MULTIFRACTAL ANALYSIS AND WAVELETS

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Abstract. In this course, we give the basics of the part of multifractal theory that intersects wavelet theory. We start by characterizing the pointwise Hölder exponents by some decay rates of wavelet coefficients. Then, we give some examples of wavelet series having a multifractal behavior, and we explain how to build wavelet series with prescribed pointwise Hölder exponents. Next we develop the problematics of multifractal formalism, going from the intuitive formula by Frisch and Parisi to explicit and exploitable formulas. We prove that "multifractals are everywhere", in the sense that typical functions in Besov spaces or typical measures are multifractal in the sense of Baire's categories. We finish by some well-known examples of multifractal wavelet series, random and deterministic, focusing on the influence of certain adaptive threshold procedures to the multifractal properties of signals.

1. Introduction

In the context of functional analysis, multifractal analysis is concerned with the local regularity and the scaling behavior of functions: it is an attempt to describe the geometric and statistic distribution of the singularities of a function. One major motivation for going inside multifractal theory is that multifractal studies have direct connections with many mathematical fields (harmonic and functional analysis, probability theory and stochastic processes, dynamical systems and ergodic theory, geometric measure theory and even number theory), and simultaneously they have many natural application fields (physics, biology and physiology, amongst many other upcoming examples) based on the developments of new numerical procedures in signal and image processing.

In this course, I develop the basic facts about multifractal analysis of functions, the tools being mainly geometric measure theory and wavelets.

Let us start by recalling how the local regularity of a locally bounded function is quantified.

Definition 1.1. *Let $f \in L^\infty_{loc}(\mathbb{R}^d)$, and $x_0 \in \mathbb{R}^d$. Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$.*

The function f is said to belong to $C^\alpha(x_0)$ if there exist two positive constants $C > 0$, $M > 0$, a polynomial P with degree less than $\lfloor \alpha \rfloor$ (the integer part of α), such that when $|x - x_0| \le M$,

$$|f(x) - P(x - x_0)| \le C|x - x_0|^\alpha. \qquad (1)$$

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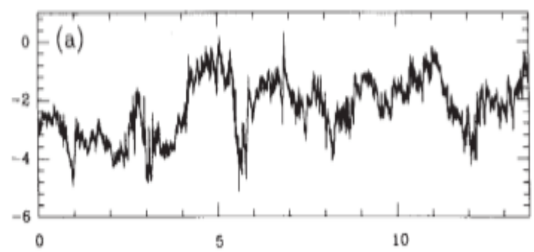

FIGURE 1. 1D-signal of the velocity of a turbulent fluid [18].

Then, the pointwise Hölder exponent of f at x *is*

$$h_f(x_0) = \sup\{\alpha \geq 0 : \ f \in C^\alpha(x_0)\}, \tag{2}$$

If $h_f(x_0) = h$, the point x_0 is called a singularity of order h for f.

Observe that when the exponent $h_f(x_0)$ is strictly less than 1, it takes a much simpler form:

$$h_f(x_0) = \liminf_{x \to x_0} \frac{\log |f(x) - f(x_0)|}{\log |x - x_0|},$$

where by convention $\log 0 = -\infty$.

Exercise 1.1. *Prove that the polynomial P in the definition of $C^s(x)$ is unique.*

Exercise 1.2. *Prove that if $s < s'$, $C^{s'}(x) \subset C^s(x)$.*

Exercise 1.3. *Let $f \in C^s(x)$, and call F a primitive of f. Prove that $F \in C^{s+1}(x)$. Build an example where $h_f(x) = s$ and $h_F(x) = s + 2$.*

Exercise 1.4. *Let $f \in C^s(x)$, with $s > 1$. Do one always have $f' \in C^{s-1}(x)$?*

This exponent $h_f(x)$ encapsulates significant information about the local behavior of f around x: the less its value is, the more irregular the graph of the function f locally looks like.

As can be seen on real data signals (see Figure 1), or as can be computed on "pure" mathematical functions, the pointwise Hölder exponent $h_f(x)$ can be very erratic when viewed as a function of x, even for functions very easy to define. The most popular example of function whose pointwise Hölder exponent $h_f(x)$ depends highly on x (in a non-continuous manner) is certainly the "non-differentiable Riemann function", i.e. the lacunary Fourier series

$$R(x) = \sum_{n \geq 1} \frac{\sin(n^2 \pi x)}{n^2}.$$

It took almost 140 years to complete the multifractal analysis of R, i.e. to compute the pointwise exponent of R at every x and to fully describe the

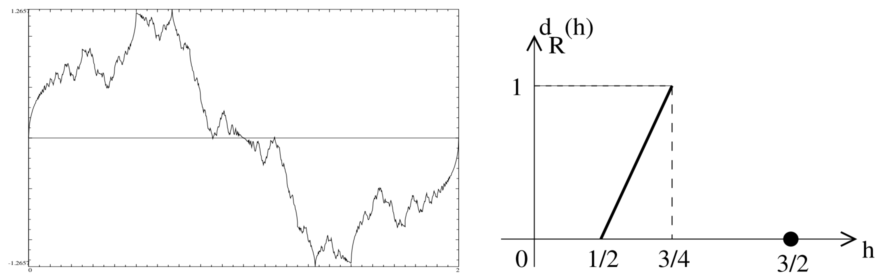

FIGURE 2. "Non-differentiable" Riemann function, and its multifractal spectrum.

geometric distribution of these singularities!! The graph of R is plotted Figure 2.

Even if one is able to compute the pointwise Hölder exponent of a function f at every point x, the knowledge of all these exponents does not necessarily give a concrete idea of what the graph of the function looks like, or of which the most significant singularities (or the most frequent ones) are. In order to describe the diversity of the local behaviors of f, one focuses on the iso-Hölder sets associated with the pointwise Hölder exponents.

Definition 1.2. *For every $h \in \mathbb{R}^+ \cup \{+\infty\}$, the iso-Hölder set $E_f(h)$ is the set*

$$E_f(h) = \{x \in \mathbb{R}^d : h_f(x) = h\}$$

of all singularities of pointwise Hölder exponent for f equal to h.

A single iso-Hölder set $E_f(h)$ may be concentrated around one region of $\mathbb{R}^d$, or spread all over the space. One thus needs a way to compare the sizes of the sets $E_f(h)$. It turns out that the right notion to distinguish them in the Hausdorff dimension, the main reason being the following: if one keeps in mind that the models we are interested in are built using procedures involving either random construction or dynamical systems, then our intuition (based on the law of large numbers or the Birkhoff ergodic theorem, depending on the context) makes us expect that there is a single value h_s such that Lebesgue-almost every point $x \in \mathbb{R}^d$ has a pointwise Hölder exponent h_s for f (the same value h_s for Lebesgue every x!). So the Lebesgue measure is not the appropriate tool to measure the size of the iso-Hölder sets, since one $E_f(h)$ will have full Lebesgue measure, and all the other ones will have measure 0. It is natural idea to compare their "fractal" dimension. Actually, "fractal" dimension does not exist, it is either box (also called Minkowski) dimension, Hausdorff dimension or less frequently packing dimension. It appears that for many natural functions or sample paths of stochastic processes, the sets $E_f(h)$ are fractal (whatever this means!) and often dense in the support of the corresponding function. Unfortunately, the box dimension gives full dimension (i.e. dimension d) to any dense set, so it does not distinguish them. This is one of the heuristic reasons that

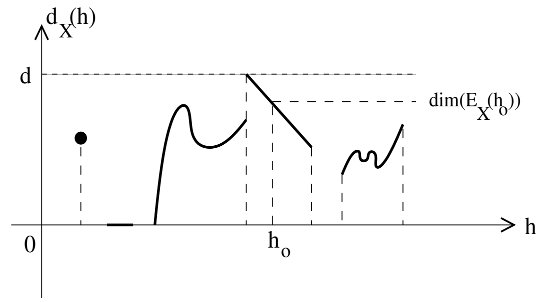

FIGURE 3. Example of multifractal spectrum

explains the choice of the Hausdorff dimension (there are other explanations based on theoretical results, as will be explained below), and leads to the definition of the main object of study of this course.

Definition 1.3. *The multifractal spectrum (also called the spectrum of singularities) of a locally bounded function* $f : \mathbb{R}^d \to \mathbb{R}$ *is the mapping* $d_f : \mathbb{R}^+ \cup \{+\infty\} \to [0, d] \cup \{-\infty\}$ *defined by*

$$h \longmapsto d_f(h) := \dim_{\mathcal{H}} E_f(h),$$

where $\dim_{\mathcal{H}}$ *stands for the Hausdorff dimension, and where by convention* $\dim_{\mathcal{H}} \emptyset = -\infty$.

The multifractal spectrum of a function contains informations regarding the geometric distribution of the singularities of f. Of course, at first sight this quantity may seem difficult to compute, even more to estimate on real signals or images. Indeed, before accessing to the value of $d_f(h)$, many limits are needed, and the successive approximations would lead to results that are certainly meaningless. This is where the notion of multifractal formalism arises. I do not develop it here, just indicating the main idea of it: for "homogeneous" functions (i.e. functions which at least statistically have the same scaling behavior in any region of $\mathbb{R}^d$), following some heuristics from turbulence and thermodynamic formalism, it is reasonable to expect that the multifractal spectrum should be a concave function and should satisfy an equality having the following form:

$$d_f(h) = \inf_{q \in \mathbb{R}} (qh - \zeta_f(q) + d), \tag{3}$$

where $\zeta_f(q)$ is some global quantity computed from f, called the scaling function associated with f. In this situation, the multifractal spectrum is thus obtained as the Legendre transform of the scaling function, hence leading to the concave shape for d_f that I mentioned before. For continuous functions, a possible definition for $\zeta_f(q)$ is

$$\zeta_f(q) := \sup \left\{ s > 0 : f \in B^{s/q,\infty}_{q,loc}(\mathbb{R}^d) \right\}.$$

Of course the precise value of the scaling function $\zeta_f(q)$ may depend on the context, nevertheless, in all cases, when formula (3) (or an analog of it) holds true for a function f and an exponent h, one says that **the multifractal formalism holds for f at h**. See Section 4 for all details.

The intuition that the multifractal formalism holds for nice models is supported by numerous representative examples: many stochastic processes (Lévy processes, wavelet series, ...) obey the multifractal formalism, as well as "typical" functions in many functional spaces (the set of monotone functions, Hölder and Besov spaces for instance). Moreover, even if the multifractal formalism does not hold, the Legendre spectrum

$$\zeta_f^*(h) := \inf_{q\in\mathbb{R}}(qh - \zeta_f(q) + d)$$

is meaningful since it encapsulates information about the histograms of oscillations or of wavelet coefficients associated with f. The key point is that the scaling function $\zeta_f(q)$ as we defined it just above, and thus its Legendre transform $\zeta_f^*(h)$, is accessible by numerical methods (using log-log diagrams for instance), while d_f is not. Hence, the Legendre spectrum $\zeta_f^*(h)$ is a quantity that can be estimated on every signal or image, and its form can be interpreted in terms of presence/relevance/density of the singularities of the object under consideration. The reader is referred to the course of P. Abry and S. Jaffard in the same volume to learn about efficient numerical procedures to estimate various scaling functions (see also [1, 2]).

Wavelets constitute a natural tool to study the multifractal nature of a function. For, there are two main reasons: the first one is that the pointwise Hölder exponent can be characterized by size estimates on the wavelet coefficients (see Theorem 3.1 below). The second one is that many functional spaces (Hölder and Besov spaces for instance) can also be characterized by decay rates of the wavelet coefficients. Also, the fact that a wavelet basis is self-similar by construction (all the functions $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$ are obtained through a translation and dilation of a same initial function ψ) is a priori an advantage to study "fractal"-like properties, but this could be discussed since it is self-similar with very specific ratio (powers of 2) while one aims at studying any irregular function. Anyway, wavelets are very important tools in this course, and some prior knowledge about their construction is advised, although we will only use their basic properties (vanishing moments, space and frequency localization).

The course is organized as follows. Section 2 contains the necessary materials for the rest of the course: wavelet coefficients, Hausdorff dimension and some geometric measure theory, local dimensions of measures. In Section 3, I prove the characterization of the pointwise Hölder exponent by size estimates of the wavelet coefficients, or by size estimates of the wavelet leaders. I also explain how to build a function with prescribed local regularity, and give some examples of multifractal wavelet series. In Section 4, I develop the intuitive notion of multifractal formalism, and then give some theoretical results on multifractals; for instance I explain how to obtain a priori upper bounds for multifractal spectra for Besov function and measures. In Section 5 it is proved that typical functions or measures (in the sense of Baire's category) in suitable functional spaces are multifractal. There, I use methods described in Section 2 to effectively compute the Hausdorff dimensions of the iso-Hölder sets of some functions. Finally Section 6 contains examples of multifractal functions built as wavelet series (the proofs are essentially written as long exercises).

2. Recalls on wavelets and geometric measure theory

2.1. **Wavelets.** I recall very briefly the basics of multiresolution wavelet analysis (for details see for instance [34, 15]). For an arbitrary integer $N \geq 1$ one can construct compactly supported functions $\Psi^0 \in C^N(\mathbb{R})$ (called the scaling function) and $\Psi^1 \in C^N(\mathbb{R})$ (called the mother wavelet), with Ψ^1 having at least $N+1$ vanishing moments (i.e. $\int_{\mathbb{R}} x^n \Psi^1(x)dx = 0$ for $n \in \{0, \dots, N\}$), and such that the set of functions

$$\Psi^1_{j,k} : x \mapsto \Psi^1(2^j x - k)$$

for $j \in \mathbb{Z}, k \in \mathbb{Z}$ form an orthogonal basis of $L^2(\mathbb{R})$ (note that we choose the L^∞ normalization, not L^2). In this case, the wavelet is said to be N-regular.

Let us introduce the notations

$$0^d := (0, 0, \cdots, 0), \qquad 1^d := (1, 1, \cdots, 1), \qquad L^d = \{0, 1\}^d \backslash 0^d.$$

An orthogonal basis of $L^2(\mathbb{R}^d)$ is then obtained by tensorization. For every $\lambda = (j, \mathbf{k}, \mathbf{l}) \in \mathbb{Z} \times \mathbb{Z}^d \times L^d$, let us define the tensorized wavelet

$$\Psi_\lambda(x) = \Psi^{\mathbf{l}}(2^j x - \mathbf{k}) := \prod_{i=1}^d \Psi^{l_i}_{j,k_i}(x_i),$$

with obvious notations: $\mathbf{k} = (k_1, k_2, \cdots, k_d)$ and $\mathbf{l} = (l_1, l_2, \cdots, l_d)$.

Any function $f \in L^2(\mathbb{R}^d)$ can be written (the equality being true in $L^2(\mathbb{R}^d)$)

$$f(x) = \sum_{\lambda = (j,\mathbf{k},\mathbf{l}) : \, j \in \mathbb{Z}, \, \mathbf{k} \in \mathbb{Z}^d, \, \mathbf{l} \in L^d} d_\lambda \Psi_\lambda(x), \tag{4}$$

where

$$d_\lambda = d_\lambda(f) := 2^{jd} \int_{R^d} f(x) \Psi_\lambda(x) \, dx. \tag{5}$$

It is implicit in (5) that the wavelet coefficients depend on f. Observe that in the wavelet decomposition (4), no wavelet Ψ_λ such that $\mathbf{l} = 0^d$ (where $\lambda = (j, \mathbf{k}, \mathbf{l})$) appears.

