

FIXED POINTS FOR THE MULTIFRACTAL SPECTRUM APPLICATION

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ABSTRACT. For all continuous function g having a specific form that we call *with increasing visibility*, we construct a function f whose multifractal spectrum is such that $d_f = g \circ f$. The function f is obtained as an infinite superposition of piecewise C^1 functions, is also with increasing visibility, and is homogeneously multifractal, i.e. its restriction on any subinterval of $[0, 1]$ has the same multifractal spectrum as the function f itself. In particular, we can construct a function f which is its own multifractal spectrum i.e. $f = d_f$.

1. INTRODUCTION

The goal of this article is to investigate the possible forms that a multifractal spectrum can take. We construct a function f which has a specific multifractal spectrum which is a function of f itself. More precisely, for a large class of continuous functions g , we give a technical construction of a continuous function f satisfying $g \circ f = d_f$. The function f will be obtained as an infinite superposition of piecewise affine functions. We do not obtain a characterization of the functions f satisfying this property, but this could certainly be studied.

In particular, if we apply this construction to the function $g(x) = x$, we obtain a function f which is a fixed point for the multifractal spectrum application $f \mapsto d_f$, i.e. $f = d_f$. It is a natural question to ask for the existence of such a function, and very satisfactory to have a positive answer.

Let us start by recalling the notion of local regularity and multifractal analysis.

Definition 1.1. *Let $f \in L_{loc}^\infty(\mathbb{R})$, $x_0 \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. The function f is said to belong to the space $\mathcal{C}^\alpha(x_0)$, if there exists a polynomial P of degree at most $[\alpha]$ and a constant $C > 0$ such that locally around x_0 :*

$$(1) \quad |f(x) - P(x - x_0)| \leq C |x - x_0|^\alpha.$$

The local regularity of f at x_0 is measured by the pointwise Hölder exponent :

$$h_f(x_0) = \sup\{\alpha \geq 0 : f \in \mathcal{C}^\alpha(x_0)\}.$$

Remark that if $\alpha < 1$ the condition (1) is, in fact :

$$(2) \quad |f(x) - f(x_0)| \leq C |x - x_0|^\alpha.$$

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Given a function f , this exponent $h_f(x_0)$ typically varies erratically with x_0 , and the relevant information is then provided by the multifractal spectrum d_f of f , defined as the mapping

$$d_f : h \in [0, \infty] \longmapsto \dim E_f(h),$$

where $E_f(h)$ is the iso-Hölder set

$$E_f(h) := \{x_0 \in \mathbb{R} : h_f(x_0) = h\}.$$

In the definition of the multifractal spectrum, \dim stands for the Hausdorff dimension, and by convention $\dim \emptyset = -\infty$. The spectrum of singularities d_f describes the geometrical distribution of the singularities of f .

In order to state the main theorem, one first needs a definition.

Definition 1.2. *Let $0 \leq a < b \leq +\infty$. A function $f : [a, b] \rightarrow \mathbb{R}^+$ is with increasing-visibility when f is continuous at a and when the function*

$$x \mapsto \frac{f(x)}{x}$$

is increasing on $(a, b]$. We call $V_{a,b}$ the set of functions with increasing visibility on the interval $[a, b]$.

Functions with increasing visibility are studied later. Our main result is:

Theorem 1.1. *Let $0 < \eta < 1$ and let $g \in V_{0,1}$ be a function belonging to $C^\eta(\mathbb{R})$ with increasing visibility such that $g(0) = 0$ and $g(1) = 1$. There exists a continuous function $f : [0, 1] \rightarrow \mathbb{R}^+$ satisfying*

$$(3) \quad \forall x \in [0, 1], \quad g(f(x)) = d_f(x).$$

In particular, one obtains :

