Dynamique de type Pisot et fractales de Rauzy

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SCAM-Novembre 2013
- Discrepancy and bounded remainder sets
- Pisot substitutions and dynamics
- The Pisot substitution conjecture
- Continued fractions and generalized Pisot property
Discrepancy
and
bounded remainder sets
Bounded remainder sets and Kronecker sequences

Let \( \alpha = (\alpha_1, \ldots, \alpha_d) \in [0, 1]^d \)
with \( 1, \alpha_1, \ldots, \alpha_d \) \( \mathbb{Q} \)-linearly independent

We consider the \textbf{Kronecker sequence}

\[
(\{n\alpha_1\}, \ldots, \{n\alpha_d\})_n
\]
We consider the Kronecker sequence

$$(\{n\alpha_1\}, \ldots, \{n\alpha_d\})_n$$

associated with the translation over $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

$$R_\alpha : \mathbb{T}^d \mapsto \mathbb{T}^d, \quad x \mapsto x + \alpha$$

$$\alpha = (\alpha_1, \ldots, \alpha_d)$$
Bounded remainder sets and Kronecker sequences

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\(R_\alpha : \mathbb{T}^d \mapsto \mathbb{T}^d, \ x \mapsto x + \alpha\)

\(\alpha = (\alpha_1, \cdots, \alpha_d)\)

Discrepancy

\[\Delta_N = \sup_B \text{box } |\text{Card } \{0 \leq n \leq N; R^n_\alpha(0) \in B\} - N \cdot \mu(B)|\]
Bounded remainder sets and Kronecker sequences

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associated with the translation over \( \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \)

\[ R_\alpha : \mathbb{T}^d \mapsto \mathbb{T}^d, \; x \mapsto x + \alpha \]

\[ \alpha = (\alpha_1, \cdots, \alpha_d) \]

Discrepancy

\[ \Delta_N = \sup_{B \text{ box}} |\text{Card} \{0 \leq n \leq N; R_\alpha^n(0) \in B\} - N \cdot \mu(B)| \]

Bounded remainder set A set \( X \) for which there exists \( C > 0 \) s.t. for all \( N \)

\[ |\text{Card}\{0 \leq n \leq N; R_\alpha^n(0) \in X\} - N\mu(X)| \leq C \]
Bounded remainder sets

Case \( d = 1 \)

**Theorem [Kesten’66]** Intervals that are bounded remainder sets are the intervals with length in \( \mathbb{Z} + \alpha \mathbb{Z} \)
Bounded remainder sets

Case $d = 1$

Theorem [Kesten’66] Intervals that are bounded remainder sets are the intervals with length in $\mathbb{Z} + \alpha\mathbb{Z}$

General dimension $d$

Theorem [Liardet’87] There are no nontrivial boxes that are bounded remainder sets
Bounded remainder sets

Case $d = 1$

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General dimension $d$

Theorem [Liardet’87] There are no nontrivial boxes that are bounded remainder sets

Boxes are not bounded remainder sets

How well can one approximate a box by bounded remainder sets?
A symbolic approach

We consider a partition \( \{X_1, \cdots, X_k\} \) of \( \mathbb{T}^d \)

\[
\mathbb{T}^d = \bigcup_{1 \leq i \leq k} X_i, \quad \mu(X_i \cap X_j) = 0, \text{ for all } i \neq j
\]

We code the trajectory of \( x \) under the action of \( R_\alpha : x \mapsto x + \alpha \) as follows

\[
x \rightsquigarrow (u_n)_n \in \{1, 2, \ldots, k\}^\mathbb{N}
\]

\[
u_n = i \quad \text{if and only if} \quad R^n_\alpha(x) = x + n\alpha \in X_i
\]
A symbolic approach

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\[
u_n = i \quad \text{if and only if} \quad R^n_{\alpha}(x) = x + n\alpha \in X_i
\]

Questions Which information on \( R_\alpha \) can we get from the combinatorial properties of the sequence \( (u_n) \)? What is a good coding?
Pisot substitutions and dynamics
A substitution on words: the Tribonacci substitution

**Definition**  A substitution $\sigma$ is a **morphism** of the free monoid

Non-negative morphism of the free group, no cancellations

$$\sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1$$

$$\sigma^\infty(1) = 12131211213121213 \cdots$$
A substitution on words: the Tribonacci substitution

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A substitution on words: the Tribonacci substitution

\[
\sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1
\]

\[
\sigma^\infty (1) = 1213121121312121312131213 \cdots
\]

The incidence matrix \( M_\sigma \) of \( \sigma \) is defined by

\[
M_\sigma = (|\sigma(j)|i)_{(i,j) \in \mathcal{A}^2},
\]

where \( |\sigma(j)|_i \) counts the number of occurrences of the letter \( i \) in \( \sigma(j) \).

