the focal radius of ∂D . Let $\partial D_{=2\rho} = \{x \in D \mid d_M(x, \partial D) = 2\rho\}$ and $\partial D_{\leq 2\rho} = \{x \in D \mid d_M(x, \partial D) \leq 2\rho\}$. Consider a maximal system of disjoint 2ρ -balls of M centered in $\partial D_{=2\rho}$ and denote by x_i the centers of these balls. Note that $B(x_i, 2\rho)$ lies in D . Since the system of balls is maximal, the balls $B(x_i, 4\rho)$ cover $\partial D_{=2\rho}$. As ρ is smaller than half the focal radius of ∂D , every point of ∂D is at distance 2ρ from some point of $\partial D_{=2\rho}$. Thus, the balls $B(x_i, 6\rho)$ cover the 2ρ -neighborhood $\partial D_{\leq 2\rho}$ of ∂D in D .

Now, we complete the collection of disjoint balls $B(x_i, 2\rho)$ of M lying in D into a maximal system of disjoint 2ρ -balls of M lying in D . We will still denote by x_i the centers of the balls thus obtained. The Voronoi cell

$$V_i = \{x \in D \mid d(x, x_i) \leq d(x, x_j) \text{ for every } j \neq i\}$$

centered at x_i clearly contains the ball $B(x_i, 2\rho)$. It also lies in the ball $B(x_i, 6\rho)$. Otherwise, we could find a point $x \in D$ at distance at least 6ρ from any point x_j . From the previous paragraph, this implies that the point x is at distance at least ρ from ∂D . Thus, the ball $B(x, 2\rho)$ lies in D and is disjoint from the other balls $B(x_j, 2\rho)$. This contradicts the construction of the points x_i .

A consequence of these inclusions and the Gauss lemma is that the preimage \hat{V}_i of V_i by the exponential map \exp_{x_i} based at x_i lies in $B^n(6\rho)$ and contains $B^n(2\rho)$. This preimage is quasi-isometric to the convex polyhedral Voronoi cell \mathcal{E}_i formed of the vectors u of $T_{x_i}M$ whose distance to the origin is less or equal to the distance from u to the preimages of the x_j under \exp_{x_i} for every $j \neq i$. Furthermore, the factor of the quasi-isometry $V_i \rightarrow \hat{V}_i \rightarrow \mathcal{E}_i$ tends to 1 when ρ goes to 0, *cf.* Definition 2.1, item (3). (In the following, all the quasi-isometries will satisfy this property.)

We would like to glue the convex Euclidean polyhedra \mathcal{E}_i together to construct a simplicial n -complex P quasi-isometric to D . Two adjacent Voronoi cells V_i and V_j in D meeting along an $(n - 1)$ -face give rise to two convex Euclidean polyhedra \mathcal{E}_i and \mathcal{E}_j along with two corresponding $(n - 1)$ -faces $e_{i,j}$ and $e_{j,i}$ in \mathcal{E}_i and \mathcal{E}_j . If $e_{i,j}$ and $e_{j,i}$ were isometric, we could glue the convex Euclidean polyhedra \mathcal{E}_i and \mathcal{E}_j together. Unfortunately, this is not always the case. Indeed, the $(n - 1)$ -faces $e_{i,j}$ and $e_{j,i}$, though combinatorially equivalent, are only quasi-isometric.

To get around this technical problem, we consider a quasi-geodesic triangulation of the convex Euclidean polyhedra \mathcal{E}_i such that the induced triangulations on the Voronoi cells V_i in D agree on their common faces. This gives rise to a triangulation \mathcal{T} of D (refining the CW complex structure of D given by the Voronoi cells) where each simplex σ is quasi-isometric to a Euclidean simplex σ' .

Now, we need to deform these Euclidean simplices σ' in order to glue them together. Let e be an edge of \mathcal{T} . Denote by σ'_j , where $j = 1, \dots, p$, the Euclidean simplices with an edge e'_j corresponding to e . Replace the edges e'_j of σ'_j with an edge of the same length equal to the average of the lengths of e'_1, \dots, e'_p . As the edges e'_j are quasi-isometric to e , the same goes for the new edges.

By applying this averaging argument to every edge of the triangulation \mathcal{T} , we obtain new Euclidean simplices σ'' quasi-isometric to the initial Euclidean simplices σ' and so to the simplex σ . Furthermore, the $(n - 1)$ -faces of these new Euclidean simplices σ'' corresponding to common faces of adjacent simplices in D are isometric since the lengths of their edges agree.

Therefore, we can replace each simplex σ in D with a quasi-isometric Euclidean simplex σ'' so that the resulting space is a simplicial n -complex P quasi-isometric to D , *cf.* Definition 2.1, item (2). By construction, the simplices of P corresponding to the Euclidean simplices of the triangulation of \mathcal{E}_k form an n -cell Δ_k quasi-isometric to \mathcal{E}_k , *cf.* Definition 2.1, item (2). Since the factors of all these quasi-isometries tends to 1 when ρ goes to 0, this yields the desired result. \square

We will also need the following result.

Lemma 2.3. *Fix $\rho > 0$ and $\lambda > 1$ close enough to 1. Let Δ be an n -cell of a CW complex structure with quasi-convex cells of size (ρ, λ) on a bounded domain of a compact Riemannian n -manifold. There exists a retraction*

$$\pi_\Delta : \Delta \setminus B(\lambda\rho) \rightarrow \partial\Delta$$

of $\partial\Delta$ with Lipschitz constant at most $36\lambda^2$.

Proof. By definition, the n -cell Δ is λ -quasi-isometric to a convex Euclidean polyhedron \mathcal{E} lying in $B^n(6\rho)$ and containing $B^n(2\rho)$. The radial projection from $\mathcal{E} \setminus B^n(\rho)$ to $\partial\mathcal{E}$ is Lipschitz. A sharp upper bound on its Lipschitz constant is given by the limit from the right at $t = 0$ of the length ratio $\frac{A'B'}{AB}$, where the points $A, A', B, B' \in \mathbb{R}^2$ are defined as follows, cf. Figure 1.

- The coordinates of A and A' are $(\rho, 0)$ and $(6\rho, 0)$.
- The point B lies in the same vertical line as A and satisfies $\widehat{AOB} = t$.
- The point B' is the intersection point of (OB) with the line tangent at A to the circle of radius ρ centered at the origin.

