DIOPHANTINE APPROXIMATION BY ORBITS OF EXPANDING MARKOV MAPS

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Abstract. Given a dynamical system $([0,1],T)$, the distribution properties of the orbits of real numbers $x \in [0,1]$ under T constitute a longstanding problem. In 1995, Hill and Velani introduced the "shrinking targets" theory, which aims at investigating precisely the Hausdorff dimensions of sets whose orbits are close to some fixed point. In this paper, we study the sets of points well-approximated by orbits $\{T^n x\}_{n\geq 0}$, where T is an expanding Markov map with finite partitions supported by the whole interval $[0,1]$. The values of the dimensions of sets of well-approximable points are described using the multifractal properties of Gibbs measures invariant under the action of T. This study can be viewed as a moving shrinking targets problem.

1. Introduction

Let (X,d) be a compact metric space and $T : X \to X$ a piecewise continuous transformation. For $x \in X$, let $\mathcal{O}(x) = \{T^n x : n \in \mathbb{N}\}$ be the orbit of x under the action of T. The (equi-)repartition of the points of the orbit of x, in particular its density over X, is a historical issue, which goes back to Poincaré's results. In 1995, Hill and Velani in [18] introduced the theory of shrinking targets, which aims at investigating the Hausdorff dimensions of sets of points whose orbits contains points arbitrary close to some fixed point. Precisely, for a fixed point $y \in X$, they studied the following set

$$\Big\{x \in X : \quad T^n x \in B(y, r_n) \quad \text{for infinitely many integers } n \in \mathbb{N} \Big\}, \qquad (1{\cdot}1)$$

where $B(x,r)$ stands for the ball of radius $r > 0$ centered at $x \in X = [0,1]$ and $(r_n)_{n\geq 1}$ is a sequence of positive real numbers converging to 0. In this article, we adopt a complementary point of view: we fix a point $x \in X$ and consider the set of points y well-approximated by the orbit $\mathcal{O}(x)$ of x, i.e. we focus on the following set

$$\Big\{y \in X : \quad T^n x \in B(y, r_n) \quad \text{for infinitely many integers } n \in \mathbb{N} \Big\}, \qquad (1{\cdot}2)$$

which can also be written as

$$\limsup_{n\to\infty} B(T^n x, r_n) = \bigcap_{N\geq 1} \bigcup_{n\geq N} B(T^n x, r_n).$$

In fact, many questions can be asked about the sets (1·1) and (1·2): for what choice of sequence $(r_n)_{n\geq 1}$ do they cover the whole interval $[0,1]$? When $[0,1]$ is not fully covered, what are their Hausdorff dimensions (denoted by $\dim_H$)?

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And of course, can we quantify the dependence on x? Answering these questions for the set (1·2) provides us with a very precise description of the distribution properties of the orbit of x under the action of T (for instance, if x is a fixed point of T, the set (1·2) is reduced to $\{x\}$ whatever the sequence $(r_n)_{n\geq 1}$ is).

When T is an expanding Markov map with finite partitions supported by the whole interval $[0,1]$, we will compute the value of the Hausdorff dimensions of sets (1·2) for real numbers x which are typical points for T-invariant Gibbs measures associated with any Hölder potential.

Such questions have been investigated in several contexts, and can be interpreted as general Diophantine approximation problems. Indeed, the classical Diophantine questions concern the dimension of the set

$$\mathcal{S}(\delta) = \left\{ y \in [0,1] : \left| y - \frac{p}{q} \right| \leq \frac{1}{q^{2\delta}} \text{ for infinitely many couples } (p,q)=1 \right\}, \tag{1·3}$$

which can also be written as a limsup set like (1·2)

$$\limsup_{q\to+\infty} \bigcup_{p\in\mathbb{Z}} B(p/q, 1/q^{2\delta}).$$

The work [18] is precursor on this subject in the dynamical setting, and thereafter, many people studied sets of the form (1·1) (see for instance [21] for the case where T is an irrational rotation on the torus $\mathbb{T}^1$). In the literature, one often refers to these results as shrinking targets problem or dynamical Borel-Cantelli Lemma. The paper [16] by Fan, Schmeling and Troubetzkoy, where the doubling map on $\mathbb{T}^1$ is studied, is the first one to consider the set (1·2). These studies are also related to many other famous works concerned with metric theory of Diophantine approximation (see [11, 13, 19, 22, 23, 5, 14] and references therein).

In this work, we focus on the study of the set (1·2) when T is an expanding Markov map of the interval $[0,1]$ with finite partitions (Markov map, for short). It appears that for Markov maps, the relevant choice for the sequence $(r_n)_{n\geq 1}$ is $r_n = 1/n^\delta$, for $\delta > 0$. We thus introduce the sets

$$\begin{aligned} \mathcal{L}^\delta(x) &:= \limsup_{n\to\infty} B(T^n x, n^{-\delta}), \\ \mathcal{F}^\delta(x) &:= [0,1] \setminus \mathcal{L}^\delta(x), \end{aligned}$$

which are respectively the set of points covered by infinitely many intervals $B(T^n x, n^{-\delta})$, and its complement. We study the size of the sets $\mathcal{L}^\delta(x)$ and $\mathcal{F}^\delta(x)$ in terms of their Hausdorff dimension. Such questions may be called "moving shrinking targets" problem (or dynamical Diophantine approximation in the vocabulary of [16]). Before stating our main result (Theorem 1.6), some definitions and recalls are needed.

Definition 1.1 (Expanding Markov maps)**.** *A transformation $T : [0,1] \to [0,1]$ is an expanding Markov map with finite partitions if there is a subdivision $\{a_i\}_{0\leq i\leq Q}$ of $[0,1]$ (denoted by $I(k) =]a_k, a_{k+1}[$ for $0 \leq k \leq Q-1$) such that:*

(1) *(Expanding property) there is a positive integer n and a real number ρ such that*

$$|(T^n)'| \geq \rho > 1,$$

(2) *(Piecewise monotonicity)* T *is strictly monotonic and can be extended to a* C^2 *function on each* $\overline{I(i)}$,
(3) *(Markov property) if* $I(j) \cap T(I(k)) \neq \emptyset$, *then* $I(j) \subset T(I(k))$,
(4) *(Mixing) there is an integer* R *such that* $I(j) \subset \cup_{n=1}^{R} T^n(I(k))$ *for every* k *and* j,
(5) *(Rényi's condition) For every* $k \in \{0, \cdots, Q-1\}$,

$$\sup_{(x,y,z)\in I(k)^3} \frac{|T''(x)|}{|T'(y)||T'(z)|} < \infty.$$

With an expanding Markov map are associated generations of *basic* intervals, coded by the alphabet $\{0, 1, \cdots, Q-1\}$.

Definition 1.2. *Let* $\mathcal{A} = \{0, 1, \cdots, Q-1\}$. *For every integer* $n \geq 1$, *we denote by* $\mathcal{G}_n$ *the set of* basic intervals *of generation* n *defined by*

$$\text{if } (i_1 i_2 \cdots i_n) \in \mathcal{A}^n, \quad I_{i_1 i_2 \cdots i_n} = I(i_1) \cap T^{-1}(I(i_2)) \cap \cdots \cap T^{-n+1}(I(i_n)).$$

The following distortion property between intervals will be crucial: there is a constant $L > 1$ such that for every integer $n \geq 2$, for every $(i_1 i_2 \cdots i_n) \in \mathcal{A}^n$,

$$1 \leq \frac{|I_{i_1 i_2 \cdots i_{n-1}}|}{|I_{i_1 i_2 \cdots i_{n-1} i_n}|} \leq L. \tag{1·4}$$

Obviously, the Hausdorff dimension (and also the Lebesgue measure) of $\mathcal{L}^\delta(x)$ depends on δ and on x. We aim at describing the possible values of $\dim_H \mathcal{L}^\delta(x)$ and $\dim_H \mathcal{F}^\delta(x)$ efficiently. As in [16], we provide a description of these dimensions for μ_ϕ-almost every $x \in [0,1]$, where μ_ϕ is a Gibbs state associated with a Hölder potential ϕ. Such measures always exist, as stated in the following well-known theorem (see Bowen [7] and Walters [33]).

Theorem 1.3. *Let* $T : I \to I$ *be an expanding Markov map. Then for any Hölder continuous function* $\phi : I \to \mathbb{R}$, *there exists a unique equilibrium state* μ_ϕ *which satisfies the following Gibbs property: there exist constants* $\gamma > 0$ *and* $P(\phi)$ *(topological pressure associated to* ϕ*), such that for any basic interval* $I_n \in \mathcal{G}_n$,

$$\gamma^{-1} \leq \frac{\mu_\phi(I_n)}{e^{S_n\phi(x) - nP(\phi)}} \leq \gamma, \quad \forall x \in I_n, \tag{1·5}$$

where $S_n\phi(x) = \phi(x) + \cdots + \phi(T^{n-1}x)$.

Such Gibbs measures have exponential decay of correlations (see Section 4), which will be crucial hereafter.

As said above, the Hausdorff dimensions of $\mathcal{L}^\delta(x)$ and $\mathcal{F}^\delta(x)$ will be given for the points x which are typical for the Gibbs measure μ_ϕ. Such Gibbs measures μ_ϕ have been extensively studied, in particular from the multifractal standpoint. It is striking that the multifractal properties of μ_ϕ are important to state our results. Let us recall some standard facts on multifractal analysis of Borel measures.

Definition 1.4. *For any Borel probability measure* μ *on* $[0,1]$, *let us define the lower (*resp. *upper) local dimension* $\underline{d}_\mu(y)$ *(*resp. $\overline{d}_\mu(y)$*) of* μ *at* $y \in [0,1]$ *by*

$$\underline{d}_\mu(y) := \liminf_{r\to 0} \frac{\log \mu(B(y,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(y) := \limsup_{r\to 0} \frac{\log \mu(B(y,r))}{\log r}.$$

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When $\underline{d}_\mu(y) = \overline{d}_\mu(y)$*, their common value is denoted by* $d_\mu(y)$*, and is simply called the local dimension of* μ *at* y*.*

We then consider the level sets of the local dimension, i.e.

$$\text{for every } \alpha \geq 0, \quad \mathcal{E}_\mu(\alpha) = \{y \in [0,1] : d_\mu(y) = \alpha\}, \tag{1·6}$$

and the multifractal spectrum *of* μ*, defined as the application*

$$D_\mu : \alpha \geq 0 \ \longmapsto \ \dim_H \mathcal{E}_\mu(\alpha).$$

By an extensive literature (Collet, Lebowitz and Porzio [12], Rand [30], Brown, Michon and Peyrière [8], Simpelaere [32], Barreira, Pesin and Schmeling [4], Pesin and Weiss [26, 27]), the multifractal analysis of μ_ϕ can be achieved, i.e. the multifractal spectrum of μ_ϕ can be computed (see Section 2.3 for more details).

Denote by $\mathcal{M}_{\text{inv}}$ the set of T-invariant probability measures on $[0,1]$. The dimension of a Borel probability measure μ is defined as

$$\dim_H \mu = \inf\{\dim_H E : E \text{ Borel set} \subset [0,1] \text{ and } \mu(E) > 0\}.$$

Theorem 1.5. *The multifractal spectrum* D_{μ_ϕ} *of* μ_ϕ *is a concave analytic map on the interval* $]\alpha_-, \alpha_+[$*, where*

$$\alpha_- := \min_{\mu \in \mathcal{M}_{\text{inv}}} \frac{\int_{[0,1]} (-\phi)\, d\mu}{\int_{[0,1]} \log|T'|\, d\mu} \quad \text{and} \quad \alpha_+ := \max_{\mu \in \mathcal{M}_{\text{inv}}} \frac{\int_{[0,1]} (-\phi)\, d\mu}{\int_{[0,1]} \log|T'|\, d\mu}.$$

The spectrum D_{μ_ϕ} *reaches its maximum value 1 at a unique exponent* $\alpha_{\max}$ *defined by*

$$\alpha_{\max} := \frac{\int_{[0,1]} (-\phi)\, d\mu_{\max}}{\int_{[0,1]} \log|T'|\, d\mu_{\max}}, \tag{1·7}$$

where $\mu_{\max}$ *is the Gibbs measure associated with the potential* $\psi = -\log|T'|$ *(*$\mu_{\max}$ *is known to be equivalent to the Lebesgue measure).*

Finally, the graph of D_{μ_ϕ} *and the first bisector intersect at a unique point which is* $(\dim_H \mu_\phi, \dim_H \mu_\phi)$*. Moreover,* $\dim_H \mu_\phi$ *satisfies*

$$\dim_H \mu_\phi = \frac{\int_{[0,1]} (-\phi)\, d\mu_\phi}{\int_{[0,1]} \log|T'|\, d\mu_\phi}.$$

We also recall that this spectrum can be computed as the Legendre transform of the scaling function (also called the partition function or the L^q-spectrum) of μ_ϕ, but we do not need these properties yet (see Section 2.3 for further details).

We are now ready to state our main theorem. Denote by *Leb* the one-dimensional Lebesgue measure.