Corollary 1.1. *There exists a continuous function $f : [0, 1] \rightarrow \mathbb{R}^+$ satisfying*

$$(4) \quad \forall x \in [0, 1], \quad f(x) = d_f(x).$$

The function we build is monotone increasing, satisfying $f(0) = 0$ and $f(1) = 1$, thus it can also be viewed as a Borel probability measure on $[0, 1]$. In particular, it is known [6] that for such functions, one necessarily has $d_f(h) \leq h$ for every $h \in [0, 1]$. This in turn implies that the graph of f itself is below the diagonal. The function f we build has also a very specific form, it is with increasing visibility, as the function g itself. This shape is due to its construction: f is obtained as series of monotone increasing functions, which are piecewise affine. Each finite sum is thus piecewise affine, but the infinite sum, though Lebesgue almost-differentiable by the Lebesgue theorem, is multifractal, and exhibits a whole range of exponents between 0 and 1. On top of that, the function f we build is homogeneously multifractal in the sense of [3] or [2], i.e. its restriction on any subinterval of $[0, 1]$ has the same multifractal spectrum as the function f itself. But, of course, the restriction does not satisfy (4) any more!

As said before, we do not obtain a characterization of the functions f satisfying (3), and this does not seem reachable nor reasonable. Indeed, for example, being a fixed point

for the multifractal spectrum application is extremely unstable with respect to any norm. This is the reason why we do not hope to find general answers to such questions. But our work is a part of a general scheme of investigation of the possible shapes that a multifractal spectrum can take, see for instance [7] or [1].

2. FUNCTIONS WITH INCREASING VISIBILITY

We are going to build a function which enjoys a specific property, that we call *increasing-visibility*, recall Definition 1.2. Observe that a function with increasing visibility is necessarily non-decreasing and that for every $x \in (a, b]$,

$$\frac{f(x)}{x} \geq \sup_{a < x' \leq x} \frac{f(x')}{x'}.$$

It explains geometrically this denomination "visible": every point $(x, f(x))$ of the graph of the function is visible from the origin $(0, 0)$, i.e. the segment joining these two points lies above the graph of f .

2.1. First properties of functions with increasing visibility. A natural way to construct functions with increasing visibility is the following.

Lemma 2.1. *Let $0 \leq a < b \leq +\infty$. For every function $f : [a, b] \rightarrow \mathbb{R}^+$, let us construct the function $\tilde{f} : [a, b] \rightarrow \mathbb{R}^+$ as follows: $\tilde{f}(a) = f(a)$ and*

$$\forall x \in (a, b], \quad \tilde{f}(x) = x \sup_{a < x' \leq x} \frac{f(x')}{x'}.$$

The function \tilde{f} is non-decreasing on its support, and one always has $f \leq \tilde{f}$. Then

$$V_{a,b} = \left\{ f : [a, b] \rightarrow \mathbb{R}^+ \text{ such that } f = \tilde{f} \right\}.$$

The proof is immediate.

Lemma 2.2. *Every $f \in V_{0,b}$ satisfies necessarily $f(0) = 0$ and is differentiable at 0.*

Proof. The ratio $\frac{f(x)}{x}$ is positive and non-decreasing, hence it has a limit when $x \rightarrow 0$. \square

Lemma 2.3. *For any a, b , the set $V_{a,b}$ is stable under addition, positive multiplication and max, and $V_{0,b}$ is also stable under positive dilation. It is closed using the supremum norm.*

Proof. Let $f, g \in V_{a,b}$, $\lambda \in \mathbb{R}^+$. It is obvious to check that $\lambda f(\cdot)$, $f(\lambda \cdot)$ and $\max(f, g)$ belong to $V_{a,b}$. To get the other properties, recall that $f = \tilde{f}$ and $g = \tilde{g}$.

Set $s = f + g$. On one side, one always has $s \leq \tilde{s}$, hence $s = f + g = \tilde{f} + \tilde{g} \leq \tilde{s}$. On the other side, for every x ,

$$\begin{aligned} s(x) &= f(x) + g(x) = \tilde{f}(x) + \tilde{g}(x) = x \left(\sup_{a < x' \leq x} \frac{f(x')}{x'} + \sup_{a < x' \leq x} \frac{g(x')}{x'} \right) \\ &\geq x \sup_{a < x' \leq x} \frac{f(x') + g(x')}{x'} = \tilde{s}(x). \end{aligned}$$

Hence $s = \tilde{s}$.

