INHOMOGENEOUS COVERINGS OF TOPOLOGICAL MARKOV SHIFTS

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Abstract. Let $\mathscr{S}$ be an irreducible topological Markov shift, and let μ be a shift-invariant Gibbs measure on $\mathscr{S}$. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with common law μ. In this article, we focus on the size of the covering of $\mathscr{S}$ by the balls $B(X_n, n^{-s})$. This generalizes the original Dvoretzky problem by considering random coverings of fractal sets by non-homogeneously distributed balls. We compute the almost sure dimension of $\limsup_{n\to+\infty} B(X_n, n^{-s})$ for every $s \geq 0$, which depends on s and the multifractal features of μ. Our results include the inhomogeneous covering of $\mathbb{T}^d$ and Sierpinski carpets.

1. Introduction

The Dvoreztky covering problem has drawn the interest of many mathematicians in the past years. The original question can be resumed as follows: let $(l_n)_{n\geq 1}$ be a non-increasing positive sequence converging to zero, and $(X_n)_{n\geq 1}$ be an i.i.d. sequence of random variables uniformly distributed on the one-dimensional torus $\mathbb{T}$: is the torus $\mathbb{T}$ entirely covered (infinitely many times) by the balls $B(X_n, l_n)$?

This question initiated by Dvoretzky [10] has been studied, amongst many other mathematicians, by P. Lévy, J.P. Kahane, P. Erdös; L. Shepp found finally a necessary and sufficient condition on the sequence $(l_n)_{n\geq 1}$ for almost sure full covering [22]. The Dvoretzky problem has regained recently some attention with numerous interesting generalizations. For instance, refinements on the covering frequencies have been obtained by Barral and Fan [5], the probability of hitting a given analytic set has been studied in [17], coverings with arbitrary sets with non-empty interiors (not necessarily balls) in Ahlfors regular metric spaces are considered in [15, 20, 16], and finally in [13] the authors focus on coverings of smooth Riemann manifolds M by measurable sets distributed according to probability measures not purely singular with respect to the natural measure on M (see also the references in the mentioned papers). A related problem concerns the random cut-out sets with inhomogeneous densities in [18].

The answers are now well identified in the "homogeneous" context, i.e. when the $(X_n)_{n\geq 1}$ are more or less uniformly distributed over the considered ambiant metric space E. Focusing on the sequence $l_n = n^{-s}$ and the limsup set

$$\mathcal{L}^s := \limsup_{n\to+\infty} B(X_n, n^{-s}), \tag{1}$$

the expected result in an Ahlfors regular set E of Hausdorff dimension d is that the Hausdorff dimension $\dim_H \mathcal{L}^s$ of $\mathcal{L}^s$ equals $\min(d, s^{-1})$, almost surely, see Figure 1. While

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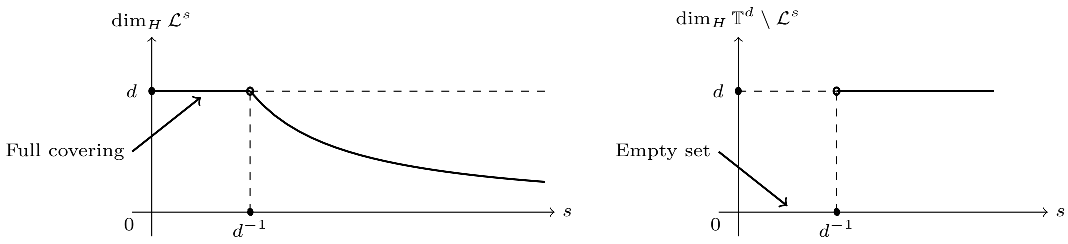

FIGURE 1. Homogeneous case in $\mathbb{T}^d$: **Left:** Almost sure Hausdorff dimension of $\mathcal{L}^s$. **Right:** Almost sure Hausdorff dimension of $\mathbb{T}^d \setminus \mathcal{L}^s$.

proving the upper bound for the Hausdorff dimension of $\mathcal{L}^s$ is relatively straightforward, the arguments used to get the corresponding lower bound (and so the sharp value for the dimension) are often very delicate and rely on generalized Borel-Cantelli lemmas or mass transference principles. Moreover, there exists usually a critical value s_c (which most of the time is $s_c = d^{-1}$) such that there is full covering $\mathcal{L}^s = E$ when $s < s_c$, and $\dim_H \mathcal{L}^s < \dim_H E$ as soon as $s > s_c$.

In this article, the problem of inhomogeneous covering is addressed, in the case where the $(X_n)_{n\geq 1}$ are i.i.d. random variables with common law a Gibbs measure μ invariant by the left-shift σ, supported by an irreducible topological Markov shift $\mathscr{S}$ (see just below for the notations and definitions). We focus on those Gibbs measures that are purely singular with respect to the Hausdorff measure $\mathcal{H}^{\dim_H \mathscr{S}}$ on $\mathscr{S}$. We prove that the dimension of $\mathcal{L}^s$, as a function of s, has a very different behavior than in the uniform case (see below Theorem 1).

First the formula $\dim_H \mathcal{L}^s = s^{-1}$ holds only when $s \geq (\dim_H \mu)^{-1}$. There is a phase transition phenomenon at this critical value $(\dim_H \mu)^{-1}$, and a different formula prevails for $s < (\dim_H \mu)^{-1}$. The heuristic reason is that the random points accumulate in the support of μ, and are not uniformly distributed any more: the corresponding balls overlap a lot. The lower bound for $\dim_H \mathcal{L}^s$ comes from an heterogeneous mass transference principle [1, 2, 3].

Second, there are two critical values $s_c^1 < s_c^2$ depending on μ, such that almost surely:

- $\dim_H \mathcal{L}^s < \dim_H \mathscr{S}$ when $s > s_c^2$,
- $\dim_H \mathcal{L}^s = \dim \mathscr{S}$ when s ranges in the non-trivial interval $[s_c^1, s_c^2]$, but **there is not full covering**: $\mathcal{L}^s \neq \mathscr{S}$,
- there is full covering $\mathcal{L}^s = \mathscr{S}$ when $s < s_c^1$.

These additional difficulties to cover the whole set $\mathscr{S}$ are in sharp contrast with the homogeneous situation. We also investigate the Hausdorff dimension of the complementary set $\mathscr{S} \setminus \mathcal{L}^s$. Let us mention that results were obtained for Gibbs distributed points on the one-dimensional torus in [23, 24]; the method developed in our paper is rather different, and covers Sierpinski carpets in any finite dimensional case, as canonical projections of subshifts of finite type.

Let us fix some notations: One works with the d-dimensional dyadic tree. The set Σ_j is the set of words of length $j \geq 1$ over the alphabet $\{0,1\}^d$ (the generalization to the b-adic alphabet is trivial):

$$\Sigma_j = \Big\{ (w_1 w_2 \cdots w_j) : \forall\, k \in \{1, ..., j\},\ w_k = (w_k^{(1)}, w_k^{(2)}, , ..., w_k^{(d)}) \in \{0,1\}^d \Big\}.$$

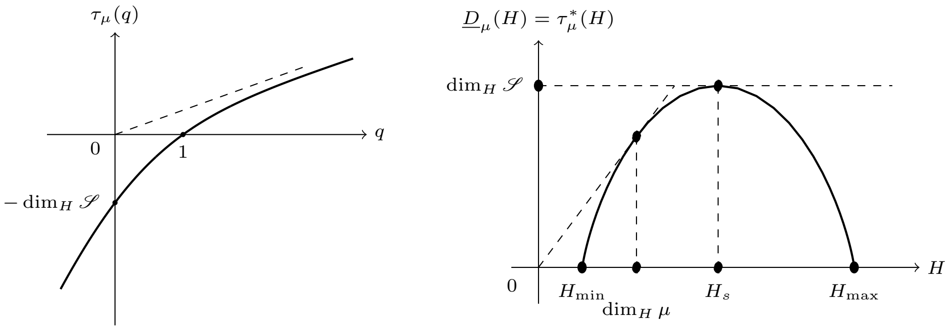

FIGURE 2. Free energy and multifractal spectrum of a Gibbs measure μ on $\mathscr{S}$. The values of D_μ at $H_{\min}$ and $H_{\max}$ may be strictly positive.