Assumption: We always assume that the wavelet has a number of vanishing moments larger than the index of regularity that we are looking at. Typically, if we focus on singularities on order h, we assume that Ψ^1 has at least $\lfloor h \rfloor + 1$ vanishing moments.

The reason for this assumption is that wavelets with enough vanishing moments can be used to characterize Hölder functions.

Theorem 2.1. *For $s \in \mathbb{R}_+ \setminus \mathbb{N}$, a function f belongs to $C^s(\mathbb{R}^d)$ if and only if there exists a constant $C > 0$ such that*

$$\forall \lambda \in \mathbb{Z} \times \mathbb{Z}^d \times \{0,1\}^d \qquad |d_\lambda(f)| \leq C 2^{-js}. \tag{6}$$

I do not prove Theorem 2.1 here, it is a good exercise, the proof of which can be achieved by adapting the proof of Theorem 3.1 given later.

Wavelets can also be used to characterize functions in a Besov space, see Section 4.3.

2.2. **Localization of the problem.** We are interested in the local behavior of functions, hence when focusing on a point $x_0 \in \mathbb{R}^d$, what happens far from x_0 should not interfere with our questions. Moreover, again because we concentrate on local phenomena, the low frequency terms have no importance in our analysis. This is why we focus only on functions supported by $[0,1]^d$, and when we deal with wavelets, we assume that the function we deal with have a wavelet decomposition like

$$f = \sum_{\substack{\lambda=(j,\mathbf{k},\mathbf{l}):\\ j\geq 0,\, \mathbf{k}2^{-j}\in[0,1]^d,\, \mathbf{l}\in L^d}} d_\lambda \Psi_\lambda(x). \tag{7}$$

2.3. **Hausdorff and box dimension.** Two notions of dimensions of sets in $\mathbb{R}^d$ will be used below: the Hausdorff dimension and the upper box dimension. We recall them quickly.

Let X be a bounded set in $\mathbb{R}^d$. For every $\varepsilon > 0$, denote by $N_\varepsilon(X)$ the minimal number of balls of diameter ε needed to fully cover the set X. The lower box dimension of X, denoted by $\underline{\dim}_B(X)$, is then the real number $\in [0,d]$ defined as

$$\underline{\dim}_B(X) = \liminf_{\varepsilon \to 0^+} \frac{\log N_\varepsilon(X)}{-\log \varepsilon}. \tag{8}$$

Similarly, the upper box dimension of X is

$$\overline{\dim}_B(X) = \limsup_{\varepsilon \to 0^+} \frac{\log N_\varepsilon(X)}{-\log \varepsilon}. \tag{9}$$

Exercise 2.1. *Prove that $N_\varepsilon(X)$ is well-defined, and that $\overline{\dim}_B(X) \leq d$.*

Exercise 2.2. *Build a set X such that $\underline{\dim}_B(X) < \overline{\dim}_B(X)$*

When $\underline{\dim}_B(X) = \overline{\dim}_B(X)$, one denotes by $\dim_B(X)$ their common value.

Exercise 2.3. *Prove that:*

(1) *the box dimension of an open set is d.*
(2) *the box dimension of the triadic Cantor set is $\log 2 / \log 3$.*

(3) *the box dimension of a set X dense in $[0,1]^d$ is d.*

I also recall the definition of the Hausdorff dimension.

Definition 2.1. *Let $s \geq 0$. The s-dimensional Hausdorff measure of a set X, $\mathcal{H}^s(X)$, is defined as*

$$\mathcal{H}^s(X) = \lim_{r \searrow 0} \mathcal{H}^s_r(X), \quad \text{with } \mathcal{H}^s_r(X) = \inf \Big\{ \sum_i |X_i|^s \Big\},$$

the infimum being taken over all the countable families of sets X_i such that $|X_i| \leq r$ and $X \subset \bigcup_i X_i$. Then, the Hausdorff dimension of X, $\dim_{\mathcal{H}} X$, is defined as

$$\dim_{\mathcal{H}} X = \inf\{s \geq 0 : \mathcal{H}^s(X) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(X) = +\infty\}.$$

It is a good exercise to prove that for any bounded set $E \subset \mathbb{R}^d$, we have

$$0 \leq \dim_{\mathcal{H}}(E) \leq \underline{\dim}_B(E) \leq d.$$

In order to find an upper bound for the Hausdorff dimension of X, the most common method is the following: First guess what the dimension should be, call δ this expected value. Then, fix $s > \delta$, and find a covering $(X_i)_{i\in\mathbb{N}}$ of X such that

$$\mathcal{H}^s \left(\bigcup_{i\in\mathbb{N}} X_i \right) < +\infty.$$

This implies, by the definition of $\dim_{\mathcal{H}} X$, that $s \geq \dim_{\mathcal{H}} X$. This being true for any $s > \delta$, one deduces that $\delta \geq \dim_{\mathcal{H}} X$, hence the upper bound.

Obtaining a lower bound is in most cases much more difficult. Let us mention the mass distribution principle, on which most methods are based.

Theorem 2.2. *Let $X \subset \mathbb{R}^d$ be a borelian set, and assume that there exists a positive finite measure μ supported by X satisfying the following scaling property: there exists a positive real number $s > 0$ and a constant $C > 0$ such that*

$$\text{for any } x \in X, \text{ for any } 0 < r < 1, \quad \mu(B(x,r)) \leq Cr^s.$$

Then $\mathcal{H}^s(X) > \frac{\mu(X)}{C}$, and thus $\dim_{\mathcal{H}} X \geq s$.

Exercise 2.4. *Prove that the Hausdorff dimension of the triadic Cantor set is* $\log 2/\log 3$ (Hint: apply Theorem 2.2 with the uniform measure on the Cantor set).

Another theorem that often allows one to compute Hausdorff dimension of iso-Hölder sets in multifractal analysis is the following theorem by Beresnevich and Velani [11]:

Theorem 2.3. *Let $(x_n)_{n\geq 1}$ be a sequence of points in $[0,1]^d$, and let $(l_n)_{n\geq 1}$ be a positive non-increasing sequence of radii. If*

$$\mathcal{L}^d\Big(\limsup_{n\to+\infty} B(x_n, l_n)\Big) = \mathcal{L}^d\Big([0,1]^d\Big),$$

($\mathcal{L}^d$ is the d-dimensional Lebesgue measure), then for every $\xi > 1$, one has

$$\mathcal{H}^{d/\xi}\Big(\limsup_{n\to+\infty} B(x_n, (l_n)^\xi)\Big) = +\infty.$$

Exercise 2.5. *Let ξ_x be the approximation rate of an irrational number $x \in [0,1]$ by the dyadic numbers, defined by*

$$\xi_x = \sup\{\xi \geq 0 : |x - k2^{-j}| \leq 2^{-j\xi} \text{ for i.m. couples } (j,k),\ j\geq 1,\ k \text{ odd}\}.$$

1. *Prove that for every irrational number $x \in [0,1]$, $\xi_x \geq 1$.*
2. *Let $S_\xi = \{x : |x - k2^{-j}| \leq 2^{-j\xi}$ for i.m. couples (j,k), $j \geq 1$, k odd$\}$ and $\widetilde{S}_\xi = \{x : \xi_x = \xi\}$. Prove that*

$$\widetilde{S}_\xi = \left(\bigcap_{\xi'<\xi} S_{\xi'}\right) \setminus \left(\bigcup_{\xi'>\xi} S_{\xi'}\right).$$

3. *Prove that for $\xi \geq 1$, $\dim_{\mathcal{H}}(S_\xi) \leq 1/\xi$ and $\dim_{\mathcal{H}}(\widetilde{S}_\xi) \leq 1/\xi$.*
4. *Prove that for $\xi > 1$, $\mathcal{H}^{1/\xi}(S_\xi) = \mathcal{H}^{1/\xi}(\widetilde{S}_\xi) = +\infty$.*
5. *Deduce the value of the Hausdorff dimension of S_ξ and $\widetilde{S}_\xi$, for every $\xi \geq 1$.*

Exercise 2.6. *Let ξ_x be the Diophantine approximation rate of an irrational number $x \in [0,1]$ by the rationals, defined by*

$$\xi_x = \sup\{\xi \geq 0 : |x - p/q| \leq q^{-2\xi} \text{ for infinitely many } q \geq 1 \text{ and } p \text{ with } p \wedge q = 1\}.$$

1. *Prove Dirichlet's theorem: for every irrational number $x \in [0,1]$, $\xi_x \geq 1$. (Hint: Use a counting argument).*
2. *Let $S_\xi = \{x : |x - p/q| \leq q^{-2\xi}$ for infinitely many $q \geq 1$ and p with $p \wedge q = 1\}$ and $\widetilde{S}_\xi = \{x : \xi_x = \xi\}$. Prove that*

$$\widetilde{S}_\xi = \left(\bigcap_{\xi'<\xi} S_{\xi'}\right) \setminus \left(\bigcup_{\xi'>\xi} S_{\xi'}\right).$$

3. *Prove that for $\xi \geq 1$, $\dim_{\mathcal{H}}(S_\xi) \leq 1/\xi$ and $\dim_{\mathcal{H}}(\widetilde{S}_\xi) \leq 1/\xi$.*
4. *Prove that for $\xi > 1$, $\mathcal{H}^{1/\xi}(S_\xi) = \mathcal{H}^{1/\xi}(\widetilde{S}_\xi) = +\infty$.*
5. *Deduce the value of the Hausdorff dimension of S_ξ and $\widetilde{S}_\xi$, for every $\xi \geq 1$.*

Other methods will be probably used hereafter, I will mention them along the proofs.

2.4. **Local dimensions of measures.** Recall that the support of a Borel positive measure, denoted by $\text{Supp}(\mu)$, is the smallest closed set $E \in \mathbb{R}^d$ such that $\mu(\mathbb{R}^d \setminus E) = 0$.

Definition 2.2. *Let μ be a positive measure supported on $\mathbb{R}^d$ at $x_0 \in$ Supp(μ). The (lower) local dimension $h_\mu(x_0)$ (also called local Hölder exponent) is*

$$h_\mu(x_0) = \liminf_{r \to 0^+} \frac{\log \mu(B(x_0, r))}{\log r}, \tag{10}$$

where $B(x_0, r)$ denotes the open ball with center x_0 and radius r. When $x_0 \notin$ Supp(μ), by convention we set $h_\mu(x_0) = +\infty$.

Of course, in $\mathbb{R}$, there is a correspondence between the local dimension of a measure and the pointwise Hölder exponent of its primitive $F(x) = \int_0^x d\mu$.

Exercise 2.7. *Prove that if $h_\mu(x_0) \notin \mathbb{N}$, then $h_F(x_0) = h_\mu(x_0)$. Is the converse true? (Hint: consider the Lebesgue measure).*

Iso-Hölder sets and multifractal spectrum are quantities that can be defined for measures using the same ideas: One sets

$$E_\mu(h) = \{x \in \mathbb{R}^d : h_\mu(x) = h\}$$

and

$$d_\mu : h \longmapsto d_\mu(h) := \dim_{\mathcal{H}} E_\mu(h).$$

Contrarily to what happens for measures, there is a strong constraint valid for all measures: for every $h \geq 0$, for every positive Borel measure on $\mathbb{R}^d$, one has (see next Sections)

$$d_\mu(h) \leq \min(h, d).$$

This is one major difference between measures and functions from the multifractal standpoint (essentially due to the fact that measures have bounded variations).

2.5. **Legendre transform.** The Legendre transform appears in many places in analysis, I recall the properties that are needed in the following.

Definition 2.3. *Let $L : \mathbb{R} \to \mathbb{R}$ be a concave increasing function. The Legendre transform of L is the mapping $L^* : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ defined by*

$$h \mapsto L^*(h) := \inf_{q \in \mathbb{R}} \big(qh - L(q)\big).$$

The assumption that L is increasing is not necessary for the definition of the Legendre transform, but it will be the case in our context in the following. In our cases, the function L satisfies $L(0) < 0$, and in this case one shall keep in mind the following properties:

- The support of L^* is included in the smallest interval containing $L'(\mathbb{R})$, the extreme points may or may not belong to the support, depending on L.

- L^* is concave on its support.
- If $h = L'(q)$ (i.e. L is differentiable at q), then $L^*(h) = qh - L(q)$.
- When $L'(0^+)$ exists, the Legendre transform L^* reaches its maximum at $h = L'(0^+)$, and $L^*(L'(0^+)) = -L(0)$.
- L^* is increasing on the interval when $h \leq L'(0^+)$, and is decreasing when $h \geq L'(0^+)$ (again, the extreme points may not belong to the support).

I draw the attention of the reader that L is not necessarily continuously differentiable, and that difficulties may appear to find the precise range of real numbers h such that $L^*(h) \geq 0$. These problems occur in many contexts, too numerous to list them in details now.

Exercise 2.8. *Prove each of the preceding items.*

3. Pointwise Hölder exponent

3.1. **Characterization by decay rate of wavelet coefficients.** Recall the Definition 2 of the pointwise Hölder exponent of a locally bounded function f at a point $x_0 \in \mathbb{R}^d$. The definition of $h_f(x)$ involves some functional spaces $C^s(x)$, which can be (almost) characterized by the decay rate of the coefficients located around x_0, as stated by the next theorem of Jaffard [23].

Theorem 3.1. *Let $s \in \mathbb{R}^+ \setminus \mathbb{N}$, and let $f \in L^2(\mathbb{R}^d)$.*

Assume that f belongs to $C^s(x_0)$. Then, there exist two constants $M > 0$ and $C > 0$ such that for every $\lambda = (j, \mathbf{k}, \mathbf{l})$ such that $j \geq 0$, $|x_0 - \mathbf{k}2^{-j}| \leq M$, and for every $\mathbf{l} \in L^d$, one has

$$|d_\lambda| \leq C(2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^s. \tag{11}$$

Reciprocally, if the wavelet coefficients of a function $f \in \bigcap_{\varepsilon>0} C^\varepsilon([0,1]^d)$ satisfies (11), then $f \in C^s_{\log}(x_0)$.

Recall that $f \in C^s_{\log}(x_0)$ when locally around x_0, there exists a polynomial P of degree less than $\lfloor s \rfloor$ such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^s |\log|x - x_0||. \tag{12}$$

As usual, the symbol $A \lesssim B$ means that inequality $A \leq CB$ holds for some constant C independent of the parameters involved in the formula.

Proof. We start by the direct implication. Assume that $f \in C^s(x_0)$, and let us call P the unique polynomial such that if $|x - x_0| \leq M$ (for some constant M), (1) holds true.