Let $(f_n)_{n \geq 1}$ be a sequence of functions in $V_{a,b}$ converging uniformly to a function f . For every $x \in (a, b]$, one sets a' such that $a < a' < x$, one has

$$\begin{aligned} \left| \tilde{f}(x) - f(x) \right| &= \left| x \sup_{a < x' \leq x} \frac{f(x')}{x'} - f(x) \right| \\ &\leq \left| x \sup_{a < x' \leq x} \frac{f(x') - f_n(x')}{x'} \right| + \left| x \sup_{a < x' \leq x} \frac{f_n(x')}{x'} - f(x) \right| \\ &= \left| x \sup_{a < x' \leq x} \frac{f(x') - f_n(x')}{x'} \right| + \left| \tilde{f}_n(x) - f(x) \right| \\ &\leq \left| x \sup_{a < x' \leq a'} \frac{f(x') - f_n(x')}{x'} \right| + \left| x \sup_{a' < x' \leq x} \frac{f(x') - f_n(x')}{x'} \right| \\ &\quad + |f_n(x) - f(x)|, \end{aligned}$$

since $\tilde{f}_n = f_n$. Letting n go to $+\infty$ and a' go to a gives $f = \tilde{f}$. \square

Lemma 2.4. *For every function $f \in V_{0,1}$ such that $f(1) \leq 1$ and every continuous function $g \in V_{0,1}$, one has $g \circ f \in V_{0,1}$.*

Proof. Let $f, g \in V_{0,1}$ with $f(1) \leq 1$ and $g \in V_{0,1}$ a continuous function. Let $0 < x \leq x' \leq 1$. Using the increasing visibility of f and g , one has $\frac{g \circ f(x)}{f(x)} \leq \frac{g \circ f(x')}{f(x')}$. Then, one gets $\frac{g \circ f(x)}{f(x)} \cdot \frac{f(x)}{x} \leq \frac{g \circ f(x')}{f(x')} \cdot \frac{f(x')}{x'}$, so that $g \circ f \in V_{0,1}$. \square

2.2. Construction of a discontinuous function with increasing visibility on \mathbb{R} .

Definition 2.1. *Let $\widehat{\xi} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function defined as follows. We set $\widehat{\xi}(0) = 0$, $\widehat{\xi}(1) = 1$, and for every $n \geq 2$,*

$$\frac{\widehat{\xi}(n)}{n} = \frac{\widehat{\xi}(n-1)}{n-1} + \frac{1}{n}.$$

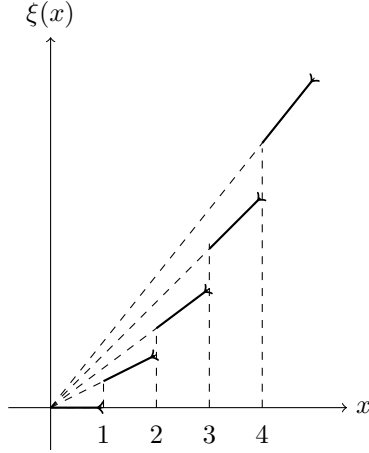
Then, for every $n \in \mathbb{N}$ and $x \in (n, n+1)$, we set

$$\widehat{\xi}(x) = \begin{cases} x & \text{if } n = 0 \\ \frac{\widehat{\xi}(n)}{n} \times x & \text{if } n \geq 1. \end{cases}$$

Geometrically, $\widehat{\xi}$ is a piecewise affine function with jump between two linear pieces of slope 1 (see figure 2.2). Observe that $\widehat{\xi}$ is left-discontinuous exactly at each integer $n \geq 1$, with a positive jump of size 1.

Lemma 2.5. *The function $\widehat{\xi}$ is with increasing visibility on \mathbb{R}^+ , $\frac{\widehat{\xi}(n)}{n} = \sum_{k=1}^n \frac{1}{k}$ and*

$$n \ln n \leq \widehat{\xi}(n) \leq n(1 + \ln n).$$

FIGURE 1. Allure of the function $\widehat{\xi}$

The lemma is clear and left to the reader. In order to prove Theorem 1.1, we are going to use a sequence of functions $(\xi_n)_{n \geq 1} \in V_{0,1}$ built from $\widehat{\xi}$ as follows.