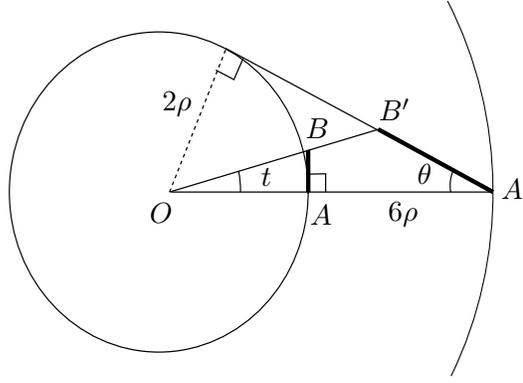


FIGURE 1. Extremal case of the projection

A straightforward computation shows that $AB = \rho \tan t$, $A'B' = 6\rho \frac{\sin t}{\sin(\theta + t)}$ and $\sin \theta = \frac{1}{6}$ where $\theta = \widehat{B'A'O}$. Hence,

$$\frac{A'B'}{AB} = \frac{6 \cos t}{\sin(\theta + t)} \rightarrow 36.$$

We derive the desired Lipschitz constant after pre- and post-composition with a λ -quasi-isometry. \square

3. SPLITTING MANIFOLDS

The goal of this section is to prove the following result.

Theorem 3.1. *Let M be a complete Riemannian n -manifold with nonnegative Ricci curvature. For every bounded domain D of M , there exists a smooth hypersurface S of D with boundary lying in ∂D which decomposes D into two domains D_1 and D_2 with $\text{vol}_n(D_i) \geq \alpha_n \text{vol}_n(D)$ such that*

$$\text{vol}_{n-1}(S) \leq A_n \text{vol}_n(D)^{\frac{n-1}{n}}$$

where A_n and α_n are explicit positive constants depending only on n .

Remark 3.2. The conclusion of Theorem 3.1 (and Theorem 4.1 below) fails if one drops the curvature condition. For instance, this is the case on a closed Riemannian n -manifold M when $D = M$ if the Cheeger constant $h(M)$ of M satisfies

$$h(M) \gg \frac{1}{\text{vol}_n(M)^{\frac{1}{n}}} \quad (3.1)$$

where the Cheeger constant of M is defined as

$$h(M) = \inf_S \frac{\text{vol}_{n-1}(S)}{\min\{\text{vol}_n(M_1), \text{vol}_n(M_2)\}}$$

with S running over all hypersurfaces decomposing M into two domains M_1 and M_2 . For $n \geq 3$, concrete examples are given by the Riemannian n -tori with arbitrarily small asymptotic isoperimetric constant presented in [BI98]. Strictly speaking the authors do not state that the metrics they construct satisfy the relation (3.1), but one can adapt their arguments to derived the desired relation. For these metrics, the conclusion of Theorem 3.1 fails.

The case $n = 2$ is special. Indeed, the classical inequalities

$$\lambda_1(M) \geq \frac{h(M)^2}{4}$$

and

$$\lambda_1(M) \text{ area}(M) \leq 24\pi(g + 1)$$

established in [Ch70] and [LY82], where $\lambda_1(M)$ represents the first nonzero eigenvalue of the Laplacian on a closed Riemannian surface M of genus g , show that the relation (3.1) does not hold since

$$h(M) \leq \left(\frac{96\pi(g + 1)}{\text{area}(M)} \right)^{\frac{1}{2}}.$$

When M is a nonnegatively curved two-sphere, the conclusion of Theorem 3.1 (and of Theorems 1.1 and 1.2) follows from [Tr85] since M can be isometrically embedded as a smooth convex surface in \mathbb{R}^3 by Alexandrov's isometric embedding theorem. Alternately, it can be derived from [CC92] as in the proof of Corollary 1.3, *cf.* Section 6. In fact, one can show that Theorem 3.1 holds on any closed Riemannian surface M of genus g without any curvature condition if one allows A_n to depend also on g , *cf.* [BS10].

Before proceeding to the proof of Theorem 3.1, let us recall the Bishop-Gromov inequality, which plays a crucial role in the study of manifolds with nonnegative Ricci curvature.

Let M be a complete Riemannian n -manifold with nonnegative Ricci curvature. Then for every $0 < r < R$ and $x \in M$, we have

$$\frac{\text{vol}_n(B(x, R))}{\text{vol}_n(B(x, r))} \leq \left(\frac{R}{r} \right)^n. \quad (3.2)$$

In particular, the volume of every R -ball in M is at most $\omega_n R^n$, where ω_n is the volume of a unit ball in the Euclidean n -space.

A consequence of the Bishop-Gromov inequality is the following classical packing estimate.

Lemma 3.3. *Let $r > 0$. Every ball of radius $2r$ in a complete Riemannian n -manifold M with nonnegative Ricci curvature can be covered by $c_n = 9^n$ balls of radius r .*

Proof. Let $B(x, 2r)$ be a ball of radius $2r$ in M . Consider a maximal system of disjoint $\frac{r}{2}$ -balls in M with centers x_i lying in $B(x, 2r)$. Clearly, the balls $B(x_i, \frac{r}{2})$ are contained in $B(x, \frac{5}{2}r)$. From the Bishop-Gromov inequality (3.2), the number of elements in such maximal system of disjoint balls is at most

$$\frac{\text{vol}_n(B(x, \frac{5}{2}r))}{\text{vol}_n(B(x_i, \frac{r}{2}))} \leq \frac{\text{vol}_n(B(x_i, \frac{9}{2}r))}{\text{vol}_n(B(x_i, \frac{r}{2}))} \leq 9^n.$$

On the other hand, by maximality, the balls $B(x_i, r)$ cover $B(x, 2r)$. \square

The following result about the existence of capacitors can be found in [CMa08, Lemma 2.2] (see also [Ko93, GY99, GNY04]). We include a proof for the sake of completeness.

Lemma 3.4. *Let D be a bounded domain of a complete Riemannian n -manifold M with nonnegative Ricci curvature. Let $r = \left(\frac{\text{vol}_n(D)}{\omega_n(1+2c_n)}\right)^{\frac{1}{n}}$ where $c_n = 9^n$. Then there exist two domains A_1 and A_2 in D with $\text{vol}_n(A_i) \geq \frac{1}{1+2c_n} \text{vol}_n(D)$ at distance at least r from each other in M .*

Proof. Given a subset A of M , let $\text{vol}_D(A) = \text{vol}_n(A \cap D)$. Fix $\lambda_n = \frac{1}{1+2c_n}$. For $m \in \mathbb{N}^*$, consider the function $\Psi_m : M^m \rightarrow \mathbb{R}$ defined on the m -fold product M^m of M as

$$\Psi_m(x_1, \dots, x_m) = \text{vol}_D \left(\bigcup_{i=1}^m B(x_i, r) \right).$$