Fix $\lambda = (j, \mathbf{k}, \mathbf{l})$ such that $j \geq 0$ and

$$|x_0 - \mathbf{k}2^{-j}| \leq \widetilde{M} := M/2. \tag{13}$$

One has

$$d_\lambda = 2^{dj} \int_{\mathbb{R}^d} f(x)\Psi_\lambda(x)dx = 2^{dj} \int_{\mathbb{R}^d} (f(x) - P(x - x_0))\Psi_\lambda(x)dx,$$

where we used the vanishing moments up to the order $\lfloor s \rfloor$ to introduce the polynomial in the integral. Then,

$$\begin{aligned} |d_\lambda| &\le 2^{dj} \int_{|x-x_0|\le M} |f(x) - P(x-x_0)||\Psi_\lambda(x)|dx \\ &+2^{dj} \int_{|x-x_0|\ge M} |f(x)||\Psi_\lambda(x)|dx \\ &+2^{dj} \int_{|x-x_0|\ge M} |P(x-x_0)||\Psi_\lambda(x)|dx. \end{aligned}$$

Let us call I_M, J_M and K_M the last three terms. Using (1), the first term is bounded above by

$$\begin{aligned} I_M &\lesssim 2^{dj} \int_{|x-x_0|\le M} |x-x_0|^s |\Psi^{\mathbf{1}}(2^j x - \mathbf{k})|dx \\ &\lesssim \int_{|u|\le M2^j} |2^{-j}(u+\mathbf{k}) - x_0|^s |\Psi^{\mathbf{1}}(u)|du. \end{aligned}$$

Since each Ψ_λ is continuous and compactly supported, say, with support included in $[-M', M']^d$, it is uniformly bounded (independently of λ, due to the choice of the L^∞-normalization for the wavelet's family) and one gets

$$\begin{aligned} I_M &\lesssim \int_{[-M',M']^d} |2^{-j}(u+\mathbf{k}) - x_0|^s du \\ &\lesssim \int_{[-M',M']^d} |2^{-j}u|^s + |x_0 - \mathbf{k}2^{-j}|^s du \\ &\lesssim (2^{-js} + |x_0 - \mathbf{k}2^{-j}|^s) \lesssim (2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^s, \end{aligned}$$

where we used the double-sided inequality $(x+y)^s \le 2^s(x^s+y^s) \le 2^{s+1}(x+y)^s$.

Let us now treat the second term. By the Cauchy-Schwarz inequality and using that $f \in L^2(\mathbb{R}^d)$, one obtains

$$\begin{aligned} J_M &\lesssim 2^{dj} \left(\int_{|x-x_0|\ge M} |f(x)|^2 dx \right)^{1/2} \left(\int_{|x-x_0|\ge M} |\Psi_\lambda(x)|^2 dx \right)^{1/2} \\ &\lesssim 2^{dj} \left(\int_{|x-x_0|\ge M} |\Psi^{\mathbf{1}}(2^j x - \mathbf{k})|^2 dx \right)^{1/2} \\ &\lesssim 2^{dj/2} \left(\int_{|2^{-j}(u+\mathbf{k})-x_0|\ge M} |\Psi^{\mathbf{1}}(u)|^2 du \right)^{1/2}. \end{aligned} \tag{14}$$

Observe that our choice (13) for λ imposes that

$$\{u : |2^{-j}(u+\mathbf{k}) - x_0| \ge M\} \subset \{u : |u| \ge \widetilde{M}2^j\}. \tag{15}$$

The wavelets Ψ^0 and Ψ^1 being compactly supported, (15) tells us that the integral in (14) is 0 for j large enough.

The third term is treated almost similarly. The polynomial P being of degree at most $\lfloor s \rfloor$, one can write

$$\begin{aligned} K_M &\lesssim 2^{dj} \left(\int_{|x-x_0|\geq M} \left(\frac{|P(x-x_0)|}{1+|x-x_0|^{s+d+2}} \right)^2 dx \right)^{1/2} \\ &\quad \times \left(\int_{|x-x_0|\geq M} (1+|x-x_0|^{s+d+2})^2 |\Psi_\lambda(x)|^2 dx \right)^{1/2} \\ &\lesssim 2^{dj} \left\| \frac{|P(.)|}{1+|.|^{s+d+2}} \right\|_{L^2(\mathbb{R}^d)} \\ &\quad \times \left(\int_{|x-x_0|\geq M} (1+|x-x_0|^{s+d+2})^2 |\Psi^{\mathbf{l}}(2^j x - \mathbf{k})|^2 dx \right)^{1/2} \\ &\lesssim 2^{dj/2} \left(\int_{|u|\geq \widetilde{M} 2^j} (1+ \ |2^{-j}(u+\mathbf{k}) - x_0|^{s+d+2})^2 |\Psi^{\mathbf{l}}(u)|^2 du \right)^{1/2}. \end{aligned}$$

where (15) has been used. Again, the last integral is zero when j becomes large. Hence the first assertion.

Exercise 3.1. *Prove that the same holds when the wavelets are not compactly supported (Hint: use their rapid decay at infinity).*

Let us move to the reciprocal, which is more delicate to handle with.

Assume that (11) holds for every $\lambda = (j, \mathbf{k}, \mathbf{l})$ such that $j \geq 0$, $|x_0 - \mathbf{k}2^{-j}| \leq M$, and $\mathbf{l} \in L^d$.

We start from the decomposition (7). Since each Ψ_λ is at least $C^{\lfloor s \rfloor +1}(\mathbb{R}^d)$, this is also true for every function f_j defined as the sum over each fixed generation j of the wavelet coefficients of f, i.e.

$$f_j(x) = \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d,\, \mathbf{l}\in L^d} d_\lambda \Psi_\lambda(x), \tag{16}$$

The partial derivatives of f_j are: for every $\alpha := (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ such that $|\alpha| := \alpha_1 + ... + \alpha_d \leq \lfloor s \rfloor + 1$, one has

$$\partial^\alpha f_j(x) = \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d,\, \mathbf{l}\in L^d} d_\lambda \, \partial^\alpha \Psi_\lambda(x),$$

Each partial derivative of Ψ^0 and Ψ^1 is compactly supported, hence they satisfy the inequalities for all $v \in \mathbb{R}^d$

$$|\partial^\alpha \Psi^{\mathbf{l}}(v)| \leq \frac{C}{1+|v|^{2s+2d+4}} \tag{17}$$

where the constant C is uniform in α ranging in the set of indices such that $|\alpha| \leq \lfloor s \rfloor + 1$. Since $\Psi_\lambda(x) = \Psi^{\mathbf{l}}(2^j x - \mathbf{k})$, the last upper bound yields

$$|\partial^\alpha \Psi_\lambda(x)| \leq \frac{C 2^{j|\alpha|}}{1 + |2^j x - \mathbf{k}|^{2s+2d+4}}.$$

From this we deduce that

$$|\partial^\alpha f_j(x)| \leq \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d,\, \mathbf{l}\in L^d} |d_\lambda| \frac{C 2^{j|\alpha|}}{1 + |2^j x - \mathbf{k}|^{2s+2d+4}},$$

Observe now that, up to a modification of the constant C, (11) is also true for $\lambda = (j, \mathbf{k}, \mathbf{l})$ such that $j \geq 0$, $|x_0 - \mathbf{k}2^{-j}| \leq 1$ (not only for $|x_0 - \mathbf{k}2^{-j}| \leq M$), since the sequence of the wavelet coefficients of f are necessarily bounded by $\|f\|_{L^2}$, thus when for all λ such that $|x_0 - \mathbf{k}2^{-j}| \geq M$, one has $|d_\lambda| \leq \|f\|_{L^2} \lesssim |x_0 - \mathbf{k}2^{-j}|^s$ (with uniform constants). This yields the upper bound

$$\begin{aligned}
|\partial^\alpha f_j(x)| &\lesssim \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d,\, \mathbf{l}\in L^d} \frac{2^{j|\alpha|}(2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^s}{1 + |2^j x - \mathbf{k}|^{2s+2d+4}} \\
&\lesssim \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d,\, \mathbf{l}\in L^d} \frac{2^{j|\alpha|}(2^{-js} + |x - x_0|^s + |x - \mathbf{k}2^{-j}|^s)}{1 + |2^j x - \mathbf{k}|^{2s+2d+4}}.
\end{aligned}$$

The first two terms in the last sum are independent of $\mathbf{k}$, and thus the corresponding sums are bounded above by $2^{j|\alpha|}(2^{-js} + |x - x_0|^s)$. It is easy to see that the last one satisfies

$$\sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\ \mathbf{k}2^{-j}\in[0,1]^d,\, \mathbf{l}\in L^d} \frac{2^{j|\alpha|}|x - \mathbf{k}2^{-j}|^s}{1 + |2^j x - \mathbf{k}|^{2s+2d+4}} \lesssim 2^{-j(s+d+2)}.$$

We finally get the inequality

$$|\partial^\alpha f_j(x)| \lesssim 2^{j|\alpha|}(2^{-js} + |x - x_0|^s). \tag{18}$$

Since the wavelets are compactly supported, it is easily seen that f_j has same uniform Hölder regularity as Ψ^0 and Ψ^1, i.e. f_j belongs at least to $C^{\lfloor s \rfloor + 1}(\mathbb{R}^d)$. Using the Taylor polynomial P_j of order $\lfloor s \rfloor$ of each f_j, one has

$$\begin{aligned}
|f_j(x) - P_j(x - x_0)| &\leq |x - x_0|^{\lfloor s \rfloor + 1} \sup_{|\alpha| = \lfloor s \rfloor + 1} |\partial^\alpha f_j(x)| \\
&\lesssim |x - x_0|^{\lfloor s \rfloor + 1} 2^{j(\lfloor s \rfloor + 1)} (2^{-js} + |x - x_0|^s). \qquad (19)
\end{aligned}$$

It is time now to construct the polynomial associated with f. Obviously one has the decomposition $f(x) = \sum_{j \geq 0} f_j(x)$, hence it is natural to consider the polynomial $P = \sum_{j \geq 0} P_j$ as potential candidate. From the above estimates one easily sees that this polynomial is well defined.

We fix some x close to x_0. Let us call j_0 the unique integer such that

$$2^{-j_0} \leq |x - x_0| < 2^{-j_0+1}. \tag{20}$$

Recall that f is supposed to belong to $C^\eta(\mathbb{R}^d)$, for some $\eta > 0$. We then introduce the integer

$$j_1 = \lfloor j_0 \frac{s}{\eta} \rfloor > j_0, \tag{21}$$

where we can assume that $j_1 > j_0$ since η can be taken as small as we want. It remains us to bound above the difference $|f(x) - P(x - x_0)|$ by the desired quantity, i.e. $|x - x_0|^s |\log |x - x_0||$. Let us split this quantity into four terms, depending on the generations of the associated wavelet decomposition. More precisely,

$$\begin{aligned} f(x) - P(x - x_0) &= \sum_{j=0}^{j_0} (f_j(x) - P_j(x - x_0)) + \sum_{j=j_0+1}^{j_1} f_j(x) \\ &\quad + \sum_{j \geq j_1+1} f_j(x) - \sum_{j=j_0+1}^{+\infty} P_j(x - x_0). \end{aligned}$$

We call S_1, S_2, S_3 and S_4 the four sums above.
First, we have by (19)

$$\begin{aligned} |S_1| &\lesssim \sum_{j=0}^{j_0} |f_j(x) - P_j(x - x_0)| \\ &\lesssim \sum_{j=0}^{j_0} |x - x_0|^{\lfloor s \rfloor + 1} 2^{j(\lfloor s \rfloor + 1)} (2^{-js} + |x - x_0|^s) \\ &\lesssim |x - x_0|^{\lfloor s \rfloor + 1} \sum_{j=0}^{j_0} \left(2^{j(\lfloor s \rfloor + 1 - s)} + 2^{j(\lfloor s \rfloor + 1)} |x - x_0|^s \right) \\ &\lesssim |x - x_0|^{\lfloor s \rfloor + 1} (2^{j_0(\lfloor s \rfloor + 1 - s)} + 2^{j_0(\lfloor s \rfloor + 1)} |x - x_0|^s) \\ &\lesssim |x - x_0|^s, \end{aligned}$$

where the last "miracle" follows from (20). Then, by (18) with $\alpha = 0^d$,

$$\begin{aligned} |S_2| &\lesssim \sum_{j=j_0+1}^{j_1} |f_j(x)| \lesssim \sum_{j=j_0+1}^{j_1} (2^{-js} + |x - x_0|^s) \\ &\lesssim (j_1 - j_0)(2^{-j_0 s} + |x - x_0|^s) \lesssim |x - x_0|^s |\log |x - x_0||, \end{aligned}$$

since $j_1 - j_0 \sim j_0(1 - s/\eta) \sim \log |x - x_0||$ by (13). Further, recalling that the function $f \in C^\eta(\mathbb{R}^d)$, the wavelet coefficients of f_j satisfy $|d_\lambda| \lesssim 2^{-j\eta}$, one sees that $\|f_j\|_\infty \lesssim 2^{-j\eta}$ (here the assumption that the wavelets are compactly supported makes the computations easier). Hence,

$$|S_3| \lesssim \sum_{j=j_1+1}^{+\infty} |f_j(x)| \lesssim \sum_{j=j_1+1}^{+\infty} 2^{-j\eta} \lesssim 2^{-j_1 \eta} \lesssim 2^{-j_0 s} \lesssim |x - x_0|^s.$$

Finally, each polynomial P_j has the form

$$P_j(x-x_0) = \sum_{n=0}^{\lfloor s \rfloor} \sum_{|\alpha|=n} c_\alpha \partial^\alpha f_j(x_0)(x-x_0)^n,$$

for some universal coefficients c_α. Hence it can be bounded above as follows using (18)

$$\begin{aligned} |P_j(x-x_0)| &\lesssim \sum_{n=0}^{\lfloor s \rfloor} \sum_{|\alpha|=n} c_\alpha |\partial^\alpha f_j(x_0)||x-x_0|^n \\ &\lesssim \sum_{n=0}^{\lfloor s \rfloor} 2^{jn}|x-x_0|^n(2^{-js}+|x-x_0|^s) \\ &\lesssim 2^{j\lfloor s \rfloor}|x-x_0|^{\lfloor s \rfloor}(2^{-js}+|x-x_0|^s), \end{aligned}$$

where we used that $j \geq j_0$, implying $2^j|x-x_0| > 1$. Finally,

$$|S_4| \lesssim \sum_{j=j_0+1}^{+\infty} |P_j(x)| \lesssim \sum_{j=j_0+1}^{+\infty} 2^{j\lfloor s \rfloor}|x-x_0|^{\lfloor s \rfloor}(2^{-js}+|x-x_0|^s) \lesssim |x-x_0|^s.$$

This concludes the proof. □

Let us end this section with an important remark: Theorem 3.1 tells us that to find the value of $h_f(x_0)$, it is not enough to look at the wavelet coefficients that lie inside the "cone of influence" around x_0, i.e. the λ such that $|\mathbf{k}2^{-j} - x_0| \leq M2^{-j}$. The cone of influence contains the wavelet coefficients whose value is influenced by the value of f at x, and one may believe that they are the only ones that play a role in the value of the pointwise Hölder exponent of f at x. In fact, when the largest coefficients are located within the cone of influence of x_0, x_0 is a cusp.

But it may happen that coefficients located very far from the cone of influence are the most important ones, in the sense the inequality (11) is saturated for the coefficients. Actually, when (11) is saturated for wavelet coefficients d_λ satisfying $|x-x_0| \sim 2^{-j\rho}$ for some $\rho < 1$, one can prove that x_0 is an oscillating singularity with singularity exponent $1/\rho - 1$ (see the next sections for more details). So it is definitely not enough to concentrate on the cone of influence, especially when building local regularity algorithms. This is also one main motivation for introducing wavelet leaders.

Exercise 3.2. *Construct a wavelet series in $\mathbb{R}$ such that all its wavelet coefficients are either 0 or equal to $2^{-j\alpha}$ and such that $h_f(0) = 2\alpha$, $h_f(x) = \alpha$ if $x \neq 0$.*

Is is possible to have $h_f(x) = 2\alpha$ if $x \in \mathbb{Q}$, $h_f(x) = \alpha$ if $x \notin \mathbb{R} \setminus \mathbb{Q}$? What if one inverses the values α and 2α?