Theorem 1.6. *Let* $T : [0,1] \to [0,1]$ *be an expanding Markov map. Let* ϕ *be a Hölder continuous potential and let* μ_ϕ *be the corresponding Gibbs measure. For* μ_ϕ*-almost every* $x \in [0,1]$*:*

(1) *The Hausdorff dimension of* $\mathcal{L}^\delta(x)$ *satisfies*

$$\dim_H \mathcal{L}^\delta(x) = \begin{cases} \frac{1}{\delta} & \text{if } 0 < \frac{1}{\delta} \leq \dim_H \mu_\phi, \\ D_{\mu_\phi}(\frac{1}{\delta}) & \text{if } \dim_H \mu_\phi < \frac{1}{\delta} \leq \alpha_{\max}, \\ 1 & \text{if } \frac{1}{\delta} > \alpha_{\max}. \end{cases} \tag{1·8}$$

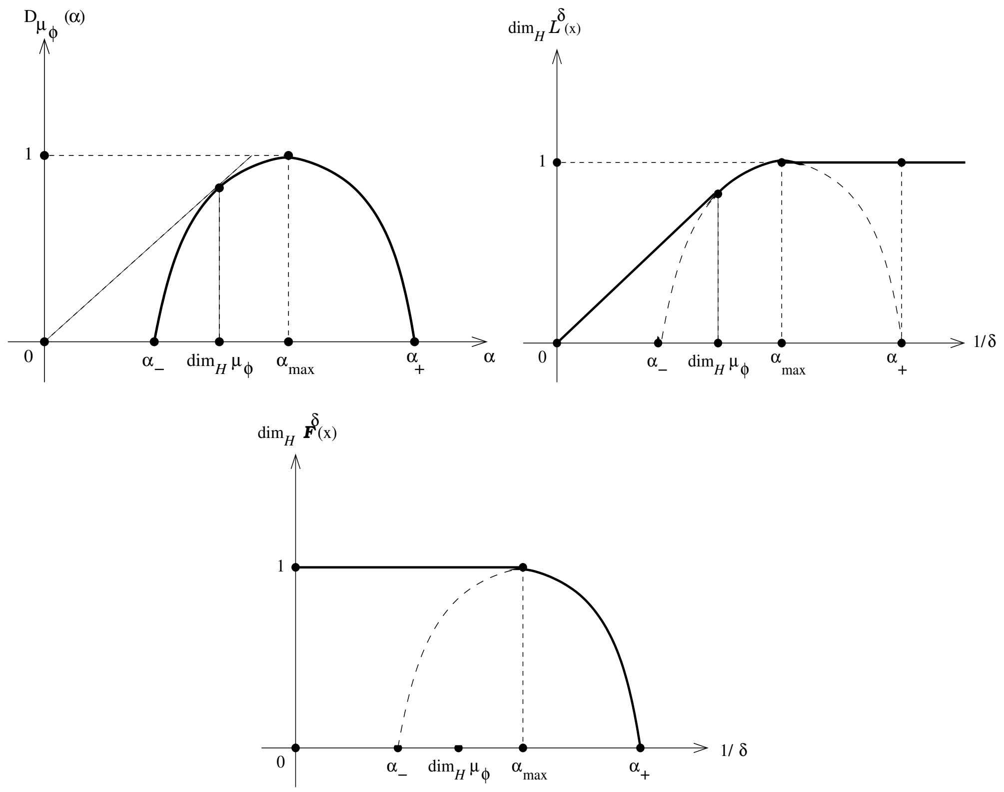

FIGURE 1. Multifractal spectrum of the measure μ_ϕ, and the two maps $1/\delta \mapsto \dim_H \mathcal{L}^\delta(x)$ and $1/\delta \mapsto \dim_H \mathcal{F}^\delta(x)$

(2) *The Hausdorff dimension of $\mathcal{F}^\delta(x)$ satisfies*

$$\dim_H \mathcal{F}^\delta(x) = \begin{cases} 1 & \text{if } 0 < \frac{1}{\delta} \le \alpha_{\max}, \\ D_{\mu_\phi}(\frac{1}{\delta}) & \text{if } \frac{1}{\delta} > \alpha_{\max}. \end{cases} \tag{1·9}$$

(3) *Concerning the Lebesgue measure of $\mathcal{L}^\delta(x)$ and $\mathcal{F}^\delta(x)$, we have:*

$$Leb(\mathcal{L}^\delta(x)) = 1 - Leb(\mathcal{F}^\delta(x)) = \begin{cases} 0 & \text{if } 0 < \frac{1}{\delta} < \alpha_{\max}, \\ 1 & \text{if } \frac{1}{\delta} > \alpha_{\max}. \end{cases} \tag{1·10}$$

(4) *If $1/\delta > \alpha_+$, then $\mathcal{F}^\delta(x) = \emptyset$ and hence $\mathcal{L}^\delta(x) = [0,1]$.*

Remark 1.7. *For the critical point $1/\delta = \alpha_{\max}$, we see that the dimensions of $\mathcal{L}^\delta(x)$ and $\mathcal{F}^\delta(x)$ are 1, but their Lebesgue measures are not determined in this paper. Similarly, if $1/\delta = \alpha_+$, we are able to prove that Leb($\mathcal{L}^\delta(x)$) = 1, and $\dim_H \mathcal{F}^\delta(x) \le \lim_{1/\delta \to \alpha_+} D_{\mu_\phi}(1/\delta)$, but we can not judge whether $\mathcal{L}^\delta(x) = [0,1]$ or not.*

The mapping $1/\delta \longmapsto \dim_H \mathcal{L}^\delta(x)$ exhibits clearly four distinct behaviors (see Figure 1), that we denote respectively by Part I (for $\frac{1}{\delta} \le \dim_H \mu_\phi$), Part II (for $\dim_H \mu_\phi < \frac{1}{\delta} \le \alpha_{\max}$), Part III (for $\alpha_{\max} < \frac{1}{\delta} \le \alpha_+$) and finally Part IV (for $\frac{1}{\delta} > \alpha_+$).

Let us make some comments on the results of Theorem 1.6. The behavior of $\mathcal{L}^\delta(x)$, for μ_ϕ-typical x, possesses two remarkable characteristics when compared to classical Diophantine approximation results:

- the map $1/\delta \longmapsto \dim_H \mathcal{L}^\delta(x)$ may have a strictly concave part (Part II),
- the smallest δ for which $Leb\,(\mathcal{L}^\delta(x)) = 1$ and the smallest δ for which $\mathcal{L}^\delta(x) = [0,1]$ do not coincide (Part III).

This is in sharp contrast with the classical results on Diophantine approximation, especially with the approximation by rational numbers. In this (historical) context, as said above, the analog of the sets $\mathcal{L}^\delta(x)$ are the sets $S(\delta)$ defined by (1·3). In this case, the Dirichlet theorem ensures that $\mathcal{S}(1) = [0,1]$, and it is well-known [20, 6] that for every $\delta > 1$, $\dim_H \mathcal{S}(\delta) = 1/\delta$. In particular, the dimension of $\mathcal{S}(\delta)$ decreases linearly with respect to $1/\delta$, and as soon as the Lebesgue measure of $\mathcal{S}(\delta)$ reaches one, it instantaneously covers the whole interval $[0,1]$. Comparable results hold for sets of numbers approximated by other families (see for instance [2, 3, 9].

To our opinion, the two characteristics of the sets $\mathcal{L}^\delta(x)$ mentioned above must be interpreted by the fact that, although the orbits $\{T^n x\}_{n\geq 1}$ of μ_ϕ-typical points x are dense, they are not as regularly distributed when n tends to infinity as the rational numbers are. The exponents $\dim_H \mu_\phi$, $\alpha_{\max}$ and α_+ characterize this "distortion".

The paper is organized as follows. Section 2 contains the necessary material in multifractal analysis and the definition of the key notion of hitting time associated with a dynamical system. Section 3 describes the relation between sets of points having given hitting times, the limsup sets $\mathcal{L}^\delta(x)$, the sets $\mathcal{F}^\delta(x)$, and the local dimensions of μ_ϕ. From these relations we will give a direct proof of item (3) of Theorem 1.6. In Section 4, two key lemmas are proved. They illustrate the fact that intervals which have a small local dimension for μ_ϕ (or equivalently, which have a large μ_ϕ-mass) are hit by the balls $B(T^n x, 1/n^\delta)$ with big probability, and *vice-versa*. Then, Sections 5, 6, 7 and 8 contains the proofs of the upper and lower bounds for the Hausdorff dimensions of $\mathcal{L}^\delta(x)$ and $\mathcal{F}^\delta(x)$ for Parts I, IV, III and II, respectively.

Remark 1.8. *Theorem 1.6 is similar to the results of Fan, Schmeling and Troubetzkoy in* [16] *for the doubling map on $\mathbb{T}^1$. But there are many differences:*

- *For $x \mapsto 2x$, since the Lyapunov exponents are constant, the intervals of generation n have same lengths, while for the Markov maps their lengths may be of very different order. In* [16]*, the authors focus on the Bowen's topological entropy spectrum (using techniques of words combinatorics). In our case, the non-constant Lyapunov exponents bring many difficulties.*
- *The notions of local Hölder exponent and hitting time for the doubling map involve only cylinders, while we need centered balls in the definitions of the similar quantities on the interval.*

Some arguments (Lemmas 4.2, 4.3 and 4.5) are adapted from those of [16] *to the context of Markov maps, but several others do not apply at all. The best example is the difficult lower bound for $\dim_H \mathcal{L}^\delta(x)$ when $0 < \frac{1}{\delta} \leq \dim_H \mu_\phi$.*

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2. First definitions, and recalls on multifractal analysis

2.1. Covering of $[0,1]$ by basic intervals.

Recall that we have the distortion property (1·4): there is a constant $L > 1$ such that for every integer $n \geq 2$, for every $(i_1 i_2 \cdots i_n) \in \mathcal{A}^n$,

$$1 \leq \frac{|I_{i_1 i_2 \cdots i_{n-1}}|}{|I_{i_1 i_2 \cdots i_{n-1} i_n}|} \leq L.$$

It is obvious that the intervals $\mathcal{G}_n$ of a given generation n form a covering of $[0,1]$. This covering of $[0,1]$ is not composed of intervals of same length.

But using (1·4), for every real number $0 < r < 1$, one easily shows that there is a finite family of basic intervals $J_1, J_2, \cdots, J_N$ (not belonging to the same $\mathcal{G}_n$'s) such that:

- $\bigcup_{j=1}^N J_j = [0,1]$
- for every $j \neq j'$, the intersection of interiors $\mathring{J}_j \cap \mathring{J}_{j'}$ is empty,
- for every $j \in \{1, 2, \cdots, N\}$,
$$L^{-1} r \leq |J_j| \leq Lr, \tag{2·11}$$
for the same constant L as in (1·4).

For $n \in \mathbb{N}$, we denote by $\mathcal{C}_n$ one possible corresponding collection of intervals such that (2·11) holds for $r = 2^{-n}$. From the above considerations, one deduces that there exists a number $L' > 1$ such that the generation n_J of a basic interval $J \in \mathcal{C}_n$ satisfies

$$(L')^{-1} n \leq n_J \leq L' n. \tag{2·12}$$

The sequence of sets of intervals $(\mathcal{C}_n)_{n \geq 1}$ will be used often in the sequel.

2.2. Hitting times.

Hitting times will play a major role along the proof of our main theorem. These quantities are related both to the local dimension of Gibbs measures, and to the covering properties of $\mathcal{L}^\delta(x)$.

For every $x \in [0,1]$, consider the orbits of x

$$\mathcal{O}(x) := \{T^n x : n \geq 0\}, \quad \mathcal{O}^+(x) := \mathcal{O}(x) \setminus \{x\}.$$

Definition 2.1. *For every $(x, y) \in [0,1]^2$ and $r > 0$, we define the hitting time (first entrance time) of the orbit of x in the ball $B(y, r)$ by*

$$\tau_r(x, y) := \inf\{n \geq 1 : T^n x \in B(y, r)\}.$$

Then we set

$$\underline{R}(x, y) := \liminf_{r \to 0} \frac{\log \tau_r(x, y)}{-\log r}. \tag{2·13}$$

By convention $\tau_r(x, y) = +\infty$ if $T^n x$ never meets $B(y, r)$, and for such a couple (x, y), $\underline{R}(x, y) = +\infty$. These quantities can of course be defined for any dynamical system (X, T).

We will also define the hitting time of a basic interval by a point x. Let $m \geq 1$ and let $C \in \mathcal{G}_m$. We set

$$\tau(x, C) := \inf\{n \geq 0 : T^n x \in C\}. \tag{2·14}$$

If $T^n x \notin C$ for every integer $n \geq 0$, by convention we set $\tau(x, C) = +\infty$.

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The following sets will be key in the sequel.