Definition 2.2. Let $(j_n)_{n \geq 1}$ be an increasing sequence of integers (that will be fixed later on). We set for every $n \geq 1$

$$\forall x \in [0, 1], \quad \xi_n(x) = \frac{\widehat{\xi}(2^{j_n}x)}{\widehat{\xi}(2^{j_n})2^n}.$$

Obviously, by the properties of functions with increasing visibility, $\xi_n \in V_{0,1}$, and

$$\xi_n(x) \leq \frac{x}{2^n}, \quad \forall x \in [0, 1],$$

the inequality being strict when $x \in (0, 1)$. In fact, it is easy to see that $\xi_n(x) \sim \frac{x}{2^n} + \frac{x \log x}{2^n j_n \log 2}$. Moreover, since $\widehat{\xi}$ has a jump of size 1 at every integer, ξ_n has a jump of size $\frac{1}{\widehat{\xi}(2^{j_n})2^n}$ at every dyadic number $k2^{-j_n}$.

Lemma 2.6. For every $n \geq 1$, for every $k \in \{0, \dots, 2^{j_n} - 1\}$, one has

$$\frac{\ln k}{2^{j_n+n}(1+j_n)} \leq \xi_n((k+1)2^{-j_n}) - \xi_n(k2^{-j_n}) \leq \frac{9}{2^{j_n+n}}.$$

Proof. One has, for $k \geq 1$

$$\xi_n((k+1)2^{-j_n}) - \xi_n(k2^{-j_n}) = \frac{\widehat{\xi}(k+1) - \widehat{\xi}(k)}{\widehat{\xi}(2^{j_n})2^n} = \frac{\frac{\widehat{\xi}(k)}{k}(k+1) + 1 - \widehat{\xi}(k)}{\widehat{\xi}(2^{j_n})2^n} = \frac{\frac{\widehat{\xi}(k)}{k} + 1}{\widehat{\xi}(2^{j_n})2^n}$$

Using Lemma 2.5, one has :

$$\begin{aligned} \frac{\ln k + 1}{2^{j_n}(1 + \ln 2^{j_n})2^n} &\leq \xi_n((k+1)2^{-j_n}) - \xi_n(k2^{-j_n}) \leq \frac{1 + \ln k + 1}{2^{j_n} \ln 2^{j_n} 2^n} \\ \frac{\ln k + 1}{2^{j_n+n}(1 + j_n \ln 2)} &\leq \xi_n((k+1)2^{-j_n}) - \xi_n(k2^{-j_n}) \leq \frac{2}{2^{j_n+n} \ln 2} + \frac{\ln k}{2^{j_n+n} \ln 2^{j_n}}. \end{aligned}$$

Hence, since $k \leq 2^{j_n}$, one has

$$(5) \quad \frac{\ln k}{2^{j_n+n}(1 + j_n)} \leq \xi_n((k+1)2^{-j_n}) - \xi_n(k2^{-j_n}) \leq \frac{9}{2^{j_n+n}}.$$

□

3. CONSTRUCTION OF THE FUNCTION f

As said in the introduction, the function f will be obtained as a sum of an infinite positive functions Φ_n , which will be defined iteratively.

We start our induction with $j_1 = 1$, and the function $f_1(x) = \Phi_1(x) = \frac{x}{2}$. We explain how to construct f_2 , and then one proves the induction by constructing the functions f_n .

3.1. **Construction of f_2 .** We set $\alpha_{2,1} = \frac{1}{2}$ and choose $j_2 \gg j_1$ so that

$$(6) \quad 2 \left[j_2 \left(1 - \frac{g \circ f_1(\alpha_{2,1})}{\alpha_{2,1}} \right) \right] \geq 2^2.$$

Definition 3.1. The set $\mathcal{A}_{2,1}$ is the set of those integers $k \in \{1, \dots, 2^{j_2}\}$ of the form

$$k = 1 + p 2 \left[j_2 \left(1 - \frac{g \circ f_1(\alpha_{2,1})}{\alpha_{2,1}} \right) \right], \text{ where } p \in \left\{ 1, \dots, 2 \left[j_2 \frac{g \circ f_1(\alpha_{2,1})}{\alpha_{2,1}} \right] - 1 \right\}.$$

The dyadic numbers $k2^{-j_2}$, where $k \in \mathcal{A}_{2,1}$, are uniformly distributed in the interval $[0, 1]$. This is key in our construction.