Exercise 3.3. *Prove that, under the same assumption, it is enough for the reconstruction part to assume that* (11) *holds only for those wavelet* *coefficients* d_λ *such that the corresponding* $\lambda = (j, \mathbf{k}, \mathbf{l})$ *satisfies* $|x_0 - \mathbf{k}2^{-j}| \leq 2^{-j/\log j}$.

Exercise 3.4. *Consider the continuous wavelet transform defined for* $a > 0$ *and* $b \in \mathbb{R}$ *and for a* L^2 *function* $f : \mathbb{R}^d \to \mathbb{R}$ *by*

$$W_f(b, a) = \frac{1}{a^{d/2}} \int_{\mathbb{R}^d} f(t)\psi\left(\frac{t-b}{a}\right) dt.$$

Prove an analog of Theorem 3.1 for this wavelet transform.

Exercise 3.5. *Consider the Riemann series*

$$R(x) = \sum_{n \geq 1} \frac{\sin(\pi n^2 x)}{n^2}.$$

and the wavelet $\psi(x) = \frac{1}{(x+i)^2}$.

(1) *Compute the continuous wavelet transform of* R*, and relate it to the Theta Jacobi function* $\Theta(z) = \sum_{n \in Z} e^{i\pi n^2 z}$*. (Hint: use the a residue formula.)*
(2) *Prove that* R *is at least* $C^{1/2}(x)$ *at every* x *(difficult).*
(3) *To learn more about* R*, see* [21, 20, 26, 13, 35].

Exercise 3.6. *Let* $0 < H < 1$*, and consider the Weierstrass function*

$$W_H(x) = \sum_{n \geq 1} 2^{-nH} \sin(2^n x).$$

(1) *Prove that* $W_H \in C^H(\mathbb{R})$ *(Hint: directly prove that* $|W_H(x) - W_H(y)| \leq C|x-y|^H$*.)*
(2) *Using a suitable wavelet* ψ *(for instance assuming that its Fourier transform* $\widehat{\psi}$ *has support in* $[1/2, 2]$*), prove that for every* $x \in \mathbb{R}$*,* $h_{W_H}(x) = H$*. Hence,* W_H *is monofractal.*

3.2. **Characterization by decay rate of wavelet leaders.** Wavelet leaders are a theoretical tool introduced by S. Jaffard in [29] essentially for numerical reasons. The main idea comes from the fact that in multifractal analysis (see next Section for the details), it is natural to consider sums of wavelet coefficients like

$$\sum_{\lambda : |\lambda| = j} |d_\lambda|^q$$

for a varying parameter $q \in \mathbb{R}$. In particular, as we will explain, the behavior of the sum when $j \to +\infty$ for $q < 0$ is related to the decreasing part of the multifractal spectrum of functions. It is thus natural to try to estimate the values of such sums. Unfortunately numerical experiments show that this quantity is extremely unstable due to the presence of small wavelet coefficients, which, when they are taken to a negative power, can be extremely large. Wavelet leaders have been thought to stabilize these sums, and they are in fact related to multifractal analysis of capacities [31].

Definition 3.1. *For every $\lambda = (j, \mathbf{k}, \mathbf{l})$, one defines the dyadic cube I_λ associated with λ by*

$$I_\lambda = [\mathbf{k}2^{-j}, \mathbf{k}2^{-j}) := [k_1 2^{-j}, k_1 2^{-j} + 2^{-j}) \times ... \times [k_d 2^{-j}, k_d 2^{-j} + 2^{-j}),$$

where $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{Z}^d$.

Let f be a function of the form (7)*. For every $\lambda = (j, \mathbf{k}, \mathbf{l})$ with $j \geq 0$ such that $\mathbf{k}2^{-j} \in [0,1]^d$, one defines the wavelet leader D_λ by*

$$D_\lambda = \sup \left\{ |d_{\lambda'}| : I_{\lambda'} \subset \bigcup_{i \in \{-1,0,1\}^d} I_\lambda + i 2^{-j} \right\}.$$

In other words, the wavelet leader D_λ is in fact the maximal value (in absolute value) amongst all the wavelet coefficients $d_{\lambda'}$ such that the corresponding cube $I_{\lambda'}$ lies inside I_λ or inside one of its $3^d - 1$ immediate neighbors.

Exercise 3.7. *Prove that for every $f \in L^2$ each wavelet leader is a maximum (not only a supremum).*

It is immediate that if $I_{\lambda'} \subset I_\lambda$, $D_{\lambda'} \leq D_\lambda$. Hence, instead of having wavelet coefficients that may be sparse, we end up with leader coefficients that enjoy a nice decreasing property (the set of wavelet leaders forms a *capacity* as a function of the dyadic cubes, i.e. a decreasing set function on the dyadic wavelet tree). Multifractal analysis of capacities has been studied in [31, 5] for instance.

Definition 3.2. *For every $x_0 \in [0,1]^d$ and $j \geq 0$, let us denote by $\lambda_j(x_0)$ the unique cube (up to the value of $\mathbf{l} \in L^d$) such that $x_0 \in \lambda$ with $|\lambda| = j$, and we set*

$$D_j(x_0) = D_{\lambda_j(x_0)} \quad \text{and} \quad I_j(x_0) = I_{\lambda_j(x_0)}.$$

We also set $\lambda_j(x_0) = (j, \mathbf{k}_j(x_0), \mathbf{l})$ (the index $\mathbf{l}$ has no importance here, only the location matters).

The main theorem relating wavelet leaders and pointwise regularity is the following.

Theorem 3.2. *Let f be locally bounded of the form* (7)*. Then*

$$h_f(x_0) = \liminf_{j \to +\infty} \frac{\log D_j(x_0)}{\log 2^{-j}}, \tag{22}$$

where $\log 0 = -\infty$ by convention.

Proof. The proof is rather quick and is based on Theorem 3.1. Let $h := h_f(x_0)$.

Let $\varepsilon > 0$. Inequality (11) implies that for large j, all wavelet coefficients around x_0 satisfy

$$|d_\lambda| \leq C(2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^{h-\varepsilon}.$$

Let $J \geq 0$ and $\lambda = (J, \mathbf{K}, \mathbf{L})$ be such that $|\mathbf{K}2^{-J} - x_0| \leq M$ (the constant M being the one such that (11) holds).

Let $\lambda' = (j, \mathbf{k}, \mathbf{l})$ be such that $I_{\lambda'} \subset I_\lambda + i2^{-J}$, for some $i \in \{-1, 0, 1\}^d$. Obviously one has $|\mathbf{k}2^{-j} - x_0| \leq 2 \cdot 2^{-J}$, thus

$$|d_{\lambda'}| \lesssim (2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^{h-\varepsilon}| \lesssim 2^{-J(h-\varepsilon)}.$$

We deduce that $|D_\lambda| \leq 2^{-J(h-\varepsilon)}$, and thus that

$$\liminf_{j\to+\infty} \frac{\log D_j(x_0)}{\log 2^{-j}} \geq h - \varepsilon.$$

Letting ε go to zero gives one inequality in (22).

Moving to the converse inequality, we know that (11) must be saturated for some coefficients. Let $\varepsilon > 0$, and consider one coefficient $\lambda = (j, \mathbf{k}, \mathbf{l})$ such that

$$|d_\lambda| \geq (2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^{h+\varepsilon}.$$

There are infinitely many such coefficients.

Let J be the unique integer such that $2^{-J-1} \leq |x_0 - \mathbf{k}2^{-j}| < 2^{-J}$. Then, by construction, $I_\lambda \subset I_J(x_0) + i2^{-J}$ for some $i \in \{-1, 0, 1\}^d$. This yields that

$$D_J(x_0) \geq |d_\lambda| \geq (2^{-j} + |x_0 - \mathbf{k}2^{-j}|)^{h+\varepsilon} \geq 2^{-J(h+2\varepsilon)}.$$

Taking liming of both sides, we get

$$\liminf_{j\to+\infty} \frac{\log D_j(x_0)}{\log 2^{-j}} \leq h + \varepsilon,$$

and the result follows when ε tends to zero. □

Exercise 3.8. *Prove that there must exist (an infinite number of) λ such that $D_\lambda = |d_\lambda|$.*

3.3. **Prescription of Hölder exponents.** As said in the introduction, the exponent mapping $x \mapsto h_f(x)$ of a locally bounded function may not be regular, and one may wonder what form this mapping can take. It is also natural, for practical purposes, to try to build functions with prescribed Hölder regularity. This problem is completely solved.

Exercise 3.9. *Let g be a strictly positive and continuous function. Build a function f such that its pointwise Hölder exponents are exactly $h_f(x) = g(x)$ at every x (Hint: modify the Weierstrass function W_H introduced in Exercise 3.6).*

Proposition 3.1. *Let $f \in C^\eta(\mathbb{R}^d)$ for some $\eta > 0$. Then the mapping $x \mapsto h_f(x)$ is the liminf of a sequence of continuous functions.*

Reciprocally, if $(g_n)_{n\geq1}$ is a sequence of continuous functions satisfying $g_n \geq \eta$, then there exists a function $f \in C^\eta(\mathbb{R}^d)$ such that $h_f(x) = \liminf_{n\to+\infty} g_n(x)$.

Proof. Let us start by remarking that any function f has the same pointwise Hölder exponents everywhere as the sum $f+g$ where g is the wavelet series whose wavelet coefficients are all equal to $\pm 2^{-j^2}$ for $j \geq 0$ (there is no need to precise how the signs are chosen). Hence, up to a modification that does not affect the pointwise Hölder coefficients, one may assume that the wavelet coefficients of f satisfy $|d_\lambda| \geq 2^{-j^2}$ for every $j \geq 2$.

Then, (11) implies that

$$h_f(x) = \liminf_{j\to+\infty, |\lambda|=j} \frac{\log|d_\lambda|}{\log(2^{-j}+|x-\mathbf{k}2^{-j}|)}.$$

Let us denote by g_λ the map $x \mapsto \frac{\log|d_\lambda|}{\log(2^{-j}+|x-\mathbf{k}2^{-j}|)}$. It is obviously continuous with respect to x. Hence, $h_f(x)$ is indeed the liminf of a sequence of continuous functions.

Reciprocally, consider a sequence $(g_n)_{n\geq 1}$ of continuous functions greater than $\eta > 0$. We work only on the cube $[0,1]^d$, the extension to $\mathbb{R}^d$ is immediate by concatenation. We build iteratively a wavelet series by the following method.

Let us first start by remarking that we can assume that each function g_n is C^1. Otherwise we replace g_n by any C^1 function $\tilde{g}_n$ such that $|\|g_n - \tilde{g}_n\|_\infty \leq 1/2^n$. Then it is obvious that

$$g(x) = \liminf_{n\to+\infty} g_n(x) = \liminf_{n\to+\infty} \tilde{g}_n(x).$$

We first construct a sequence J_n as follows.

Fix $J_0 = 1$, and assume that J_n is found. To find J_{n+1}, consider g_{n+1}. By uniform continuity, there exists $\tilde{J}_{n+1}$ such that $|x-y| \leq 2^{-\tilde{J}_{n+1}}$ implies $|g_{n+1}(x) - g_{n+1}(y)| \leq 2^{-(n+1)}$. We also assume that

$$2^{-J_n \inf\{g_n(x):x\in[0,1]^d\}} \geq 2^{-\tilde{J}_{n+1}(\sup\{g_{n+1}(x):x\in[0,1]^d\})}.$$

Finally, we choose J_{n+1} as the integer

$$J_{n+1} = \max(J_n + n, \tilde{J}_{n+1}, \sup\{|\nabla g_{n+1}(x)| : x \in [0,1]\}). \tag{23}$$

Then, we prescribe the wavelet coefficients d_λ for all $\lambda = (j, \mathbf{k}, \mathbf{l})$ as follows: if $J_n < j < J_{n+1}$, then $d_\lambda = 0$, and if $j = J_n$ for some $n \geq 1$, one sets

$$d_\lambda = 2^{-jg_n(\mathbf{k}2^{-j})}.$$

Fix $x_0 \in [0,1]^d$, and recall the Definition 3.2 of $\lambda_j(x_0)$ and $\mathbf{k}_j(x_0)$. It is a trivial matter to check that our construction implies that for every J_n, for the wavelet leader $D_{J_n}(x_0)$,

$$2^{-J_n/2^n} \leq \frac{D_{J_n}(x_0)}{2^{-J_n g_n(\mathbf{k}2^{-J_n})}} \leq 2^{J_n/2^n}.$$

By (22), one has $h_f(x_0) \leq \liminf_{n\to+\infty} g_n(\mathbf{k}_{J_n}(x_0)2^{-J_n})$. But our construction implies that for every n, $|g_n(\mathbf{k}_{J_n}(x_0)2^{-J_n}) - g_n(x_0)| \leq 1/2^n$, hence

$$\liminf_{n\to+\infty} g_n(\mathbf{k}_{J_n}(x_0)2^{-J_n}) = \liminf_{n\to+\infty} g_n(x_0),$$

from which we deduce that $h_f(x_0) \leq g(x_0)$.

Conversely, by Exercise 3.3, it is enough to consider those wavelet coefficients around x_0 that satisfy $|x_0 - \mathbf{k}2^{-J_n}| \leq 2^{-J_n/\log J_n}$. One sees that for such a coefficient $\lambda = (J_n, \mathbf{k}, \mathbf{l})$,

$$|d_\lambda| = 2^{-J_n g_n(\mathbf{k}2^{-J_n})} \leq 2^{-J_n\left(g_n(x_0) - \sup\{|\nabla g_n(x)|: x\in[0,1]\}\cdot|x_0 - \mathbf{k}2^{-J_n}|\right)}.$$

But our construction implies that

$$0 \leq \sup\{|\nabla g_{n+1}(x)| : x \in [0,1]\} \cdot |x_0 - \mathbf{k}2^{-J_n}| \leq J_n 2^{-J_n/\log J_n},$$

which tends to zero when n tends to infinity. This yields

$$|d_\lambda| \leq 2 \cdot 2^{-J_n g_n(x_0)}$$

for n large. In particular, $h_f(x_0) \geq \liminf_{n\to+\infty} g_n(x_0) = g(x_0)$. This gives the converse inequality. □

Exercise 3.10. *Extend last Proposition to functions that are only continuous, or only with bounded variations.*

I finish this section by drawing the attention of the reader that what we have achieved for functions is not known for measures: one does not know what the possible forms of the local dimension map of a measure are like. The situation is much more complicated, as proved by next exercise [12].

Exercise 3.11. *Consider the local dimension mapping $x \mapsto h_\mu(x)$ associated with a probability measure μ on $\mathbb{R}^d$. Prove that if it is continuous on an open set Ω, then it is constant and equal to d on Ω.*

3.4. **Other exponents.** The pointwise Hölder exponent does not fully describe the local behavior of a continuous function. For instance, it does not reflect the local oscillatory behavior: the functions $f_1(x) = |x|^{1/4}$ and $f_2(x) = |x|^{1/4}\sin(|x|^{-1})$ have the same exponent 1/4 at 0, but they exhibit obviously a different behavior.

There are many other local regularity exponents that allows one to distinguish functions with the same pointwise Hölder exponent. Let us mention two of them.

Definition 3.3. *The local Hölder exponent of f at x_0 is defined as*

$$h^l_f(x_0) = \lim_{\varepsilon\to0} \sup\{\alpha \geq 0 : f \in C^\alpha(B(x_0,\varepsilon))\}, \tag{24}$$

where $B(x_0,\varepsilon)$ stands for the ball (using any norm) centered at x_0 of radius ε.