The notation $|w| = j$ stands for the length of $w \in \Sigma_j$. The sets of all finite words and infinite words over $\{0,1\}^d$ are $\Sigma^* = \bigcup_{j\geq 1} \Sigma_j$ and $\Sigma = (\{0,1\}^d)^{\mathbb{N}_+}$, which are endowed with the ultra-metric distance $d(w,w') = 2^{-\inf\{j\geq 1: w_j = w'_j\}}$ for $w = (w_j)_{j\geq 1}$ and $w' = (w'_j)_{j\geq 1}$.

The left-shift on Σ is defined by

$$\sigma(w_1 w_2 ... w_n ...) = (w_2 w_3 ...).$$

Let A be an irreducible $2^d \times 2^d$ matrix of zeros and ones, i.e. A^M contains only positive entries for some integer $M \geq 1$. Writing the elements of $\{0,1\}^d$ as $\{m_1, m_2, ..., m_{2^d}\}$, one considers the associated irreducible subshift of finite type $(\mathscr{S}, \sigma)$, i.e. the subset $\mathscr{S}$ of Σ, invariant by σ, constituted by those words $w = (w_k)_{k\geq 1} \in \Sigma$ such that for every $k \geq 1$, if $w_k = m_i$ and $w_{k+1} = m_j$ with $\{m_i, m_j\} \subset \{m_1, m_2, ..., m_{2^d}\}$, then $A_{ij} = 1$. It is standard that

$$\dim_B \mathscr{S} = \dim_H \mathscr{S} = \frac{h_{\mathscr{S}}}{\log 2^d},$$

i.e. the box and Hausdorff dimension of $\mathscr{S}$ coincide, up to the factor $\log 2^d$, with the topological entropy $h_{\mathscr{S}}$ of $(\mathscr{S}, \sigma)$.

As said before, one considers inhomogeneous coverings of $\mathscr{S}$. Let μ be a shift-invariant Gibbs measure μ on $\mathscr{S}$ associated with a Hölder continuous potential $\varphi : \mathscr{S} \to \mathbb{R}$. One shall keep in mind that the potential is not cohomologous to a constant, so that the associated Gibbs measure μ enjoys multifractal properties (see Definition 2 in Section 2.2). These multifractal features of μ play a key role in our result.

Recall that $B(x,r)$ is the ball of center x with radius r, and that the support of a measure μ is

$$\mathrm{Supp}(\mu) = \{x \in \mathscr{S} : \mu(B(x,r)) > 0, \ \forall\, r > 0\}.$$

Definition 1. *Let μ be a probability measure with support included in $\mathscr{S}$. For $x \in Supp(\mu)$, the lower local dimension of μ at x is defined as*

$$\underline{\dim}(\mu, x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}.$$

When the above liminf is a limit, the value is denoted by $\dim(\mu, x)$*. For $H \in \mathbb{R}$, set*

$$\underline{E}_\mu(H) = \{x \in \mathscr{S} : \underline{\dim}(\mu,x) = H\} \quad \text{and} \quad E_\mu(h) = \{x \in \mathscr{S} : \dim(\mu,x) = H\}.$$

The distribution of the singularities of μ is described via its multifractal spectrum, which is the mapping (with the convention that $\dim_H \emptyset = -\infty$)

$$\underline{D}_\mu: \ H \in \mathbb{R} \longmapsto \dim_H \underline{E}_\mu(H) \ \in [0,d] \cup \{-\infty\}.$$

For Gibbs measures on irreducible topological Markov shifts associated with a Hölder continuous potential φ, the shape of the multifractal spectrum of $\underline{D}_\mu$ is well known, and is related to the *free energy* $\tau_\mu(q)$ of μ as follows. For $q \in \mathbb{R}$, one has

$$\tau_\mu(q) = \liminf_{j\to\infty} \tau_{\mu,j}(q), \quad \text{where} \ \ \tau_{\mu,j}(q) := \frac{-1}{j} \log_2 \sum_{w\in\Sigma_j : \mu(I_w)>0} \mu(I_w)^q. \tag{2}$$

The multifractal properties of Gibbs measures are now standard results [21, 8, 6, 14, 11], and can be resumed as follows (see Figure 2). Let $H_{\min} = \tau'_\mu(+\infty) \leq H_s := \tau'_\mu(0) \leq H_{\max} = \tau'_\mu(-\infty)$.

(1) The function τ_μ is analytic, increasing, strictly concave on $\mathbb{R}$, and its Legendre transform

$$\tau^*_\mu(H) := \inf_{q\in\mathbb{R}} \big(Hq - \tau_\mu(q)\big). \tag{3}$$

is non-negative on its domain $[H_{\min}, H_{\max}] \subset \mathbb{R}^*_+$, strictly concave, analytic on $(H_{\min}, H_{\max})$. The maximum of τ^*_μ is $\dim_H \mathscr{S}$, and is reached at $H = H_s$.

(2) For all $H \geq 0$, $\underline{D}_\mu(H) = \dim_H E_\mu(H) = \tau^*_\mu(H)$. In this situation, one says that μ **satisfies the multifractal formalism at** H. The case where $\tau^*_\mu(H) = -\infty$ corresponds to $\underline{E}_\mu(H) = \emptyset$.

(3) The dimension of μ defined by

$$\dim_H \mu = \inf\{\dim_H E : \mu(E) = 1\}$$

equals $\tau'_\mu(1)$, and coincides with the metric entropy of μ.

The main theorem of this paper is now easy to state, see Figure 3 for an illustration.

Theorem 1. *Let μ be an invariant Gibbs measure on an irreducible topological Markov shift $\mathscr{S}$, associated with a Hölder continuous potential $\varphi : \mathscr{S} \to \mathbb{R}$. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables, such that $X_n \sim \mu$. With probability one:*

(1) *the limsup set $\mathcal{L}^s$ satisfies:*

$$\dim \mathcal{L}^s = \begin{cases} s^{-1} & \text{when } s \geq (\dim_H \mu)^{-1} \\ \underline{D}_\mu(s^{-1}) & \text{when } H_s^{-1} \leq s < (\dim_H \mu)^{-1} \\ \dim_H \mathscr{S} & \text{when } s < H_s^{-1}. \end{cases} \tag{4}$$

(2) *the complementary set of the limsup set $\mathcal{L}^s$ in $\mathscr{S}$ satisfies:*

$$\dim \mathscr{S} \setminus \mathcal{L}^s = \begin{cases} \dim_H \mathscr{S} & \text{when } s \geq H_s^{-1} \\ \underline{D}_\mu(s^{-1}) & \text{when } H_{\max}^{-1} < s < H_s^{-1} \\ -\infty & \text{when } s < H_{\max}^{-1}. \end{cases} \tag{5}$$

An interesting corollary is when $s \in (H_{\max}^{-1}, H_s^{-1})$, $\mathcal{L}^s$ covers $\mathcal{H}^{\dim_H \mathscr{S}}$-almost every point of $\mathscr{S}$, but not $\mathscr{S}$ itself. One needs to wait until $s < H_{\max}^{-1}$ to fully cover $\mathscr{S}$. This constitutes a striking difference with the homogeneous case where, as soon as $\mathcal{L}^s$ covers μ-almost every point, $\mathcal{L}^t$ (where $t < s$) covers the whole set. The non-homogeneity of the distribution μ