Definition 3.2. For every $k \in \{0, \dots, 2^j\}$ and every $\alpha \in (0, 1)$ one sets

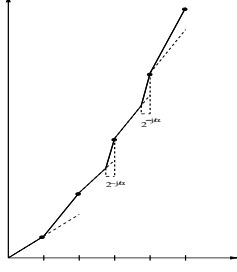
$$\begin{aligned} I_{j,k} &= [k2^{-j}, (k+1)2^{-j}], \\ x_{j,k}(\alpha) &= (k+1)2^{-j} - 2^{-\frac{j}{\alpha}}, \\ I_{j,k}(\alpha) &= [x_{j,k}(\alpha), (k+1)2^{-j}]. \end{aligned}$$

Definition 3.3. One introduces the function Φ_2 as follows (see figure 2):

1. If $k \in \mathcal{A}_{2,1}$, then

- when $x \in I_{j_2,k} \setminus I_{j_2,k}(\alpha_{2,1})$, $\Phi_2(x) = \xi_2(x)$,
- when $x \in I_{j_2,k}(\alpha_{2,1})$, one sets

$$\Phi_2(x) = \frac{\xi_2((k+1)2^{-j_2}) - \xi_2(x_{j_2,k}(\alpha_{2,1}))}{2^{-j_2/\alpha_{2,1}}} (x - (k+1)2^{-j_2}) + \xi_2((k+1)2^{-j_2}).$$

FIGURE 2. Allure of the function Φ_2 .

2. If $k \notin \mathcal{A}_{2,1}$, then Φ_2 is affine on $I_{j_2,k}$ and satisfies

$$\Phi_2(k2^{-j_2}) = \xi_2(k2^{-j_2}) \quad \text{and} \quad \Phi_2((k+1)2^{-j_2}) = \xi_2((k+1)2^{-j_2}).$$

Geometrically, Φ_2 is a piecewise affine function that we construct using ξ_2 as follows:

1. If $k \in \mathcal{A}_{2,1}$, then

- on $[k2^{-j_2}; x_{j_2,k}(\alpha_{2,1})]$, Φ_2 is equal to the function ξ_2 ,
- on $[x_{j_2,k}(\alpha_{2,1}); (k+1)2^{-j_2}]$, we draw the segment joining the point of the graph of the function ξ_2 with abscissa $x_{j_2,k}(\alpha_{2,1})$ with the point with abscissa $(k+1)2^{-j_2}$.

2. If $k \notin \mathcal{A}_{2,1}$, we draw the segment joining the point of the graph of the function ξ_2 with abscissa $k2^{-j_2}$ with the point with abscissa $(k+1)2^{-j_2}$.

Heuristically, Φ_2 is a continuous version of ξ_2 , where the jumps at the dyadics $k2^{-j_2}$ are either erased if $k \notin \mathcal{A}_{2,1}$, or replaced by a rapid increasing affine map on the small interval $I_{j_2,k}(\alpha_{2,1})$ when $k \in \mathcal{A}_{2,1}$. Observe that by construction, when $k \in \mathcal{A}_{2,1}$, the oscillation of the increasing function Φ_2 satisfies :