The matrix \( M_\sigma \) has nonnegative entries \( \rightsquigarrow \) Perron-Frobenius theory.
A substitution on words: the Tribonacci substitution

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \]

\[ \sigma^\infty(1) = 12131211213121213 \cdots \]

Its incidence matrix is

\[
M_\sigma = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

Its characteristic polynomial is \( X^3 - X^2 - X - 1 \). Its Perron-Frobenius eigenvalue \( \beta > 1 \) is a Pisot number.

It is primitive: there exists a power of \( M_\sigma \) which contains only positive entries.

Perron-Frobenius \( \rightsquigarrow \) one expanding eigendirection

a contracting eigenplane
Pisot substitution

Pisot-Vijayaraghavan number An algebraic integer is a Pisot number if its algebraic conjugates \( \lambda \) (except itself) satisfy

\[ |\lambda| < 1 \]

Let \( \sigma \) be a substitution over the alphabet \( \mathcal{A} \)

Pisot substitution \( \sigma \) is primitive and its Perron–Frobenius eigenvalue is a Pisot number
Tribonacci dynamics

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \]

Theorem [Rauzy’82] \((X_\sigma, S)\) is measure-theoretically isomorphic to the translation \(R_\beta\) on the two-dimensional torus \(\mathbb{T}^2\)

\[ R_\beta : \mathbb{T}^2 \to \mathbb{T}^2, \ x \mapsto x + (1/\beta, 1/\beta^2) \]
Triboonacci dynamics

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\[ R_\beta : \mathbb{T}^2 \rightarrow \mathbb{T}^2, \ x \mapsto x + (1/\beta, 1/\beta^2) \]

Markov partition for the toral automorphism

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]
Substitutions

- Substitutions on **words** and symbolic dynamical systems
- Substitutions on **tiles** : inflation/subdivision rules, **tilings** and point sets
Substitutions

- Substitutions on words and symbolic dynamical systems
- Substitutions on tiles: inflation/subdivision rules, tilings and point sets

Tilings Encyclopedia http://tilings.math.uni-bielefeld.de/

[E. Harriss, D. Frettlöh]
Substitutions produce hierarchical ordered structures (infinite words, point sets, tilings) that display strong self-similarity properties.

Substitutions are closely related to induction (first return maps, Rokhlin towers, renormalization etc.)

A system is **self-induced** if there is a subset with positive measure for which the induced system is isomorphic to the original system.

We consider substitutions that create hierarchical structure with a significant amount of **long range order**.
The Pisot substitution conjecture

Substitutive structure + Algebraic assumption (Pisot) = Order (discrete spectrum)

Discrete spectrum = translation on a compact group
Let $\sigma$ be a primitive substitution over $\mathcal{A}$. Let $\omega = (\omega_n)$ with $\sigma(\omega) = \omega$ be an infinite word generated by $\sigma$. Let $S$ be the shift

$$S((\omega_n)_n) = (\omega_{n+1})_n$$

The symbolic dynamical system generated by $\sigma$ is $(X_\sigma, S)$

$$X_\sigma := \{S^n(\omega); \ n \in \mathbb{N}\} \subset \mathcal{A}^\mathbb{N}$$
Substitutive dynamical systems

Let $\sigma$ be a primitive substitution over $A$. The symbolic dynamical system generated by $\sigma$ is $(X_\sigma, S)$

$$X_\sigma := \{S^n(\omega); \ n \in \mathbb{N}\} \subset A^\mathbb{N}$$

Question Under which conditions is it possible to give a geometric representation of a substitutive dynamical system as a translation on a compact abelian group? (discrete spectrum)

$$X_\sigma \xrightarrow{S} X_\sigma$$

Necessary condition: the expansion $\beta$ is a Pisot number [Solomyak]
Let $\sigma$ be a primitive substitution over $\mathcal{A}$. The symbolic dynamical system generated by $\sigma$ is $(X_\sigma, S)$

$$X_\sigma := \{ S^n(\omega); \, n \in \mathbb{N} \} \subset \mathcal{A}^\mathbb{N}$$