Since D is bounded, the function Ψ_m attains its maximum at some point $\mathbf{x}_m = (x_{1,m}, \dots, x_{m,m})$ in M^m . Clearly, $\max(\Psi_1) = \Psi_1(\mathbf{x}_1) \leq \omega_n r^n = \lambda_n \text{vol}_n(D)$ from the Bishop-Gromov inequality (3.2) and $\max(\Psi_m) = \Psi_m(\mathbf{x}_m) = \text{vol}_n(D)$ for m large enough. Let $m_0 \in \mathbb{N}^*$ be the smallest integer such that $\max(\Psi_{m_0}) = \Psi_{m_0}(\mathbf{x}_{m_0}) \geq \lambda_n \text{vol}_n(D)$. Define $A_1 = \bigcup_{i=1}^{m_0} B(x_{i,m_0}, r) \cap D$. Note that

$$\text{vol}_n(A_1) \geq \lambda_n \text{vol}_n(D).$$

We also have

$$\begin{aligned} \text{vol}_n(A_1) &\leq \text{vol}_D \left(\bigcup_{i=1}^{m_0-1} B(x_{i,m_0}, r) \right) + \text{vol}_D(B(x_{m_0,m_0}, r)) \\ &\leq \max(\Psi_{m_0-1}) + \max(\Psi_1) \\ &\leq \lambda_n \text{vol}_n(D) + \lambda_n \text{vol}_n(D) = 2\lambda_n \text{vol}_n(D). \end{aligned}$$

By Lemma 3.3, the set $A'_1 = \bigcup_{i=1}^{m_0} B(x_{i,m_0}, 2r) \cap D$ can be covered by $m_0 c_n$ balls of radius r in M . The union of these r -balls decomposes into the union of $c_n = 9^n$ sets of the form $\bigcup_{i=1}^{m_0} B(x_i, r)$ with $x_i \in M$. Since

$$\text{vol}_D \left(\bigcup_{i=1}^{m_0} B(x_i, r) \right) \leq \text{vol}_D \left(\bigcup_{i=1}^{m_0} B(x_{i,m_0}, r) \right) = \text{vol}_n(A_1),$$

we derive

$$\text{vol}_n(A'_1) \leq c_n \text{vol}_n(A_1) \leq 2\lambda_n c_n \text{vol}_n(D).$$

Thus, for $A_2 = D \setminus A'_1$, we obtain

$$\text{vol}_n(A_2) \geq (1 - 2\lambda_n c_n) \text{vol}_n(D) = \lambda_n \text{vol}_n(D).$$

By construction, we have $d(A_1, A_2) \geq r$. Note that we can also smooth out the boundary of A_i . \square

We can now derive Theorem 3.1.

Proof of Theorem 3.1. Let A_1 and A_2 be as in Lemma 3.4. Denote by $A_1(r)$ the r -neighborhood of A_1 in D . Note that A_2 lies in the complementary set of $A_1(r)$ in D . Define a Lipschitz function $f : D \rightarrow \mathbb{R}$ as

$$f(x) = \begin{cases} 1 & \text{if } x \in A_1 \\ 1 - \frac{d(x, A_1)}{r} & \text{if } x \in A_1(r) \setminus A_1 \\ 0 & \text{if } x \notin A_1(r) \end{cases}$$

The function f is not a Morse function. However, it can be approximated by a Morse function $f_\varepsilon : D \rightarrow \mathbb{R}$ with $\varepsilon > 0$ such that

- $f_\varepsilon(x) > 1$ if $x \in A_1$;
- $f_\varepsilon(x) < 0$ if $x \notin A_1(r)$;
- $|df_\varepsilon| \leq |df| + \varepsilon$.

Hence, from the coarea formula [BZ88, 13.4], we obtain

$$\begin{aligned} \int_0^1 \text{vol}_{n-1}(f_\varepsilon^{-1}(t)) dt &\leq \int_D |df_\varepsilon| dv \\ &\leq \left(\frac{1}{r} + \varepsilon\right) \text{vol}_n(D) \\ &\leq (\omega_n(1 + 2c_n))^{\frac{1}{n}} \text{vol}_n(D)^{\frac{n-1}{n}} + \varepsilon \text{vol}_n(D). \end{aligned}$$

To avoid burdening the argument by epsilontics, we will assume that $\varepsilon = 0$. Thus, there exists a regular value $t_0 \in (0, 1)$ such that

$$\text{vol}_{n-1}(f_\varepsilon^{-1}(t_0)) \leq (\omega_n(1 + 2c_n))^{\frac{1}{n}} \text{vol}_n(D)^{\frac{n-1}{n}}.$$

The preimage $S_\varepsilon = f_\varepsilon^{-1}(t_0)$ is a smooth hypersurface of D with boundary lying in ∂D which decomposes D into two domains $D_1 = f_\varepsilon^{-1}((-\infty, t_0])$ and $D_2 = f_\varepsilon^{-1}([t_0, \infty))$. Since $A_i \subset D_i$, we obtain

$$\text{vol}_n(D_i) \geq \frac{1}{1 + 2c_n} \text{vol}_n(D).$$

The theorem follows with $A_n = (\omega_n(1 + 2c_n))^{\frac{1}{n}}$ and $\alpha_n = \frac{1}{1+2c_n}$. \square

4. SPLITTING MANIFOLDS PRESERVING THE CW COMPLEX STRUCTURE

In this section, we establish a version of Theorem 3.1 which preserves the CW complex structure of the manifold with quasi-convex cells.

Theorem 4.1. *Fix $\rho > 0$ and $\lambda > 1$ close enough to 1. Let M be a complete Riemannian n -manifold with nonnegative Ricci curvature. Consider a bounded domain D in M endowed with a CW complex structure with quasi-convex cells of size (ρ, λ) . Suppose D has more than one n -cell. Then there exists an $(n - 1)$ -chain S of D with boundary lying in ∂D which*

decomposes D into two n -dimensional CW subcomplexes D_1 and D_2 with $\text{vol}_n(D_i) \geq \alpha'_n \text{vol}_n(D)$ such that

$$\text{vol}_{n-1}(S) \leq A'_n \text{vol}_n(D)^{\frac{n-1}{n}} \quad (4.1)$$

where A'_n and α'_n are explicit positive constants depending only on n .

Proof. For the sake of simplicity, we will assume that $\lambda = 1$, that is, the n -cells Δ of the CW complex decomposition of D are flat. The general case only requires technical changes to keep track of the bi-Lipschitz factor.

From Theorem 3.1, the domain D decomposes into two domains D_1 and D_2 such that their common boundary $S = \partial D_1 = \partial D_2$ satisfies

$$\text{vol}_{n-1}(S) \leq A_n \text{vol}_n(D)^{\frac{n-1}{n}}.$$

Without loss of generality, we can assume that no connected component of D_1 and D_2 lies in a single n -cell of the CW complex structure of D . This can be achieved by translating such connected components if necessary.

We want to deform the hypersurface S into the $(n-1)$ -skeleton of the CW complex decomposition of D while controlling its $(n-1)$ -volume and the n -volume of the two domains of D it bounds. By perturbing the hypersurface S if necessary, we can assume that it intersects transversally the $(n-1)$ -cells of D and decomposes its n -cells into finitely many domains. Thus, the hypersurface S decomposes into finitely many pieces $(S_i)_{i \in I}$ obtained as the connected components of the intersection of S with the n -cells of the CW complex decomposition of D . For every $i \in I$, denote by Δ_i the n -cell of D in which S_i lies. By construction, ∂S_i lies in $\partial \Delta_i$. We also have

$$\text{vol}_{n-1}(S) = \sum_{i \in I} \text{vol}_{n-1}(S_i).$$

Consider a volume-minimizing $(n-1)$ -chain S'_i in Δ_i with the same boundary as S_i .