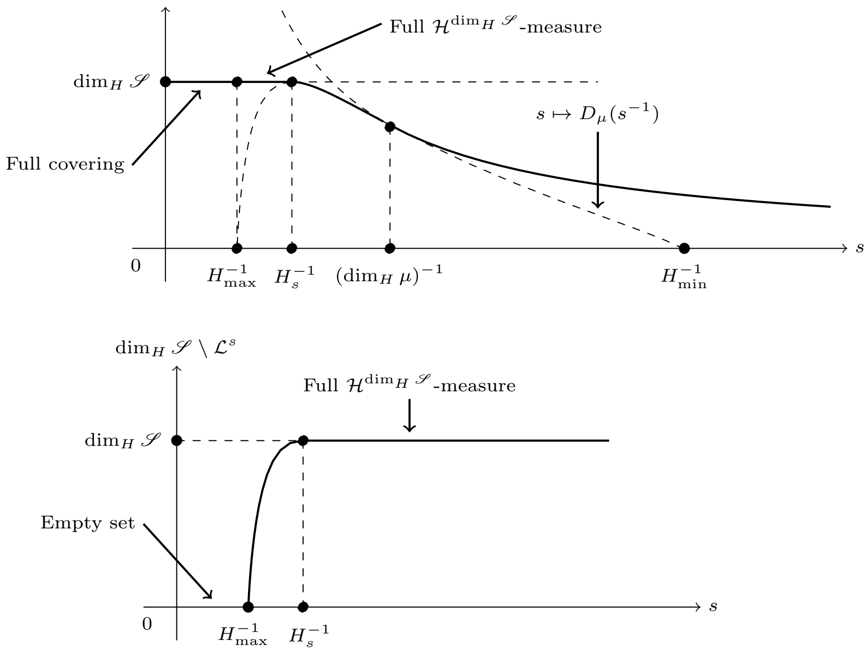

FIGURE 3. Inhomogeneous case: **Up:** Almost sure Hausdorff dimension of $\mathcal{L}^s$. **Bottom:** Almost sure Hausdorff dimension of $\mathscr{S} \setminus \mathcal{L}^s$.

(i.e. the fact that it is purely singular with respect to $\mathcal{H}^{\dim \mathscr{S}}$) is obviously a key issue here.

The full covering property at $s = H_{\max}^{-1}$ is not automatic. It depends on the emptiness (or not) of $\mathscr{S} \setminus \mathcal{L}^{H_{\max}^{-1}}$, which relies on properties of the potential $\varphi : \mathscr{S} \to \mathbb{R}$ that would be worth to be studied.

In Section 2, additional properties of Gibbs measures are given. Section 3 contains a fine study of the distribution of the random points $(X_n)_{n\geq 1}$. The Hausdorff dimension of $\mathcal{L}^s$ is computed in Sections 4 and 5. Finally, we focus one the complementary set $\mathscr{S} \setminus \mathcal{L}^s$ in Section 6.

We finish by mentioning that, using the canonical projection of $\mathscr{S}$ onto the d-dimensional torus $\mathbb{T}^d$, our results apply to the covering of Sierpinski carpets by balls $B(X_n, n^{-s})$ distributed according to a Gibbs measure with full support in $\mathscr{S}$, and also to the specific case where $\mathscr{S} = \Sigma$ and so the inhomogeneous covering of the full torus.

It would be interesting to study the covering frequencies associated with these distributions as in [5], and also to find conditions to ensure full covering at the critical value $s = H_{\max}^{-1}$. Similar results are expected to hold for covering of deterministic and random self-similar sets (not only regular Sierpinski carpets), of self-affine sets, maybe of non-linear Cantor sets (here the Lyapunov exponent should play a role), and one could also consider possible generalizations in Riemanian manifolds using coverings by balls or more general measurable sets distributed according to suitable distributions.

2. Some notations, and recalls on Gibbs measures

2.1. **Cylinders, Box dimension of $\mathscr{S}$.** Recall that Σ_j is the set of words of length j. We define $\mathscr{S}_j = \{w \in \mathscr{S} : |w| = j\}$. If $w \in \Sigma \cup \Sigma^*$ and $j \leq |w|$, then $w_{|j}$ stands for the prefix of w of length j. With each $w \in \Sigma_j$ is associated the cylinder

$$I_w = \left\{ w' \in \Sigma \cup \Sigma^* : w'_{|j} = w \right\}.$$

In that case, j is called the generation of I_w.

For every word $w \in \mathscr{S}_j$, I_w is thus the cylinder of words with prefix w, i.e. $I_w = \{w' \in \mathscr{S} : w'_{|j} = w\}$.

As said in the introduction, it is standard that the Hausdorff dimension $\dim_H \mathscr{S}$ equals the box dimension of $\mathscr{S}$. In particular, one has for every $\varepsilon > 0$, for every sufficiently large integer j,

$$2^{j(\dim_H \mathscr{S} - \varepsilon)} \leq \#\mathscr{S}_j \leq 2^{j(\dim_H \mathscr{S} + \varepsilon)}. \tag{6}$$

2.2. **Recalls on Gibbs measures.**

Definition 2. *A Gibbs measure μ associated with a Hölder continuous potential $\varphi : \mathscr{S} \to \mathbb{R}$ is the invariant ergodic probability measure on $\mathscr{S}$ satisfying the following: there exists a constant $C > 0$ such that for every $w \in \mathscr{S}$,*

$$\frac{1}{C} \leq \frac{\mu(I_w)}{\exp(S_j\varphi(w) - nP)} \leq C,$$

where $S_j\varphi(w) = \sum_{k=0}^{j-1} \varphi(\sigma^k w)$ is the j-th Birkhoff sum of φ, and $P = P(\varphi)$ is the topological pressure of φ defined as

$$P(\varphi) = \lim_{j\to\infty} \frac{1}{j} \log \sum_{w \in \mathscr{S}_j} \sup_{t \in I_w} \exp(S_j\varphi(t)).$$

A Gibbs measure μ satisfies the quasi-Bernoulli property, i.e. there exists a constant $C_\varphi > 1$ such that for every words $w, w' \in \Sigma^*$ intersecting $\mathscr{S}$, if ww' is the word obtained by concatenation of w and w', then

$$C_\varphi^{-1} \mu(I_w)\mu(I_{w'}) \leq \mu(I_{ww'}) \leq C_\varphi \mu(I_w)\mu(I_{w'}).$$

In particular, one can ensure that for another constant still denoted $C_\varphi > 2$, for every cylinder I_w with $w \in \mathscr{S}_j$ and any of its sub-cylinder $I_{w'}$ with $w' \in \mathscr{S}_{j+1}$ one has

$$C_\varphi^{-1} \leq \frac{\mu(I_{w'})}{\mu(I_w)} \leq 1 - C_\varphi^{-1}. \tag{7}$$

In addition, the Gibbs measure is doubling: for a suitable choice for C_φ, one also has for every $x \in \mathscr{S}$ and every $r > 0$, $\mu\big(B(x, 2r)\big) \leq C_\varphi \mu\big(B(x, r)\big)$. This property implies that one can also choose the constant C_φ so that any cylinders I_w and $I_{w'}$ associated with words w, w' belonging to $\mathscr{S}_j$ such that $w_{|j-1} = w'_{|j-1}$ (in other words, I_w and $I_{w'}$ share the same "father" cylinder), satisfy

$$C_\varphi^{-1} \leq \frac{\mu(I_{w'})}{\mu(I_w)} \leq C_\varphi. \tag{8}$$

2.3. **Multifractal properties of Gibbs measures.** Next proposition gathers information about the multifractal features of Gibbs measures. Recall that the Legendre transform τ_μ^*, given by (3), is concave and is called Legendre spectrum of μ.