The Pisot substitution conjecture Dates back to the 80’s

[Bombieri-Taylor, Rauzy, Thurston]

If $\sigma$ is a Pisot irreducible substitution, then $(X_\sigma, S)$ has discrete spectrum
Substitutive dynamical systems

Let $\sigma$ be a primitive substitution over $A$.
The symbolic dynamical system generated by $\sigma$ is $(X_\sigma, S)$

$$X_\sigma := \{S^n(\omega); \; n \in \mathbb{N}\} \subset A^\mathbb{N}$$

Example In the Fibonacci case

$$\sigma: a \mapsto ab, b \mapsto a$$

$(X_\sigma, S)$ is isomorphic to $(\mathbb{R}/\mathbb{Z}, R_{1+\sqrt{5}}^2)$

$$R_{1+\sqrt{5}}^2 : x \mapsto x + \frac{1 + \sqrt{5}}{2} \mod 1$$
Substitutive dynamical systems

Let $\sigma$ be a \textbf{primitive} substitution over $A$.
The \textbf{symbolic dynamical system} generated by $\sigma$ is $(X_\sigma, S)$

$$X_\sigma := \{ S^n(\omega); \; n \in \mathbb{N} \} \subset A^\mathbb{N}$$

The Pisot substitution conjecture
If $\sigma$ is a \textbf{Pisot irreducible} substitution, then $(X_\sigma, S)$ has discrete spectrum

The conjecture is proved for two-letter alphabets

[Host, Barge-Diamond, Hollander-Solomyak]
Substitutive dynamical systems

Let $\sigma$ be a primitive substitution over $\mathcal{A}$. The symbolic dynamical system generated by $\sigma$ is $(X_\sigma, S)$

$$X_\sigma := \{ S^n(\omega); \; n \in \mathbb{N} \} \subset A^\mathbb{N}$$

Remark  Measure-theoretic discrete spectrum and topological discrete spectrum are equivalent for primitive substitutive dynamical systems [Host]
Tribonacci’s substitution [Rauzy ’82]

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \]

\[ 12131211213121213 \cdots \]

**Question**  Is it possible to give a geometric representation of the associated substitutive dynamical system \( X_\sigma \) as a translation on an abelian compact group?

**Yes!**  \((X_\sigma, S)\) is isomorphic to a translation on the two-dimensional torus

**Question**  How to produce explicitly a fundamental domain for this translation?

**Rauzy fractal**  G. Rauzy introduced in the 80’s a compact set with fractal boundary that tiles the plane which provides a geometric representation of \((X_\sigma, S)\)

\(\Leftrightarrow\) **Thurston** for beta-numeration
The Rauzy fractal as a geometric representation

Consider the Tribonacci substitution

$$\sigma : 1 \mapsto 12, \ 2 \mapsto 3, \ 3 \mapsto 1$$

1213121121312131211213 \cdots

\pi projection along the expanding eigenline onto the contracting plane of the incidence matrix of $M_\sigma$

\(\pi(\vec{e}_3)\) \quad \pi(\vec{e}_1) \quad \pi(\vec{e}_2)
The Rauzy fractal as a geometric representation

Consider the Tribonacci substitution

\[ \sigma: 1 \mapsto 12, \ 2 \mapsto 3, \ 3 \mapsto 1 \]

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\[ \pi (\vec{e}_1) \]

\[ \pi \text{ projection along the expanding eigenline onto the contracting plane of the incidence matrix of } M_\sigma \]

\[ \pi (\vec{e}_2) \]

\[ \pi (\vec{e}_3) \]
The Rauzy fractal as a geometric representation

Consider the Tribonacci substitution

\[ \sigma : 1 \mapsto 12, \quad 2 \mapsto 3, \quad 3 \mapsto 1 \]

\[ 121312112131212131211213 \cdots \]

\[ \pi (\vec{e}_1 + \vec{e}_2) \]

\[ \pi \text{ projection along the expanding eigenline onto the contracting plane of the incidence matrix of } M_\sigma \]

\[ \pi (\vec{e}_1) \quad \pi (\vec{e}_2) \quad \pi (\vec{e}_3) \]
The Rauzy fractal as a geometric representation

Consider the Tribonacci substitution

\[ \sigma: 1 \mapsto 12, \ 2 \mapsto 3, \ 3 \mapsto 1 \]

\[ 121312112131212131211213 \ldots \]

\[ \pi\left(\vec{e}_1 + \vec{e}_2 + \vec{e}_1\right) \]

\[ \pi \text{ projection along the expanding eigenline onto the contracting plane of the incidence matrix of } M_\sigma \]
The Rauzy fractal as a geometric representation