If S'_i is at distance at least ρ from the center of Δ_i , we replace S_i with the image S''_i of S'_i by the projection π_{Δ_i} given by Lemma 2.3. The new $(n-1)$ -chain S''_i lies in $\partial \Delta_i$, has the same boundary as S_i and satisfies

$$\text{vol}_{n-1}(S''_i) \leq 36^{n-1} \text{vol}_{n-1}(S_i)$$

from Lemma 2.3.

If some point x'_i of S'_i is at distance at most ρ from the center of Δ_i , we replace S_i with a volume-minimizing $(n-1)$ -chain S''_i in $\partial \Delta_i$ with the same boundary as S_i . Note that $\text{vol}_{n-1}(S''_i) \leq \frac{1}{2} \text{vol}_{n-1}(S^{n-1}(6\rho))$. Since S'_i is a minimal $(n-1)$ -chain of Δ_i with boundary lying in $\partial \Delta_i$, the volume of the intersection of S'_i with the ρ -ball of Δ_i centered at x'_i is at least $\omega_{n-1} \rho^{n-1}$ from the monotonicity formula, cf. [Mo09, §9]. That is,

$$\text{vol}_{n-1}(S'_i) \geq \text{vol}_{n-1}(S'_i \cap B(x'_i, \rho)) \geq \omega_{n-1} \rho^{n-1}.$$

Hence,

$$\text{vol}_{n-1}(S''_i) \leq \frac{1}{2} \text{vol}_{n-1}(S^{n-1}(6\rho)) = \frac{1}{2} n \omega_n 6^{n-1} \rho^{n-1} \leq \frac{n \omega_n 6^{n-1}}{2 \omega_{n-1}} \text{vol}_{n-1}(S'_i).$$

In both case, we obtain an $(n-1)$ -chain S_i'' in $\partial\Delta_i$ with the same boundary as S_i such that

$$\text{vol}_{n-1}(S_i'') \leq C_{n-1} \text{vol}_{n-1}(S_i)$$

where $C_{n-1} = \max\{36^{n-1}, \frac{n\omega_n 6^{n-1}}{2\omega_{n-1}}\}$.

Replacing S_i with S_i'' gives rise to an $(n-1)$ -chain $S'' = \cup_{i \in I} S_i''$ lying in the $(n-1)$ -skeleton of the CW complex decomposition of D , with boundary $\partial S''$ lying in ∂D . This $(n-1)$ -chain bounds two n -dimensional CW subcomplexes D_1'' and D_2'' in D . Furthermore, the volume of S'' is bounded from above as follows

$$\text{vol}_{n-1}(S'') \leq C_{n-1} \text{vol}_{n-1}(S) \leq C_{n-1} A_n \text{vol}_n(D)^{\frac{n-1}{n}}. \quad (4.2)$$

Hence, the bound (4.1) for $A'_n = \max\{C_{n-1} A_n, n\omega_n^{\frac{1}{n}} 3^{n-1}\}$. (The reason for the second argument in the max will become clear later.)

It remains to bound from below the volume of the two CW subcomplexes D_1'' and D_2'' bounded by S'' . By the isoperimetric inequality, the volume of the region R_i of Δ_i bounded by S_i and S_i'' satisfies

$$\begin{aligned} \text{vol}_n(R_i) &\leq I_n (\text{vol}_{n-1}(S_i) + \text{vol}_{n-1}(S_i''))^{\frac{n}{n-1}} \\ &\leq I_n (1 + C_{n-1})^{\frac{n}{n-1}} \text{vol}_{n-1}(S_i)^{\frac{n}{n-1}}, \end{aligned}$$

where $I_n = \left(\frac{1}{n^n \omega_n}\right)^{\frac{1}{n-1}}$ is the n -dimensional isoperimetric constant in \mathbb{R}^n . Since $R_i \subset \Delta_i \subset B^n(6\rho)$, we also have

$$\text{vol}_n(R_i) \leq \text{vol}_n(\Delta_i) \leq \text{vol}_n(B^n(6\rho)) = \omega_n 6^n \rho^n.$$

Now, let I_- be the set of all the indices $i \in I$ such that $\text{vol}_{n-1}(S_i) \leq \rho^{n-1}$ and $I_+ = I \setminus I_-$ be the set of all the indices $i \in I$ such that $\text{vol}_{n-1}(S_i) > \rho^{n-1}$. By subtracting up the volume change of $D_j'' \cap \Delta_i$ through the replacement process for every n -cell Δ_i of the CW complex decomposition of D , we obtain

$$\begin{aligned} \text{vol}_n(D_j'') &\geq \text{vol}_n(D_j) - \sum_{i \in I_-} I_n (1 + C_{n-1})^{\frac{n}{n-1}} \text{vol}_{n-1}(S_i)^{\frac{n}{n-1}} - \sum_{i \in I_+} \omega_n 6^n \rho^n \\ &\geq \text{vol}_n(D_j) - \sum_{i \in I_-} I_n (1 + C_{n-1})^{\frac{n}{n-1}} \rho \text{vol}_{n-1}(S_i) - \sum_{i \in I_+} \omega_n 6^n \rho \text{vol}_{n-1}(S_i) \\ &\geq \text{vol}_n(D_j) - K_n \rho \text{vol}_{n-1}(S) \end{aligned}$$

where $K_n = \max\{I_n (1 + C_{n-1})^{\frac{n}{n-1}}, \omega_n 6^n\}$.

Suppose the number N of n -cells in the CW complex decomposition of D is greater or equal to $N_0 = \frac{1}{\omega_n} \left(\frac{K_n A'_n}{\alpha_n}\right)^n$. From the volume estimate (2.1), we have $\rho \leq \frac{1}{2} \left(\frac{\text{vol}_n(D)}{N_0 \omega_n}\right)^{\frac{1}{n}} = \frac{\alpha_n}{2K_n A'_n} \text{vol}_n(D)^{\frac{1}{n}}$. Combined with the bound (4.2), this implies that

$$K_n \rho \text{vol}_{n-1}(S'') \leq \frac{\alpha_n}{2} \text{vol}_n(D) \leq \frac{1}{2} \text{vol}_n(D_j).$$

Thus,

$$\text{vol}_n(D_j'') \geq \frac{1}{2} \text{vol}_n(D_j) \geq \frac{\alpha_n}{2} \text{vol}_n(D).$$