Definition 2.2. *Let $s \geq 0$ be a real number. We define the sets*

$$\mathcal{R}_{\geq s}(x) = \{y \in [0,1] : \underline{R}(x,y) \geq s\} \quad \text{and} \quad \mathcal{R}_{\leq s}(x) = \{y \in [0,1] : \underline{R}(x,y) \leq s\}. \tag{2·15}$$

Similarly, when the inequalities are strict, we define

$$\mathcal{R}_{>s}(x) = \{y \in [0,1] : \underline{R}(x,y) > s\} \quad \text{and} \quad \mathcal{R}_{<s}(x) = \{y \in [0,1] : \underline{R}(x,y) < s\}. \tag{2·16}$$

2.3. **Multifractal analysis of μ_ϕ.**

Consider an expanding Markov map $T : [0,1] \to [0,1]$ and a Hölder continuous potential $\phi : [0,1] \to \mathbb{R}$ that we suppose *normalized*, i.e. the topological pressure $P(\phi)$ in (1·5) is equal to 0 (in other words, we replace ϕ by $\phi - P(\phi)$). All the following results are standard results, and can be found in the references we cited in the precedent section: [12, 30, 8, 32, 4, 26, 27].

For every $q \in \mathbb{R}$, there is a unique real number $\eta_\phi(q)$ such that the topological pressure $P(-\eta_\phi(q)\log|T'| + q\phi)$ associated with the Hölder potential $\phi_q := -\eta_\phi(q)\log|T'| + q\phi$ equals 0. Such a number exists since the map $P : t \mapsto (-t\log|T'| + q\phi)$ is real-analytic and decreasing in t. The resulting function $q \mapsto \eta_\phi(q)$ is real-analytic and concave. We denote by

$$\mu_q := \mu_{\phi_q}, \quad \text{where } \phi_q := -\eta_\phi(q)\log|T'| + q\phi \tag{2·17}$$

the Gibbs measure associated with the potential ϕ_q. Observe that $\eta_\phi(0) = 1$, and $\eta_\phi(1) = 0$. The measures μ_0 and $\mu_1(= \mu_\phi)$ are associated with the potentials $\phi_0 = -\log|T'|$ and $\phi_1 = \phi$ respectively. By a folklore theorem, the Lebesgue measure *Leb* is equivalent to the Gibbs state μ_0.

For every $q \in \mathbb{R}$, we introduce the exponent

$$\alpha(q) = \frac{\int_{[0,1]}(-\phi)\,d\mu_q}{\int_{[0,1]}\log|T'|\,d\mu_q}. \tag{2·18}$$

By the Gibbs property of μ_ϕ and the ergodicity of μ_q, the measure μ_q is supported by the level set $\mathcal{E}_{\mu_\phi}(\alpha(q))$ (defined by (1·6)), i.e. by

$$\left\{y : \lim_{r\to o}\frac{\log\mu_\phi(B(y,r))}{\log r} = \alpha(q)\right\}. \tag{2·19}$$

We deduce that

$$D_{\mu_\phi}(\alpha(q)) = \dim_H \mu_q = \eta_\phi(q) + q\alpha(q).$$

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The function $q \mapsto \alpha(q)$ is decreasing, and

$$\lim_{q\to+\infty} \alpha(q) = \inf_{\nu\in\mathcal{M}_{\rm inv}} \frac{\int_{[0,1]}(-\phi)d\nu}{\int_{[0,1]}\log|T'|d\nu} = \alpha_- , \tag{2·20}$$

$$\alpha(1) = \frac{\int_{[0,1]}(-\phi)d\mu_\phi}{\int_{[0,1]}\log|T'|d\mu_\phi} = \dim_H \mu_\phi , \tag{2·21}$$

$$\alpha(0) = \frac{\int_{[0,1]}(-\phi)d\mu_0}{\int_{[0,1]}\log|T'|d\mu_0} = \alpha_{\max} , \tag{2·22}$$

$$\lim_{q\to-\infty} \alpha(q) = \sup_{\nu\in\mathcal{M}_{\rm inv}} \frac{\int_{[0,1]}(-\phi)d\nu}{\int_{[0,1]}\log|T'|d\nu} = \alpha_+ . \tag{2·23}$$

Notice that $\mu_0 = \mu_{\max}$, the measure described in Theorem 1.5. We write

$$\alpha \mapsto q(\alpha) \tag{2·24}$$

($\alpha \in]\alpha_-, \alpha_+[$) for the inverse function of $q \mapsto \alpha(q)$.

Denote by $\mathcal{M}_{\rm erg}$ the set of ergodic T-invariant probability measures on $[0,1]$. The following fact from multifractal analysis will be used.

Lemma 2.3. *We have*

$$\sup_y \ d_{\mu_\phi}(y) = \sup_{v\in\mathcal{M}_{\rm erg}} \frac{-\int_{[0,1]}\phi\, d_\nu}{\int_{[0,1]}\log|T'|d\nu} = \alpha_+,$$

where the supremum is taken over all y's for which the limit $d_{\mu_\phi}(y)$ exists.

Again, as in Definition 2.2, some sets will be repeatedly used in the sequel.

Definition 2.4. *Let $s \geq 0$ be a real number. We define the sets*

$$\mathcal{E}_{\geq s} = \{y\in[0,1] : d_{\mu_\phi}(y) \geq s\} \ \ \textit{and} \ \ \mathcal{E}_{\leq s} = \{y\in[0,1] : d_{\mu_\phi}(y) \leq s\}. \tag{2·25}$$

Similarly, when the inequalities are strict, we define

$$\mathcal{E}_{> s} = \{y\in[0,1] : d_{\mu_\phi}(y) > s\} \ \ \textit{and} \ \ \mathcal{E}_{< s} = \{y\in[0,1] : d_{\mu_\phi}(y) < s\}. \tag{2·26}$$

From the standard large deviations theory, we get the useful upper bounds for the Hausdorff dimensions of the sets $\mathcal{E}$ in Definition 2.4 (see [8] among many references). These upper bounds differ whether s is located in the increasing or in the decreasing part of the multifractal spectrum of μ_ϕ.

Proposition 2.5. *Let ϕ be a Hölder potential and let μ_ϕ be its associated Gibbs measure. Then:*

1. *For every $s < \alpha_{\max}$, $\dim_H \left(\mathcal{E}_{<s}\right) = \dim_H \left(\mathcal{E}_{\leq s}\right) = D_{\mu_\phi}(s)$.*
2. *For every $s > \alpha_{\max}$, $\dim_H \left(\mathcal{E}_{>s}\right) = \dim_H \left(\mathcal{E}_{\geq s}\right) = D_{\mu_\phi}(s)$.*

Actually, the above inequalities hold for any Borel probability measure μ if $D_{\mu_\phi}(s)$ is replaced by the Legendre transform of the L^q-spectrum of μ. Nevertheless, since the Gibbs measures we consider satisfy a *multifractal formalism*, the two quantities coincide. To avoid unnecessary definitions, we will only use the formulas as stated in Proposition 2.5.

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3. Hitting times

3.1. **Orbits and hitting times.**

In this subsection we discuss the relationship between the orbit of a point x and hitting times.

Lemma 3.1. *The following three assertions are equivalent:*

(1) *There exists an integer $n_0 \geq 1$ such that $y = T^{n_0}x$ (i.e. $y \in \mathcal{O}^+(x)$).*
(2) *The hitting time $\tau_r(x, y)$ is bounded for all $r > 0$.*
(3) *There is a sequence $r_i \to 0$ such that $\tau_{r_i}(x, y)$ is bounded.*

The proof is left to the reader. Next lemmas investigate the relationship between the set $\mathcal{L}^\delta(x)$ and hitting times.

Lemma 3.2. *For every $\delta > 0$, we have the two embedding properties:*

$$\mathcal{R}_{<1/\delta}(x) \setminus \mathcal{O}^+(x) \subset \quad \mathcal{L}^\delta(x) \quad \subset \mathcal{R}_{\leq 1/\delta}(x) \tag{3·27}$$

$$\mathcal{R}_{>1/\delta}(x) \subset \quad \mathcal{F}^\delta(x) \quad \subset \mathcal{R}_{\geq 1/\delta}(x) \cup \mathcal{O}^+(x). \tag{3·28}$$

Proof. We prove (3·27), since (3·28) is deduced by taking complements.

For the first inclusion, consider y such that $\underline{R}(x, y) < \frac{1}{\delta}$ and $y \notin \mathcal{O}^+(x)$. Choose $\varepsilon > 0$ such that $\underline{R}(x, y) < \frac{1}{\delta} - \varepsilon$. Then by definition of $\underline{R}(x, y)$, there is a sequence of positive numbers (r_i) such that $\tau_{r_i}(x, y) < \left(\frac{1}{r_i}\right)^{\frac{1}{\delta}-\varepsilon}$. Consider the sequence of integers $n_i := \tau_{r_i}(x, y)$, for all $i \geq 1$. Since $y \notin \mathcal{O}^+(x)$, Lemma 3.1 yields that $(n_i)_{i\geq 1}$ is not bounded. Let us remark then that, by construction,

$$r_i < (n_i)^{-1/(\frac{1}{\delta}-\varepsilon)} < n_i^{-\delta}.$$

Thus $T^{n_i}x \in B(y, n_i^{-\delta})$, or equivalently $y \in B(T^{n_i}x, n_i^{-\delta})$, for infinitely many increasing integers n_i. Subsequently, $y \in \mathcal{L}^\delta(x)$.

For the second inclusion of (3·27), consider $y \in \mathcal{L}^\delta(x)$. By definition, $T^{n_i}x \in B(y, n_i^{-\delta})$ for infinitely many integers $(n_i)_{i\geq 1}$. Hence, for these n_i, we have $\tau_{1/n_i^\delta}(x, y) \leq n_i$, which implies that

$$\underline{R}(x, y) \leq \liminf_{i\to\infty} \frac{\log \tau_{1/n_i^\delta}(x, y)}{-\log(1/n_i^\delta)} \leq \liminf_{n_i\to\infty} \frac{\log n_i}{\delta \log n_i} = \frac{1}{\delta}.$$

This completes the proof. □

Lemma 3.3. *Suppose that the orbit of x is not finite (i.e. x is not eventually periodic). If $y \in \mathcal{O}^+(x)$, then we have:*

$$y \in \mathcal{L}^\delta(x) \ \text{ if } \underline{R}(y, y) < \frac{1}{\delta} \qquad \text{and} \qquad y \in \mathcal{F}^\delta(x) \ \text{ if } \underline{R}(y, y) > \frac{1}{\delta}.$$

Observe that the case where $\underline{R}(y, y) = 1/\delta$ is not determined yet.

Proof. Suppose that $y \in \mathcal{O}^+(x)$. Since x is not eventually periodic, there exists a unique positive integer n_0 such that $T^{n_0}x = y$. Then $y \notin \mathcal{O}^+(y)$. The rest of the proof is the same as that of Lemma 3.2. □

Lemma 3.4. *Suppose that μ is an invariant measure with respect to T, and that μ has no atoms. Then for μ-almost all x, we have*

$$\mathcal{O}^+(x) \subset \mathcal{L}^\delta(x) \text{ if } \frac{1}{\delta} > \dim_H \mu \quad \text{and} \quad \mathcal{O}^+(x) \subset \mathcal{F}^\delta(x) \text{ if } \frac{1}{\delta} < \dim_H \mu.$$

Proof. Remark that the set of eventually periodic points is a countable set, hence it has a μ-measure equal to zero. By Ornstein-Weiss Theorem [25], for μ-almost all x, we have the property that

$$\text{for every } n \geq 1, \qquad \underline{R}(T^n x, T^n x) = \dim_H \mu. \tag{3·29}$$

Hence, for a μ-typical x (which is not eventually periodic), consider $y \in \mathcal{O}^+(x)$. By the same argument as above, $y = T^{n_0}x$ for some unique integer $n_0 \geq 0$. By (3·29), $\underline{R}(y,y) = \dim_H \mu$, and applying Lemma 3.3, we find that if $\frac{1}{\delta} > \underline{R}(y,y) = \dim_H \mu$ (*resp.* $\frac{1}{\delta} < \underline{R}(y,y)$) then $y \in \mathcal{L}^\delta(x)$ (*resp.* $y \in \mathcal{F}^\delta(x)$). □

3.2. **Local dimension and hitting times.**

As said in the introduction, Gibbs measures enjoy exponential decay of correlations (see Ruelle [31], Parry and Pollicott [28], Liverani, Saussol and Vaienti [24], and Baladi [1]). More precisely, we have the following theorem.

Theorem 3.5. *Suppose that $f : [0,1] \to [0,1]$ has bounded variations and $g : [0,1] \to [0,1]$ is integrable. Then there exist constants $0 < \beta < 1$ and $\Theta > 0$, such that for every integer $n \geq 1$,*

$$\left| \int fg \circ T^n d\mu_\phi - \int f d\mu_\phi \int g d\mu_\phi \right| \leq \Theta\beta^n \left(\int |f| d\mu_\phi + \mathrm{var}(f) \right) \int |g| d\mu_\phi,$$

where $\mathrm{var}(f)$ *stands for the total variation of f on $[0,1]$. In particular, if $f = \mathbf{1}_A$ and $g = \mathbf{1}_B$ where A is an interval and B is a measurable set, then for every n,*

$$\left| \mu_\phi(A \cap T^{-n}B) - \mu_\phi(A)\mu_\phi(B) \right| \leq \Theta\beta^n \left(\mu_\phi(A) + 2 \right) \mu_\phi(B). \tag{3·30}$$

Theorem 3.5 allows us to use the following theorem borrowed from [17], which gives some clues about the relationship between hitting time and local dimension of invariant measures.