Consider the Tribonacci substitution

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 3, \ 3 \mapsto 1 \]

\[
\begin{align*}
121312112131212131211213 \cdots \\
\pi(\vec{e}_1 + \vec{e}_2 + \vec{e}_1 + \vec{e}_3)
\end{align*}
\]

\[ \pi \text{ projection along the expanding eigenline onto the contracting plane of the incidence matrix of } M_\sigma \]

\[ \pi(\vec{e}_1), \pi(\vec{e}_2), \pi(\vec{e}_3) \]
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The Rauzy fractal as a geometric representation

Consider the Tribonacci substitution

$$\sigma: 1 \mapsto 12, \ 2 \mapsto 3, \ 3 \mapsto 1$$

$$121312112131212131211213 \cdots$$

$$\pi(e_1 + e_2 + e_1 + e_3 + e_1 + e_2 + e_1 + \cdots)$$

$$\pi(e_3) \quad \Rightarrow \quad \pi(e_1)$$

$$\pi(e_2) \quad \Rightarrow \quad \pi(e_1)$$

$$\pi$$ projection along the expanding eigenline onto the contracting plane of the incidence matrix of $M_\sigma$
The Rauzy fractal as a geometric representation

Consider the Tribonacci substitution

$$\sigma : 1 \mapsto 12, \ 2 \mapsto 3, \ 3 \mapsto 1$$

121312112131212131212131211213 \cdots

$$\pi(\vec{e}_1 + \vec{e}_2 + \vec{e}_1 + \vec{e}_3 + \vec{e}_1 + \vec{e}_2 + \vec{e}_1 + \cdots)$$

$$\pi$$ projection along the expanding eigenline onto the contracting plane of the incidence matrix of $$M_\sigma$$

$$\pi(\vec{e}_3)$$

$$\pi(\vec{e}_2)$$

$$\pi(\vec{e}_1)$$
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\[ \pi \text{ projection along the expanding eigenline onto the contracting plane of the incidence matrix of } M_\sigma \]

\[ \pi(\vec{e}_1), \ \pi(\vec{e}_2), \ \pi(\vec{e}_3) \]
The Rauzy fractal as a geometric representation

Consider the Tribonacci substitution

$$\sigma : 1 \mapsto 12, \ 2 \mapsto 3, \ 3 \mapsto 1$$

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$\pi$ projection along the expanding eigenline onto the contracting plane of the incidence matrix of $M_{\sigma}$
The Rauzy fractal as a geometric representation

Consider the Tribonacci substitution

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The Rauzy fractal as a geometric representation

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\( \pi \) projection along the expanding eigenline onto the contracting plane of the incidence matrix of \( M_\sigma \)

\( \pi(\vec{e}_1), \ \pi(\vec{e}_2), \ \pi(\vec{e}_3) \)
Rauzy fractal and dynamics

One first defines an exchange of pieces acting on the Rauzy fractal.
The subtiles are disjoint in measure (the proof uses the associated graph-directed Iterated Function System).

π projection along the expanding eigenline onto the contracting plane of the incidence matrix $M_\sigma$

The translation vectors are the projections of the canonical basis vectors $\pi(\vec{e}_i)$.
Rauzy fractal and dynamics

One first defines an *exchange of pieces* acting on the Rauzy fractal.

This exchange of pieces factorizes into a translation of $\mathbb{T}^2$. This is due to the fact that the Rauzy fractal tiles periodically the plane.
Pisot dynamics

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \]

Theorem [Rauzy’82] \((X_\sigma, S)\) is measure-theoretically isomorphic to the translation \(R_\beta\) on the two-dimensional torus \(\mathbb{T}^2\)

\[ R_\beta : \mathbb{T}^2 \to \mathbb{T}^2, \ x \mapsto x + (1/\beta, 1/\beta^2) \]

Pisot substitution conjecture Let \(\sigma\) be a Pisot irreducible substitution, then \((X_\sigma, S)\) has pure discrete spectrum
Quasi-crystals and Pisot dynamics

Bounded remainder set A set $X$ for which there exists $C > 0$ s.t. for all $N$

$$\left| \text{Card}\{0 \leq n \leq N; R^n_\alpha(0) \in X\} - N\mu(X) \right| \leq C$$