If N is less than N_0 , we can take for D_1'' any n -cell of D and for D_2'' the union of the remaining n -cells. In this case, still from (2.1), we deduce that

$$\text{vol}_{n-1}(\partial D_j'') \leq \text{vol}_{n-1}(S^{n-1}(6\rho)) = n \omega_n 6^{n-1} \rho^{n-1} \leq n \omega_n^{\frac{1}{n}} 3^{n-1} \text{vol}_n(D)^{\frac{n-1}{n}}$$

where $S^{n-1}(6\rho)$ is the 6ρ -sphere centered at the origin of \mathbb{R}^n . By our choice of A_n' , the bound (4.1) is still satisfied. We also derive from (2.1) that

$$\text{vol}_n(D_j'') \geq \text{vol}_n(B^n(\rho)) = \omega_n \rho^n \geq \frac{1}{6^n N_0} \text{vol}_n(D) = \omega_n \left(\frac{\alpha_n}{6K_n A_n'} \right)^n \text{vol}_n(D).$$

In both cases, the result follows by renaming D_j'' into D_j with $\alpha_n' = \min \left\{ \frac{\alpha_n}{2}, \omega_n \left(\frac{\alpha_n}{6K_n A_n'} \right)^n \right\}$. \square

Remark 4.2. By the simplicial approximation theorem, we can assume that S is an $(n-1)$ -dimensional simplicial chain of a simplicial subdivision of D with boundary lying in ∂D which decomposes D into two n -dimensional simplicial subcomplexes D_1 and D_2 satisfying the same geometric estimates.

5. MERGING SWEEP-OUTS

For the next result, we need to introduce a min-max value for sweep-outs defined in terms of PL Morse functions.

Definition 5.1. Let D be a piecewise flat simplicial n -complex homeomorphic to a compact n -manifold with (possibly empty) boundary. A function $f : D \rightarrow \mathbb{R}$ is a *PL Morse function* if it is linear in restriction to every simplex of D and takes pairwise distinct values at the vertices of D . Note that a PL Morse function is uniquely determined by its values on the set of vertices of D .

The $(n-1)$ -volume width of a PL Morse function $f : D \rightarrow \mathbb{R}$ is defined as

$$\mathcal{W}_{n-1}(f) = \sup_t \text{vol}_{n-1}(f^{-1}(t)).$$

Similarly, the $(n-1)$ -volume width of D is defined as

$$\mathcal{W}_{n-1}(D) = \inf_f \mathcal{W}_{n-1}(f) = \inf_f \sup_t \text{vol}_{n-1}(f^{-1}(t))$$

where f runs over all PL Morse functions on some simplicial subdivision of D .

When D is a compact Riemannian n -manifold with (possibly empty) piecewise smooth boundary and no fixed underlying simplicial complex structure, we extend the notion of PL Morse function as follows. A function $f : D \rightarrow \mathbb{R}$ is a (*generalized*) *PL Morse function* if there exists a simplicial complex structure on D for which f is a PL Morse function.

The following estimate on the $(n-1)$ -volume width of a bounded domain in terms of the $(n-1)$ -volume widths of the subdomains it is made of results from a cut-and-paste argument.

Proposition 5.2. *Let D be a piecewise flat simplicial n -complex homeomorphic to a compact n -manifold with (possibly empty) boundary. Let S be an $(n - 1)$ -chain of D with boundary lying in ∂D which decomposes D into two n -dimensional simplicial subcomplexes D_1 and D_2 . Then*

$$\mathcal{W}_{n-1}(D) \leq \max\{\mathcal{W}_{n-1}(D_1), \mathcal{W}_{n-1}(D_2)\} + 2n \operatorname{vol}_{n-1}(S). \quad (5.1)$$

Remark 5.3. One might wonder if the last term involving the volume of S in the right-hand side of the inequality (5.1) is really necessary. The following example shows it is indeed required. Consider a long thin cylinder with a cone attached to one of its boundary components. Glue three copies $\Delta_1, \Delta_2, \Delta_3$ of the resulting piecewise flat disk to obtain a piecewise flat sphere D shaped like a tripod. The sphere D splits into two domains $D_1 = \Delta_1$ and $D_2 = \Delta_2 \cup \Delta_3$ along a simple loop S . We can show that $\mathcal{W}_1(D) = 2 \operatorname{length}(S)$ and $\mathcal{W}_1(D_i) = \operatorname{length}(S)$. Thus, the inequality (5.1) does not hold without the term involving the volume of S . Note however that with extra care in the proof, it seems likely that the $2n$ factor can be dropped the inequality.

Proof of Proposition 5.2. The boundary of the $(n-1)$ -chain S decomposes ∂D into two simplicial subcomplexes S_1 and S_2 with S_i lying in ∂D_i . Therefore, the boundary of D_i decomposes into $\partial D_i = S \cup S_i$.

Let $f_i : D_i \rightarrow \mathbb{R}$ be a PL Morse function defined on a simplicial subdivision of D_i . By adding constants to f_1 and f_2 if necessary, we can assume that

$$\max_{D_1} f_1 < -1 < 1 < \min_{D_2} f_2.$$

We want to construct a PL Morse function $g : D \rightarrow \mathbb{R}$ defined on a simplicial subdivision of D by deforming f_1 and f_2 in the neighborhood of S into two other PL Morse functions which agree on S . The new PL Morse function $g : D \rightarrow \mathbb{R}$ should satisfy

$$\operatorname{vol}_{n-1}(g^{-1}(t)) \leq \operatorname{vol}_{n-1}(f_i^{-1}(t)) + \operatorname{vol}_{n-1}(S) + \delta$$

for every $t \in \mathbb{R}$, where δ is an arbitrarily small error term.

Let $\varepsilon \in (0, 1)$ small enough. From [Sp95, p. 122], we can assume that the simplicial subdivisions on D_1 and D_2 arise from the same simplicial subdivision of D . The n -simplices of D_i meeting S form a (closed) neighborhood \mathcal{U}_i of S in D_i . Let e be an edge of this neighborhood with exactly one endpoint, say p , lying in S . Denote by p_e the point of e such that $d(p, p_e) = \varepsilon \operatorname{length}(e)$. Note that for ε small enough, we can assume that the values taken by f_i at the points p_e are pairwise distinct.

Now, we truncate every n -simplex σ of \mathcal{U}_i along the hyperplane passing through the points p_e where e is an edge of σ . This gives rise to a CW complex structure of \mathcal{U}_i (finer than the one given by the initial triangulation) whose n -cells are convex Euclidean polyhedra, namely truncated n -simplices.

Denote by \mathcal{V}_i the (closed) neighborhood of S in \mathcal{U}_i formed of the n -cells (*i.e.*, truncated n -simplices) of D_i meeting S . Every n -cell of \mathcal{V}_i meets S along a k -simplex with $k \leq n - 1$. An n -cell Δ of \mathcal{V}_i is said to be *big* if it meets S along an $(n - 1)$ -simplex $\Delta \cap S$ and *small* otherwise. Observe that every hyperplane intersects a small n -cell (viewed as a convex Euclidean polyhedron) of \mathcal{V}_i along a region of $(n - 1)$ -volume at most δ_ε , where $\delta_\varepsilon \rightarrow 0$ when ε goes to zero. Similarly, every hyperplane intersects a big n -cell Δ

of \mathcal{V}_i along a region of $(n-1)$ -volume at most $\eta_\varepsilon \text{vol}_{n-1}(\Delta \cap S)$, where $\eta_\varepsilon \rightarrow 1$ when ε goes to zero.