Theorem 3.6 (Galatolo [17]). *If (X, T, μ) has superpolynomial decay of correlations and if $d_\mu(y)$ exists, then for μ-almost every x we have*

$$\underline{R}(x,y) = d_\mu(y).$$

We return to the study of the expanding Markov map T on the interval $[0,1]$.

Corollary 3.7. *Let μ_ϕ be a T-invariant Gibbs probability measure on $[0,1]$ associated with a normalized Hölder potential ϕ (i.e. $P(\phi) = 0$ in (1·5)). For any invariant ergodic measure ν, we have*

$$\text{for } \mu_\phi\text{-a.e. } x, \ \text{ for } \nu\text{-a.e. } y, \ \underline{R}(x,y) = d_{\mu_\phi}(y) = \frac{-\int_{[0,1]} \phi \, d\nu}{\int_{[0,1]} \log |T'| \, d\nu}.$$

Proof. Let ν be a T-invariant ergodic measure on $[0,1]$. For ν-almost every y, $\frac{1}{n} S_n\phi(y)$ tends to $\int_{[0,1]} \phi \, d\nu$. Hence, by definition of d_{μ_ϕ} and the Gibbs property of μ_ϕ, for ν-almost every y, $d_{\mu_\phi}(y)$ exists and is equal to

$$\frac{-\int_{[0,1]} \phi \, d\nu}{\int_{[0,1]} \log |T'| \, d\nu}.$$

Thus by Theorem 3.6, for $\nu-$almost every y,

$$\underline{R}(x,y) = d_{\mu_\phi}(y) = \frac{-\int_{[0,1]} \phi \, d\nu}{\int_{[0,1]} \log |T'| \, d\nu} \qquad \text{for } \mu_\phi\text{-almost every } x.$$

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The lemma then follows by applying the Fubini Theorem. □

Corollary 3.8. *Let μ_ϕ, μ_ψ be two T-invariant Gibbs probability measures on $[0,1]$ associated with normalized Hölder potentials ϕ and ψ. Then*

$$\text{for } \mu_\phi \times \mu_\psi\text{−almost every } (x,y), \quad \underline{R}(x,y) = d_{\mu_\phi}(y) = \frac{\int_{[0,1]}(-\phi)d\mu_\psi}{\int_{[0,1]} \log|T'|d\mu_\psi}.$$

3.3. First results on covering.

The above considerations lead us to introduce the real number

$$\delta(\phi,\psi) := \sup\left\{\delta : \mu_\psi(\mathcal{L}^\delta(x)) = 1 \text{ for } \mu_\phi - \text{almost every } x\right\}.$$

The following proposition, which summarizes the results of the previous sections, will be useful when proving the lower bound for Part I of Theorem 1.6. We can also give a direct proof for the item (3) of Theorem 1.6.

Recall the notations $q(\alpha)$ and $\phi_{q(\alpha)}$ in Section 2.3.

Proposition 3.9. *For any normalized Hölder potentials ϕ and ψ, we have*

$$\delta(\phi,\psi) = \frac{\int_{[0,1]} \log|T'|\, d\mu_\psi}{\int_{[0,1]} (-\phi)\, d\mu_\psi}.$$

In particular, for every $\alpha \in]\alpha_-, \alpha_+[$,

$$\delta(\phi, \phi_{q(\alpha)}) = \frac{1}{\alpha}, \tag{3·31}$$

i.e.

$$\sup\left\{\delta : \mu_{q(\alpha)}(\mathcal{L}^\delta(x)) = 1 \text{ for } \mu_\phi - a.e.\ x\right\} = \frac{1}{\alpha}.$$

Proof. Combine Lemma 3.2, Corollary 3.8 and the definition of $\alpha(q)$ (formula (2·18)). □

Now we are able to give a direct proof for the item (3) of Theorem 1.6.

Proof. [Direct proof for the item (3) of Theorem 1.6] Take the potential $\psi := \phi_0 = -\log|T'|$. As we have already observed, the corresponding Gibbs measure μ_0 is an invariant measure which is equivalent to the Lebesgue measure. Subsequently, μ_0-almost everywhere is equivalent to Lebesgue-almost everywhere, hence $\delta(\phi,\psi)$ is also equal to

$$\sup\left\{\delta : \text{Leb}(\mathcal{L}^\delta(x)) = 1 \text{ for } \mu_\phi - a.e.\ x\right\}.$$

From Proposition 3.9, applying (3·31) with the measure μ_0, the exponent $\delta(\phi,\psi)$ coincides with $\frac{1}{\alpha_{\max}}$ (defined by (1·7)). This concludes the proof. □

Now, we investigate other exponents, which will be used for proving the lower bound for Part I of Theorem 1.6.

- By Proposition 3.9, for any $\varepsilon > 0$, and $\delta \in]1/\alpha_+, 1/\alpha_-[$, for μ_ϕ-almost every x, we have
$$\mu_{q\left(1/\delta\right)}(\mathcal{L}^{\delta-\varepsilon}(x)) = 1. \tag{3·32}$$

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- For $1/\delta = \alpha(1) = \dim_H \mu_\phi$, we have $q(1/\delta) = 1$ and $\mu_q = \mu_1 = \mu_\phi$. Hence, applying Proposition 3.9 and (2·21), we get
$$\sup\left\{\delta : \mu_\phi(\mathcal{L}^\delta(x)) = 1 \text{ for } \mu_\phi - a.e.\ x\right\} = \frac{1}{\dim_H \mu_\phi},$$
and thus for every $\delta < \frac{1}{\dim_H \mu_\phi}$, we have $\mu_\phi(\mathcal{L}^\delta(x)) = 1$ for μ_ϕ-almost every x.

4. Multiple-quasi-Bernoulli inequalities and hitting lemmas

4.1. Multiple-quasi-Bernoulli inequalities.

Recall that the potential ϕ is normalized, i.e. $P(\phi) = 0$. In this case, the Gibbs property (1·5) of μ_ϕ can be written as

$$\forall\, x \in [0,1],\ \forall\, n \geq 1, \qquad \frac{1}{\gamma} e^{S_n\phi(x)} \leq \mu_\phi(I_n(x)) \leq \gamma e^{S_n\phi(x)}, \tag{4·33}$$

where $S_n\phi(y) = \sum_{j=0}^{n-1} \phi(T^j y)$ is the Birkhoff sum associated with ϕ.

It is classical that the Gibbs property (4·33) implies the following so-called quasi-Bernoulli property of μ_ϕ.

Lemma 4.1. *For any couple of basic intervals A and B of respective generation n_A and n_B, we have*

$$\frac{1}{\gamma^3}\mu_\phi(A)\mu_\phi(B) \leq \mu_\phi(A \cap T^{-n_A}B) \leq \gamma^3 \mu_\phi(A)\mu_\phi(B). \tag{4·34}$$

Proof. Consider any $x \in A \cap T^{-n_A}B$. Applying (4·33) three times, we get

$$\mu_\phi(A \cap T^{-n_A}B) \geq \frac{1}{\gamma} e^{S_{n_A}\phi(x) + S_{n_B}(T^{n_A}x)} \geq \frac{1}{\gamma^3}\mu_\phi(A)\mu_\phi(B), \quad \text{and}$$
$$\mu_\phi(A \cap T^{-n_A}B) \leq \gamma e^{S_{n_A}\phi(x) + S_{n_B}(T^{n_A}x)} \leq \gamma^3\mu_\phi(A)\mu_\phi(B).$$

□

Furthermore, by applying the exponential decay of correlations (3·30), we have the following multiple-quasi-Bernoulli inequalities (the same inequalities were referred to as *multi-relation* for the doubling maps in [16]).

Recall in Subsection 2.1 that $\mathcal{C}_n$ is a covering of the interval $[0,1]$ constituted of basic intervals with comparable lengths of order 2^{-n}.

Lemma 4.2. *Let μ_ϕ be the Gibbs measure associated with a normalized potential ϕ. Let $n \in \mathbb{N}, n \geq 1$ and let $C_0, C_1, \dots, C_k$ be $(k+1)$ basic intervals in $\mathcal{C}_n$.*

There exist an integer ω large enough and a constant $M > 0$ (which are independent of the choice of n) such that

$$\frac{1}{\gamma^3}(1 - M\beta^{\omega n})^{k-1} \leq \frac{\mu_\phi\Big(C_0 \cap \bigcap_{j=1}^{k} T^{-2j\omega n} C_j\Big)}{\prod_{j=0}^{k} \mu_\phi(C_j)} \leq \gamma^3(1 + M\beta^{\omega n})^{k-1}, \tag{4·35}$$

where β is the constant appearing in (3·30).

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Proof. Let n_0 be the generation of C_0, and let ω be an integer so large that $n_0 - 2\omega n \leq -1$. Observe that

$$C_0 \cap \bigcap_{j=1}^{k} T^{-2j\omega n} C_j = C_0 \cap T^{-n_0}\mathcal{B}$$

where $\mathcal{B} = \bigcap_{j=1}^{k} T^{n_0-2j\omega n} C_j$ is a finite union of disjoint basic intervals, that we denote by B_i's. Applying the quasi-Bernoulli property (4·34) to $A = C_0$ and to each $B = B_i$, we obtain

$$\frac{1}{\gamma^3}\mu_\phi(C_0)\mu_\phi(B_i) \leq \mu_\phi(C_0 \cap T^{-n_0}B_i) \leq \gamma^3\mu_\phi(C_0)\mu_\phi(B_i).$$

Then, summing over all the B_i's, we get

$$\frac{1}{\gamma^3}\mu_\phi(C_0)\mu_\phi(\mathcal{B}) \leq \mu_\phi(C_0 \cap T^{-n_0}\mathcal{B}) \leq \gamma^3\mu_\phi(C_0)\mu_\phi(\mathcal{B}). \tag{4·36}$$

The invariance of μ_ϕ implies that

$$\mu_\phi(\mathcal{B}) = \mu_\phi\left(\bigcap_{j=1}^{k} T^{-2j\omega n} C_j\right).$$

Thus, in order to get (4·35), we need only to prove that for some constant M,

$$(1 - M\beta^{\omega n})^{k-1} \leq \frac{\mu_\phi\left(\bigcap_{j=1}^{k} T^{-2j\omega n} C_j\right)}{\prod_{j=1}^{k}\mu_\phi(C_j)} \leq (1 + M\beta^{\omega n})^{k-1}. \tag{4·37}$$

Recalling the exponential decay of correlation (3·30), for every choice of two basic intervals A, B, and for every integer m, we have

$$\left|\mu_\phi(A \cap T^{-m}B) - \mu_\phi(A)\mu_\phi(B)\right| \leq \Theta\beta^m\left(\mu_\phi(A) + 2\right)\mu_\phi(B),$$

which can be rewritten as

$$\left(1 - \Theta\beta^m\frac{\mu_\phi(A)+2}{\mu_\phi(A)}\right) \leq \frac{\mu_\phi(A \cap T^{-m}B)}{\mu_\phi(A)\mu_\phi(B)} \leq \left(1 + \Theta\beta^m\frac{\mu_\phi(A)+2}{\mu_\phi(A)}\right). \tag{4·38}$$

Consider the intervals $C_1, \cdots, C_k$ and observe that

$$\bigcap_{j=1}^{k} T^{-2j\omega n} C_j = \left(T^{-2\omega n}C_1\right) \bigcap \left(T^{-2\omega n}\Big(\bigcap_{j=2}^{k} T^{-2(j-1)\omega n}C_j\Big)\right). \tag{4·39}$$

Iterating (4·39), we apply the double-sided inequality (4·38) inductively to obtain

$$\prod_{j=1}^{k-1}\left(1 - \Theta\beta^{2\omega n}\frac{\mu_\phi(C_j)+2}{\mu_\phi(C_j)}\right) \tag{4·40}$$

$$\leq \frac{\mu_\phi(\bigcap_{j=1}^{k} T^{-2j\omega n}C_j)}{\prod_{j=1}^{k}\mu_\phi(C_j)} \leq \prod_{j=1}^{k-1}\left(1 + \Theta\beta^{2\omega n}\frac{\mu_\phi(C_j)+2}{\mu_\phi(C_j)}\right).$$

By the Gibbs property (4·33), we have for $1 \leq j \leq k$,

$$\frac{\mu_\phi(C_j)+2}{\mu_\phi(C_j)} \leq \frac{3}{\mu_\phi(C_j)} \leq 3\gamma \cdot e^{-S_{n_j}\phi(x)} \leq 3\gamma \cdot e^{-n_j(\min_{x\in[0,1]}(\phi(x)))}.$$

Notice that $\min_{x\in[0,1]}(\phi(x))$ is negative since ϕ is normalized. Recalling (2·12), we find that

$$\frac{\mu_\phi(C_j)+2}{\mu_\phi(C_j)} \leq 3\gamma \cdot e^{-L' n (\min_{x\in[0,1]}(\phi(x)))}.$$

Choose ω sufficiently large so that $e^{-L'(\min_{x\in[0,1]}(\phi(x)))} < \beta^{-\omega}$. Then each term in the product on the right-side of (4·40) can be bounded from above by

$$1 + \Theta\beta^{2\omega n} \cdot 3\gamma\beta^{-n\omega} \leq 1 + M\beta^{\omega n},$$

where M is some constant depending on γ and Θ. Then (4·37) is obtained by multiplying $k-1$ identical terms.