$\sigma$: $1 \mapsto 12$, $2 \mapsto 3$, $3 \mapsto 1$

Theorem The pieces of the Rauzy fractal are bounded remainder sets
Quasi-crystals and Pisot dynamics

Cut and project scheme: projection of a linear slicing through the lattice $\mathbb{Z}^d$ (cf. Y. Meyer's lecture)

$$\sigma: 1 \mapsto 12, \ 2 \mapsto 1$$

$$1 \mapsto \tilde{e}_1, \ 2 \mapsto \tilde{e}_2$$

The one-dimensional tiling associated with the infinite word $\sigma^\infty(1)$ is a regular model set.
The Tribonacci fractal as an acceptance window

Consider the Tribonacci substitution $\sigma: 1 \mapsto 12, \ 2 \mapsto 3, \ 3 \mapsto 1$
One represents $\sigma^\infty(1)$ as a broken line via

$$1 \mapsto \vec{e}_1, \ 2 \mapsto \vec{e}_2, \ 3 \mapsto \vec{e}_3$$

that we will be projected according to the eigenspaces of $M_\sigma$. 
The Tribonacci fractal as an acceptance window

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**Theorem** The one-dimensional tiling associated with the infinite word $\sigma^\infty(1)$ is a regular model set
Best approximations

Let \( \vec{v} \) be a vector.
The increasing sequence of positive integers \( (q_n) \) is said to be the sequence of best approximations of the vector \( \vec{v} \) for the norm \( || \cdot || \) if for each integer \( n \), one has

\[
||q_{n+1} \vec{v}|| < ||q_n \vec{v}||
\]

and for every \( q < q_{n+1}, q \neq q_n \), then

\[
||q_n \vec{v}|| < ||q \vec{v}||
\]
Tribonacci best approximations

Theorem [Chekhovaya-Hubert-Messaoudi]

\[ \beta^3 = \beta^2 + \beta + 1, \quad \beta > 1 \text{ is a Pisot number} \]

The Tribonacci sequence \((T_n)\)

\[ \forall n, \ T_{n+3} = T_{n+2} + T_{n+1} + T_n \]

is the sequence of best approximations of the vector \((1/\beta, 1/\beta^2)\) for the so-called Rauzy norm

Extended to the case of non-totally real Pisot numbers with property (F) [Hubert-Messaoudi]
Variations around Rauzy fractals

One can define **Rauzy fractals** for substitutions over

- Delone sets/cut-and-project schemes [Lee, Moody, Solomyak, Sing, Frettlöh, Baake etc.]
- trees [Bressaud, Jullian]
- on the free group [Arnoux, B., Hillion, Siegel, Coulbois]

and for **numeration dynamical systems** defined in terms of Pisot numbers

- beta-numeration [Thurston, Akiyama, Ei-Ito-Rao, B.-Siegel etc.]
- abstract numerations [B., Rigo]
- Shift Radix Systems [B., Siegel, Steiner, Surer, Thuswaldner]
Variations around Rauzy fractals

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and for numeration dynamical systems defined in terms of Pisot numbers
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and even
- in codimension 2 [Arnoux, Furukado, Harris, Ito]
- Pisot families [Akiyama-Lee, Barge-Stimac-Williams]
- nonalgebraic parameters \( \rightsquigarrow S\)-adic Rauzy fractals
Beyond the Pisot substitution conjecture
How to reach nonalgebraic parameters?

Theorem [Rauzy’82] \((X_\sigma, S)\) is measure-theoretically isomorphic to the translation \(R_\beta\) on the two-dimensional torus \(\mathbb{T}^2\)

\[
R_\beta : \mathbb{T}^2 \rightarrow \mathbb{T}^2, \ x \mapsto x + (1/\beta, 1/\beta^2)
\]

- We want to find symbolic realizations for toral translations
- We want to reach nonalgebraic parameters by considering convergent products of matrices
- We want to consider not only a substitution but a sequence of substitutions. Non-stationary dynamics
How to reach nonalgebraic parameters?

Theorem [Rauzy’82] \((X_\sigma, S)\) is measure-theoretically isomorphic to the translation \(R_\beta\) on the two-dimensional torus \(\mathbb{T}^2\):

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- We want to find symbolic realizations for toral translations
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\[\sim\text{ Multidimensional continued fractions algorithms}\]
Multidimensional continued fractions

There is no canonical generalization of continued fractions to higher dimensions.