$$\begin{aligned} \Phi_2((k+1)2^{-j_2}) - \Phi_2(x_{j_2,k}(\alpha_{2,1})) &= \xi_2((k+1)2^{-j_2}) - \xi_2(x_{j_2,k}(\alpha_{2,1})) \\ &= \frac{\widehat{\xi}(k+1) - \widehat{\xi}(x_{j_2,k}(\alpha_{2,1})2^{j_2})}{\widehat{\xi}(2^{j_2})2^2} \\ &= \frac{1}{\widehat{\xi}(2^{j_2})2^2} \left(\frac{\widehat{\xi}(k)}{k}(k+1) + 1 - \frac{\widehat{\xi}(k)}{k}(x_{j_2,k}(\alpha_{2,1})2^{j_2}) \right) \\ &= \frac{1}{\widehat{\xi}(2^{j_2})2^2} \left(\frac{\widehat{\xi}(k)}{k}(k+1 - x_{j_2,k}(\alpha_{2,1})2^{j_2}) + 1 \right) \\ &= \frac{1}{\widehat{\xi}(2^{j_2})2^2} \left(\frac{\widehat{\xi}(k)}{k}(2^{-j_2/\alpha_{2,1}}2^{j_2}) + 1 \right) \\ &\geq \frac{1}{\widehat{\xi}(2^{j_2})2^2} \geq \frac{1}{2^{j_2+2}(1 + \ln 2^{j_2})}. \end{aligned}$$

Hence,

$$\Phi_2((k+1)2^{-j_2}) - \Phi_2(x_{j_2,k}(\alpha_{2,1})) \geq \frac{1}{4(1+j_2)} |(k+1)2^{-j_2} - x_{j_2,k}(\alpha_{2,1})|^{\alpha_{2,1}}$$

Further,

$$\begin{aligned} \Phi_2((k+1)2^{-j_2}) - \Phi_2(x_{j_2,k}(\alpha_{2,1})) &\leq \xi_2((k+1)2^{-j_2}) - \xi_2(k2^{-j_2}) \leq \frac{9}{2^{j_2}2^2} \\ &\leq \frac{9}{2^2} |(k+1)2^{-j_2} - x_{j_2,k}(\alpha_{2,1})|^{\alpha_{2,1}}. \end{aligned}$$

where the upper bound (5) has been used.

To summarize the above, one has

$$(7) \quad \frac{1}{4(j_2+1)} \leq \frac{\Phi_2((k+1)2^{-j_2}) - \Phi_2(x_{j_2,k}(\alpha_{2,1}))}{|(k+1)2^{-j_2} - x_{j_2,k}(\alpha_{2,1})|^{\alpha_{2,1}}} \leq \frac{9}{4}.$$

Remark that, by construction of Φ_2 , using ξ_2 , which is a function with increasing visibility, one has $\Phi_2(x) \leq \xi_2(1)x \leq \frac{x}{2^2}$.

One is now able to define the next step in the construction.

Definition 3.4. *The function f_2 is simply $f_2 := f_1 + \Phi_2$.*

Remark that f_2 is a function with increasing visibility as sum of two functions with increasing visibility and that $f_2(x) = f_1(x) + \Phi_2(x) \leq \frac{x}{2} + \frac{x}{2^2} \leq (1 - \frac{1}{2^2})x$.

3.2. Construction of Φ_{n+1} and f_{n+1} . One introduces for every $n \geq 2$ and $l \in \{1, \dots, n\}$,

$$\alpha_{n+1,l} = \frac{l}{n+1}.$$

Assume that f_1, f_2, \dots, f_n have been constructed, as well as integers $j_1 < j_2 < \dots < j_n$ satisfying the following properties:

- for every $p \in \{2, \dots, n\}$, for every $l \in \{1, \dots, p-1\}$,

$$(8) \quad 2 \left[j_p \left(1 - \frac{g \circ f_{p-1}(\alpha_{p,l})}{\alpha_{p,l}} \right) \right] \geq 2^p, \quad j_{n+1} > j_n n^2 \quad \text{and} \quad 2^{(p-1)j_{p-1}} \leq \frac{j_p}{p^2}.$$

- for every $p \in \{2, \dots, n\}$, $f_p(x) \leq (1 - 2^{-p})x$ on $[0, 1]$, and the inequality is strict when $x \in (0, 1)$.
- for every $p \in \{2, \dots, n\}$, $f_p = f_{p-1} + \Phi_p$, where Φ_p is build from the procedure described Definitions 3.3 and 3.6.

Let $j_{n+1} \gg j_n$ be such that (8) is satisfied for $p = n+1$.