One gets the lower bound in (4·37) by applying similar computations. □

4.2. **Big hitting probability lemma.**

Lemma 4.3 illustrates the fact that intervals with small local dimension for μ_ϕ are hit by the balls $B(T^n x, 1/n^\delta)$ with big probability.

Recall the definition (2·14) of the hitting time $\tau(x, C)$ of a basic interval C by $x \in [0,1]$.

Lemma 4.3. *Let h and ε be two positive real numbers. Consider N distinct basic intervals $C_1, \cdots, C_N$ in $\mathcal{C}_n$ satisfying $\mu_\phi(C_i) \geq |C_i|^{h-\varepsilon}$.*

Let us define the set

$$\mathcal{C}_{n,N,h} = \Big\{x \in [0,1] : \exists\, C \in \{C_i\}_{i=1,\dots,N} \text{ such that } \tau(x,C) > |C|^{-h}\Big\}.$$

There exists an integer $n_h \in \mathbb{N}$ independent of N such that

$$\text{for every } n \geq n_h, \quad \mu_\phi\big(\mathcal{C}_{n,N,h}\big) \leq 2^{-n}.$$

Remark 4.4. *The independence with respect to the integer N in Lemma 4.3 follows from the fact that this number N of basic intervals of generation n is upper bounded by $(M'')^n$ (for some integer M'' depending on the expanding Markov map only) and that these intervals do not overlap.*

Proof. Fix one interval $\widetilde{C}$ among the N basic intervals $C_1, \cdots, C_N$. Let

$$\mathcal{X}_{\widetilde{C}} := \Big\{x \in [0,1] : \forall\, n \leq |\widetilde{C}|^{-h}, \quad T^n x \notin \widetilde{C}\Big\}.$$

Obviously we have the embedding property

$$\mathcal{C}_{n,N,h} \subset \bigcup_{i=1}^{N} \mathcal{X}_{C_i},$$

so we are going to bound from above each $\mu_\phi(\mathcal{X}_{\widetilde{C}})$. Pick up an integer ω such that $2\omega > L$ (where L is the constant appearing in (2·11)), and set $m_{\widetilde{C}} = \lfloor |\widetilde{C}|^{-h}/(2\omega n)\rfloor$. Then by definition of $m_{\widetilde{C}}$, we have in particular that

$$\mathcal{X}_{\widetilde{C}} \subset \bigcap_{j=0}^{m_{\widetilde{C}}} \Big\{x \in [0,1] : T^{2j\omega n} x \notin \widetilde{C}\Big\} = \bigcap_{j=0}^{m_{\widetilde{C}}} \Big([0,1] \setminus T^{-2j\omega n}(\widetilde{C})\Big).$$

Recalling now the covering property of $\mathcal{C}_n$, we know that the union of the intervals belonging to $\mathcal{C}_n$ is the whole interval $[0,1]$. Observe also that the cardinality of $\mathcal{C}_n$ is of order 2^n.

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Let us denote by $\mathcal{C}_n(\widetilde{C})$ the subset of $\mathcal{C}_n$ constitued by the basic intervals disjoint from $\widetilde{C}$.

Since $2\omega > L$, the definition of $\mathcal{X}_{\widetilde{C}}$ implies that for any real number $x \in \mathcal{X}_{\widetilde{C}}$, there is a choice of $m_{\widetilde{C}}+1$ basic intervals $(D_0, \dots, D_{m_{\widetilde{C}}})$ all belonging to $\mathcal{C}_n(\widetilde{C})$, such that $x \in D_0 \cap T^{-2\omega n} D_1 \cap \cdots \cap T^{-2m_{\widetilde{C}}\omega n} D_{m_{\widetilde{C}}}$. From this we deduce that

$$\mu_\phi(\mathcal{X}_{\widetilde{C}}) \leq \sum_{(D_0,\dots,D_{m_{\widetilde{C}}})\in(\mathcal{C}_n(\widetilde{C}))^n} \mu_\phi(D_0 \cap T^{-2\omega n} D_1 \cap \cdots \cap T^{-2m_{\widetilde{C}}\omega n} D_{m_{\widetilde{C}}}).$$

We choose ω so large that Lemma 4.2 can be applied. Inequality (4·35) yields

$$\begin{aligned}
\mu_\phi(\mathcal{X}_{\widetilde{C}}) &\leq \gamma^3 (1+M\beta^{\omega n})^{m_{\widetilde{C}}} \sum_{(D_0,\dots,D_{m_{\widetilde{C}}})\in(\mathcal{C}_n(\widetilde{C}))^n} \prod_{j=0}^{m_{\widetilde{C}}} \mu_\phi(T^{-2j\omega n} D_i) \\
&= \gamma^3 (1+M\beta^{\omega n})^{m_{\widetilde{C}}} \sum_{(D_0,\dots,D_{m_{\widetilde{C}}})\in(\mathcal{C}_n(\widetilde{C}))^n} \prod_{j=0}^{m_{\widetilde{C}}} \mu_\phi(D_i) \\
&= \gamma^3 (1+M\beta^{\omega n})^{m_{\widetilde{C}}} \left(\sum_{D\in\mathcal{C}_n(\widetilde{C})} \mu_\phi(D) \right)^{m_{\widetilde{C}}+1}.
\end{aligned}$$

This last sum can be simplified, by the fact that the intervals of $\mathcal{C}_n$ have their interiors disjoint:

$$\sum_{D\in\mathcal{C}_n(\widetilde{C})} \mu_\phi(D) \leq 1 - \mu_\phi(\widetilde{C}).$$

Hence

$$\begin{aligned}
\mu_\phi(\mathcal{X}_{\widetilde{C}}) &\leq \gamma^3 (1+M\beta^{\omega n})^{m_{\widetilde{C}}} (1-\mu_\phi(\widetilde{C}))^{m_{\widetilde{C}}+1} \\
&\leq \frac{\gamma^3}{1+M\beta^{\omega n}} \Big((1+M\beta^{\omega n})(1-\mu_\phi(\widetilde{C})) \Big)^{m_{\widetilde{C}}+1}.
\end{aligned}$$

If ω is chosen large enough,

$$(1+M\beta^{\omega n})(1-\mu_\phi(\widetilde{C})) \leq 1 - \frac{1}{2}\mu_\phi(\widetilde{C}), \tag{4·41}$$

thus we finally obtain

$$\mu_\phi(\mathcal{X}_{\widetilde{C}}) \leq \frac{\gamma^3}{1+M\beta^{\omega n}} \left(1 - \frac{1}{2}\mu_\phi(\widetilde{C}) \right)^{m_{\widetilde{C}}+1}. \tag{4·42}$$

Here, we emphasize that ω can be chosen so large that (4·41) (and thus (4·42)) can be realized simultaneously for all $\widetilde{C}$ and for all n. In fact, from the Gibbs property (1·5), for the measure μ_ϕ, there exists a maximal exponent $H > 0$ such that for every basic interval $\widetilde{C}$ of any generation n,

$$\mu_\phi(\widetilde{C}) \geq |\widetilde{C}|^H \geq L^{-H} 2^{-nH},$$

where L is the constant of (2·11). Thus, for all $\widetilde{C} \in \mathcal{C}_n$,

$$\frac{1-\frac{1}{2}\mu_\phi(\widetilde{C})}{1-\mu_\phi(\widetilde{C})} \geq \frac{1-\frac{1}{2}L^{-H}2^{-nH}}{1-L^{-H}2^{-nH}}.$$

So, we can choose suitably the integer ω so that

$$1 + M\beta^{\omega n} \le \frac{1 - \frac{1}{2}L^{-H}2^{-nH}}{1 - L^{-H}2^{-nH}} \le \frac{1 - \frac{1}{2}\mu_\phi(\widetilde{C})}{1 - \mu_\phi(\widetilde{C})},$$

which implies that (4·41) holds for all $\widetilde{C}$.

Now, summing over all $\widetilde{C} \in \{C_1, \dots, C_N\}$, by (4·42) we have (recall that $m_{\widetilde{C}} = \lfloor |\widetilde{C}|^{-h}/(2\omega n) \rfloor$ and $\mu_\phi(\widetilde{C}) \ge |\widetilde{C}|^{h-\varepsilon}$ by assumption)

$$\begin{aligned}
\mu_\phi\Big(\mathcal{C}_{n,N,h}\Big) &\le \frac{\gamma^3}{1 + M\beta^{\omega n}} \sum_{\widetilde{C}} \left(1 - \frac{1}{2}\mu_\phi(\widetilde{C})\right)^{m_{\widetilde{C}}+1} \\
&\le \frac{\gamma^3}{1 + M\beta^{\omega n}} \sum_{\widetilde{C}} \left(1 - \frac{1}{2}\mu_\phi(\widetilde{C})\right)^{|\widetilde{C}|^{-h}/(2\omega n)} \\
&\le \frac{\gamma^3}{1 + M\beta^{\omega n}} \sum_{\widetilde{C}} \left(1 - \frac{1}{2}\mu_\phi(\widetilde{C})\right)^{|\widetilde{C}|^{-\varepsilon}/(2\omega n \mu_\phi(\widetilde{C}))} \\
&= \frac{\gamma^3}{1 + M\beta^{\omega n}} \sum_{\widetilde{C}} \exp\left(\frac{|\widetilde{C}|^{-\varepsilon}}{2\omega n \mu_\phi(\widetilde{C})} \log\left(1 - \frac{1}{2}\mu_\phi(\widetilde{C})\right)\right) \\
&\le \frac{\gamma^3}{1 + M\beta^{\omega n}} \sum_{\widetilde{C}} \exp\left(\frac{-|\widetilde{C}|^{-\varepsilon}}{4\omega n}\right).
\end{aligned}$$

We can observe in this formula the independence of the result with respect to the integer N.

Now, recalling again (2·11), we have $|\widetilde{C}|^{-\varepsilon} \ge L^{-\varepsilon}2^{\epsilon n}$. Since the number N of possible choices for $\widetilde{C}$ is less than $L \cdot 2^n$, we have

$$\begin{aligned}
&\mu_\phi\Big(\Big\{x : \exists C \in \{C_i\}_{i=1,\dots,N} \text{ such that } \tau(x, C) > |C|^{-h}\Big\}\Big) \\
&\le \frac{\gamma^3}{1 + M\beta^{\omega n}} \sum_{\widetilde{C}} \exp\left(\frac{-2^{\epsilon n}}{4\omega n L^\varepsilon}\right) \\
&\le \frac{\gamma^3}{1 + M\beta^{\omega n}} \cdot L \cdot 2^n \exp\left(\frac{-2^{\epsilon n}}{4\omega n L^\varepsilon}\right) = \frac{L\gamma^3}{1 + M\beta^{\omega n}} 2^{\frac{1}{\log 2}\left(n \log 2 - \frac{2^{\epsilon n}}{4\omega n L^\varepsilon}\right)}.
\end{aligned}$$

Obviously, this last term is less than 2^{-n} for sufficiently large n. □

4.3. **Small hitting probability lemma.**

As a complement to Lemma 4.3, we now study the probability of hitting points with high local dimension for μ_ϕ. As expected, next lemma expresses that these points are not encountered many times. The arguments are close to those of [16].

Lemma 4.5. *Let $(a, b) \in (0, 1)^2$ and let $0 < c < b$ and $\eta > b - c$. Consider 2^{bn} different basic intervals $C_1, \cdots, C_{2^{bn}}$ in $\mathcal{C}_n$. Assume that for every $j \in \{1, \cdots, 2^{bn}\}$,*

$$\mu_\phi(C_j) \le 2^{-(a+\eta)n}.$$

Let us define

$$\mathcal{X}_{a,b,c} := \Big\{x : \tau(x, C_i) \le 2^{an} \text{ for } 2^{cn} \text{ distinct intervals among the } \{C_i\}_{i=1,\dots,2^{bn}}\Big\}.$$

There exists an integer $n_{a,b,c} \in \mathbb{N}$ such that as soon as $n \geq n_{a,b,c}$,

$$\mu_\phi(\mathcal{X}_{a,b,c}) \leq 2^{-n}.$$

Proof. Let us denote $K := 2^{an}, P := 2^{bn}, N := 2^{cn}$.

When $x \in \mathcal{X}_{a,b,c}$, there exist N integers $0 < \ell_1 < \ell_2 < \cdots < \ell_N \leq K$ and N different basic intervals $C_{i_1}, C_{i_2}, \cdots, C_{i_N}$ such that

$$T^{\ell_1}x \in C_{i_1}, \quad T^{\ell_2}x \in C_{i_2}, \quad \cdots, \quad T^{\ell_N}x \in C_{i_N}. \tag{4·43}$$

Let $N' := \lfloor N/(2\omega n) \rfloor$ and let $(t_p)_{p=1}^{N'}$ be a subset of $(\ell_j)_{j\in\{1,\cdots,N\}}$ defined by

$$t_p = \ell_{2\omega np}.$$

Denote by j_p the (unique) index i such that $T^{t_p}x \in C_i$ in (4·43).