Without introducing new vertices, we define a new triangulation of D_i by subdividing the n -cells (convex Euclidean polyhedra) of \mathcal{U}_i into n -simplices. In the process, big n -cells (which have exactly $2n$ vertices) are split into n simplices of maximal dimension and small n -cells (which have at most n^2 vertices) are split into at most n^2 simplices of maximal dimension.

We define a PL Morse function $g : D \rightarrow \mathbb{R}$ with respect to the new triangulation of D which agrees with f_i on the vertices of $D_i \setminus S$ and takes the vertices of S to pairwise distinct values close to zero in $(-1, 1)$.

The level sets of g are transverse to the $(n-1)$ -faces of the triangulation of D and intersect each n -simplex of $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ along a (possibly empty) hyperplane. The volume estimates on these hyperplanes and the bounds on the maximal number of n -simplices in an n -cell yield the following estimate

$$\begin{aligned} \text{vol}_{n-1}(g^{-1}(t) \cap \mathcal{V}) &\leq \sum_{\sigma \text{ big}} \text{vol}_{n-1}(g^{-1}(t) \cap \sigma) + \sum_{\sigma \text{ small}} \text{vol}_{n-1}(g^{-1}(t) \cap \sigma) \\ &\leq \sum_{\sigma \text{ big}} n \eta_\varepsilon \text{vol}_{n-1}(S \cap \sigma) + \sum_{\sigma \text{ small}} n^2 \delta_\varepsilon \\ &\leq 2n \eta_\varepsilon \text{vol}_{n-1}(S) + n^2 N_s \delta_\varepsilon \\ &\lesssim 2n \text{vol}_{n-1}(S) \end{aligned} \tag{5.2}$$

where the first sum is over the n -simplices lying in a big n -cell of \mathcal{V} , the second sum is over the n -simplices lying in a small n -cell of \mathcal{V} and N_s is the number of n -simplices lying in a small n -cell of \mathcal{V} .

The level sets of g satisfy the following properties for different values of t . For $t < -1$, the level set $g^{-1}(t) \setminus \mathcal{V}$ lies in D_1 and agrees with $f_1^{-1}(t) \setminus \mathcal{V}_1$, while for $t > 1$, it lies in D_2 and agrees with $f_2^{-1}(t) \setminus \mathcal{V}_2$. Finally, for $t \in [-1, 1]$, the level set $g^{-1}(t)$ lies in \mathcal{V} .

From this, we immediately derive

$$\text{vol}_{n-1}(g^{-1}(t)) \leq \max\{\text{vol}_{n-1}(f_1^{-1}(t)), \text{vol}_{n-1}(f_2^{-1}(t))\} + \text{vol}_{n-1}(g^{-1}(t) \cap \mathcal{V}).$$

Combined with (5.2), we obtain

$$\mathcal{W}_{n-1}(g) \leq \max\{\mathcal{W}_{n-1}(f_1), \mathcal{W}_{n-1}(f_2)\} + 2n \eta_\varepsilon \text{vol}_{n-1}(S) + n^2 N \delta_\varepsilon.$$

The desired result follows by letting ε go to zero, namely

$$\mathcal{W}_{n-1}(D) \leq \max\{\mathcal{W}_{n-1}(D_1), \mathcal{W}_{n-1}(D_2)\} + 2n \text{vol}_{n-1}(S).$$

□

6. MAIN THEOREM

We can now prove the main theorem of this article.

Theorem 6.1. *Let M be a complete Riemannian n -manifold with nonnegative Ricci curvature. For every bounded domain D in M , there exists a PL Morse function $f : D \rightarrow \mathbb{R}$ such that*

$$\sup_t \text{vol}_{n-1}(f^{-1}(t)) \leq C_n \text{vol}_n(D)^{\frac{n-1}{n}}$$

where C_n is an explicit positive constant depending only on n .

Remark 6.2. When M is a closed Riemannian n -manifold and D agrees with the whole manifold M , we recover Theorem 1.2 from the introduction.

Proof of Theorem 6.1. Fix $\rho > 0$ small enough so that there exists a CW complex structure on D with quasi-convex cells of size (ρ, λ) with $\lambda > 1$ close enough to 1, cf. Proposition 2.2. As in the proof of Theorem 4.1, we will assume that $\lambda = 1$ for the sake of simplicity. That is, the quasi-convex n -cells of the CW complex structure of D are flat. In particular, D is a piecewise flat simplicial n -complex. We argue by induction on the number of quasi-convex n -cells in the CW complex structure of D .

Suppose that D is composed of a single n -cell. Every polyhedron lying in $B^n(6\rho)$ can be swept out by parallel hyperplanes of volume at most $\text{vol}_{n-1}(B^{n-1}(6\rho)) = \omega_{n-1}6^{n-1}\rho^{n-1}$, which correspond to the level sets of some PL Morse function. Now, the volume of D is greater or equal to $\text{vol}_n(B^n(2\rho)) = \omega_n 2^n \rho^n$. Thus,

$$\mathcal{W}_{n-1}(D) \leq 6^{n-1} \omega_{n-1} \omega_n^{-\frac{n-1}{n}} \text{vol}_n(D)^{\frac{n-1}{n}}.$$

In the general case, from Theorem 4.1, there exists an $(n-1)$ -chain S of D with boundary lying in ∂D which decomposes D into two CW sub-complexes D_1 and D_2 with $\text{vol}_n(D_i) \geq \alpha'_n \text{vol}_n(D)$ such that

$$\text{vol}_{n-1}(S) \leq A'_n \text{vol}_n(D)^{\frac{n-1}{n}}, \quad (6.1)$$

Both D_1 and D_2 have fewer n -cells in their CW complex decomposition than D . Therefore, we can assume that

$$\frac{\mathcal{W}_{n-1}(D_i)}{\text{vol}_n(D_i)^{\frac{n-1}{n}}} \leq \frac{\mathcal{W}_{n-1}(D)}{\text{vol}_n(D)^{\frac{n-1}{n}}}$$

otherwise the result follows by induction.