Exercise 3.12. *Prove that the formula (24) makes sense, and that the value does not depend on the choice of the norm.*

The local Hölder exponent is always lower than the pointwise Hölder exponent (Exercise: prove it!). This other exponent is often used when studying local regularity of stochastic processes, for which it is often difficult to obtain results that are valid almost surely for all points (while it is often easy to get an exact value for every point almost surely). For instance, for a multifractional Brownian motion (see [30, 10] for definitions), one can compute almost surely the value of every $h^l_f(x)$, and sometimes the value of $h_f(x)$ is not known (only for every x almost surely, not almost surely for every x: nevertheless under some conditions the pointwise Hölder exponent is known everywhere almost surely).

Another exponent encapsulates the oscillatory behavior of a function.

Definition 3.4. *For every $\varepsilon > 0$, let f^ε be a fractional primitive of f of order ε. Then the oscillating exponent of f at x_0 is defined as*

$$\beta_f(x_0) = \lim_{\varepsilon \to 0} \left(\frac{\partial h_{f^\varepsilon}(x_0)}{\partial \varepsilon} \right)_{|\varepsilon=0} - 1. \tag{25}$$

Recall that a fractional primitive f^ε of order ε of, say, a L^2 function can be defined via the formula

$$f^\varepsilon(x) = (-\Delta)^{\varepsilon/2}(f)(x),$$

or via its Fourier transform by

$$\widehat{f^\varepsilon}(\xi) = \frac{\hat{f}(\xi)}{(1+|\xi|^2)^{\varepsilon/2}}.$$

Exercise 3.13. *Prove that for every $\varepsilon > 0$, $h_{f^\varepsilon}(x_0) \geq h_f(x_0) + \varepsilon$. Deduce that the formula* (25) *makes sense.*

As an example, it is quite easy to see that for the functions f_1 and f_2 introduced at the beginning of this section, $\beta_{f_1}(0) = 0$ while $\beta_{f_2}(0) = 1$. The term $\sin(|x|^{-1})$ is responsible for the value $\beta_{f_2}(0) = 1$, and one can prove that the oscillatory content is indeed contained in $\beta_f(x)$.

When $\beta_f(x) > 0$, x is called an *oscillating singularity* of f. It is also often referred to as a *chirp*, on the opposite to the case where $\beta_f(x) = 0$, where the singularity is called a *cusp*.

Detecting oscillatory singularities is an important issue in signal processing, one knows that many phenomena occur only on such points (for instance, dissipation of energy in turbulent fluids may be due to this kind of singularities).

3.5. **An example.** Let μ be a positive Borel probability measure on $[0,1]^d$. Let us construct the wavelet series F_μ by prescribing its wavelet coefficients as follows: for every λ, we set

$$d_\lambda = \mu(I_\lambda). \tag{26}$$

Assume that the measure is uniformly regular, in the sense that there exists a constant $C > 0$ and an exponent $h_{\min} > 0$ such that for every ball with center x and radius $0 < r < 1$, one has

$$\mu(B(x,r)) \leq C r^{h_{\min}}.$$

These assumptions can be weakened.

Proposition 3.2. *Under the assumption above, the wavelet series F_μ converges, $F_\mu \in C^{h_{\min}}(\mathbb{R}^d)$, and for every $x \in [0,1]^d$, one has*

$$h_{F_\mu}(x) = h_\mu(x).$$

In particular, $d_\mu \equiv d_F$.

Proof. I let the proof as an exercise. The idea is essentially to prove that there is a universal constant $C > 1$ such that for every $x_0 \in \mathrm{Supp}(\mu)$, for every $j \geq 0$, one has

$$C^{-1} \leq \frac{D_j(x_0)}{\mu(B(x_0, 2^{-j}))} \leq C.$$

□

Exercise 3.14. *Let (ξ_λ) be a family of i.i.d random variables with common law the normal Gaussian law. Consider the (random) wavelet series $\tilde{F}$ whose wavelet coefficients are*

$$d_\lambda = \mu(I_\lambda)\xi_\lambda.$$

This is a random modification of (26) *and of F.*

(1) *Prove that, almost surely, for every x, $h_F(x) = h_{\tilde{F}}(x)$, and thus $d_F \equiv d_{\tilde{F}}$.*
(2) *Can one weaken the i.i.d. assumption on the random coefficients? (the answer is yes, but to what extend...)*

Exercise 3.15. *What happens if* (26) *is replaced by*

$$d_\lambda = 2^{-j\alpha}\mu(I_\lambda)^\beta$$

for some $\alpha, \beta > 0$?

4. Multifractal formalism

4.1. **The intuition of U. Frisch and G. Parisi.** Multifractal analysis and formalism for functions were introduced by physicists in order to interpret some experimental observations related to Kolmogorov's theory of fully developed turbulence. Since the 1940's, Kolmogorov emphasized the role in fluid mechanics played by the scaling function associated with the fluid's velocity, defined as follows. Let $v(x)$ be the velocity at time t and position x of a turbulent fluid contained in a bounded domain Ω. For every $q \in \mathbb{R}$, one studies the q-th moment of v defined by

$$S(q, l) = \int_\Omega |v(x + l) - v(x)|^q dx.$$

In his K41 model, Kolmogorov models the small fluctuations of the velocity by a fractional Brownian motion with Hurst exponent $H = 1/3$, for which one can prove $S(q, l)$ enjoys a nice scaling behavior of the form, for every $q \geq 0$,

$$S(q, l) \sim |l|^{qH} \quad \text{when } |l| \text{ tends to } 0.$$

But very quickly, some experiments showed that in reality

$$S(q,l) \sim |l|^{\zeta(q)} \quad \text{when } |l| \text{ tends to } 0, \tag{27}$$

where the mapping $q \longmapsto \zeta(q)$, called the *scaling function* of the velocity, is a strictly concave, increasing function. This has been definitely confirmed by experiments that took place at the ONERA in Modane by Y. Gagne [18] (see Figure 1 for the one-dimensional trace of the 3D velocity of a turbulent fluid).

Uriel Frisch and Georgio Parisi had this insightful idea that the non-linearity of the scaling fonction shall be a conséquence of the multifractality of the velocity, i.e. the fact that there are different pointwise Hölder exponents occurring at different places, the corresponding iso-Hölder sets $E_v(h) = \{x \in \Omega : h_v(h) = h\}$ having non-zero Hausdorff dimension, whose value depends on h.

Their heuristics was the following: Assume that there are many exponents h such that their associated iso-Hölder set $E_v(h)$ is non-empty, with Hausdorff dimension $\dim E_v(h) = d_v(h) > 0$. Intuitively, around every point $x \in \mathbb{R}^3$ with $h_v(x) = h$, one has

$$|v(x+l) - v(x)| \sim |l|^h.$$

Since $\dim E_v(h) = d_v(h) > 0$, there are approximately $|l|^{-d_v(h)}$ cubes of size length $|l|$ (hence, of volume $|l|^3$) that contain points x whose exponent is h. Hence,

$$S(q,l) = \int_\Omega |v(x+l) - v(x)|^q dx \sim \int_h |l|^{qh} |l|^{-d_v(h)} |l|^3 dh \sim \int_h |l|^{qh-d_v(h)+3} dh.$$

When $l \to 0$, the most important contribution in the integral comes from the smallest possible value for the exponent $qh - d_v(h) + 3$. Combining this with (27), one deduces that

$$\zeta(q) = \inf_h \, (qh - d_v(h) + 3).$$

This expression is a Legendre transform, which explains a priori the concavity of the scaling function ζ. Moreover, by inverse Legendre transform, one gets

$$d_v(h) = \inf_{q \in \mathbb{R}} \, (qh - \zeta(q) + 3), \tag{28}$$

which suggests us that the multifractal spectrum of v should also have a concave shape.

The remarkable, and surprising, point is that despite the successive approximations made along this proof, the formula (28) (or resembling formulas) holds true for many mathematical objects, from self-similar functions and measures to generic functions. In fact, as soon as the function enjoys some nice scaling properties, this kind of formula is expected to hold.

Definition 4.1. *When a formula like* (28) *holds true, one says that* the multifractal formalism is true for the function f at the exponent h.

The definition is volontarily imprecise, since the right formulation for the scaling function and for the range of q's ($q \in \mathbb{R}$, or $q \in \mathbb{R}^+$, ...) may depend on the context (the support of the function, the functional space, ...).

It is important at this point to emphasize once again that this multifractal formalism is the main reason for the use of multifractals in applications. Indeed, as said in the introduction, it is useless to try to estimate directly the multifractal spectrum of a signal or an image, too many limits are involved. Nevertheless, when the object under consideration enjoys some specific scaling properties (deterministic or statistical self-similarity, independence or stationarity of increments,...), it is natural to look for a multifractal formalism-like formula involving a scaling function $\zeta(q)$, which is hopefully easy to estimate numerically.

4.2. **A serious formulation of the multifractal formalism.** It is possible to give an effective meaning to the multifractal formalism in many contexts. The easiest one is obtained through a scaling function associated with the wavelet leaders.

Definition 4.2. *For every $q \in \mathbb{R}$, one considers the leader scaling function of the function f defined by*

$$L_f(q) = \liminf_{j \to +\infty} \frac{1}{-j} \log_2 \left(\sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\mathbf{k}2^{-j} \in [0,1]^d} |D_\lambda|^q \right).$$

From this value one deduces an upper bound a priori for the multifractal spectrum of f.

Theorem 4.1. *For every function $f \in C^\eta(\mathbb{R}^d)$ for some $\eta > 0$, one has*

$$d_f(h) \leq L_f^*(h) := \inf_{q \in \mathbb{R}} (qh - L_f(q)).$$

L_f^* is called the Leader Legendre Spectrum of f.

Actually we will prove a much stronger result:

- for every $h \leq L^*(L'(0^+))$, i.e. in the increasing part of the Leader Legendre spectrum,

$$\dim_{\mathcal{H}} \{x : h_f(x) \leq h\} \leq L_f^*(h), \tag{29}$$

- on the decreasing part $h \geq L^*(L'(0^+))$, one has

$$\dim_{\mathcal{H}} \{x : \overline{h_f}(x) \geq h\} \leq L_f^*(h), \tag{30}$$

  where $\overline{h_f(x)}$ is the limsup exponent defined by

$$\overline{h_f}(x) = \limsup_{j \to +\infty} \frac{\log D_j(x_0)}{\log 2^{-j}}. \tag{31}$$

From (29) and (30) one easily deduces Theorem 4.1, since in the increasing part

$d_f(h) = \dim_{\mathcal{H}} E_f(h) = \dim_{\mathcal{H}}\{x \in [0,1] : h_f(x) = h\} \leq \dim_{\mathcal{H}}\{x : h_f(x) \leq h\}$,

and in the decreasing part, one has

$d_f(h) = \dim_{\mathcal{H}} E_f(h) = \dim_{\mathcal{H}}\{x \in [0,1] : h_f(x) = h\} \leq \dim_{\mathcal{H}}\{x : \overline{h_f}(x) \geq h\}$.

Proof. This is actually a standard proof coming from large deviations theory, which is only based on formula (22)

$$h_f(x_0) = \liminf_{j\to+\infty} \frac{\log D_j(x_0)}{\log 2^{-j}},$$

and on a counting argument.

From the recalls on the Legendre transform in Section 2.5, one knows that L_f^* reaches its maximum at $L_f^*(L'(0^+))$. It is obvious that this maximum is equal to $L_f(0) = d$.

Let $h \leq L_f^*(L'(0^+))$. We are going to prove (29). In that case, the maximal value of $L^*(h)$ is reached for a positive value of q.

Let $\varepsilon > 0$ be small.

Since $f \in C^\eta(\mathbb{R}^d)$, it is enough to consider $h \geq \eta$, and the set $\widetilde{E}_f(h) = \{x \in [0,1]^d : h_f(x) \leq h\}$. For every x in this set, by (22), there exists an infinite number of generations j such that

$$2^{-j(h+\varepsilon)} \leq D_j(x_0). \tag{32}$$

Let us denote by $N_j(h,\varepsilon)$ the number of wavelet leaders D_λ of generation j such that (32) holds for D_λ (instead of $D_j(x_0)$). From Definition 4.2 of L_f, there exists a generation J_ε such that for every $j \geq J_\varepsilon$,

$$\sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\mathbf{k}2^{-j}\in[0,1]^d} |D_\lambda|^q \leq 2^{-j(L_f(q)-\varepsilon)}.$$

One deduces that when $q > 0$,

$$2^{-j(L_f(q)-\varepsilon)} \geq \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):\mathbf{k}2^{-j}\in[0,1]^d} |D_\lambda|^q \geq N_j(h,\varepsilon)2^{-qj(h+\varepsilon)}.$$

In particular,

$$N_j(h,\varepsilon) \leq 2^{j(qh-L_f(q)+\varepsilon(1+q))}.$$

From (32), the set $\widetilde{E}_f(h)$ is included in

$$\widetilde{E}_f(h) \subset \bigcap_{J\geq 1} \bigcup_{j\geq J} \bigcup_{\lambda=(j,\mathbf{k},\mathbf{l}):D_\lambda\geq 2^{-j(h+\varepsilon)}} B(\mathbf{k}2^{-j}, 2.2^{-j}).$$

Hence a covering of $\widetilde{E}_f(h)$ by sets of diameter less than $\delta > 0$ is given by the union

$$\bigcup_{j\geq J} \bigcup_{\lambda=(j,\mathbf{k},\mathbf{l}):D_\lambda\geq 2^{-j(h+\varepsilon)}} B(\mathbf{k}2^{-j}, 2.2^{-j}),$$

where J is such that $4.2^{-J} \le \delta$. Let $s > qh - L_f(q) + \varepsilon(1+q)$. We use this covering to bound from above the $\mathcal{H}^s_\eta$-Hausdorff pre-measure of $\widetilde{E}_f(h)$ as follows:

$$\begin{aligned}\mathcal{H}^s_\eta(\widetilde{E}_f(h)) &\le \sum_{j\ge J} \sum_{\lambda=(j,\mathbf{k},\mathbf{l}):D_\lambda \ge 2^{-j(h+\varepsilon)}} \left|B(\mathbf{k}2^{-j}, 2.2^{-j})\right|^s \\ &\lesssim \sum_{j\ge J} N_j(h,\varepsilon) 2^{-js} \le \sum_{j\ge J} 2^{j(qh - L_f(q) + \varepsilon(1+q) - s)},\end{aligned}$$

which is finite by our choice of s. Hence, $\dim_{\mathcal{H}} \widetilde{E}_f(h) \le s$, and letting s tend to $qh - L_f(q) + \varepsilon(1+q)$ and then ε to zero, one deduces that

$$\dim_{\mathcal{H}} \widetilde{E}_f(h) \le qh - L_f(q).$$

This holds true for every $q > 0$, hence

$$\dim_{\mathcal{H}} \widetilde{E}_f(h) \le \inf_{q\ge 0} qh - L_f(q).$$

Finally. as said above, the positive q's are the only one that matter in the range $h \le L^*_f(L'(0^+))$, hence (29).

Inequality (30) is obtained similarly, by inverting liminf and limsup and replacing $h_f(x)$ by the limsup exponent (31) $\overline{h_f}(x)$. □

Exercise 4.1. *Prove* (30).

Theorem 4.1 yields an (adaptive) upper bound for the multifractal spectrum of every function f. This is of course important for the applications, since the Legendre transform of the Leader scaling function is estimable numerically, at least if the data set is large enough (i.e. there are many generations j available).