Proposition 1. *Let μ be a Gibbs measure on $\mathscr{S}$ as defined before. Consider the sets*

$$\underline{E}_\mu^{\leq}(H) = \{x \in \mathscr{S} : \underline{\dim}(\mu, x) \leq H\} \quad \text{and} \quad \underline{E}_\mu^{\geq}(H) = \{x \in \mathscr{S} : \underline{\dim}(\mu, x) \geq H\}.$$

(1) *For every $H \geq 0$, one has $\underline{D}_\mu(H) = \dim_H E_\mu(H) = \tau_\mu^*(H)$, with $\underline{E}_\mu(H) = \emptyset$ if and only if $\underline{D}_\mu(H) = -\infty$.*
(2) *For every $H \in [H_{\min}, H_s]$, $\dim_H \underline{E}_\mu^{\leq}(H) = \underline{D}_\mu(H)$.*
(3) *For every $H \in [H_s, H_{\max}]$, $\dim_H \underline{E}_\mu^{\geq}(H) = \underline{D}_\mu(H)$.*

This is deduced from [6, 19, 4, 11]. Item (1) of the last proposition says in particular that the Hausdorff dimension of the sets of $x \in \mathscr{S}$ at which the liminf $\underline{\dim}(\mu, x)$ is H is the same as the Hausdorff dimension of the set of points at which the limit $\dim(\mu, x)$ is H.

2.4. **Large deviations for Gibbs measures.** The asymptotical statistical distribution of μ is also described via the large deviations theory.

Definition 3. *Let μ be a Gibbs measure on $\mathscr{S}$. For every $I \subset \mathbb{R}^+$ and $j \geq 1$, set*

$$\mathcal{E}_\mu(j, I) = \Big\{ w \in \mathscr{S}_j : \frac{\log_2 \mu(I_w)}{-j} \in I \Big\}.$$

For $H \geq 0$, $\varepsilon > 0$, let

$$\mathcal{E}_\mu(j, H \pm \varepsilon) = \Big\{ w \in \mathscr{S}_j : \frac{\log_2 \mu(I_w)}{-j} \in [H - \varepsilon, H + \varepsilon] \Big\}.$$

The large deviations spectrum of μ is (when it exists)

$$f_\mu(H) = \lim_{\varepsilon \to 0} \lim_{j \to +\infty} \frac{\log_2 \# \mathcal{E}_\mu(j, H \pm \varepsilon)}{j}.$$

The last quantity is not always defined, but it is again a folklore result that $f_\mu(H)$ is well defined for a Gibbs measure for all $H \geq 0$ [11].

Proposition 2. *If μ is a Gibbs measure on $\mathscr{S}$ as in Definition 2, one has:*

(1) *for all $H \geq 0$, $f_\mu(H) = \underline{D}_\mu(H)$.*
(2) *For every $\varepsilon > 0$ and every interval $I \subset \mathbb{R}_+$, there exists an integer $J_{I,\varepsilon}$ such that for every $j \geq J_{I,\varepsilon}$,*

$$\left| \frac{\log_2 \mathcal{E}_\mu(j, I)}{j} - \sup_{h \in I} \underline{D}_\mu(h) \right| \leq \varepsilon.$$

(3) *For every $\varepsilon > 0$, there exists an integer J_ε such that $j \geq J_\varepsilon$ implies that every word $w \in \mathscr{S}_j$ satisfies $|I|^{H_{\max}+\varepsilon} \leq \mu(I) \leq |I|^{H_{\min}-\varepsilon}$.*

One needs to keep in mind that when j is large, $\# \mathcal{E}_\mu(j, H \pm \varepsilon) \approx 2^{j \underline{D}_\mu(H)}$.

3. Distribution of the X_n

In this section, the distribution of the random points $(X_n)_{n\geq 1}$ over $\mathscr{S}$ is further investigated. The results are essentially based on the following proposition, which provides us with a family of disjoint cylinders covering $\mathscr{S}$, with approximately same μ-mass.

Proposition 3. *Let μ be a Gibbs measure on $\mathscr{S}$, such that Supp(μ)=$\mathscr{S}$.*

For every real number $H \in (H_{\min}, H_{\max})$, every $\varepsilon > 0$, and every integer $j \geq 1$, there exists a finite μ-adapted family of cylinders $\mathcal{C}_\mu(j, H \pm \varepsilon) = \{I_i\}_{i=1,...,N}$ such that :

(1) *the number of cylinders satisfies*

$$C_\varphi^{-1} 2^{j(H-\varepsilon)} \leq N = \#\mathcal{C}_\mu(j, H \pm \varepsilon) \leq C_\varphi^4 2^{j(H+\varepsilon)}. \tag{9}$$

(2) *each I_i is a cylinder belonging to $\bigcup_{j\geq 1} \mathscr{S}_j$,*
(3) *$\mathcal{C}_\mu(j, H \pm \varepsilon)$ contains all the cylinders in $\mathcal{E}_\mu(j, H \pm \varepsilon)$,*
(4) *the cylinders $\{I_i\}_{i=1,...,N}$ are pairwise disjoint,*
(5) *they form a covering of $\mathscr{S}$, i.e. $\bigcup_{i=1}^N I_i = \mathscr{S}$,*
(6) *for every $i \geq 1$,*

$$(C_\varphi)^{-3} 2^{-j(H+\varepsilon)} \leq \mu(I_i) \leq C_\varphi 2^{-j(H-\varepsilon)}. \tag{10}$$

Observe that in the last double-sided inequality, the generation of the cylinder I_i may not be j.

Proof. One starts with the cylinders of generation j in $\mathscr{S}_j$. Since μ has full support in $\mathscr{S}$, $\mu(I) > 0$ for all $I \in \mathscr{S}_j$. Various situations may occur, one deals with them in the following order:

1- First put all the cylinders $\mathcal{E}_\mu(j, H \pm \varepsilon)$ in the family $\mathcal{C}_\mu(j, H \pm \varepsilon)$.

2- Consider one cylinder $I \in \mathscr{S}_j$ satisfying $\mu(I) > C_\varphi 2^{-j(H-\varepsilon)}$ (if such a cylinder exists). One proceeds by recursion as follows. Define $I_0 := I$.

Step 1. Let j_0 be the generation of I_0. Split I_0 into its sub-cylinders I_1, I_2, ..., I_M (with $M \leq 2^d$) belonging to $\mathscr{S}_{j_0+1}$.

Step 2. For every $i = 1, ..., M$, apply the following procedure: if I_i satisfies

$$(C_\varphi)^{-1} 2^{-j(H+\varepsilon)} \leq \mu(I_i) \leq C_\varphi 2^{-j(H-\varepsilon)}, \tag{11}$$

then keep it in the family $\mathcal{C}_\mu(j, H \pm \varepsilon)$, and stop the recursion for I_i. Otherwise, apply Step 1 to the cylinder $I_0 := I_i$.

Observe that :

- by (7), the ratio between the value of $\mu(I_0)$ and any of its subcylinder is lower bounded by a constant strictly greater than 1. Hence each time one goes back from Step 2. to Step 1., the value of $\mu(I_0)$ is multiplied by a factor strictly less than 1, and an infinite iteration of this would make the value of $\mu(I_0)$ tend to zero.
- if a cylinder $\widetilde{I}$ satisfies $\mu(\widetilde{I}) > C_\varphi 2^{-j(H-\varepsilon)}$, then by (7) any of its sub-cylinders $\widetilde{\widetilde{I}}$ with non-zero mass satisfies $\mu(\widetilde{\widetilde{I}}) \geq (C_\varphi)^{-1} 2^{-j(H+\varepsilon)}$. One deduces that, along the construction, it is not possible to skip the range $[(C_\varphi)^{-1} 2^{-j(H+\varepsilon)}, C_\varphi 2^{-j(H-\varepsilon)}]$ for the values of $\mu(I_i)$.