Thus, since $\text{vol}_n(D_i) \leq (1 - \alpha'_n) \text{vol}_n(D)$, we deduce that

$$\mathcal{W}_{n-1}(D_i) \leq (1 - \alpha'_n)^{\frac{n-1}{n}} \mathcal{W}_{n-1}(D).$$

Combined with Proposition 5.2 and the bound (6.1), we obtain

$$\begin{aligned} \mathcal{W}_{n-1}(D) &\leq \frac{2n}{1 - (1 - \alpha'_n)^{\frac{n-1}{n}}} \text{vol}_{n-1}(S) \\ &\leq \frac{2n A'_n}{1 - (1 - \alpha'_n)^{\frac{n-1}{n}}} \text{vol}_n(D)^{\frac{n-1}{n}} \end{aligned}$$

Hence, the theorem holds with $C_n = \max \left\{ \frac{2n A'_n}{1 - (1 - \alpha'_n)^{\frac{n-1}{n}}}, 6^{n-1} \omega_{n-1} \omega_n^{-\frac{n-1}{n}} \right\}$. \square

Given a closed Riemannian n -manifold M , we deduce from the min-max principle of [DT13] the existence of an embedded closed minimal hypersurface in M with a singular set of Hausdorff dimension at most $n - 8$ and volume at most $\mathcal{W}_{n-1}(M)$. Combined with Theorem 1.2, this yields Theorem 1.1.

Similarly, Corollary 1.3 is a consequence of Theorem 1.2.

Proof of Corollary 1.3. From Theorem 1.2, a closed Riemannian manifold M with nonnegative Ricci curvature and dimension $n \geq 3$ decomposes into two non-necessarily connected domains D_1 and D_2 with the same volume such that $\text{vol}_{n-1}(\partial D_i) \leq C_n \text{vol}_n(M)^{\frac{n-1}{n}}$. Slightly perturbing the domains D_i and connecting their connected components with thin tubes if necessary, we can assume that the domains D_i are connected. In dimension two, it follows from [CC92] that any two-sphere with nonnegative curvature can be swept out by a one-parameter family of disjoint simple loops of length at most the length of the shortest closed geodesic. Since the length of the shortest closed geodesic is bounded in terms of the area of the sphere, cf. [Cr88], we immediately obtain the desired decomposition. \square

7. A CURVATURE-FREE COUNTER-EXAMPLE

In this section, we show that the curvature condition in Theorem 1.2 cannot be dropped in general.

Before going further, we need to briefly review the Almgren-Pitts min-max principle. For more details, we refer the reader to [Pi81, SS81, CD03, DT13].

Let M be a closed Riemannian n -manifold. Fix $\mathbb{k} = \mathbb{Z}$ if M is orientable and $\mathbb{k} = \mathbb{Z}/2\mathbb{Z}$ otherwise. Denote by $\mathcal{Z}_k(M)$ the k -cycle space of M over \mathbb{k} , that is, the space of k -dimensional integral currents of M over \mathbb{k} with zero boundary. The mass of a k -cycle $z \in \mathcal{Z}_k(M)$ defined as

$$\mathbf{M}(z) = \sup \left\{ \int_z \omega \mid \omega \text{ smooth } k\text{-form on } M \text{ with } \|\omega\| \leq 1 \right\}$$

extends the notion of volume for k -submanifolds.

The homotopy groups of the k -cycle space endowed with the flat norm topology have been determined by F. Almgren [Al62, Pi81]. More precisely, there is a natural isomorphism between $H_p(\mathcal{Z}_k(M); \mathbb{k}), \{0\})$ and $H_{p+k}(M; \mathbb{k})$. In particular,

$$\pi_1(\mathcal{Z}_{n-1}(M; \mathbb{k}), \{0\}) \simeq H_n(M; \mathbb{k}) \simeq \mathbb{k}.$$

This isomorphism allows us to apply the Almgren-Pitts min-max principle to the $(n-1)$ -cycle space of M as follows.

A sweep-out of M by $(n-1)$ -cycles is defined as a one-parameter family of $(n-1)$ -cycles $(z_t)_{0 \leq t \leq 1}$ starting and ending at the null-cycle which induces a generator of $\pi_1(\mathcal{Z}_{n-1}(M; \mathbb{k}), \{0\})$. An example of sweep-out is given by the level sets $f^{-1}(t)$ of a PL Morse function $f : M \rightarrow [0, 1]$.

The $(n-1)$ -volume width of M defined in terms of $(n-1)$ -cycles is given by the following min-max value

$$\mathcal{W}_{\mathcal{Z}_{n-1}}(M) = \inf_{(z_t)} \sup_{0 \leq t \leq 1} \mathbf{M}(z_t)$$

where (z_t) runs over the sweep-outs of M . Clearly,

$$\mathcal{W}_{\mathcal{Z}_{n-1}}(M) \leq \mathcal{W}_{n-1}(M).$$

From [BS10], Theorem 1.2 holds true for every Riemannian two-sphere without any curvature assumption. It actually holds true for every closed Riemannian surface without any curvature assumption if one allows the multiplicative constant to depend on the genus of the surface (and examples show the multiplicative constant has to depend on the genus).

On the other hand, L. Guth [Gu07, §5] deduced from a construction of [BI98] that every n -sphere has a Riemannian metric with unit n -volume and $\mathcal{W}_{z_{n-1}}$ arbitrarily large for $n \geq 3$.

The next proposition shows this result extends to every closed manifold of dimension at least three.

Proposition 7.1. *Let M be a closed n -manifold with $n \geq 3$. There exists a Riemannian metric on M with unit n -volume and arbitrarily large $(n - 1)$ -volume width. In particular, every PL Morse function on M has a fiber of arbitrarily large $(n - 1)$ -volume for this metric.*

Proof. Consider an n -sphere S^n endowed with a Riemannian metric with unit n -volume and $\mathcal{W}_{z_{n-1}}(S^n)$ arbitrarily large. Given a domain D of M diffeomorphic to a ball, we consider a degree one map $\varphi : M \rightarrow S^n$ taking D to some point $p \in S^n$ and $M \setminus D$ to $S^n \setminus \{p\}$. We also define a metric on M such that the volume of $M \setminus D$ is arbitrarily small and the restriction of φ to D is an isometry onto $S^n \setminus \{p\}$. Now, since the map φ has degree one, the image by φ of a sweep-out (z_t) of M by $(n - 1)$ -cycles induces a sweep-out of S^n . From the geometry of S^n , one of the $(n - 1)$ -cycles of this sweep-out of S^n has an arbitrarily large mass. As the map φ is distance, and so mass, nonincreasing, the same holds for the $(n - 1)$ -cycles (z_t) . This shows that $\mathcal{W}_{z_{n-1}}(M)$ can be arbitrarily large while M has unit n -volume. \square

Remark 7.2. Working with level sets does not allow us to conclude in the proof of Proposition 7.1. In order to connect M to the sphere S^n , we need the flexibility provided by the general Almgren-Pitts min-max principle defined on the $(n - 1)$ -cycle space.