4.3. **Upper bounds for the multifractal spectrum of functions in classical functional spaces.** In the previous section we found an upper bound for the multifractal spectrum, but this upper bound is not related directly to "classical" functional spaces. In other words, the value of the Leader scaling function of f is not equivalent to the fact that f belongs to a Sobolev or a Besov space. Stéphane Jaffard introduced new functional spaces, that he named "Oscillation spaces", which are naturally associated to the leader scaling function; I refer the reader to [29] for further details.

I explain now how to obtain a priori upper bounds for the multifractal spectrum of a function f that belongs to a Hölder or a Besov space, or when $f \in \mathcal{M}([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ monotone}\}$.

1. For the Hölder space $C^s(\mathbb{R}^d)$: it is quite straightforward.

Exercise 4.2. *Prove that for every $f \in C^s(\mathbb{R}^d)$, $d_f(h) = -\infty$ if $h < s$, and $d_f(h) \le d$ if $h \ge s$.*

Exercise 4.3. *Construct a function $f \in C^s(\mathbb{R}^d)$ for which $d_f(h) = d \cdot \mathbf{1}_{\{s\}}(h)$.*

Exercise 4.4. *Construct a function* $f \in C^s(\mathbb{R}^d)$ *for which* $d_f(h) = d \cdot \mathbf{1}_{\{h\geq s\}\cap\mathbb{Q}}(h)$.

Exercise 4.5. *Construct a function* $f \in C^s(\mathbb{R}^d)$ *for which* $d_f(h) = d \cdot \mathbf{1}_{\{h\geq s\}}(h)$.

2. For a Besov space: it is more tricky. Let $0 < s < \infty$, $0 < p, q \leq \infty$. Assume that the wavelets Ψ^0 and Ψ^1 are at least $[s+1]$-regular. The $B^s_{p,q}([0,1]^d)$ Besov norm (quasi-norm when $p < 1$ or $q < 1$) of a distribution f on $[0,1]^d$ (with wavelet coefficients d_λ) is

$$\|f\|_{B^s_{p,q}} = \left(\sum_{j\geq 1} \left(2^{(sp-d)j} \sum_{|\lambda|=j} |d_\lambda|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \tag{33}$$

with the obvious modifications when $p = \infty$ or $q = \infty$. The Besov space $B^s_{p,q}([0,1]^d)$ is the set of functions with finite norm. It is a complete metrizable space, normed when p and $q \geq 1$, separable when both are finite.

The following standard embeddings are easy to deduce from (33): For any $0 < s < \infty$, $0 < p \leq \infty$, $0 < q < q' \leq \infty$, $\varepsilon > 0$,

$$B^s_{p,q}([0,1]^d) \hookrightarrow B^s_{p,q'}([0,1]^d) \hookrightarrow B^{s-\varepsilon}_{p,q}([0,1]^d) \tag{34}$$

We prove the result of Jaffard [25]: belonging to a Besov space yields an upper bound on the multifractal spectrum.

Theorem 4.2. *Let* $0 < p < \infty$ *and* $d/p < s < \infty$. *For every* $f \in B^s_{p,\infty}([0,1]^d)$ *and every* $h \geq s - d/p$,

$$d_f(h) \leq \min\big(d, d + (h-s)p\big), \tag{35}$$

and $E_f(h) = \emptyset$ *if* $h < s - d/p$.

Remark 4.1. *The results have been stated for Besov spaces with* $q = \infty$ *but it is clear from classical Besov embeddings* (34) *that they hold identically for any* $q > 0$.

Theorem 4.2 is not only optimal, the upper bound is actually an *almost sure* equality in $B^s_{p,q}([0,1]^d)$ in the sense of genericity or prevalence, as explained next Section 5.

Proof. The proof follows the same lines as the one of Theorem 4.1. I indicate the main steps, and let the reader complete the missing parts as exercises. Let $f \in B^s_{p,q}([0,1]^d)$. Hence $\|f\|_{B^s_{p,q}} < +\infty$.

(1) The Sobolev embedding $B^s_{p,q}([0,1]^d) \hookrightarrow C^{s-d/p}([0,1]^d)$ implies that $E_f(h) = \emptyset$ for all $h < s - d/p$.

(2) The inequality (35) is trivial when $h \geq s$, hence we fix $h \in [s-d/p, s)$. Then, for every $h' \leq h$, one has

$$N_j(h') = \#\{\lambda : |\lambda| = j \ \text{ and } \ |d_\lambda| \geq 2^{-jh'}\} \leq C 2^{j(ph'-ps+d)},$$

this inequality following from the fact that $\|f\|_{B^s_{p,q}} < +\infty$.

(3) Let $\lambda = (j, \mathbf{k}, \mathbf{l})$ and D_λ be a wavelet leader such that $D_\lambda \geq 2^{-jh}$. This means that there exists $\lambda' = (j', \mathbf{k}', \mathbf{l}')$ such that $j' \geq j$, $I_{\lambda'} \subset \bigcup_{i\in\{-1,0,1\}^d} I_\lambda + i2^{-j}$ and $|d_{\lambda'}| \geq 2^{-jh}$. For every $j' \geq j$, the number of wavelet coefficients satisfying $|d_{\lambda'}| \geq 2^{-jh} = 2^{-j'\frac{j}{j'}h}$ is less than

$$N_{j'}\left(\frac{j}{j'}h\right) \leq 2^{j'p\left(\frac{j}{j'}h-s+d/p\right)} = 2^{jph-j'ps+j'd}.$$

Hence,

$$\begin{aligned}\#\{\lambda = (j, \mathbf{k}, \mathbf{l}) : D_\lambda \geq 2^{-jh}\} &\lesssim \sum_{j'=j}^{+\infty} 2^{jph-j'ps+j'd} \\ &\lesssim 2^{j(ph-ps+d)}.\end{aligned}$$

(4) The last argument implies that the leader scaling function L_f associated with f satisfies

$$L_f(p) \geq ps - d.$$

at the specific value p associated with the Besov space $B^s_{p,q}([0,1]^d)$ we have chosen. Indeed, one has

$$\sum_\lambda |D_\lambda|^p \geq \sum_{\lambda: D_\lambda \geq 2^{-jh}} |D_\lambda|^p \geq 2^{j(ph-ps+d)} 2^{-jph} = 2^{j(-ps+d)}.$$

Finally, apply Theorem 4.1 to get

$$d_f(h) \leq ph - ps + d.$$

□

3. For monotone functions: There are many constraints on the multifractal spectrum of a monotone function $f \in \mathcal{M}([0,1]) = \{f : [0,1] \to \mathbb{R} : \ f$ monotone$\}$. The main reason is the fact that a monotone function has (obviously) bounded variations, and that f is the integral of a positive measure. I will work with measures rather than functions, but the two are equivalent.

The first constraint on the multifractal spectrum is due to the famous Lebesgue theorem on Lebesgue density, which implies that for every positive and finite Borel measure μ, for Lebesgue-almost every $x \in \mathbb{R}^d$,

$$\lim_{\varepsilon\to 0} \frac{\mu(B(x,\varepsilon))}{\varepsilon^d} \in [0, +\infty).$$

This obviously implies that for Lebesgue-almost every $x \in \mathbb{R}^d$, $h_\mu(x) \geq d$.

Theorem 4.3. *For every probability measure μ supported on $[0,1]^d$,*

$$d_\mu(h) \le \min(h,d).$$

Proof. Fix a measure μ, and an exponent $0 < h < d$. As in the preceding proofs, we find an upper bound for $\widetilde{E}_\mu(h) = \{x : h_\mu(x) \le h\}$.

Fix $\varepsilon > 0$, and $\eta > 0$.

Let $x \in \widetilde{E}_\mu(h)$. By definition of the local dimension of a measure, there exists $0 < r_x < \eta$ such that $\mu(B(x,r)) \ge (r_x)^{h+\varepsilon}$.

Th set of balls $(B(x,r_x))_{x\in\widetilde{E}_\mu(h)}$ forms a covering of $\widetilde{E}_\mu(h)$ by balls centered at points belonging to it. By the Besicovich's covering lemma, there exists $Q(d)$ disjoint families F^1, ..., $F^{Q(d)}$, each of them being constituted by pairwise disjoint balls $F^j = (B(x_i^j, r_i^j))_{i\in\mathbb{N}}$, such that

$$\widetilde{E}_\mu(h) \subset \bigcup_{j=1}^{Q(d)} \bigcup_{i\in\mathbb{N}} B(x_i^j, r_i^j).$$

Let us estimate by above the $\mathcal{H}_\eta^{h+\varepsilon}$ Hausdorff pre-measure of $\widetilde{E}_\mu(h)$ using this covering. One gets

$$\begin{aligned} \mathcal{H}_\eta^{h+\varepsilon}(\widetilde{E}_\mu(h)) &\le \sum_{j=1}^{Q(d)} \sum_{i\in\mathbb{N}} |B(x_i^j, r_i^j)|^{h+\varepsilon} \le \sum_{j=1}^{Q(d)} \sum_{i\in\mathbb{N}} \mu(B(x_i^j, r_i^j)) \\ &\le \sum_{j=1}^{Q(d)} \mu([0,1]^d) = Q(d), \end{aligned}$$

the last inequality following from the fact the the balls constituting one family F^j are pairwise disjoint. Hence $\dim_{\mathcal{H}} \widetilde{E}_\mu(h) \le h+\varepsilon$, for every $\varepsilon > 0$. □

Exercise 4.6. *Let $\mathcal{G}_j$ be the partition of $[0,1)^d$ into dyadic boxes that I denote I_λ where $\lambda = (j, \mathbf{k})$ (in this section the index $\mathbf{l}$ (due the the wavelets) does not exist).*

The L^q-spectrum of a measure $\mu \in \mathcal{M}([0,1]^d)$, which is the analog of the scaling function associated with functions, is the mapping defined for any $q \in \mathbb{R}$ by

$$\tau_\mu(q) = \liminf_{j\to\infty} -\frac{1}{j}\log_2 s_j(q) \quad \textit{where} \quad s_j(q) = \sum_{|\lambda|=j,\ \mu(I_\lambda)\neq 0} \mu(I_\lambda)^q.$$

Prove that for every probability measure on $[0,1]^d$,

$$d_\mu(h) \le (\tau_\mu)^*(h) := \inf_{q\in\mathbb{R}} \ (qh - \tau_\mu(q)).$$

Hint: use the same ideas as the one developed in Theorems 4.1 and 4.3.

Exercise 4.7. *Consider the measure*

$$\nu = \sum_{j\geq 1} \frac{1}{j^2} \sum_{k \ odd} 2^{-j} \delta_{k2^{-j}},$$

where δ_x *is the Dirac mass at* x. *Let*

$$\xi_x = \sup\{\xi \geq 0 : |x - k2^{-j}| \leq 2^{-j\xi} \textit{ for infinitely many } j \geq 1 \textit{ and odd } k\}.$$

ξ_x *is called the approximation rate of* x *by the dyadic numbers.*

(1) *Prove that* $\xi_x \in [1, +\infty]$ *for every* $x \in [0,1]$.
(2) *For every* $\xi \in [1, +\infty]$, *construct a real number* $x \in [0,1]$ *such that* $\xi_x = \xi$ *(Hint: use a dyadic decomposition).*
(3) *Prove that for every* x, $h_\nu(x) \leq 1/\xi_x$.
(4) *Using that the dyadic numbers are well distributed in* $[0,1]$, *prove that one has* $h_\nu(x) = 1/\xi_x$.
(5) *Conclude that the support of the multifractal spectrum of* ν *is exactly the interval* $[0,1]$.
(6) *(Difficult) Prove that for every* $h \in [0,1]$, $d_\nu(h) = h$.
Hint: One can:
- *either prove it directly by computing* $\dim_{\mathcal{H}} E_\nu(h) = \dim_{\mathcal{H}}\{x : \xi_x = 1/h\}$ *and use the mass distribution principle (Theorem 2.2),*
- *or use the theorem by Beresnevich and Velani* [11], *recalled before (Theorem 2.3 and Exercise 2.5). The method consists in applying Theorem 2.3 to the family* $\Big((\mathbf{k}2^{-j}, 2^{-j})\Big)_{j\geq 1, \mathbf{k}2^{-j}\in[0,1]^d}$ *of dyadic balls in* $[0,1]^d$, *to prove that* $\dim_{\mathcal{H}}\{x : \xi_x = 1/h\} = 1/(1/h) = h$.

Exercise 4.8. *Build a measure* μ *supported on* $[0,1]$ *such that for Lebesgue-almost every* $x \in [0,1]$, $h_\mu(x) \geq 2$. *(Hint: build a devil's staircase.)*

4.4. **Another multifractal spectrum: The large deviations spectrum.** The *large deviations* spectrum d_f^{ld} of a function is related on the asymptotic histogram of wavelet coefficients, see [4, 7, 32] for a complete study of this spectrum an an application to heart beat rates analysis. It is also relatively easy to estimate, in practical cases.

Definition 4.3. *Let* f *of the form* (7), *and let* $\varepsilon > 0$. *For every* λ *such that* $|\lambda| = j$ *and* $\mathbf{k}2^{-j} \in [0,1]^d$, *let* $h_\lambda = -j^{-1}\log_2|d_\lambda|$ *(we set* $h_\lambda = +\infty$ *if* $d_\lambda = 0$*). We set*

$$N_j^\varepsilon(h) = \#\left\{\lambda : |h_\lambda - h| \leq \varepsilon\right\}. \tag{36}$$

and $d_f^{ld}(\varepsilon, h) = \limsup_{j\to+\infty} j^{-1}\log_2 N_j^\varepsilon(h)$.

The large deviations spectrum $d_f^{ld}(h)$ *is defined as the mapping* $d_f^{ld}(h) = \lim_{\varepsilon\to 0} d_f^{ld}(\varepsilon, h)$.

Exercise 4.9. *Prove that the definition makes sense, and that the mapping* $h \mapsto d_f^{ld}(h)$ *is lower semi-continuous.*

The large deviations spectrum clearly depends on the choice of the wavelet ψ. While one always has $d_f^{ld}(h) \leq (\eta_f - d)^*(h)$ (the Legendre transform of the wavelet scaling function), there is no general relationship between d_f^{ld} and d_f. The examples we later consider illustrate this statement.

5. Generic results for the multifractality of functions

In the previous section, we obtained upper bounds for the multifractal spectrum of many functions, based on the functional spaces to which these functions belong. It is a natural question to ask whether these bounds are optimal. In the cases developed before, they are indeed. Even more, one can show that "almost every function" in these spaces realizes the upper bound. This can be interpreted by the fact that the worst regularity is the most common one, since the iso-Hölder sets for typical functions have the greatest possible dimension, as we will see.

Let us start by recalling how one can talk about "almost every" element in infinite dimensional spaces.

Definition 5.1. *A property* $\mathcal{P}$ *is said to be* generic *in a complete metric space* E *when it holds on a residual set, i.e. a set with a complement of first Baire category. A set is of first Baire category if it is the union of countably many nowhere dense sets. As it is often the case, it is enough to build a residual set which is a countable intersection of dense open sets in* E*.*

Genericity is essentially a topological notion, and this is the one that we are going to use in this course.

Exercise 5.1. *Prove that a generic set in* $\mathbb{R}$ *must be dense uncountable.*

Exercise 5.2. *Find a generic set in* $\mathbb{R}$ *of Lebesgue measure 0.*

There is another notion for describing the "size" of a set. Prevalence theory is used to supersede the Lebesgue measure in any topological vector space E. This notion was proposed by Christensen [14] and later by Hunt [22]. The space E is endowed with its Borel σ-algebra $\mathcal{B}(E)$.