Then $x \in \mathcal{X}_{a,b,c}$ implies necessarily that

$$T^{t_1}x \in C_{j_1}, \quad T^{t_2}x \in C_{j_2}, \quad \cdots, \quad T^{t_{N'}}x \in C_{j_{N'}}, \tag{4·44}$$

where $C_{j_1}, \ldots, C_{j_{N'}}$ are N' different basic intervals among the intervals $C_1, \ldots, C_P$.

Fix now N' basic intervals $C_{j_1}, \ldots, C_{j_{N'}}$ among the intervals $C_1, \ldots, C_P$ and fix also the integers $t_1 < \ldots < t_{N'} \leq K$, and consider the set $\widetilde{\mathcal{X}}$ of real numbers x such that (4·44) is satisfied. This set $\widetilde{\mathcal{X}}$ depends on a, b, c, and on the intervals and the integers we have chosen. As said above, $\mathcal{X}_{a,b,c} \subset \bigcup \widetilde{\mathcal{X}}$, where the union is taken over all possible choices of parameters $C_{j_1}, \ldots, C_{j_{N'}}$ and $t_1 < \ldots < t_{N'}$. In order to bound from above the μ_ϕ-measure of $\mathcal{X}_{a,b,c}$, we will first study the μ_ϕ-measure of one set $\widetilde{\mathcal{X}}$.

Applying (4·37) again and using the same arguments as in Lemma 4.3, we see that the μ_ϕ-measure of $\widetilde{\mathcal{X}}$ (and thus of $\mathcal{X}_{a,b,c}$) is bounded from above by

$$\mu_\phi(\widetilde{\mathcal{X}}) \leq \max_{1\leq i\leq L} \mu_\phi(C_i)^{N'}(1 + M\beta^{2\omega n})^{N'}. \tag{4·45}$$

It remains us to estimate the maximal number of choices for the associated intervals $C_{j_1}, \ldots, C_{j_{N'}}$ and integers $(t_1, \ldots, t_{N'})$.

First, we have $\binom{P}{N'}$ possible choices for the N' different basic intervals among the list of P intervals $C_1, \ldots, C_P$, and there are at most $\binom{K}{N'}$ choices for the integers $t_1 < t_2 < \cdots < t_{N'} < K$. Finally there are $N'!$ ways to arrange the N' intervals.

Combining these informations with (4·45), we find that

$$\mu_\phi(\mathcal{X}_{a,b,c}) \leq \sum_{\widetilde{\mathcal{X}}} \mu_\phi(\widetilde{\mathcal{X}}) \leq \binom{P}{N'}\binom{K}{N'} \cdot N'! \cdot \max_{C_i} \mu_\phi(C_i)^{N'} \cdot (1 + M\beta^{\omega n})^{N'}.$$

Since

$$\binom{P}{N'}\binom{K}{N'} \cdot N'! = \frac{P!}{(P-N')!} \cdot \frac{K!}{(K-N')!} \cdot \frac{1}{N'!},$$

and using the estimates $\frac{P!}{(P-N')!} \leq P^{N'}$, $\frac{K!}{(K-N')!} \leq K^{N'}$, $\frac{1}{N'!} \leq \xi \cdot \frac{e^{N'}}{N'^{N'}}$ for some universal constant ξ, we conclude that

$$\mu_\phi(\mathcal{X}_{a,b,c}) \quad \leq \quad \xi \cdot P^{N'} \cdot K^{N'} \cdot e^{N'} \cdot N'^{-N'} \cdot (\max_{C_i} \mu_\phi(C_i))^{N'} \cdot (1 + M\beta^{\omega n})^{N'}.$$

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Recalling that $\mu_\phi(C_i) \leq 2^{-(a+\eta)n}$ by assumption, and replacing all constants K, P, N by their values, we get

$$\mu_\phi(\mathcal{X}_{a,b,c}) \ \leq \ \xi \cdot \Big(2^{bn} \cdot 2^{an} \cdot e \cdot (N')^{-1} \cdot 2^{-(a+\eta)n} \cdot (1 + M\beta^{\omega n})\Big)^{N'} .$$

By definition of N', we have $(N')^{-1} \leq \frac{2\omega n}{N} = 2\omega n 2^{-cn}$ when ω is large enough. Subsequently, the last inequality yields

$$\begin{aligned}\mu_\phi(\mathcal{X}_{a,b,c}) \ &\leq \ \xi \cdot \Big(2^{bn} \cdot 2^{an} \cdot e \cdot 2\omega n 2^{-cn} \cdot 2^{-(a+\eta)n} \cdot (1 + M\beta^{\omega n})\Big)^{N'} \\ &\leq \ \xi \cdot \Big(e \cdot 2\omega n \cdot (1 + M\beta^{\omega n}) \cdot 2^{(b-c-\eta)n}\Big)^{N'} .\end{aligned}$$

By assumption we have $\eta > b - c$, so the quantity between brackets tends to zero exponentially fast, and in particular it is less than 1/2. Using that $N' \geq \frac{2^{cn}}{2\omega n}$, we deduce that

$$\mu_\phi(\mathcal{X}_{a,b,c}) \ \leq \ \xi \cdot 2^{-\frac{2^{cn}}{2\omega n}}.$$

It is easy to see now that the right term in the above inequality is less than 2^{-n} when n becomes large. □

5. Part I of the spectrum: Exponents δ such that $1/\delta < \alpha(1) = \dim_H \mu_\phi$

5.1. Upper bound for $\dim_H \mathcal{L}^\delta(x)$.

We start by bounding from above the sets of y with given return times, for any $x \in [0,1]$.

Proposition 5.1. *For every $0 < s \leq \dim_H \mu_\phi$, for every $x \in [0,1]$, we have*

$$\dim_H \big(\mathcal{R}_{\leq s}(x)\big) \leq s. \tag{5·46}$$

Proof. Notice that in the definition of $\underline{R}(x,y)$, one can replace the limit process of $r \to 0$ by the sequence 2^{-n} with $n \to \infty$. Then for $x \in [0,1]$, and for any real number $a > s$,

$$\mathcal{R}_{\leq s}(x) \subset \limsup_{n\to\infty} \{y \ : \ \tau_{2^{-n}}(x,y) \leq 2^{an}\}.$$

In other words, given $y \in \mathcal{R}_{\leq s}(x)$, for infinitely many integers n, $y \in B(T^{k_n}x, 2^{-n})$ for some integer k_n such that $1 \leq k_n \leq 2^{an}$. Assume that the sequence of integers (k_n) tends to infinity (in other words, $y \notin \mathcal{O}(x)$). Using that $2^{-n} \leq (k_n)^{-1/a}$ for such a couple of integers (k_n, n), we deduce that $y \in B(T^{k_n}x, (k_n)^{-1/a})$ for infinitely many integers k_n.

Hence

$$\mathcal{R}_{\leq s}(x) \subset \limsup_{n\to\infty} B(T^k x, k^{-1/a}).$$

We deduce that for each integer n, the set of balls $\{B(T^k x, k^{-1/a})\}_{k\geq n}$ is a cover of $\mathcal{R}_{\leq s}(x)$ by intervals of length smaller than $n^{-1/a}$. Let $\mathcal{H}^a_\varepsilon$ stand for the a-Hausdorff pre-measure obtained by using coverings by balls of size less than ε (see for instance [15] for the definition of Hausdorff measures and dimension). Using $\{B(T^k x, k^{-1/a})\}_{k\geq n}$ as covering, we see that for any $a' > a$,

$$\mathcal{H}^{a'}_{n^{-1/a}}(\mathcal{R}_{\leq s}(x)) \leq \sum_{k\geq n} |B(T^k x, k^{-1/a})|^{a'} \leq 2^{a'/a} \sum_{k \geq n} k^{-a'/a} \leq \xi' \, n^{1-a'/a},$$

which tends to zero when n tends to infinity (ξ' is a universal constant). We deduce that the a'-Hausdorff measure of $\mathcal{R}_{\leq s}(x)$ is necessarily zero, and thus $\dim_H \mathcal{R}_{\leq s}(x) \leq a'$. Since this holds for any $a' > a$, and then for any $a > s$, we deduce (5·46). □

Remark 5.2. *Observe that the upper bound* $\dim_H \big(\mathcal{R}_{\leq s}(x)\big) \leq s$ *holds in fact for any* $0 \leq s \leq 1$. *Nevertheless it is relevant for us only when* $s \leq \dim_H \mu_\phi$*, since the multifractal spectrum of* μ_ϕ *becomes strictly concave when* $s > \dim_H \mu_\phi$.

5.2. Lower bound for $\dim_H \mathcal{L}^\delta(x)$ when $s < \dim_H \mu_\phi$.

We are going to apply to μ_ϕ the theorem of heterogeneous ubiquity developed in [2]. The heterogeneous ubiquity theorem allows to find lower bound for limsup sets of the form

$$\mathcal{L}_\zeta := \bigcap_{N\geq 1} \bigcup_{n\geq N} B(x_n, (l_n)^\zeta)$$

under the assumption that $\mu_\phi(\mathcal{L}_{\zeta_0}) = \mu_\phi$, for some $\zeta_0 > 0$, for any sequence (x_n) and any non-increasing positive sequence (l_n). In order to apply this theorem, some assumptions need to be checked for μ_ϕ. We refer to Definition 2 of [2] for the precise description of the assumptions to be satisfied, and we explain now why these assumptions are fulfilled in our frame.

From Theorem 1.11(2) of V. Baladi [1] and Theorem 7.1 of Philipp and Stout [29], we deduce the following properties for μ_ϕ.

Theorem 5.3. *Assume that the potential* ϕ *associated with* μ_ϕ *is Hölderian.*

There exists a non-decreasing continuous function χ *defined on* $\mathbb{R}_+$ *with the following properties:*

- $\chi(0) = 0$*,* $r \mapsto r^{-\chi(r)}$ *is non-increasing near* 0^+*,*
- $\lim_{r\to 0^+} r^{-\chi(r)} = +\infty$*, and* $\forall\, \varepsilon > 0$*,* $r \mapsto r^{\varepsilon-\chi(r)}$ *is non-decreasing near 0,*

and such that for μ_ϕ*-almost every* $y \in [0,1]^d$*, there exists* $r(y) > 0$*, such that for all* $0 < r \leq r(y)$*, we have*

$$r^{\dim_H \mu_\phi + \chi(r)} \leq \quad \mu_\phi\big(B(y,r)\big) \quad \leq r^{\dim_H \mu_\phi - \chi(r)}. \tag{5·47}$$

Property (5·47) shall be viewed as the illustration of the iterated logarithm law for invariant measures, and by the theorems of [1] and [29], the map χ can be taken equal to

$$\chi(0) = 0 \quad \text{and} \quad \chi : r \mapsto \sqrt{\frac{\log\log|\log(r)|}{|\log r|}} \quad \text{if } r > 0. \tag{5·48}$$

In the previous section, we also proved the following: for any δ such that $1/\delta > \dim_H \mu_\phi = \alpha(1)$, for μ_ϕ-almost every $x \in [0,1]$,

$$\mu_\phi(\mathcal{L}^\delta(x)) = 1.$$

Theorem 5.3 and the quasi-Bernoulli property of the measures μ_ϕ and μ_q imply that the conditions of Definition 2 of [2] are fulfilled for μ_ϕ-almost every $x \in [0,1]$. We can then apply the heterogeneous ubiquity theorem of [2] (Theorem 4), which yields the following lower bound.

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Theorem 5.4. *For any δ such that $1/\delta > \dim_H \mu_\phi = \alpha(1)$, for μ_ϕ-almost every $x \in [0,1]$, we have*

$$\text{for every } \zeta > 1, \qquad \dim_H(\mathcal{L}^{\zeta\cdot\delta}(x)) \geq \frac{\dim_H \mu_\phi}{\zeta}. \tag{5·49}$$

Immediately, by considering an increasing countable sequence (δ_n) tending to $\delta_0 = 1/\dim_H \mu_\phi$ and applying Theorem 5.4 to each δ_n, we get:

Corollary 5.5. *For μ_ϕ-almost every x, for every $\zeta > 1$,*

$$\dim_H(\mathcal{L}^{\zeta\cdot\delta_0}(x)) \geq \frac{\dim_H \mu_\phi}{\zeta} = \frac{1}{\zeta\cdot\delta_0}.$$

In other words, for every δ such that $1/\delta < \dim_H \mu_\phi$, we have the lower bound

$$\dim_H(\mathcal{L}^{\delta}(x)) \geq \frac{1}{\delta}.$$

6. Part IV of the spectrum: $1/\delta > \alpha_+$

We start by a relationship between the set $\mathcal{R}_{\geq s}(x)$ and the set $\mathcal{E}_{\geq s}$.

Proposition 6.1. *Let $s \geq 0$. For μ_ϕ-almost every x,*

$$\mathcal{R}_{\geq s}(x) \subset \mathcal{E}_{\geq s}. \tag{6·50}$$

Moreover, for any ergodic T-invariant probability measure ν on $[0,1]$, for μ_ϕ-almost every $x \in [0,1]$, we have

$$\mathcal{R}_{\geq s}(x) \overset{\nu}{=} \mathcal{E}_{\geq s},$$

where the equality means that the two sets differ from a set of ν-measure zero.