Definition 3.5. *For every $i \in \{1, \dots, n\}$, the set $\mathcal{A}_{n+1,i}$ is the set of those integers $k \in \{1, \dots, 2^{j_{n+1}}\}$ of the form*

$$k = i + p 2 \left[j_{n+1} \left(1 - \frac{g \circ f_n(\alpha_{n+1,i})}{\alpha_{n+1,i}} \right) \right], \quad \text{where } p \in \left\{ 1, \dots, 2 \left[j_{n+1} \frac{g \circ f_n(\alpha_{n+1,i})}{\alpha_{n+1,i}} \right] - 1 \right\}.$$

As for $\mathcal{A}_{2,1}$, the dyadics $k2^{-j_n}$, where $k \in \mathcal{A}_{n+1,i}$, are uniformly distributed in $[0, 1]$.

Lemma 3.1. *If $i \neq j$, then $\mathcal{A}_{n+1,i} \cap \mathcal{A}_{n+1,j} = \emptyset$.*

Proof. Assume towards a contradiction that there is a k belongs at $\mathcal{A}_{n+1,i} \cap \mathcal{A}_{n+1,j}$ with $i < j$. Then

$$k = i + p 2^{\left\lceil j_{n+1} \left(1 - \frac{g \circ f_n(\alpha_{n+1,i})}{\alpha_{n+1,i}} \right) \right\rceil} = j + p' 2^{\left\lceil j_{n+1} \left(1 - \frac{g \circ f_n(\alpha_{n+1,j})}{\alpha_{n+1,j}} \right) \right\rceil},$$

where p and p' are integers. Assume that $2^{\left\lceil j_{n+1} \left(1 - \frac{g \circ f_n(\alpha_{n+1,i})}{\alpha_{n+1,i}} \right) \right\rceil} \geq 2^{\left\lceil j_{n+1} \left(1 - \frac{g \circ f_n(\alpha_{n+1,j})}{\alpha_{n+1,j}} \right) \right\rceil}$. Then

$$j - i = 2^{\left\lceil j_{n+1} \left(1 - \frac{g \circ f_n(\alpha_{n+1,i})}{\alpha_{n+1,i}} \right) \right\rceil} \left(p - p' 2^{\left\lceil j_{n+1} \left(1 - \frac{g \circ f_n(\alpha_{n+1,j})}{\alpha_{n+1,j}} \right) \right\rceil - \left\lceil j_{n+1} \left(1 - \frac{g \circ f_n(\alpha_{n+1,i})}{\alpha_{n+1,i}} \right) \right\rceil} \right),$$

which is impossible since $j - i \leq n$, while (8) implies $2^{\left\lceil j_{n+1} \left(1 - \frac{g \circ f_n(\alpha_{n+1,i})}{\alpha_{n+1,i}} \right) \right\rceil} \geq 2^{n+1}$. \square

Next we construct the function whose oscillations around dyadics are controlled.

Definition 3.6. *One introduces the function Φ_{n+1} as follows:*

1. for all $k \in \{0, 1, \dots, 2^{j_{n+1}}\}$, $\Phi_{n+1}(k2^{-j_{n+1}}) = \xi_{n+1}(k2^{-j_{n+1}})$.
2. If $k \in \mathcal{A}_{n+1,l}$ for some $l \in \{1, \dots, n\}$, then

- when $x \in I_{j_{n+1},k} \setminus I_{j_{n+1},k}(\alpha_{n+1,l})$, $\Phi_{n+1}(x) = \xi_{n+1}(x)$,
- when $x \in I_{j_{n+1},k}(\alpha_{n+1,l})$, one sets

$$\begin{aligned} \Phi_{n+1}(x) &= \frac{\xi_{n+1}((k+1)2^{-j_{n+1}}) - \xi_{n+1}(x_{j_{n+1},k}(\alpha_{n+1,l}))}{2^{-j_{n+1}/\alpha_{n+1,l}}} \times (x - (k+1)2^{-j_{n+1}}) \\ &\quad + \xi_{n+1}((k+1)2^{-j_{n+1}}). \end{aligned}$$

3. If k does not belong to any $\mathcal{A}_{n+1,l}$, then Φ_{n+1} is affine on $I_{j_{n+1},k}$. Where $\xi_{n+1}(x)$ is given in Definition 2.2.