Definition 5.2. *A Borel set* $A \subset E$ *is said to be* shy *if there exists a positive Borel measure* μ*, supported on some compact subset* K *of* E*, such that*

$$\text{for every } x \in E, \quad \mu(A + x) = 0.$$

A set that is included in a shy Borel set is also called shy.
The complement of a shy set A *in* E *is called prevalent.*

Prevalent sets are stable under translation, dilation, union and countable intersection. Moreover, when E has finite dimension, being prevalent in E is equivalent to have full Lebesgue measure. This justifies that a prevalent set A is referred to as a "large" set in E and extends reasonably the notion of full Lebesgue measure to infinite dimensional spaces.

Exercise 5.3. *Prove the above claims about prevalent sets.*

In what follows, I essentially deal with multifractal properties of generic functions (i.e. of all functions in a generic set of some functional space), but most of the time, these results also hold true for prevalent functions (i.e. for all functions in a prevalent set). The reader can have a look at the numerous results on the subject for further details.

5.1. **Hölder spaces.**

Theorem 5.1. *There exists a dense open set (hence, generic) $\mathcal{R} \in C^s([0,1]^d)$ such that for every $f \in \mathcal{R}$ and every $x \in [0,1]^d$, $h_f(x) = s$.*

In particular, generic functions in $C^s([0,1]^d)$ are monofractal, i.e. $E_f(h) = \emptyset$ if $h \neq s$.

Proof. Let us recall that for any $f \in C^s([0,1]^d)$, there exists a constant $C > 0$ such that

$$f = \sum_{\lambda:|\lambda|\geq 1} d_\lambda \Psi_\lambda(x) \qquad \text{with } \ |d_\lambda| \leq C2^{-js} \tag{37}$$

and $\|f\|_{C^s} = \inf\{C > 0 : (37) \text{ is satisfied for all } \lambda\}$ is a Banach norm on $C^s(\mathbb{R}^d)$.

For each integer $N \geq 1$, let us introduce the sets:

$$\begin{aligned} \mathcal{E}_N &= \Big\{ f \in C^s([0,1]^d) \ : \ \forall\, \lambda,\ 2^{js+N} d_\lambda \in \mathbb{Z}^* \Big\} \\ \mathcal{F}_N &= \Big\{ g \in C^s([0,1]^d) \ : \ \exists f \in \mathcal{E}_N,\ \|f-g\|_{C^s([0,1]^d)} < 2^{-N-2} \Big\}. \end{aligned} \tag{38}$$

Lemma 5.1. *For every $N \geq 1$, all functions in $\mathcal{F}_N$ are monofractal with exponent s.*

Proof. This follows from the fact that, given $f \in \mathcal{E}_N$, all the wavelet coefficients of f satisfy

$$2^{-N-js} \leq |d_\lambda| \leq \|f\|_{C^s} 2^{-js}.$$

Thus for any function $g \in \mathcal{F}_N$ with coefficients g_λ and its associated $f \in \mathcal{E}_N$:

$$2^{-N-js} - 2^{-N-2-js} \leq |g_\lambda| \leq \|f\|_{C^s} 2^{-js} + 2^{-N-2-js}$$

i.e.

$$2^{-N-1-js} \leq |g_\lambda| \leq \left(\|f\|_{C^s} + 2^{-N-2}\right) 2^{-js}.$$

In particular, $g \in C^s(x)$ for any $x \in [0,1]^d$ and there is no $x_0 \in [0,1]^d$ and $s' > s$ such that $g \in C^{s'}(x_0)$. Indeed, (11) with $s' > s$ is not compatible when j tends to infinity with the left hand-side of the above inequality. □

We prove now that the set

$$\mathcal{R} = \bigcup_{N\geq 1} \mathcal{F}_N$$

is a dense open set in $C^s([0,1]^d)$ containing only monofractal functions with exponent s.

The preceding lemma ensures that $\mathcal{R}$ is composed of monofractal functions. According to (38), $\mathcal{F}_N$ is an open set and thus, so is $\mathcal{R}$. Let us check the density. Fix $f \in C^s([0,1]^d)$ with wavelet coefficients d_λ.

Let $\eta > 0$, choose $N \geq 1$ so that $2^{-N} < \eta$. We use the "non-zero integer part" function

$$E^*(x) = \begin{cases} 1 & \text{if } 0 \leq |x| < 2, \\ [x] & \text{if } |x| \geq 2. \end{cases}$$

Obviously $E^* : \mathbb{R} \to \mathbb{Z}^*$ and $|x - E^*(x)| \leq 1$. Let us finally define a function $g \in \mathcal{F}_N$ by its wavelets coefficients g_λ:

$$g_\lambda = 2^{-js-N} E^*\left(2^{js+N} d_\lambda\right).$$

By construction,

$$2^{js}\,|d_\lambda - g_\lambda| = 2^{-N}|2^{js+N}d_\lambda - E^*\left(2^{js+N}d_\lambda\right)| \leq 2^{-N} < \eta$$

thus $\|f-g\|_{C^s} < \eta$. This proves the density of $\mathcal{R}$ in $C^s([0,1]^d)$.

□

5.2. **Besov spaces.** One starts by constructing a measure whose multifractal spectrum is the worst possible in a given Besov space $B^s_{p,q}([0,1]^d)$.

Lemma 5.2. *Let $\beta = 1/p + 1/q$. Consider the measure ν built in Exercise 4.7, and the random series F whose wavelet coefficients F_λ are given by*

$$F_\lambda = \frac{1}{j^\beta} 2^{-j(s-d/p)} \nu(I_\lambda)^p. \tag{39}$$

Then, $F \in B^s_{p,q}([0,1]^d)$ and its multifractal spectrum is

$$\text{for every } h \in [s-d/p, s], \quad d_F(h) = p(h-s)+d,$$

and $E_F(h) = \emptyset$ if $h > s$.

Exercise 5.4. *Prove Lemma 5.2 by combining Exercises 3.15 and 4.7.*

Prove that for all λ such that $|\lambda| = j$, $|F_\lambda| \geq \frac{1}{j^\beta}2^{-js}$.

We now prove the multifractal nature of generic functions in $B^s_{p,q}([0,1]^d)$.

Theorem 5.2. . *In $B^s_{p,q}([0,1]^d)$, generic functions f are multifractal with the "as worse as possible" multifractal spectrum, i.e.*

$$\text{for every } h \in [s-d/p, s], \quad d_f(h) = p(h-s)+d,$$

and $E_f(h) = \emptyset$ if $h > s$.

Proof. The strategy to build a residual set with the desired multifractal properties is the following. Consider a dense sequence of functions $(\tilde{f}_n)_{n\geq 1}$ in the separable space $B^s_{p,q}([0,1]^d)$ (each $\tilde{f}_n$ having $(\tilde{d}^n_\lambda)$ as wavelet coefficients) and replace it by the sequence $(f_n)_{n\geq 1}$ whose wavelet coefficients (d^n_λ) are defined as follows:

$$d^n_\lambda = \begin{cases} \tilde{d}^n_\lambda & \text{if } |\lambda| < n, \\ F_\lambda & \text{if } |\lambda| \geq n. \end{cases}$$

In other words, one replaces the wavelet coefficients of $\tilde{f}_n$ by those of F for large $|\lambda|$.

It is easy to see that each f_n has the same multifractal behavior as F, since only the wavelet coefficients of large generation (corresponding to high frequencies) are important for the local behavior, and that the sequence is still dense in $B^s_{p,q}([0,1]^d)$.

Exercise 5.5. *Prove that (f_n) is indeed dense in $B^s_{p,q}([0,1]^d)$.*

Definition 5.3. *Let $\beta = 1/p + 1/q$, and $r_n = n^{-\beta}2^{-nd/p}/2$. One defines the set $\widetilde{\mathcal{R}}$*

$$\widetilde{\mathcal{R}} = \bigcap_{N\geq 1} \bigcup_{n\geq N} \mathscr{B}(f_n, r_n)$$

where $\mathscr{B}(g,r) = \{f \in B^s_{p,q}([0,1)^d) \,:\, \|f-g\|_{B^s_{p,q}([0,1]^d)} < r\}$.

The set $\widetilde{\mathcal{R}}$ is an intersection of dense open set, hence a residual set in $B^s_{p,q}([0,1]^d)$. The choice for the radius r_n is small enough to ensure that any function f in $\mathscr{B}(f_n, r_n)$ has its wavelet coefficients at generation n close to those of f_n (and thus to those of F).

Lemma 5.3. *If $f \in \mathcal{B}(f_n, r_n)$ has wavelet coefficients d_λ, then $|d_\lambda - d^n_\lambda| \leq |d^n_\lambda|/2$.*

Proof. By definition, one has $d^n_\lambda = F_\lambda$, $\forall\, \lambda$ such that $|\lambda| = n$. Hence, by definition of the Besov norm and the inclusion $\ell^q \subset \ell^\infty$:

$$\left(\sum_{\lambda:\ |\lambda| = n \quad 2^{pn(s-d/p)}} |d_\lambda - F_\lambda|^p \right)^{1/p} < r_n.$$

In particular, for any λ such that $|\lambda| = n$,

$$|d_\lambda - F_\lambda| \leq r_n 2^{-n(s-d/p)} = 2^{-ns} n^{-\beta}/2.$$

By Exercise 5.4, if $|\lambda| = j$, $|F_\lambda| \geq 2^{-js}/j^\beta$. Combining both inequalities ensures the result. $\square$

Let us now prove Theorem 5.2.

Let $f \in \widetilde{\mathcal{R}}$. There exists a strictly increasing sequence $(n_m)_{m\geq 1}$ of integers such that $f \in \mathscr{B}(g_{n_m}, r_{n_m})$.

Lemma 5.3 provides a precise estimate of the wavelet coefficients of f, namely for any $m \geq 1$: if $|\lambda| = n_m$,

$$\frac{1}{2}F_\lambda \leq |d_\lambda| \leq \frac{3}{2}F_\lambda.$$

The (almost) same proof as the one used for Exercise 4.7, Exercise 3.15 and Lemma 5.2 ensures that for any $x \in [0,1]^d$:

$$s - d/p \leq h_f(x) \leq s - d/p + d/(p\tilde{\xi}_x) \leq s,$$

where $\tilde{\xi}_x$ is the approximation rate by the family $(n_m)_{m\geq 1}$, defined by

$$\tilde{\xi}_x = \sup\{\xi \geq 0 : |x - k2^{-n_m}| \leq 2^{-n_m \xi} \text{ for infinitely many } m \geq 1 \text{ and odd } k\}.$$

The definition is almost the same as in Exercise 4.7, except that only a subsequence of the integers is used in the dyadic approximation.

Given $h \in [s - Q/p, s]$ and the unique ξ^h such that $h = s - d/p + d/(p\xi^h)$, one introduces the set

$$\mathcal{E} = \{x : \ \tilde{\xi}_x = \xi^h\} \ \setminus \ \bigcup_{i=1}^{+\infty} \{x \in [0,1]^d : h_f(x) \leq h - 1/i\}.$$

By Theorem 4.2 and the remarks thereafter, one knows that $\dim_{\mathcal{H}}\{x \in [0,1]^d : h_f(x) \leq h'\} \leq p(h' - s - d/p)$ for any $h' < h$. In particular, for every $i \geq 1$, one has:

$$\begin{aligned} \dim_{\mathcal{H}} \{x \in [0,1]^d : h_f(x) \leq h - 1/i\} &\leq p\,(h - 1/i - s - d/p) \\ &< p\,(h - s - d/p) \\ &= d/\xi^h. \end{aligned}$$

But according to Theorem 2.3 that can be applied to the subsequence of dyadic numbers

$$(\mathbf{k}2^{-n_m}, 2^{-n_m})_{m\geq 1},$$

one has $\mathcal{H}^{Q/\xi^h i}(\{x : \ \tilde{\xi}_x = \xi^h\}) = +\infty$, thus $\mathcal{H}^{Q/\xi^h}(\mathcal{E}) = +\infty$ and

$$\dim_H \mathcal{E} \geq d/\xi^h.$$

Next, one observes that $\mathcal{E} \subset E_f(h)$, since every $x \in \{x : \ \tilde{\xi}_x = \xi^h\}$ satisfies $h_f(x) \leq s - d/p + 1/(p\xi^h) = h$ and, by definition, $\mathcal{E}$ does not contains those elements x which have a pointwise Hölder exponent strictly smaller than h. One finally infers that:

$$\dim_{\mathcal{H}} E_f(h) \geq \dim_{\mathcal{H}} \mathcal{E} \geq d/\xi^h = p\,(h - s - d/p)\,.$$

The converse inequality is provided by Theorem 4.2 because $f \in B^s_{p,q}([0,1]^d)$. $\square$

5.3. **Measures (or monotone functions).** I do not give the proof of the main result of this section, I refer the reader to [12] for the details. Nevertheless I explain the context and use say that the main ideas to prove it are comparable to the one used in last Sections. I think that Theorem 5.3 is an important result because it does not require anything specific on the measures (no self-similarity or scaling behavior), it simply says that typical measures are multifractal.

We recall the notion of L^q-spectrum for a probability measure μ supported on $[0,1]^d$. I denote I_λ, where $\lambda=(j,\mathbf{k})$ and $\mathbf{k}\in\{0,1,\cdots,2^j-1\}^d$, the dyadic boxes of generation j included in $[0,1]^d$ (in this section the index $\mathbf{l}$ (due the the wavelets) does not exist).

The L^q-spectrum of a measure $\mu\in\mathcal{M}([0,1]^d)$, which is the analog of the scaling function associated with functions, is the mapping defined for any $q\in\mathbb{R}$ by

$$\tau_\mu(q)=\liminf_{j\to\infty}-\frac{1}{j}\log_2 s_j(q)\quad\text{where}\quad s_j(q)=\sum_{|\lambda|=j,\ \mu(I_\lambda)\neq 0}\mu(I_\lambda)^q.$$

To be able to talk about "generic" or "typical" measures in the sense of Baire's category, we need to define the topology on the set of probability measures on $[0,1]^d$. We endow it with the weak topology induced by the following metric: if $\mathrm{Lip}([0,1]^d)$ stand for the set of Lipschitz functions on $[0,1]^d$ with Lipschitz constant ≤ 1, and if μ and ν belong to $\mathcal{M}([0,1]^d)$, we set

$$d(\mu,\nu)=\sup\left\{\left|\int f\,d\mu-\int f\,d\nu\right| : f\in\mathrm{Lip}([0,1]^d)\right\}. \tag{40}$$

Theorem 5.3. *There is a dense G_δ set $\mathcal{R}$ included in $\mathcal{M}([0,1]^d)$ such that for every measure $\mu\in\mathcal{R}$, we have*

$$\forall\ h\in[0,d]\qquad d_\mu(h)=h, \tag{41}$$

and $E_\mu(h)=\emptyset$ if $h>d$.

In particular, for every $q\in[0,1]$, $\tau_\mu(q)=d(1-q)$, and μ satisfies the multifractal formalism at every $h\in[0,d]$, i.e. $d_\mu(h)=\tau_\mu^*(h)$.

Remark 5.1. *Theorem 5.3 has been extended by F. Bayart in* [9] *to measures supported on every compact set $K\subset\mathbb{R}^d$.*

5.4. **Traces, Slices, Projections....** Given a multifractal function $f:\mathbb{R}^d\to\mathbb{R}$, it is very natural to ask whether its traces, i.e. its restrictions on subspaces or sub manifolds of $\mathbb{R}^d$ are still multifractal. This is actually a fundamental question since, for instance, multifractality has been proved for 1D-traces of the 3D-velocity of turbulent fluids, not for the 3D-velocity itself. Only few is known, I give one theorem proved in [3] and [33].