Several approaches are possible:

- **best simultaneous approximations** but we then lose unimodularity, and the sequence of best approximations heavily depends on the chosen norm [Lagarias]
- **Klein polyhedra and sails** [Arnold]
- **unimodular** linear maps
  - Fibred systems, piecewise fractional maps e.g., Jacobi-Perron algorithm, Brun algorithm [Brentjes, Schweiger]
  - lattice reduction (LLL), [Lagarias],[Ferguson-Forcade], [Just], [Grabiner-Lagarias][Beukers]
Multidimensional Euclid’s algorithms: a zoo of algorithms

- **Jacobi-Perron**: we subtract the first one to the two other ones with \(0 \leq x_1, x_2 \leq x_3\)
  \[
  (x_1, x_2, x_3) \mapsto (x_2 - \left\lfloor \frac{x_2}{x_1} \right\rfloor x_1, x_3 - \left\lfloor \frac{x_3}{x_1} \right\rfloor x_1, x_1)
  \]

- **Brun**: we subtract the second largest and we reorder with \(x_1 \leq x_2 \leq x_3\)
  \[
  (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - x_2)
  \]

- **Poincaré**: we subtract the previous one and we reorder with \(x_1 \leq x_2 \leq x_3\)
  \[
  (x_1, x_2, x_3) \mapsto (x_1, x_2 - x_1, x_3 - x_2)
  \]

- **Selmer**: we subtract the smallest to the largest and we reorder with \(x_1 \leq x_2 \leq x_3\)
  \[
  (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - x_1)
  \]

- **Fully subtractive**: we subtract the smallest one to all the largest ones and we reorder with \(x_1 \leq x_2 \leq x_3\)
  \[
  (x_1, x_2, x_3) \mapsto (x_1, x_2 - x_1, x_3 - x_1)
  \]
Our strategy

- We apply a multidimensional continued fraction algorithm to the line in $\mathbb{R}^3$ directed by a given vector $u = (u_1, u_2, u_3)$.
- We then associate with the matrices produced by the algorithm substitutions, with these substitutions having the matrices produced by the continued fraction algorithm as incidence matrices.

\[ u = u_0 \leftarrow M_1 u_1 \leftarrow M_2 u_2 \leftarrow M_3 \ldots \leftarrow M_k u_k \]

\[ w = w_0 \leftarrow \sigma_1 w_1 \leftarrow \sigma_2 w_2 \leftarrow \sigma_3 \ldots \leftarrow \sigma_k w_k \in \{1, 2, 3\} \]

\[ u = M_1 \cdots M_k u_k \]
Applying Brun algorithm on \((7, 4, 6)\)

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}
\]

\(w_0 \mapsto 1\) \(1 \mapsto 1\) \(2 \mapsto 2\) \(3 \mapsto 13\)

\(w_1\) \(1 \mapsto 1\) \(2 \mapsto 23\) \(3 \mapsto 3\)

\(w_2\) \(1 \mapsto 1\) \(2 \mapsto 2\) \(3 \mapsto 223\)

\(w_3\) \(1 \mapsto 133\) \(2 \mapsto 2\) \(3 \mapsto 3\)

\(w_4\)
Applying Brun algorithm on \((7, 4, 6)\)

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}
\]

\((7, 4, 6) \mapsto (1, 4, 6) \mapsto (1, 4, 2) \mapsto (1, 0, 2) \mapsto (1, 0, 0)\)

\(w = w_0 = 12132131321321313\)
Applying Brun algorithm on (7, 4, 6)

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
2 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]

\[w_0 \rightarrow w_1 \rightarrow w_2 \rightarrow w_3 \rightarrow w_4\]

\[w = w_0 = 12132131321321313\]
Applying Brun algorithm on \((23, 45, 37)\)
S-adic expansions

Definition  An infinite word $\omega$ is said $S$-adic if there exist
  
  - a finite set of substitutions $S$
  
  - an infinite sequence of substitutions $(\sigma_n)_{n \geq 1}$ with values in $S$

  such that

  $$\omega = \lim_{n \to +\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(0)$$

  The terminology comes from Vershik adic transformations
Symbolic discrepancy

Take a sequence \((u_n)_n\) with values in a finite alphabet \(\mathcal{A}\).