Remark 6.2. *The full μ_ϕ-measure set concerning the first assertion of Proposition 6.1 depends on s.*

Proof. The case $s = 0$ is obvious, we assume that $s > 0$. For any integer $n \geq 1$, let $I_n(y)$ be the basic interval in $\mathcal{C}_n$ containing y (observe that a priori the generation of $I_n(y)$ is *not* n). For any real number $\varepsilon > 0$, we introduce the sets

$$\begin{aligned} \mathcal{R}_{n,s,\varepsilon}(x) &= \{y : \tau(x, I_n(y)) \geq |I_n(y)|^{s-\varepsilon}\} \\ \text{and} \qquad \mathcal{E}_{n,s,\varepsilon} &= \{y : \mu_\phi(I_n(y)) \leq |I_n(y)|^{s-2\varepsilon}\}. \end{aligned}$$

By definition of $\underline{R}(x,y)$ and $\underline{d}_{\mu_\phi}(y)$, we have

$$\mathcal{R}_{\geq s}(x) = \bigcap_{\varepsilon>0} \liminf_{n\to\infty} \mathcal{R}_{n,s,\varepsilon}(x) \quad \text{and} \quad \mathcal{E}_{\geq s} = \bigcap_{\varepsilon>0} \liminf_{n\to\infty} \mathcal{E}_{n,s,\varepsilon}.$$

In order to prove (6·50), it is sufficient to prove that for μ_ϕ-almost every x, there exists some integer $n(x)$ such that

$$\forall\, n \geq n(x), \qquad \mathcal{R}_{n,s,\varepsilon}(x) \subset \mathcal{E}_{n,s,\varepsilon}. \tag{6·51}$$

Notice that $\mathcal{E}^c_{n,s,\varepsilon}$ is the union of basic intervals C in $\mathcal{C}_n$ such that $\mu_\phi(C) > |C|^{s-2\varepsilon}$. Let $\mathcal{C}_{n,s,\varepsilon} := \{C_1, \cdots, C_N\}$ be the set of all these basic intervals. Applying Lemma 4.3 to the collection of basic intervals $\mathcal{C}_{n,s,\varepsilon}$ and to $h = s - \varepsilon$, we see that

$$P_n := \mu_\phi\Big(\Big\{x : \exists\, C \in \mathcal{C}_{n,s,\varepsilon} \text{ such that } \ \tau(x, C) \geq |C|^{s-\varepsilon}\Big\}\Big) \leq 2^{-n}$$

when n is larger than some integer $n_{s,\varepsilon}$. In particular, the sum over $n \geq n_{s,\varepsilon}$ of the P_n's being finite, we apply the Borel-Cantelli Lemma to obtain that, for

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μ_ϕ-a.e. x, there exists an integer $n(x)$ such that for every $n \geq n(x)$, for every $C \in \mathcal{C}_{n,s,\varepsilon}$, we have $\tau(x,C) < |C|^{s-\varepsilon}$ (this last inequality states precisely that $C \subset \mathcal{R}_{n,s,\varepsilon}(x)^c$). This proves that for $n \geq n(x)$, $\mathcal{E}^c_{n,s,\varepsilon} \subset \mathcal{R}_{n,s,\varepsilon}(x)^c$, which is clearly equivalent to (6·51). Then the first assertion (6·50) of Proposition 6.1 follows.

To prove the second assertion, using the ergodicity of ν, it suffices to show that for μ_ϕ-almost every x, we have

$$\nu\Big(\Big\{y \in [0,1] : \underline{d}_{\mu_\phi}(y) \geq s \ \text{ and } \ \underline{R}(x,y) < s\Big\}\Big) = 0.$$

This last statement is directly deduced from Corollary 3.7. □

We are now ready to prove some of the statements of Theorems 1.6.

Proof. [Part IV of the spectrum: Item (4) of Theorem 1.6]
By Lemma 2.3 and Proposition 6.1, for each $s > \alpha_+$ for μ_ϕ-almost every $x \in [0,1]$, we have

$$\mathcal{R}_{\geq s}(x) := \Big\{y \in [0,1] : \underline{R}(x,y) \geq s\Big\} = \emptyset,$$

i.e. there is no point with hitting times larger than $s > \alpha_+$. Then, applying formula (3·28) and Lemma 3.4, we deduce that when $1/\delta > \alpha_+$, for μ_ϕ-almost every $x \in [0,1]$, $\mathcal{F}^\delta(x) = \emptyset$ and thus $\mathcal{L}^\delta(x) = [0,1]$. But notice that as mentioned in Remark 6.2, the full μ_ϕ-measure set depends on δ. To solve this problem (i.e. to get $\mathcal{F}^\delta(x) = \emptyset$ for every δ satisfying $1/\delta > \alpha_+$), we take a sequence $(\delta_n)_{n\geq 1}$ such that $(1/\delta_n)$ is dense in $]\alpha_+, \infty[$. By taking intersection of countable full μ_ϕ-measure sets, we obtain that for μ_ϕ-almost every $x \in [0,1]$, for all n, $\mathcal{F}^{\delta_n}(x) = \emptyset$ and $\mathcal{L}^{\delta_n}(x) = [0,1]$. Finally, the case of an arbitrary δ such that $1/\delta > \alpha_+$ is obtained by using the monotonicity of the sets $\mathcal{F}^\delta(x)$ and $\mathcal{L}^\delta(x)$ with respect to δ. □

7. Part III of the spectrum: $\alpha_{\max} < 1/\delta \leq \alpha_+$

In this short section, we gather the previous results to obtain Part III of the spectrum and item (3) of Theorem 1.6 (Recall that we have given a direct proof of item (3) in Section 3).

We adopt the notations of Section 4. Let δ be such that $\alpha_{\max} < 1/\delta \leq \alpha_+$, and consider the unique real number $q(1/\delta)$ (see formula (2·24)). Then the associated invariant Gibbs measure $\mu_{q(1/\delta)}$ is supported by the level set (recall (2·19))

$$\mathcal{E}_{\mu_\phi}(1/\delta) = \Big\{y : d_{\mu_\phi}(y) = 1/\delta\Big\},$$

which has Hausdorff dimension $D_{\mu_\phi}(1/\delta)$.

We can now apply the second part of Proposition 6.1 to the invariant measure $\nu = \mu_{q(1/\delta)}$. This leads to the fact that for μ_ϕ-almost every x, the measure $\mu_{q(1/\delta)}$ is also supported by the set $\mathcal{R}_{\geq 1/\delta}(x)$. In particular, we see that $\dim_H \mathcal{R}_{\geq 1/\delta}(x) \geq \dim_H \mu_{q(1/\delta)}$.

Now, consider a countable sequence $(\delta_n)_{n\geq 1}$ such that $1/\delta_n$ is dense in the interval $[\alpha_{\max}, \alpha_+]$. The above argument applies to each δ_n, and we deduce (by taking a countable intersection of full μ_ϕ-measure sets) that there exists a set of full μ_ϕ-measure of real numbers x such that for all $n \geq 1$, $\mu_{q(1/\delta_n)}$ is also supported by the set $\mathcal{R}_{\geq 1/\delta_n}(x)$.

Let us fix δ_0 such that $\alpha_{\max} < 1/\delta_0 \leq \alpha_+$, and consider a subsequence $(\delta_{\varphi(n)})_{n\geq 1}$ decreasing to δ_0. By (3·28), for every integer n,

$$\dim_H(\mathcal{F}^{\delta_0}) \geq \dim_H \ \mathcal{R}_{\geq 1/\delta_{\varphi(n)}}(x) \geq \dim_H \ \mu_{q(1/\delta_{\varphi(n)})} = D_{\mu_\phi}(1/\delta_{\varphi(n)}).$$

Using the continuity of D_{μ_ϕ} on its support, we get that for μ_ϕ-almost every $x \in [0,1]$,

$$\dim_H(\mathcal{F}^{\delta_0}) \geq \dim_H(\mu_{q(1/\delta_0)}) = D_{\mu_\phi}(1/\delta_0).$$

Conversely, by choosing an increasing subsequence $(\delta_{\varphi(n)})_{n\geq 1}$ converging to δ_0, by (3·28) and Theorem 6.1, we have for μ_ϕ-almost every x,

$$\dim_H(\mathcal{F}^{\delta_0}) \leq \inf_n \dim_H \left(\mathcal{E}_{\geq 1/\delta_{\varphi(n)}}\right) = \inf_n D_{\mu_\phi}(1/\delta_{\varphi(n)}) = D_{\mu_\phi}(1/\delta_0).$$

This completes the proofs for the Part III and for item (3) of Theorem 1.6.

8. Part II of the spectrum: Exponents $\alpha(1) = \dim_H \mu_\phi < 1/\delta \leq \alpha_{\max}$

To finish the proof, it remains us to treat this last range of exponents. While the lower bound is easy to obtain, the upper bound for $\dim_H \mathcal{L}^\delta(x)$ turns out to be much more difficult.

8.1. Lower bound.

Proposition 8.1. *If $\dim_H \mu_\phi < s < \alpha_{\max}$ then for μ_ϕ-almost every x we have*

$$\dim_H \left(\mathcal{R}_{\leq s}(x)\right) \geq D_{\mu_\phi}(s). \tag{8·52}$$

Proof. For $\dim_H \mu_\phi < s < \alpha_{\max}$, there exists a real number $q_s > 0$ such that (recall the definition of μ_{q_s} in Section 2.3)

$$\frac{\int(-\phi)\, d\mu_{q_s}}{\int \log|T'|\, d\mu_{q_s}} = s.$$

By the Gibbs property of μ_ϕ and the ergodicity of μ_{q_s}, the measure μ_{q_s} is supported on the level set of

$$\left\{ y : \lim_{r\to 0} \frac{\log \mu_\phi(B(y,r))}{\log r} = s \right\}.$$

Then by Corollary 3.8 applied to μ_ϕ and μ_{q_s}, for μ_ϕ-almost every x we have

$$\dim_H \mathcal{R}_{\leq s}(x) \geq \dim_H \mu_{q_s} = D_{\mu_\phi}(s).$$

□

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8.2. **Upper bound.**

We finish by bounding from above the spectrum $\dim_H \mathcal{L}^\delta(x)$.

Proposition 8.2. *If* $\dim_H \mu_\phi < s < \alpha_{max}$ *then for* μ_ϕ*-almost every* x *we have*

$$\dim_H \mathcal{R}_{\leq s}(x) \leq D_{\mu_\phi}(s). \tag{8·53}$$

Proof. Fix $s \in (\dim_H \mu_\phi, \alpha_{\max})$, and let us decompose $\mathcal{R}_{\leq s}(x)$ into

$$\mathcal{R}_{\leq s}(x) = (\mathcal{R}_{\leq s}(x) \cap \mathcal{E}_{\leq s}) \bigcup (\mathcal{R}_{\leq s}(x) \cap \mathcal{E}_{>s}).$$

Since s lies in the increasing part of the spectrum, by Proposition 2.5, we have the upper bound $\dim_H \mathcal{E}_{\leq s} \leq D_{\mu_\phi}(s)$. Subsequently, in order to obtain (8·53), it suffices to prove that

$$\dim_H (\mathcal{R}_{\leq s}(x) \cap \mathcal{E}_{>s}) \leq D_{\mu_\phi}(s).$$

Recall that $\mathcal{C}_n$ forms a covering of $[0,1]$ by basic intervals of size $\sim 2^{-n}$, these intervals having disjoint interiors.

Let $0 < h' < h''$ be two real numbers. We define the subset $\mathcal{C}_n(h', h'')$ of $\mathcal{C}_n$

$$\mathcal{C}_n(h', h'') = \{C \in \mathcal{C}_n : |C|^{h''} \leq \mu_\phi(C) \leq |C|^{h'}\}$$

and the subset of $[0,1]$

$$\mathcal{Y}_n(h', h'') = \{y \in [0,1] : \exists\, C \in \mathcal{C}_n(h', h'') \text{ such that } y \in C\}.$$

We state a useful fact.

Lemma 8.3. *Let* $0 < h' < h'$. *For every* $\varepsilon > 0$*, there exists an integer* $n_{h',h'',\varepsilon}$ *large enough so that as soon as* $n \geq n_{h',h'',\varepsilon}$*,*

$$\operatorname{Card} \mathcal{C}_n(h', h'') \leq 2^{n(D_{\mu_\phi}(h'')+\varepsilon)} \quad \text{if} \quad h'' < \alpha_{\max}, \tag{8·54}$$

$$\operatorname{Card} \mathcal{C}_n(h', h'') \leq 2^{n(D_{\mu_\phi}(h')+\varepsilon)} \quad \text{if} \quad h' > \alpha_{\max}. \tag{8·55}$$

These properties, very close to Proposition 2.5, follow again from standard large deviation properties (see for instance [8]).

Let $\zeta > 0$ be a positive real number, that we will soon choose in a suitable manner. Let us set $h_1' = s$ and $h_1'' = s + \frac{1}{2}\zeta$. It is possible to cover the interval $[s + \frac{1}{2}\zeta, \alpha_+]$ by a finite number of open intervals $\{(h_i', h_i'')\}_{2 \leq i \leq \ell}$ with length less than ζ. We have the inclusion

$$\mathcal{E}_{>s} \subset \bigcup_{N=1}^{+\infty} \bigcap_{n \geq N} \bigcup_{i=1}^{\ell} \mathcal{Y}_n(h_i', h_i''). \tag{8·56}$$

This embedding property emphasizes that when $\underline{d}_{\mu_\phi}(y) > s$ for a real number $y \in [0,1]$, then necessarily $\mu_\phi(I_n(y)) < |I_n(y)|^s$ for every integer n large enough (not only for an infinite number of integers).