Theorem 5.4. *Let $1\leq d'<d$ be two integers. For every $a\in\mathbb{R}^{d-d'}$, let $\mathcal{H}_a$ be the affine space*

$$\mathcal{H}_a=\{x=(x_1,x_2,...,x_d)\in\mathbb{R}^d : x_{d'+1}=a_1,\ x_{d'+2}=a_2,...,\ x_d=a_{d-d'}\}.$$

Consider two positive real numbers s and p such that $s - d/p > 0$.

For every $f \in B^s_{p,q}(\mathbb{R}^d)$, for Lebesgue-almost every $a \in \mathbb{R}^{d-d'}$, the trace of f over $\mathcal{H}_a$, denoted by f_a, belongs to $\bigcap_{s'<s} B^{s'}_{p,\infty}(\mathbb{R}^{d'})$. If $q < p$, one has $f_a \in B^s_{p,qp/(p-q)}([0,1]^{d'})$.

Moreover, for typical functions in $B^s_{p,q}(\mathbb{R}^d)$, for Lebesgue-almost every $a \in \mathbb{R}^{d-d'}$, the trace of f over $\mathcal{H}_a$ has the following multifractal properties:

- *the exponents of f_a all belong to the interval $[s - d'/p, s]$,*
- *for every $h \in [s - d'/p, s]$, the multifractal spectrum of f_a is*

$$d_{f_a}(h) = p(h - (s - d'/p)).$$

What is surprising in this theorem is that the typical traces of f on affine substances possess a regularity which is better than the one guaranteed by the standard trace theorems (one knows that traces of functions belonging to $B^s_{p,q}(\mathbb{R}^d)$ all belong to $B^{s-(d-d'd)/p}_{p,q}(\mathbb{R}^{d'})$, but we prove that their regularity is actually better than expected). In addition, we compute their exact multifractal spectrum which **is not the worst possible regularity**, as it is the case in the preceding section. A lot is still to be done on multifractal properties of traces of functions, and also of slices and projections of multifractal measures,

6. Some examples of multifractal wavelet series

6.1. **Hierarchical wavelet series.** I come back to the example of Section 3.5, which was originally developed in [8]. Let μ be a positive Borel probability measure on $[0,1]^d$, and let F_μ be the wavelet series whose wavelet coefficients are

$$d_\lambda = 2^{-j\alpha}\mu(I_\lambda)^\beta. \tag{42}$$

Assuming that the measure is uniformly regular, i.e. there exists a constant $C > 0$ and an exponent $h_{\min} > 0$ such that for every ball with center x and radius $0 < r < 1$, one has

$$\mu(B(x,r)) \geq Cr^{h_{\min}}.$$

Then by an adaptation of Proposition 3.2, $f \in C^{h_{\min}}(\mathbb{R}^d)$, and for every $x \in [0,1]^d$, one has

$$h_F(x) = \alpha + \beta \cdot h_\mu(x).$$

In particular, $d_F(h) = d_\mu\left(\frac{h-\alpha}{\beta}\right)$.

Also, by Exercise 3.14, the multifractal spectrum is quite stable if one perturbates the wavelet coefficients by multiplying them by random variables which are "not too bad".

Such wavelet series are nice models, since they are relatively easy to simulate numerically, and there are some parameters that allow one to fit the multifractal parameters of real data, by choosing relevant values of α, β and the measure μ. In addition, the hierarchical structure of the wavelet coefficients (due to the measure, if $I_{\lambda'} \subset I_\lambda$, then $d_{\lambda'} \subset d_\lambda$) imply that large coefficients must be located around the same position through scales, which correspond to real situations (for instance, contours of images are irregular and large coefficients are located around them). Unfortunately, it has many unrealistic characteristics: there is no sparsity, and almost no small coefficients.

6.2. **Lacunary wavelet series.** A model somehow orthogonal to the previous one was introduced by S. Jaffard in [28]. In his model, all coefficients are independent, but not with common law. The model is as follows (I focus on the one-dimensional model, the extension to $\mathbb{R}^d$ is immediate): Let $\alpha > 0$, $0 < \eta < 1$ be two parameters. Let $(g_\lambda)_{\lambda:|\lambda|\geq 1}$ be a sequence of independent random variables in a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, whose law are Bernoulli laws with parameter $2^{-|\lambda|\eta}$, i.e.

$$\text{for every } \lambda \text{ such that } |\lambda| = j \geq 1, \qquad g_\lambda = \begin{cases} 1 & \text{with probability } 2^{-j\eta} \\ 0 & \text{with probability } 1 - 2^{-j\eta}. \end{cases}$$

Then consider the (random) wavelet series $R_{\alpha,\eta}$ whose wavelet coefficients d_λ are

$$\text{for every } \lambda = (j,k,l) \text{ such that } |\lambda| = j| \text{ and } k2^{-j} \in [0,1], \quad d_\lambda = 2^{-j\alpha} g_\lambda.$$

Theorem 6.1. *There exists an event of probability one in* $(\Omega, \mathcal{B}, \mathbb{P})$ *such that*

$$\textit{for every } h \in [\alpha, \alpha/\eta], \qquad d_{R_{\alpha,\eta}}(h) = \frac{\eta}{\alpha} h,$$

and $E_{R_{\alpha,\eta}}(h) = \emptyset$ *otherwise.*

The proof is written as a succession of Lemmas and exercises that are individually accessible (hopefully...).

Proof. (1) Observe that for all λ, $|d_\lambda| \leq 2^{-|\lambda|\alpha}$, hence $R_{\alpha,\eta} \in C^\alpha(\mathbb{R})$.

(2) The upper bound for the exponents is obtained thanks to the following uniform lower bound for the wavelet leaders.

Lemma 6.1. *For every* $\varepsilon > 0$*, almost surely, there exists* $J \geq 1$ *such that for all* $j \geq J$*, for all* λ *such that* $|\lambda| = j$ *and* $k2^{-j} \in [0,1]$*, the wavelet leader* D_λ *satisfies*

$$D_\lambda \geq 2^{-j(\alpha/\eta + \varepsilon)}. \tag{43}$$

Prove Lemma 6.1 using the Borel-Cantelli Lemma.

(3) By (43), one infers that almost surely, for every $x \in [0,1]$,

$$h_{R_{\alpha,\eta}}(x) \leq \alpha/\eta + \varepsilon.$$

Since (43) holds true for every $\varepsilon > 0$, one gets that almost surely, $h_{R_{\alpha,\eta}}(x) \leq \alpha/\eta$ for every $x \in [0,1]$ (observe that it is stronger than "for every $x \in [0,1]$, almost surely, we have...").

Hence we get the correct range $[\alpha, \alpha/\eta]$ for the possible exponents for $R_{\alpha,\eta}$, and it remains us to compute the Hausdorff dimension of the iso-Hölder sets.

(4) There is a relationship between the value of the pointwise Hölder exponent h of $R_{\alpha,\eta}$ at a point x and the approximation rate of x by some random family of intervals, which cover the interval $[0,1]$.

Lemma 6.2. *Let us denote by $(\lambda_n = (j_n, k_n, l_n))_{n\geq 1}$ the sequence of cubes for which $g_{\lambda_n} = 1$, re-ordered so that $j_n \leq j_{n+1}$ for every $n \geq 1$. With probability one, there exists a positive non-increasing sequence $(\varepsilon_n)_{n\geq 1}$, converging to zero, such that*

$$[0,1] \subset \limsup_{n\to+\infty} B(k_n 2^{-j_n}, 2^{-j_n(1-\varepsilon_n)}). \tag{44}$$

Prove Lemma 6.2 using Borel-Cantelli Lemma (this is a sort of refinement of Lemma 6.1).

(5) Following Lemma 6.2 and also exercises 2.5 and 2.6, let us introduce the approximation rate of a real number $x \in [0,1]$ by the random family $(k_n 2^{-j_n}, 2^{-j_n(1-\varepsilon_n)})_{n\geq 1}$ as

$$\xi_x = \sup\{\xi \geq 1 : x \in \limsup_{n\to+\infty} B(k_n 2^{-j_n}, 2^{-j_n(1-\varepsilon_n)\xi})\},$$

the associated (random) sets

$$S_\xi = \{x \in [0,1] : x \in \limsup_{n\to+\infty} B(k_n 2^{-j_n}, 2^{-j_n(1-\varepsilon_n)\xi})\}$$

and finally

$$\widetilde{S}_\xi = \{x \in [0,1] : \xi_x = \xi\}.$$

Using the same techniques as in Exercise 2.5 and 2.6, prove that almost surely, for every $\xi \geq 1$, $\dim_{\mathcal{H}} S_\xi = \dim_{\mathcal{H}} \widetilde{S}_\xi = 1/\xi$.

(6) We now find a first inequality between the approximation rate and the pointwise Hölder exponent.

Lemma 6.3. *If $\xi \in [1, 1/\eta]$ and $x \in S_\xi$, then $h_{R_{\alpha,\eta}}(x) \leq \alpha/(\eta\xi)$.*

Prove Lemma 6.3. (Hint: when $x \in B(k_n 2^{-j_n}, 2^{-j_n(1-\varepsilon_n)\xi})$, find a lower bound for the wavelet leader $D_{j_n(1-\varepsilon_n)\xi}(x)$.)

(7) Prove that if $x \notin S_\xi$, then for every $\varepsilon > 0$, for all j large enough, $D_j(x) \geq 2^{-j(\alpha/(\eta\xi)+\varepsilon)}$.

(8) Deduce that if $h = \alpha/(\eta\xi)$ with $\xi \in [1, 1/\eta]$, then $E_{R_{\alpha,\eta}}(h) = \widetilde{S}_\xi$.

(9) Conclude.

(10) Compute the almost-sure large deviations spectrum of $R_{\alpha,\eta}$. Does this spectrum satisfy a multifractal formalism?

□

A generalization of these random wavelet series is developed in [4].

6.3. **Thresholded wavelet series.** Threshold is an important method in signal an image processing. As is well known, it provides efficient methods in compression (the JPEG 2000 algorithm uses such techniques for instance). In this section, we give some connexions between multifractal properties and adaptive threshold methods, which are essentially based on results published in [36].

Theorem 6.2. *Let f be a function satisfying (7), and assume that $f \in C^\varepsilon([0,1])$ for some $\varepsilon > 0$. Assume that there exists an exponent $h > 0$ such that $d_f^{ld}(h) < d_f(h)$. Then there exists a set $E \subset E_f(h)$ of dimension $d_f(h)$ of oscillating singularities for f.*

Exercise 6.1. *Demonstrate Theorem 6.2 by proving that for $\varepsilon < d_f(h) - d_f^{ld}(h)$, there is not enough wavelet coefficients $\lambda = (j, \mathbf{k}, \mathbf{l})$ satisfying $d_\lambda \sim 2^{-|\lambda|h}$ to create a set of Hausdorff dimension $d_f(h)$ of points such that $|x - \mathbf{k}2^{-j}| \leq 2^{-j(1-\varepsilon)}$.*

Essentially, last Theorem asserts that if the multifractal and the large deviations spectra do not coincide, **there are oscillating singularities!** It is thus a simple theoretical way to detect chirp-like behaviors. Unfortunately it is not applicable, since it requires the knowledge of the multifractal spectrum.

This theorem is completed by the next one. Let us introduce an adaptive threshold to keep, at each generation, the greatest wavelet coefficients. The same can be achieved by keeping only the smallest wavelet coefficients.

Definition 6.1. *Let f be a function satisfying (7). Let $\gamma > 0$. The function series f^γ, defined by*

$$f^\gamma = \sum_{j\geq 1} \sum_{\lambda : |\lambda| = j} d_\lambda \cdot \mathbf{1}_{|u| \geq 2^{-j\gamma}}(d_\lambda) \Psi_\lambda$$

is said to be obtained from f after an adaptive threshold of order γ.

We learn from Theorem 6.2 that for any continuous enough function f, $d_f^{ld}(h) < d_f(h)$ for some exponent $h > 0$ ensures the existence of oscillating singularities for f. For such a function f, if $d_f(h) > 0$ for some $h > 0$, a threshold of order $\gamma < h$ imposes $d_{f^\gamma}^{ld}(h) = 0$. But since a threshold increases (local and global) regularity, every point $x \in E_f(h)$ has a pointwise Hölder exponent for f^γ at x which is greater than h. These points are good candidates to be oscillating singularities for f^γ.

Theorem 6.3. *Let f be a function satisfying (7), $\gamma > 0$, and assume that $f \in C^\varepsilon([0,1])$ for some $\varepsilon > 0$. Assume that $E_f^(h) \neq \emptyset$. Let f^γ be the function obtained after an adaptive threshold of f of order $\gamma < h$. Then for every $x \in E_f(h)$, either $h_{f^\gamma}(x) = +\infty$, or x is an oscillating singularity for f^γ.*

Exercise 6.2. *Prove Theorem 6.3, first by observing that necessarily $h_{f^\gamma}(x) \geq h_f(x)$, and then by estimating what is the loss of value for the wavelet leader $D_j(x)$ in terms of $h - \gamma$.*

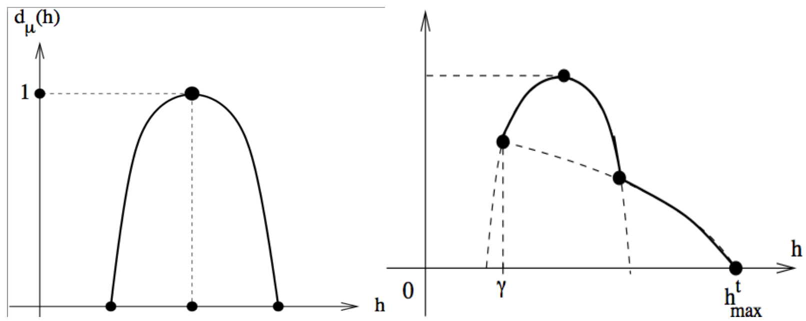

FIGURE 4. Multifractal spectrum of g (left) and of g^γ (right) when $-\log_2(1-p) < \gamma < -\log_2(p)$.

This theorem can be interpreted as a sort of Gibbs phenomenon for the adaptive threshold we proposed. It is also a very convenient method to create functions with homogenous non-concave multifractal spectra, as stated by the following (last) theorem, proved in [36], in which a concrete case is treated.

Theorem 6.4. *Let μ be the binomial measure on the interval $[0,1]$ with parameter $0 < p < 1/2$, whose multifractal spectrum ranges in $[-\log_2(1-p), -\log_2(p)]$, and consider the wavelet series g whose wavelet coefficients are built according the hierarchical model of Section 6.1, i.e. $d_\lambda = \mu(I_\lambda)$.*

Let $\omega_\gamma : [\gamma, -\log_2(p)] \to (0, +\infty)$ be the increasing function

$$u \to \gamma \, \frac{u + 1 + \log_2 p}{\gamma + \log_2 p}.$$

Let $h^\gamma_{\max} = \omega_\gamma^{-1}(-\log_2(p))$. The multifractal spectrum of g^γ ranges in $[-\log_2(p), h^\gamma_{\max}]$, and equals

$$d_{g^\gamma}(h) = \begin{cases} d_g(h) & \text{if} \quad h \in [-\log_2(1-p), \gamma], \\ d_g(\omega_\gamma^{-1}(h)) & \text{if} \quad h \in (\gamma, h^\gamma_{\max}]. \end{cases}$$

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