The frequency \(f_i\) of a letter \(i\) in \(u = (u_n)_{n \in \mathbb{N}}\) is defined as the following limit, if it exists:

\[
f_i = \lim_{n \to \infty} \frac{|u_0 \cdots u_{N-1}|_i}{N}
\]

where \(|x|_j\) stands for the number of occurrences of the letter \(j\) in the factor \(x\).
Symbolic discrepancy

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The frequency \(f_i\) of a letter \(i\) in \(u = (u_n)_{n \in \mathbb{N}}\) is defined as the following limit, if it exists

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\]

where \(|x|_j\) stands for the number of occurrences of the letter \(j\) in the factor \(x\)

Assume that each letter \(i\) has frequency \(f_i\) in \(u\)

Symbolic discrepancy

\[
\Delta_N = \max_{i \in \mathcal{A}} |u_0 u_1 \cdots u_{N-1}|_i - N \cdot f_i
\]
Our program is to associate with any translation acting on $\mathbb{T}^d$ (i.e., with any line in $\mathbb{R}^d$)

- an $S$-adic sequence

\[ \omega = \lim_{n \to +\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(0) \]

- a Rauzy fractal whose coded trajectories correspond to the $S$-adic system, i.e.,
- with finite symbolic discrepancy
- provided by a multidimensional continued fraction algorithm (e.g. Brun algorithm)
S-adic expansions

\[ \omega = \lim_{n \to +\infty} \sigma_1 \sigma_2 \cdots \sigma_n(0) \]

**Algebraically Generalized Perron–Frobenius eigendirection**

One considers an infinite product of matrices

\[ M_1 \cdots M_n \cdots \]

with entries in \( \mathbb{N} \)

Does there exist a vector \( \vec{v} \) such that

\[ \bigcap_k M_1 \cdots M_k (\mathbb{R}^d_+) = \mathbb{R}_+ \vec{v} \]
S-adic expansions

$\omega = \lim_{n \to +\infty} \sigma_1 \sigma_2 \cdots \sigma_n(0)$

**Arithmetically Weak and strong convergence of multidimensional continued fraction algorithms**

**Theorem** There exists $\delta > 0$ s.t. for almost every $(\alpha, \beta)$, there exists $n_0 = n_0(\alpha, \beta)$ s.t. for all $n \geq n_0$

$$|\alpha - p_n/q_n| < \frac{1}{q_n^{1+\delta}}, \quad |\beta - r_n/q_n| < \frac{1}{q_n^{1+\delta}}$$

where $p_n, q_n, r_n$ are produced by Brun or by Jacobi-Perron algorithm

**Brun** [Ito-Fujita-Keane-Ohtsuki ’96]

**Jacobi-Perron** [Broise-Guivarc’h ’99]
Arnoux-Rauzy words

\[ \sigma_1 : 1 \mapsto 1 \quad \sigma_2 : 1 \mapsto 12 \quad \sigma_3 : 1 \mapsto 13 \]
\[ 2 \mapsto 21 \quad 2 \mapsto 2 \quad 2 \mapsto 23 \]
\[ 3 \mapsto 31 \quad 3 \mapsto 32 \quad 3 \mapsto 3 \]

\[ \omega = \lim_{n \to \infty} \sigma_{i_0} \sigma_{i_1} \cdots \sigma_{i_n}(1) \]

and every letter in \{1, 2, 3\} occurs infinitely often in \( (i_n)_{n \geq 0} \)

Example The Tribonacci substitution and its fixed point

- The set of the letter density vectors of AR words has zero measure [Arnoux-Starosta] and even topological Hausdorff dimension <2 [Avila-Hubert-Skripchenko]
Arnoux-Rauzy words

\[\sigma_1 : \begin{array}{c} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{array}, \quad \sigma_2 : \begin{array}{c} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{array}, \quad \sigma_3 : \begin{array}{c} 1 \mapsto 13 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{array}\]

\[\omega = \lim_{n \to \infty} \sigma_{i_0} \sigma_{i_1} \cdots \sigma_{i_n}(1)\]

and every letter in \{1, 2, 3\} occurs infinitely often in \((i_n)_{n \geq 0}\)

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\[ \omega = \lim_{n \to \infty} \sigma_{i_0} \sigma_{i_1} \cdots \sigma_{i_n}(1) \]

and every letter in \( \{1, 2, 3\} \) occurs infinitely often in \( (i_n)_{n \geq 0} \)

- The set of the letter density vectors of AR words has zero measure [Arnoux-Starosta] and even topological Hausdorff dimension <2 [Avila-Hubert-Skripchenko]
- There exist AR words that do not have bounded symbolic discrepancy [Cassaigne-Ferenczi-Messaoudi]
Lyapunov exponents for $S$-adic systems