Similarly, we define the subset $\mathcal{C}_{n,a}(x)$ of $\mathcal{C}_n$

$$\mathcal{C}_{n,a}(x) := \{C \in \mathcal{C}_n : \tau(x, C) < 2^{an}\}.$$

Recall that $\mathcal{R}_{\leq s}(x) = \{y \in [0,1] : \underline{R}(x,y) \leq s\}$. Hence, for any real number $a > s$, $\mathcal{R}_{\leq s}(x) \subset \{y \in [0,1] : \underline{R}(x,y) < a\}$. Using the distorsion property (1·4) (which guarantees that $I_m(y)$ tends to zero very regularly when m tends to infinity), if $y \in \mathcal{R}_{\leq s}(x)$, there is an infinite number of integers m such that $\tau_{2^{-n}}(x,y) \leq 2^{an}$, which means that $T^p x \in B(y, 2^{-n})$ for some $p \leq 2^{an}$.

Denote by $d(y,C)$ the distance from the point y to the set C. We introduce the subsets of $[0,1]$

$$\widetilde{\mathcal{Y}}_{n,a}(x) = \{y \in [0,1] : \exists\, C \in \mathcal{C}_{n,a}(x) \text{ such that } d(y,C) \leq 2^{-n}\}.$$

Recall that $I_n(y)$ is the unique basic interval contained in $\mathcal{C}_n$ containing y. Since $T^p x \in B(y, 2^{-n})$ implies that $d(y, I_n(T^p x)) \leq 2^{-n}$, we have

$$\mathcal{R}_{\leq s}(x) \subset \bigcap_{N=1}^{+\infty} \bigcup_{n\geq N} \widetilde{\mathcal{Y}}_{n,a}(x) \tag{8·57}$$

Thus combining (8·56) and (8·57), we get

$$\begin{aligned}\mathcal{R}_{\leq s}(x) \cap \mathcal{E}_{>s} \quad &\subset \bigcap_{N=1}^{+\infty} \bigcup_{n\geq N} \left(\widetilde{\mathcal{Y}}_{n,a}(x) \cap \bigcup_{i=1}^{\ell} \mathcal{Y}_n(h_i', h_i'') \right) \\ &\subset \bigcup_{i=1}^{\ell} \bigcap_{N=1}^{+\infty} \bigcup_{n\geq N} \left(\widetilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h_i', h_i'') \right),\end{aligned}$$

the last inversion following from the fact that there is a finite number of intervals $[h_i', h_i'']$. The fact that (8·56) holds for every n large enough (for every $y \in \mathcal{E}_{>s}$) is key to obtain this inclusion. Subsequently, we need only to show that for all $1 \leq i \leq \ell$,

$$\forall\, \varepsilon > 0, \quad \dim_H \left(\limsup_{n\to\infty} \left(\widetilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h_i', h_i'') \right) \right) \leq D_{\mu_\phi}(s) + \varepsilon.$$

Let $\mathcal{C}_{n,a,h_i',h_i''}(x)$ be the subset of $\mathcal{C}_n$ constituted of the basic intervals belonging both to $\mathcal{C}_{n,a}(x)$ and $\mathcal{C}_n(h_i', h_i'')$. We are going to show the following lemma.

Lemma 8.4. *For every $a \in (s, \alpha_{\max})$, for every $\varepsilon > 0$, for each $2 \leq i \leq \ell$,*

$$\sum_n \mu_\phi \Big(\{x : \operatorname{Card} \mathcal{C}_{n,a,h_i',h_i''}(x) > 2^{n(D_{\mu_\phi}(a)+\varepsilon)}\} \Big) < \infty. \tag{8·58}$$

Paying attention to the fact that in Lemma 8.4, we do not consider the first interval $[h_1', h_1'']$. For this interval, (8·58) simply follows from (8·54), if we assume that ζ is small enough (i.e. equivalently, that $h_1'' = s + \zeta/2$ is very close to s, so that $D_{\mu_\phi}(h_1'')$ is close to $D_{\mu_\phi}(s)$).

Let us assume for a while that Lemma 8.4 holds true. Then, the Borel-Cantelli Lemma yields that for μ_ϕ-almost every x, there exists an integer $n(x)$ such that as soon as $n \geq n(x)$, we have

$$\operatorname{Card} \mathcal{C}_{n,a,h_i',h_i''}(x) \leq 2^{n(D_{\mu_\phi}(a)+\varepsilon)}.$$

In order to obtain a covering of the set $\limsup_{n\to\infty} \left(\widetilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h_i', h_i'') \right)$, by construction one may consider, for any $N \geq 1$, the union

$$\bigcup_{n\geq N} \bigcup_{C \in \mathcal{C}_{n,a,h_i',h_i''}(x)} \{y \in [0,1] : d(y,C) \leq 2^{-n}\}.$$

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Using this family of coverings, if $N \geq n(x)$, then for any $\varepsilon > 0$, the $(D_{\mu_\phi}(a) + 2\varepsilon)$-Hausdorff measure of the above limsup set is bounded by

$$\begin{aligned} \sum_{n\geq N} \sum_{C\in\mathcal{C}_{n,a,h'_i,h''_i}(x)} & \left((L+2)\cdot 2^{-n}\right)^{D_{\mu_\phi}(a)+2\varepsilon} \\ \leq & \ \Theta' \sum_{n\geq N} 2^{-n(D_{\mu_\phi}(a)+2\varepsilon)} \cdot 2^{n(D_{\mu_\phi}(a)+\varepsilon)} \\ \leq & \ \Theta' \sum_{n\geq N} 2^{-n\varepsilon} < \infty, \end{aligned}$$

where Θ' is some constant depending on L, a, μ_ϕ and ε. Hence, letting N tend to infinity, we see that the $D_{\mu_\phi}(a) + 2\varepsilon$-Hausdorff measure of the limsup set $\limsup_{n\to\infty} \left(\widetilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h'_i, h''_i)\right)$ is necessarily 0. This implies that

$$\dim_H \left(\limsup_{n\to\infty} \left(\widetilde{\mathcal{Y}}_{n,a}(x) \cap \mathcal{Y}_n(h'_i, h''_i)\right)\right) \leq D_{\mu_\phi}(a) + 2\varepsilon.$$

We finish the proof of Proposition 8.2 by letting first $\varepsilon \downarrow 0$ and then $a \downarrow s$.

It remains us to prove Lemma 8.4. For this, we will apply Lemma 4.5.

Let $a \in (s, \alpha_{\max})$. It is enough to prove Lemma 8.4 for a close to s, hence we suppose that $a < h'_2$.

We assume that the intervals $[h'_i, h''_i]$ are chosen so that except for at most one of them, either $h'_i > \alpha_{\max}$ or $h''_i < \alpha_{\max}$. In other words, we suppose that there is only integer $i \in \{2, \dots, l\}$ such that $\alpha_{\max} \in (h'_i, h''_i)$.

Recall that $a < h'_2$. We will use the following two key properties:

- The multifractal spectrum D_{μ_ϕ} is real-analytic and concave on $]\alpha_-, \alpha_+[$.
- For every exponent $h \geq a > s > \dim_H \mu_\phi$, the derivative of D_{μ_ϕ} at s is strictly less than 1, and the derivative $(D_{\mu_\phi})'(s)$ is decreasing. Hence, there is a real number $0 < \xi_a = (D_{\mu_\phi})'(a) < 1$ such that for every h in every interval $[h'_i, h''_i]$ $(i \geq 2)$,
$$\text{for every } h \geq a, \quad (D_{\mu_\phi})'(h) \leq \xi_a.$$

We distinguish three cases.

• **If** $h''_i < \alpha_{\max}$**:** Take $b = D_{\mu_\phi}(h''_i) + \varepsilon$, $c = D_{\mu_\phi}(a) + \varepsilon$, and $\eta = h'_i - a$. Then on the one hand, by (8·54), for n large enough there are at most 2^{bn} basic intervals C in $\mathcal{C}_n(h'_i, h''_i) = \mathcal{C}_n(a + \eta, h''_i)$.

On the other hand, by the mean value theorem and the fact that $D_{\mu_\phi}(\cdot)$ is increasing on $(\alpha_-, \alpha_{\max})$,

$$b - c = D_{\mu_\phi}(h''_i) - D_{\mu_\phi}(a) < \xi_a(h''_i - a) = \xi_a(h'_i - a) + \xi_a(h''_i - h'_i).$$

Since $\xi_a < 1$ and $h''_i - h'_i < \zeta$, we can choose ζ small enough such that

$$b - c < h'_i - a = \eta.$$

This choice of ζ can be uniform, i.e. valid for every index i such that $h''_i < \alpha_{\max}$.

By Lemma 4.5, for sufficiently large n,

$$\mu_\phi \left(\left\{ x : \left\{ \begin{array}{l} \tau(x, C) \leq 2^{an} \text{ for } 2^{cn} \text{ distinct} \\ \text{intervals } C \text{ among the } 2^{bn} \text{ intervals} \end{array} \right\} \right) \leq 2^{-n}. \qquad (8{\cdot}59)$$

This is equivalent to say that

$$\mu_\phi\Big(\{x : \mathrm{Card}\, \mathcal{C}_{n,a,h'_i,h''_i}(x) > 2^{n(D_{\mu_\phi}(a)+\varepsilon)}\}\Big) \leq 2^{-n}.$$

Then (8·58) follows.

• **If i is the unique integer such that $h'_i \leq \alpha_{\max} \leq h''_i$:** This occurs for one and only one interval $[h'_i, h''_i]$. Recall that the cardinality of $\mathcal{C}_n$ is less than $L2^n$. Take $b = 1$, $c = D_{\mu_\phi}(a)$, and $\eta = h'_i - a$. Then

$$b - c = 1 - D_{\mu_\phi}(a) < \xi_a(\alpha_{\max} - a) = \xi_a(h'_i - a) + \xi_a(\alpha_{\max} - h'_i).$$

Since $D'_{\mu_\phi}(a) < 1$ and $\alpha_{\max} - h'_i \leq h''_i - h'_i < \zeta$, we can choose ζ small enough such that

$$b - c < h'_i - a = \eta.$$

Thus using Lemma 4.5 and applying the same arguments as above, for sufficiently large n, we have

$$\mu_\phi\left(\left\{x : \left\{\begin{array}{l} \tau(x, C) \leq 2^{an} \text{ for } 2^{cn} \text{ distinct} \\ \text{intervals } C \text{ among } \ 2^{bn} \text{ intervals of } \mathcal{C}_n \end{array}\right.\right\}\right) \leq 2^{-n}.$$

It is not difficult to prove that we also have

$$\mu_\phi\left(\left\{x : \left\{\begin{array}{l} \tau(x, C) \leq 2^{an} \text{ for } 2^{cn} \text{ distinct} \\ \text{intervals } C \text{ among the } L \cdot 2^{bn} \text{ intervals of } \mathcal{C}_n \end{array}\right.\right\}\right) \leq 2^{-n},$$

since constants do not infer in the proofs of Lemma 4.5. In other words,

$$\mu_\phi\Big(\{x : \mathrm{Card}\, \mathcal{C}_{n,a,h'_i,h''_i}(x) > 2^{nD_{\mu_\phi}(a)}\}\Big) \leq 2^{-n},$$

and (8·58) is proved.

• **If $\alpha_{\max} < h'_i$:** Take $b = D_{\mu_\phi}(h'_i) + \varepsilon$, $c = D_{\mu_\phi}(a) + \varepsilon$, and $\eta = h'_i - a$. Then on the one hand, by (8·55), for n large enough there are at most $L = 2^{bn}$ basic intervals in $\mathcal{C}_n(h'_i, h''_i) = \mathcal{C}_n(a + \eta, h''_i)$.

On the other hand,

$$b - c = D_{\mu_\phi}(h'_i) - D_{\mu_\phi}(a) < \xi_a(h'_i - a) < (h'_i - a).$$

Thus again by Lemma 4.5, for sufficiently large n, (8·59) follows, and (8·58) is proved. □

8.3. **Conclusion.**

Combining Propositions 8.1 and 8.2, we have that for every $s \in (\dim_H \mu_\phi, \alpha_{\max})$, for μ_ϕ-almost every $x \in [0, 1]$,

$$\dim_H \mathcal{R}_{\leq s}(x) = D_{\mu_\phi}(s). \tag{8·60}$$

Then by Lemma 3.2, we have for every δ such that $\dim_H \mu_\phi < 1/\delta \leq \alpha_{\max}$, for μ_ϕ-almost every x,

$$\dim_H \mathcal{L}^\delta(x) = D_{\mu_\phi}(1/\delta). \tag{8·61}$$

As we did in proving Part III and Part IV, by noticing the monotonicity of sets $\mathcal{L}^\delta(x)$ with respect to δ and applying (8·61) to a dense countable set, we can obtain that for μ_ϕ-almost every $x \in [0, 1]$ for every δ, (8·61) holds. This concludes the proof.

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