After a finite number of steps, the process provides one with a finite family of disjoint cylinders $(\widetilde{I}_n)_{n=1,\dots,N}$, whose union form a covering of $I \cap \mathscr{S}$. Moreover, all of them satisfy (11) (which is sharper than (10)). We put all these cylinders in the family $\mathcal{C}_\mu(j, H \pm \varepsilon)$.

3- We repeat Item **2-** until there is no cylinder $I \in \mathscr{S}_j$ satisfying $\mu(I) > C_\varphi 2^{-j(H-\varepsilon)}$ not covered yet by the union of the cylinders already selected to belong to $\mathcal{C}_\mu(j, H \pm \varepsilon)$.

Remark 1. *Observe that once item* **3-** *is completed, this process produces cylinders with disjoint interiors and whose union covers* $\bigcup_{I\in\mathscr{S}_j:\mu(I)>C_\varphi^{-1}2^{-j(H+\varepsilon)}} I$.

4- Let $I \in \mathscr{S}_j$ be such that $\mu(I) \leq (C_\varphi)^{-3} 2^{-j(H+\varepsilon)}$, which is not covered yet be the cylinders we already selected to belong to $\mathcal{C}_\mu(j, H \pm \varepsilon)$. We start the recursion with $I_0 := I$.

Step 1. Let j_0 be the generation of I_0. Consider $\widetilde{I} \in \mathscr{S}_{j_0-1}$ and $\widetilde{\widetilde{I}} \in \mathscr{S}_{j_0-2}$ the unique cylinders containing I_0. Consider also the cylinders $\widetilde{I}' \in \mathscr{S}_{j_0-1}$ that are also included in $\widetilde{\widetilde{I}}$.

Step 2. If all the cylinders $\widetilde{I}'$ satisfy $\mu(\widetilde{I}') < (C_\varphi)^{-1} 2^{-j(H+\varepsilon)}$, go back to Step 1 applied with $I_0 := \widetilde{I}$.

Step 3. If one of the cylinders $\widetilde{I}'$ satisfies $\mu(\widetilde{I}') \geq (C_\varphi)^{-1} 2^{-j(H+\varepsilon)}$, keep $\widetilde{I}$ in the family $\mathcal{C}_\mu(j, H \pm \varepsilon)$.

Pay attention to the fact that $\widetilde{I}$ is kept at Step 3. One checks that the construction ensures that:

- as above, each time one goes back from Step 2 to Step 1, the value of $\mu(I_0)$ increases by a factor larger than a constant strictly larger than 1.
- the selected cylinder $\widetilde{I}$ contains the initial cylinder I.
- by (7) and (8), one has $C_\varphi^{-3} 2^{-j(H+\varepsilon)} \leq \mu(\widetilde{I}) \leq C_\varphi 2^{-j(H-\varepsilon)}$.
- no sub-cylinder of $\widetilde{I}$ may have been selected before (they all have a too small μ-mass).
- there is no overlap between the interior of this newly selected cylinder and the ones considered in items **1-**, **2-**, **3-** or in the previous iteration of **4-**.

5- We apply item **4-** until there is no cylinder in $\mathscr{S}_j$ satisfying $\mu(I) < (C_\varphi)^{-3} 2^{-j(H+\varepsilon)}$ which is not covered yet by the union of the already selected family of cylinders $\mathcal{C}_\mu(j, H \pm \varepsilon)$.

6- The remaining cylinders (if they exist) satisfy $(C_\varphi)^{-3} 2^{-j(H+\varepsilon)} \leq \mu(I) \leq (C_\varphi)^{-1} 2^{-j(H+\varepsilon)}$. We keep all of them in the family $\mathcal{C}_\mu(j, H \pm \varepsilon)$.

All theproperties we claimed for the family directly follow from the construction. Finally, (9) is an immediate consequence of (10). □

Proposition 4. *Let* μ *be a Gibbs measure on* $\mathscr{S}$, $H \in (H_{\min}, H_{\max})$ *and* $\varepsilon > 0$ *such that* $H - \varepsilon > 0$. *Let* $s < (H + \varepsilon)^{-1}$, *and* $\eta > 0$.

Let $(X_n)_{n\geq 1}$ *be an i.i.d. sequence of random variables whose common law is* μ.

Consider the set

$$\mathscr{X}_{j,s} = \{X_n : 2^{(j-1)/s} \leq n < 2^{j/s}\}. \tag{12}$$

With probability 1, for every large integer j, *each cylinder in* $\mathcal{C}_\mu(j, H \pm \varepsilon)$ *contains at least one element of* $\mathscr{X}_{j,s}$.

Proof. Let $n_j = \#\mathscr{X}_{j,s}$, and write $n_j = C_j 2^{j/s}$ where $C_j \in [2^{-1/s}, 1]$. One has

$$\begin{aligned}
&\mathbb{P}(\exists\, I \in \mathcal{C}_\mu(j, H\pm\varepsilon) : I \text{ does not contain an element of } \mathscr{X}_{j,s}) \\
&\leq \sum_{I\in\mathcal{C}_\mu(j,H\pm\varepsilon)} \mathbb{P}(I \text{ does not contain an element of } \mathscr{X}_{j,s}) \\
&\leq \sum_{I\in\mathcal{C}_\mu(j,H\pm\varepsilon)} (1-\mu(I))^{n_j} \leq \sum_{I\in\mathcal{C}_\mu(j,H\pm\varepsilon)} (1 - C_\varphi^{-3} 2^{-j(H+\varepsilon)})^{n_j} \\
&\leq 2\#\mathcal{C}_\mu(j, H\pm\varepsilon) \exp(-C_\varphi^{-3} 2^{-j(H+\varepsilon)} n_j) \\
&\leq 2C_\varphi^3 2^{j(H+\varepsilon)} \exp(-C_\varphi^{-3} 2^{-1/s} 2^{j(1/s-(H+\varepsilon))}),
\end{aligned}$$

which goes super-exponentially fast to zero when j tends to infinity. We used that $2^{-j(H+\varepsilon)} n_j$ tends to zero since $s < (H+\varepsilon)^{-1}$. Hence, Borel-Cantelli lemma ensures that for j large enough, every cylinder $I \in \mathcal{C}_\mu(j, H\pm\varepsilon)$ contains at least one element of $\mathscr{X}_{j,s}$. □

One could also prove that for every $\eta > 0$ and every large integer j, each cylinder in $\mathcal{C}_\mu(j, H\pm\varepsilon)$ contains at most $2^{j(1/s-(H-\varepsilon)+\eta)}$ elements of $\mathscr{X}_{j,s}$.

An important corollary is the following.

Corollary 1. *Under the same assumptions as in Proposition 4, if $s < (H+\varepsilon)^{-1}$, with probability 1, for every large integer j, each cylinder $I \in \mathscr{S}_j$ such that $\mu(I) \geq 2^{-j(H-2\varepsilon)}$ contains at least one element of $\mathscr{X}_{j,s}$.*

This simply follows from the fact that each cylinder $I \in \mathscr{S}_j$ (j being large) such that $\mu(I) \geq 2^{-j(H-2\varepsilon)}$ contains at least one cylinder of $\mathcal{C}_\mu(j, H\pm\varepsilon)$. This can be observed from the construction of $\mathcal{C}_\mu(j, H\pm\varepsilon)$ and the item **2-** in the proof of Proposition 3 (the presence of the "2"ε comes from the constant C_φ).