- Let $S$ be a finite set of unimodular substitutions

  $\leadsto$ log-integrability

- Let $(D, S, \nu)$ with $D \subset S^\mathbb{N}$ be an ergodic subshift equipped with a probability measure $\nu$

  $S$ is the shift acting on $D$
  A subshift is a closed shift-invariant subset of sequences

- We consider the behaviour of the matrices $A_n(s) = M_{\sigma_0} \cdots M_{\sigma_n}$ for a generic $s = (\sigma_n) \in D$
Lyapunov exponents for $S$-adic systems

- Let $S$ be a finite set of unimodular substitutions.
- Let $(D, S, \nu)$ with $D \subset S^\mathbb{N}$ be an ergodic subshift equipped with a probability measure $\nu$.
- We consider the behaviour of the matrices $A_n(s) = M_{\sigma_0} \cdots M_{\sigma_n}$ for a generic $s = (\sigma_n) \in D$.

The Lyapunov exponents $\theta_1, \theta_2, \ldots, \theta_d$ of $(D, S, \nu)$ are recursively defined by the $\mu$-a.e. limit of

$$\theta_1 + \theta_2 + \cdots + \theta_k = \lim_{n \to \infty} \frac{1}{n} \log \| \wedge^k (M_{\sigma_0} \cdots M_{\sigma_{n-1}}) \|$$

where $\wedge^k$ denotes the $k$-fold wedge product.
The Lyapunov exponents $\theta_1, \theta_2, \ldots, \theta_d$ of $(D, S, \nu)$ are recursively defined by the $\mu$-a.e. limit of

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The $S$-adic system $(D, S, \nu)$ satisfies the Pisot condition if

$$\theta_1 > 0 > \theta_2 \geq \theta_3 \geq \cdots \geq \theta_d$$
The Lyapunov exponents $\theta_1, \theta_2, \ldots, \theta_d$ of $(D, S, \nu)$ are recursively defined by the $\mu$-a.e. limit of

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where $\wedge^k$ denotes the $k$-fold wedge product.

The $S$-adic system $(D, S, \nu)$ satisfies the Pisot condition if

$$\theta_1 > 0 > \theta_2 \geq \theta_3 \geq \cdots \geq \theta_d$$

**Theorem [Avila-Delecroix]**

- The Arnoux-Rauzy $S$-adic system is Pisot
- The Brun $S$-adic system is Pisot
Theorem [B.-Steiner-Thuswaldner]

- For almost every \((\alpha, \beta) \in [0, 1]^2\), the S-adic system associated with the Brun multidimensional continued fraction algorithm of \((\alpha, \beta)\) is measurably conjugate to the translation by \((\alpha, \beta)\) on the torus \(\mathbb{T}^2\).
- For almost every Arnoux-Rauzy word, the associated S-adic system has discrete spectrum.

Proof Based on

- “adic IFS”
- and finiteness results. Finite products of Brun/Arnoux-Rauzy substitutions have discrete spectrum [B.-Bourdon-Jolivet-Siegel]
**S-adic Pisot dynamics**

**Theorem [B.-Steiner-Thuswaldner]**
- For almost every \((\alpha, \beta) \in [0, 1]^2\), the S-adic system associated with the Brun multidimensional continued fraction algorithm of \((\alpha, \beta)\) is measurably conjugate to the translation by \((\alpha, \beta)\) on the torus \(\mathbb{T}^2\).
- For almost every Arnoux-Rauzy word, the associated S-adic system has discrete spectrum.

**Conjecture**  Every unimodular S-adic Pisot system is measure-theoretically conjugate to a toral translation.
An example

\( S = \{\sigma_1, \sigma_2, \sigma_3\} \)  
(Arnoux-Rauzy)

\( \varphi : 1 \mapsto 1123, \)  
\( 2 \mapsto 23, \)  
\( 3 \mapsto 123 \)  
(Chacon substitution)

\( s = \lim_{k \to \infty} \varphi^k(1) \)

\( \omega = \lim_{n \to \infty} \sigma_{s_0} \sigma_{s_1} \cdots \sigma_{s_n}(1) \)

2-balancedness  
[B., Cassaigne, Steiner]

Theorem The symbolic dynamical system generated by \( \omega \) is measure-theoretically isomorphic to a rotation on \( \mathbb{T}^2 \)