REFERENCES

- [Al62] Almgren, F.: The homotopy groups of the integral cycle groups. *Topology* 1 (1962) 257–299.
- [Ba10] Balacheff, F.: A local optimal diastolic inequality on the two-sphere. *J. Topol. Anal.* 2 (2010), no. 1, 109–121.
- [Ba] Balacheff, F.: Measurements of Riemannian two-disks and two-spheres. *Pacific J. Math.*, to appear.
- [BS10] Balacheff, F.; Sabourau, S.: Diastolic and isoperimetric inequalities on surfaces. *Ann. Sci. Ec. Norm. Sup.* 43 (2010), no. 4, 579–605.
- [BI98] Burago, D.; Ivanov, S.: On asymptotic isoperimetric constant of tori. *Geom. Funct. Anal.* 8 (1998), no. 5, 783–787.
- [BZ88] Burago, Yu. D.; Zalgaller, V. A.: Geometric inequalities. Grundlehren der Mathematischen Wissenschaften, vol. 285, Springer-Verlag, 1988co.
- [CC92] Calabi, E.; Cao, J.: Simple closed geodesics on convex surfaces. *J. Differential Geom.* 36 (1992), no. 3, 517–549.
- [Ch70] Cheeger, J.: A lower bound for the smallest eigenvalue of the Laplacian. Problems in analysis (Papers dedicated to Salomon Bochner, 1969), pp. 195–199. Princeton Univ. Press, 1970.
- [CMa08] Colbois, B.; Maerten, D.: Eigenvalues estimate for the Neumann problem of a bounded domain. *J. Geom. Anal.* 18 (2008), no. 4, 1022–1032.
- [CD03] Colding, T.; De Lellis, C.: The min-max construction of minimal surfaces. *Surv. Differ. Geom.*, vol. VIII, 75–107, Int. Press, 2003.
- [CM05] Colding, T.; Minicozzi, W.: Estimates for the extinction time for the Ricci flow on certain 3-manifolds and a question of Perelman. *J. Amer. Math. Soc.* 18 (2005), no. 3, 561–569

- [Cr88] Croke, C.: Area and the length of the shortest closed geodesic. *J. Differential Geom.* 27 (1988), no. 1, 1–21.
- [DP10] De Lellis, C.; Pellandini, F.: Genus bounds for minimal surfaces arising from min-max constructions. *J. Reine Angew. Math.* 644 (2010) 47–99.
- [DT13] De Lellis, C.; Tasnady, D.: The existence of embedded minimal hypersurfaces. *J. Differential Geom.* 95 (2013), no. 3, 355–388.
- [Fa80] Falconer, K. J.: Continuity properties of k -plane integrals and Besicovitch sets. *Math. Proc. Cambridge Philos. Soc.* 87 (1980), no. 2, 221–226.
- [GAL] Glynn-Adey, P.; Liokumovich, Y.: Width, Ricci curvature and minimal hypersurfaces. See arXiv:1408.3656
- [GNY04] Grigor'yan, A.; Netrusov, Y.; Yau, S.-T.: Eigenvalues of elliptic operators and geometric applications. *Surv. Differ. Geom.*, vol. IX, 147–217, Int. Press, 2004.
- [GY99] Grigor'yan, A.; Yau, S.-T.: Decomposition of a metric space by capacitors. *Differential equations: La Pietra 1996 (Florence)*, 39–75, *Proc. Sympos. Pure Math.* 65, Amer. Math. Soc., 1999.
- [Gr83] Gromov, M.: Filling Riemannian manifolds. *J. Differential Geom.* 18 (1983), no. 1, 1–147.
- [Gr88] Gromov, M.: Width and related invariants of Riemannian manifolds. *Astérisque* 163-164 (1988) 93–109.
- [Gr03] Gromov, M.: Isoperimetry of waists and concentration of maps. *Geom. Funct. Anal.* 13 (2003), no. 1, 178–215.
- [Gu07] Guth, L.: The width-volume inequality. *Geom. Funct. Anal.* 17 (2007), no. 4, 1139–1179.
- [Gu09] Guth, L.: Minimax problems related to cup powers and Steenrod squares. *Geom. Funct. Anal.* 18 (2009), no 6, 1917–1987.
- [Ha02] Hatcher, A.: Algebraic topology. Cambridge University Press, Cambridge, 2002.
- [He81] Hebda, J.: Some lower bounds for the area of surfaces. *Invent. Math.* 65 (1981/82), no. 3, 485–490.
- [Ke] Ketover, D.: Degeneration of min-max sequences in 3-manifolds, See arXiv:1312.2666.
- [Ko93] Korevaar, N.: Upper bounds for eigenvalues of conformal metrics. *J. Differential Geom.* 37 (1993), no. 1, 73–93.
- [LY80] Li, P.; Yau, S.-T.: Estimates of eigenvalues of a compact Riemannian manifold, *Proc. Sympos. Pure Math.* 36 (1980) 205–239.
- [LY82] Li, P.; Yau, S.-T.: A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. *Invent. Math.* 69 (1982), no. 2, 269–291.
- [Li14] Liokumovich, Y.: Slicing a 2-sphere. *J. Topol. Anal.* 6 (2014), no. 4, 573–590.
- [Li] Liokumovich, Y.: Families of short cycles on Riemannian surfaces. See arXiv:1410.8436
- [MN12] Marques, F., Neves, A.: Rigidity of min-max minimal spheres in three-manifolds. *Duke Math. J.* 161 (2012), no. 14, 2725–2752.
- [MN14] Marques, F., Neves, A.: Min-Max theory and the Willmore conjecture. *Ann. of Math.* 179 (2014), no. 2, 683–782.
- [MN] Marques, F., Neves, A.: Existence of infinitely many minimal hypersurfaces in positive Ricci curvature. See arXiv:1311.6501.
- [Mo09] Morgan, F.: Geometric measure theory. A beginner's guide. Fourth edition. Elsevier/Academic Press, 2009.
- [Pe95] Perelman, G.: Widths of nonnegatively curved spaces. *Geom. Funct. Anal.* 5 (1995), no. 2, 445–463.
- [Pi81] Pitts, J.: Existence and regularity of minimal surfaces on Riemannian manifolds. *Mathematical Notes*, 27. Princeton University Press, 1981
- [Sa10] Sabourau, S.: Local extremality of the Calabi-Croke sphere for the length of the shortest closed geodesic. *J. Lond. Math. Soc. (2)* 82 (2010), no. 3, 549–562.
- [SS81] Schoen, R.; Simon, L.: Regularity of stable minimal hypersurfaces. *Comm. Pure Appl. Math.* 34 (1981), no. 6, 741–797.
- [Sp95] Spanier, E.: Algebraic topology. Springer-Verlag, Berlin, 1995.

- [Tr85] Treibergs, A.: Estimates of volume by the length of shortest closed geodesics on a convex hypersurface. *Invent. Math.* 80 (1985), no. 3, 481–488.
- [Zh] Zhou, X.: Min-max minimal hypersurface in (M^{n+1}, g) with $Ric_g > 0$ and $2 \leq n \leq 6$. See arXiv:1210.2112v2

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