4. Upper bound for $\dim_H \mathcal{L}^s$

One starts with an obvious upper bound. The s-Hausdorff measure of a set is denoted by $\mathcal{H}^s$, its associated pre-measure computed with coverings by sets with diameter less than η by $\mathcal{H}^s_\eta$.

Lemma 1. *For every $s > 0$, $\dim_H \mathcal{L}^s \leq \min(\dim_H \mathscr{S}, s^{-1})$.*

Proof. Observe first that $\mathcal{L}^s \subset \mathscr{S}$, since the support of μ is included in the closed set $\mathscr{S}$. This implies that $\dim_H \mathcal{L}^s \leq \dim_H \mathscr{S}$. So, when $s \leq (\dim_H \mathscr{S})^{-1}$, the lemma is obvious.

Fix $s > (\dim_H \mathscr{S})^{-1}$. Recalling (1), $\mathcal{L}^s$ is covered by $\bigcup_{n\geq N} B(X_n, n^{-s})$. Hence,

$$\mathcal{H}^{s^{-1}(1+\varepsilon)}_{2N^{-s}}(\mathcal{L}^s) \leq \sum_{n\geq N} |B(X_n, n^{-s})|^{s^{-1}(1+\varepsilon)}$$

which is the rest of a convergent series. Hence $\mathcal{H}^{s^{-1}(1+\varepsilon)}(\mathcal{L}^s) = 0$ and $\dim_H \mathcal{L}^s \leq s^{-1}(1+\varepsilon)$, for every $\varepsilon > 0$. □

The previous upper bound is not sharp when $s \in (H_s^{-1}, (\dim_H \mu)^{-1})$.

Proposition 5. *With probability 1, for every $s \in (H_s^{-1}, (\dim_H \mu)^{-1})$, $\dim_H \mathcal{L}^s \leq \underline{D}_\mu(s^{-1})$.*

Proof. Fix $s \in (H_s^{-1}, (\dim_H \mu)^{-1})$, $\varepsilon > 0$ and $s^{-1} + \varepsilon < H < H_s$.

We start by describing the distribution of the elements of the set $\mathscr{X}_{j,s}$ (recall (12)).

Let us write $\mathscr{X}_{j,s}$ as the disjoint union $\mathscr{X}_{j,s}^{\leq H} \cup \mathscr{X}_{j,s}^{>H}$, where

$$\begin{aligned} \mathscr{X}_{j,s}^{\leq H} &= \{X_n \in \mathscr{X}_{j,s} : X_n \in I \text{ with } I \in \mathcal{E}_\mu\big(j, [0, H]\big)\}, \\ \mathscr{X}_{j,s}^{>H} &= \{X_n \in \mathscr{X}_{j,s} : X_n \in I \text{ with } I \in \mathcal{E}_\mu\big(j, (H, +\infty)\big)\}. \end{aligned}$$

One shall keep in mind that when $X_n \in \mathscr{X}_{j,s}$, $2^{j-1} \leq n^s < 2^j$.

First, by Proposition 2, one knows that $\#\mathcal{E}_\mu(j, [0, H]) \leq 2^{j(\underline{D}_\mu(H)+\varepsilon)}$ when j becomes large. So $\bigcup_{X_n \in \mathscr{X}_{j,s}^1} B(X_n, n^{-s})$ is covered by at most $2^{j(\underline{D}_\mu(H)+\varepsilon)} 3^d$ cylinders of generation $j-1$, the 3^d coming from the fact that we may need to consider 3^d cylinders in $\mathscr{S}_{j-1}$ to cover $B(X_n, n^{-s})$.

Second, using Proposition 2, for every $H' \geq H$, there exists a generation $J_{H'}$ and $0 < \eta_{H'} \leq \varepsilon$ such that for every $j \geq J$,

$$\left| \frac{\log_2 \#\mathcal{E}_\mu(j, H' \pm \eta_{H'})}{j} - \underline{D}_\mu(H') \right| \leq \varepsilon.$$

In addition, when j is large, item (iii) of Proposition 2 also gives that it is enough to consider the values $H \in [H_{\min} - \varepsilon, H_{\max} + \varepsilon]$.

By compactness of $[H, H_{\max} + \varepsilon]$, one can choose an integer J_ε and a finite sequence of N real numbers $H'_1 = H$, H'_2, ..., $H'_N = H_{\max} + \varepsilon$ (together with $\eta_1 = \eta_{H'_1}$, ..., $\eta_N = \eta_{H'_N} = \eta_{H_{\max}+\varepsilon}$) such that

$$[H, H_{\max} + \varepsilon] \subset \bigcup_{i=1}^{N} [H'_i - \eta_i, H'_i + \eta_i], \tag{13}$$

and for every i, every $j \geq J_\varepsilon$,

$$\left| \frac{\log_2 \#\mathcal{E}_\mu(j, H'_i \pm \eta_i)}{j} - \underline{D}_\mu(H'_i) \right| \leq \varepsilon. \tag{14}$$

Let Y_j^i be the cardinality of

$$\{X_n \in \mathscr{X}_{j,s} : X_n \in I \text{ for some } I \in \mathcal{E}_\mu(j, H'_i \pm \eta_i)\}. \tag{15}$$

Lemma 2. *With probability one, for j large, $Y_j^i \leq 2^{j(\underline{D}_\mu(H'_i) - H'_i + 1/s + 4\varepsilon)}$.*

Proof. Using (14), and recalling that $\eta_i \leq \varepsilon$, one has

$$q_{j,i} := \mu \left(\bigcup_{I \in \mathcal{E}_\mu(j, H'_i \pm \eta_i)} I \right) \leq \#\mathcal{E}_\mu(j, H'_i \pm \eta_i) 2^{-j(H'_i - \eta_i)} \leq 2^{j(\underline{D}_\mu(H'_i) - H'_i + 2\varepsilon)}.$$

One remarks that Y_j^i is a binomial random variable with (deterministic) parameters $\#\mathscr{X}_{j,s}$ and $q_{j,i}$, with expectation

$$\mathbb{E}(Y_j^i) = \#\mathscr{X}_{j,s} q_{j,i} \leq C_j 2^{j/s} 2^{j(\underline{D}_\mu(H'_i) - H'_i + 2\varepsilon)} \leq 2^{j(\underline{D}_\mu(H'_i) - H'_i + 1/s + 3\varepsilon)}.$$

By Markov's inequality, one gets $\mathbb{P}(Y_j^i \geq 2^{j(\underline{D}_\mu(H'_i) - H'_i + 1/s + 4\varepsilon)}) \leq 2^{-j\varepsilon}$, and Borel-Cantelli's lemma gives the result. □

The cardinality of $\mathscr{X}_{j,s}^{>H}$ is necessarily less than $Y_j := \sum_{i=1}^N Y_j^i$, so the previous lemma gives that with probability one, for large integers j,

$$Y_j \leq N 2^{j(\max_{i=1,\dots,N} \underline{D}_\mu(H_i') - H_i' + 1/s + 4\varepsilon)}.$$

One observes that the mapping $H \mapsto H - \underline{D}_\mu(H)$ is an increasing map when $H \geq \dim \mu$. This follows from the fact that $H \mapsto \underline{D}_\mu(H)$ is concave and at $H = \dim_H \mu$, one has $\underline{D}_\mu'(\dim_H \mu) = 1$. Hence, the last inequality implies that

$$Y_j \leq N 2^{j(\underline{D}_\mu(1/s)+4\varepsilon)} \leq 2^{j(\underline{D}_\mu(1/s)+5\varepsilon)}.$$

Let $h > \max(\underline{D}_\mu(H) + \varepsilon, \underline{D}_\mu(s^{-1}) + 5\varepsilon)$. We use that the limsup set $\mathcal{L}^s$ is covered by the countable union $\bigcup_{j\geq J} \bigcup_{X_n \in \mathscr{X}_{j,s}} B(X_n, n^{-s})$. Observe that when $X_n \in \mathscr{X}_{j,s}$, $|B(X_n, n^{-s})| \leq C 2^{-j}$ by definition (12), for some constant $C > 0$. Hence, the h-Hausdorff measure of $\mathcal{L}^s$ is bounded above by

$$\begin{aligned}
\mathcal{H}^h_{C2^{-J}}(\mathcal{L}^s) &\leq \mathcal{H}^h_{C2^{-J}}\left(\bigcup_{j\geq J} \bigcup_{X_n \in \mathscr{X}_{j,s}^{\leq H}} B(X_n, n^{-s}) \right) + \mathcal{H}^h_{C2^{-J}}\left(\bigcup_{j\geq J} \bigcup_{X_n \in \mathscr{X}_{j,s}^{> H}} B(X_n, n^{-s}) \right) \\
&\leq C \sum_{j \geq J} 3^d 2^{j(\underline{D}_\mu(H)+\varepsilon)} 2^{-jh} + 2^{j(\underline{D}_\mu(s^{-1})+5\varepsilon)} 2^{-jh}.
\end{aligned}$$

By our choice for h, this series converges, hence $\mathcal{H}^h(\mathcal{L}^s) = 0$, and $\dim_H \mathcal{L}^s \leq h$. Since the result holds for every $h > \max(\underline{D}_\mu(H) + \varepsilon, \underline{D}_\mu(s^{-1}) + 5\varepsilon)$, $H > s^{-1}$ and $\varepsilon > 0$ small, one concludes that $\dim_H \mathcal{L}^s \leq \underline{D}_\mu(s^{-1})$. □

5. Lower bound for $\dim_H \mathcal{L}^s$

We prove (4).

Proposition 6. *With probability 1, for every $s \in [H_{\max}^{-1}, H_{\min}^{-1}]$, $\dim_H \mathcal{L}^s \geq \underline{D}_\mu(s^{-1})$.*

Proof. Let $s \in (H_{\max}^{-1}, H_{\min}^{-1})$. One applies Proposition 4, which holds with probability one simultaneously for all choices of rational numbers H, ε and η.

One chooses ε very small so that $H = (s - 2\varepsilon)^{-1} \in (H_{\max}^{-1}, H_{\min}^{-1})$. Proposition 4 states that for every large $j \geq 1$, each cylinder in $\mathcal{C}_\mu(j, H \pm \varepsilon)$ contains at least one element of $\mathscr{X}_{j,s}$. This is in particular the case for all cylinders belonging to $\mathcal{E}_\mu(j, H \pm \varepsilon)$. This means that the set

$$\mathcal{L}_j^s = \bigcup_{n \in \mathscr{X}_{j,s}} B(X_n, n^{-s}) \tag{16}$$

forms a covering of $\bigcup_{I \in \mathcal{E}_\mu(j,H\pm\varepsilon)} I$, since when $n \in \mathscr{X}_{j,s}$, $n^{-s} \geq 2^{-j}$.

Since $\mathcal{L}^s = \limsup_{j\to+\infty} \mathcal{L}_j^s$, $\mathcal{L}^s$ contains the limsup of the sets $\bigcup_{I \in \mathcal{E}_\mu(j,H\pm\varepsilon)} I$. In particular, this limsup set contains necessarily $E_\mu(H)$, which yields that $\dim_H \mathcal{L}^s \geq \dim_H E_\mu(H) = \underline{D}_\mu((s-2\varepsilon)^{-1})$ by Proposition 1. Letting ε go to zero gives the result. □

Observe that the previous lower bound is in fact useful only when $s \in [H_s^{-1}, H_{\min}^{-1}]$. Otherwise, when $s \in [H_{\max}^{-1}, H_s^{-1}]$, one has $\dim_H \mathcal{L}^s = \dim_H \mathscr{S}$ while $\underline{D}_\mu(1/s) < \dim_H \mathscr{S}$.

In particular, last proposition ensures that when $s < H_s^{-1}$, $\mathcal{L}^s$ contains $E_\mu(H_s)$, which has full $\mathcal{H}^{\dim_H \mathscr{S}}$-measure in $\mathscr{S}$, by a standard result on Gibbs measures.

The second part of (4) follows from next proposition and the heterogeneous ubiquity theorem [1, 2] which supplies the mass transference principles developed in the homogeneous situation.

Proposition 7. *With probability 1, for every* $s \geq (\dim_H \mu)^{-1}$, $\dim_H \mathcal{L}^s = s^{-1}$.

Proof. Let $s \geq (\dim_H \mu)^{-1}$. Applying the same arguments as before (Proposition 6), one sees that if $1/t \in (H_{\max}^{-1}, H_{\min}^{-1})$ is a rational number, $\mathcal{L}^t$ contains all the sets $E_\mu(H)$, for $H < 1/t$. Hence, if $1/t > \dim_H \mu$ is rational, $\mathcal{L}^t$ contains the set $E_\mu(\dim_H \mu)$. This implies that with probability one,

$$\mu\left(\limsup_{n\to+\infty} B(X_n, n^{-t})\right) = 1. \tag{17}$$

Then the theorem on heterogeneous ubiquity in [1, 2] can be applied to get the lower bound (see also [12]). More precisely, this theorem is stated as follows.

Theorem 2. *Let* μ *be a Gibbs measure on a topological Markov shift* $\mathscr{S}$ *associated with a Hölder continuous potential. Let* $(t_n, l_n)_{n\geq 1}$ *be a sequence satisfying* $t_n \in \mathscr{S}$, $(l_n)_{n\geq 1}$ *is a positive decreasing sequence, and*

$$\mu\left(\limsup_{n\geq 1} B(t_n, l_n)\right) = 1.$$

Then for every $\delta \geq 1$, $\dim_H \left(\limsup_{n\geq 1} B(t_n, l_n^\delta)\right) \geq \frac{\dim_H \mu}{\delta}$.

This theorem is deterministic, depending on the validity of (17) only. Applying Theorem 2 with $(t_n, l_n) = (X_n, n^{-t})$ and $\delta = \frac{s}{t} > 1$, since (17) is true on an event of probability one, one deduces that

$$\dim_H \mathcal{L}^s \geq \frac{\dim_H \mu}{\frac{s}{t}} = \frac{t \dim_H \mu}{s}.$$

Since (17) holds for every t rational such that $1/t > \dim_H \mu$, almost surely, one concludes that $\dim_H \mathcal{L}^s \geq s^{-1}$. □

Finally, one investigates the values of s for which there is full covering of $\mathscr{S}$.

Proposition 8. *With probability 1, for every* $s < H_{\max}^{-1}$, *one has full covering:* $\mathcal{L}^s = \mathscr{S}$.

Proof. Let $s < H_{\max}^{-1}$, and $\eta > 0$ be so small that $s + \eta < (H_{\max} + \eta)^{-1}$.

By item (3) of Proposition 2, when j is large, every word $w \in \mathscr{S}_j$ satisfies

$$|I|^{H_{\max}+\eta} \leq \mu(I) \leq |I|^{H_{\min}-\eta}. \tag{18}$$

We apply Proposition 4 with two rationals H, ε such that $[H-\varepsilon, H+\varepsilon] \supset [H_{\max}-\eta, H_{\max}+\eta]$ and $s+\eta < (H+\eta)^{-1}$. One deduces that that for every $j \geq 1$ large enough, each cylinder in $\mathcal{C}_\mu(j, H \pm \varepsilon)$ contains at least one element of $\mathscr{X}_{j,s}$.

But, observing the construction of the set $\mathcal{C}_\mu(j, H \pm \varepsilon)$, one remarks that for j large :

- $\mathcal{C}_\mu(j, H \pm \varepsilon)$ contains all the cylinders $I \in \mathscr{S}_j$ satisfying $2^{-j(H_{\max}+\eta)} \leq \mu(I) \leq 2^{-j(H_{\max}-\eta)}$,
- Every other cylinder $I \in \mathscr{S}_j$ satisfies necessarily $\mu(I) > 2^{-j(H_{\max}-\eta)}$, because of (18). Hence the construction of $\mathcal{C}_\mu(j, H \pm \varepsilon)$ ensures that each cylinder $I \in \mathscr{S}_j$ contains at least one sub-cylinder in $\mathcal{C}_\mu(j, H \pm \varepsilon)$.

So, the union $\bigcup_{X_n \in \mathscr{X}_{j,s}} B(X_n, n^{-s})$ covers completely $\mathscr{S}$.

Since this holds true for every j large, one deduces that $\mathscr{S} \subset \mathcal{L}^s$. □

6. Hausdorff dimension of $\mathscr{S}\setminus\mathcal{L}^s$

We complete the proof by proving (5).

6.1. **Upper bound for** $\dim_H \mathscr{S}\setminus\mathcal{L}^s$**.** The interesting situation is $s\in(H_{\max}^{-1}, H_s^{-1})$. Fix such an s, as well as $H,\varepsilon>0$ such that $s<(H+2\varepsilon)^{-1}<H^{-1}<H_s^{-1}$.

Corollary 1 asserts that for all integers j large enough, every cylinder $I\in\mathcal{S}_j$ satisfying $\mu(I)\geq 2^{-j(H-2\varepsilon)}$ contains an element X_n of $\mathscr{X}_{j,s}$. Hence, for the corresponding integer n, the ball $B(X_n,n^{-s})$ covers $I\in\mathcal{S}_j$. Recalling (16), one concludes that

$$\bigcup_{I\in\mathcal{S}_j:\mu(I)\geq 2^{-j(H-2\varepsilon)}} I\subset\mathcal{L}^s_j.$$

Hence, $\underline{E}_\mu^{\leq}(H-2\varepsilon)\subset\mathcal{L}^s$, and one necessarily has

$$\mathscr{S}\setminus\mathcal{L}^s\ \subset\bigcup_{H'\geq H-2\varepsilon}\underline{E}_\mu(H')=\underline{E}_\mu^{\geq}(H-2\varepsilon).$$

Since $H_s<H-2\varepsilon$, Proposition 1 ensures that $\dim\underline{E}_\mu^{\geq}(H-2\varepsilon)=\underline{D}_\mu(H-2\varepsilon)$. Letting H tend to s^{-1} and ε go to zero finally yields $\dim_H\mathscr{S}\setminus\mathcal{L}^s\leq\underline{D}_\mu(s^{-1})$.

6.2. **Lower bound for** $\dim_H \mathscr{S}\setminus\mathcal{L}^s$**.** Fix $s\in(H_{\max}^{-1}, H_s^{-1})$. We choose a small $\varepsilon>0$ and H such that

$$s^{-1}<H-5\varepsilon<H_{\max}. \tag{19}$$

Recall the notations introduced in Lemma 2: $\mathscr{X}_{j,s}=\mathscr{X}^1_{j,s}\cup\mathscr{X}^2_{j,s}$, and in particular the union (13) and equation (14). We modify our choice of $\varepsilon>0$ so small that $H_s<H_1'-\eta_1$. Recall also that with these notations , $H_1'=H$.

Let us introduce the limsup set

$$\widetilde{L}:=\limsup_{j\to+\infty}\ \bigcup_{\substack{I\in\mathcal{E}_\mu(j,H\pm\eta_1):\\ I\text{ contains some element of }\mathscr{X}_{j,s}}}\widetilde{I},$$

where $\widetilde{I}=\{w\in\mathscr{S}: d(w,I)\leq 3|I|\}$ is the set of elements of $\mathscr{S}$ at distance at most 3.2^{-j} from I. By construction, if $X_n\in\mathscr{X}_{j,s}$ and $X_n\in I$ with $I\in\mathcal{E}_\mu(j,H\pm\eta_1)$, then $B(X_n,n^{-s})\subset\widetilde{I}$. Observe that the diameter of $\widetilde{I}$ is less than $C2^{-j}$ for some constant C depending on d only.

Consequently, $\widetilde{L}$ contains all the elements of $\mathscr{S}$ that may be covered infinitely many times by those balls $B(X_n,n^{-s})$ that intersect some elements of $\mathcal{E}_\mu(j,H\pm\eta_1)$.

Let us find an upper bound for the Hausdorff dimension of $\widetilde{L}$.

By (19), one has $H-\eta_1=H_1'-\eta_1>s^{-1}+4\varepsilon$. Lemma 2 implies that with probability one that, for j large, $Y_j^1\leq 2^{j(\underline{D}_\mu(H)-H+s^{-1}+4\varepsilon)}$, where Y_j^1 was defined in (15). In other words, the number of cylinders $I\in\mathcal{E}_\mu(j,H\pm\eta_1)$ containing at least one element of $\mathscr{X}_{j,s}$ is less than $2^{j(\underline{D}_\mu(H)-H+s^{-1}+4\varepsilon)}$. A standard argument on Hausdorff dimensions shows that $\widetilde{L}$ has Hausdorff dimension less than $\underline{D}_\mu(H)-H+s^{-1}+4\varepsilon$.

Simultaneously, let us consider the set

$$\widetilde{\widetilde{L}}=\bigcup_{H'\in(H-\eta_1,H+\eta_1)}E_\mu(H'),$$

constituted by points having a limit local dimension $H' \in (H - \eta_1, H + \eta_1)$ for μ. One observes that by Proposition 1, the Hausdorff dimension of $\widetilde{\widetilde{L}}$ is equal to $\underline{D}_\mu(H-\eta_1)$, since H is located in the decreasing part of the spectrum $\underline{D}_\mu$ and $H_s < H - \eta_1$.

Our choice (19) for the parameters implies that

$$\underline{D}_\mu(H-\eta_1) > \underline{D}_\mu(H) > (\underline{D}_\mu(H) - H + s^{-1} + 4\varepsilon),$$

One deduces that

$$\dim_H \left(\widetilde{\widetilde{L}} \setminus \widetilde{L}\right) \geq \underline{D}_\mu(H).$$

A careful analysis shows that the elements of $\widetilde{\widetilde{L}} \setminus \widetilde{L}$ are not covered infinitely many times by the balls $B(X_n, n^{-s})$. Indeed, any element of $\widetilde{\widetilde{L}} \setminus \widetilde{L}$ is surrounded only by cylinders belonging to $\mathcal{E}_\mu(j, H \pm \eta_1)$ when j becomes large. So the elements of $\widetilde{\widetilde{L}} \setminus \widetilde{L}$ cannot belong to more than a finite numer of balls $B(X_n, n^{-s})$,

One concludes that $\dim_H \mathscr{S} \setminus \mathcal{L}^s \geq \dim_H(\widetilde{\widetilde{L}} \setminus \widetilde{L}) \geq \underline{D}_\mu(H)$. This holds for every $\varepsilon > 0$, and it is a trivial matter to check that when ε tends to zero and H tends to s^{-1}, one gets $\dim_H \mathscr{S} \setminus \mathcal{L}^s \geq \underline{D}_\mu(s^{-1})$.

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