Again, heuristically, Φ_{n+1} is a continuous version of ξ_{n+1} , where the jumps at the dyadics $k2^{-j_{n+1}}$ are either erased if $k \notin \mathcal{A}_{n+1,l}$ for any l , or replaced by a rapid increasing affine map on the small interval $I_{j_{n+1},k}(\alpha_{n+1,l})$ when $k \in \mathcal{A}_{n+1,l}$, and where the slope on this interval depends on the value of $\alpha_{n+1,l}$.

The same considerations as those developed for Φ_2 induce that, by construction, when $k \in \mathcal{A}_{n+1,l}$, the oscillation of the increasing function Φ_{n+1} on the small interval $I_{j_{n+1},k}(\alpha_{n+1,l})$ satisfies

$$(9) \quad \frac{1}{2^{n+1}(j_{n+1} + 1)} \leq \frac{\Phi_{n+1}((k+1)2^{-j_{n+1}}) - \Phi_{n+1}(x_{j_{n+1},k}(\alpha_{n+1,l}))}{|(k+1)2^{-j_{n+1}} - x_{j_{n+1},k}(\alpha_{n+1,l})|^{\alpha_{n+1,l}}} \leq \frac{9}{2^{n+1}}.$$

Moreover, since

$$\Phi_{n+1}((k+1)2^{-j_{n+1}}) - \Phi_{n+1}(k2^{-j_{n+1}}) = \xi_{n+1}((k+1)2^{-j_{n+1}}) - \xi_{n+1}(k2^{-j_{n+1}})$$

Using Lemma 2.6, the oscillation of Φ_{n+1} on the interval $I_{j_{n+1},k}$ satisfies

$$(10) \quad \frac{1}{2^{n+1}(j_{n+1} + 1)} \leq \frac{\Phi_{n+1}((k+1)2^{-j_{n+1}}) - \Phi_{n+1}(k2^{-j_{n+1}})}{|(k+1)2^{-j_{n+1}} - k2^{-j_{n+1}}|} \leq \frac{9}{2^{n+1}},$$

The main idea here is that the oscillations of Φ_{n+1} essentially take place on the small intervals $I_{j_{n+1},k}(\alpha_{n+1,l})$.

One is now able to define the next step in the construction.

Definition 3.7. *The function f_{n+1} is defined as $f_{n+1} = f_n + \Phi_{n+1}$.*

The function f_{n+1} is with increasing visibility, as sum of two functions with increasing visibility. Observe that, by construction,

$$(11) \quad \Phi_{n+1}(x) \leq \xi_{n+1}(1)x \leq \frac{x}{2^{n+1}}$$

and by assumption, $f_n(x) \leq (1 - \frac{1}{2^n})x$. Hence, $f_{n+1}(x) = f_n(x) + \Phi_{n+1}(x) \leq (1 - \frac{1}{2^{n+1}})x$.

3.3. Properties of the limit of f_n . By construction, the sequence $(f_n)_{n \geq 1}$ is a sequence of functions defined on $[0, 1]$ such that:

- for all n , $\frac{x}{2} \leq f_n(x) \leq (1 - 2^{-n})x$.
- the sequence is monotone increasing, i.e., for all n and $x \in (0, 1)$, $f_n(x) < f_{n+1}(x)$.
- each f_n is continuous and with increasing visibility, as finite sum of continuous functions with increasing visibility.

Proposition 3.1. *The sequence of functions $(f_n)_{n \geq 1}$ converges uniformly to a continuous function $f : [0, 1] \rightarrow \mathbb{R}^+$ satisfying $0 \leq f(x) \leq x$ with increasing visibility, and the inequalities are strict when $x \in (0, 1)$. By construction,*

$$f(x) = \sum_{n=1}^{+\infty} \Phi_n(x).$$

Proof. The convergence follows from the monotonicity and the uniform boundedness of the sequence $(f_n)_{n \geq 1}$. Hence the limit exists and is continuous. The fact that it is with increasing visibility follows from the closedness of $V_{0,1}$. \square

4. UPPER BOUND FOR THE SPECTRUM OF f .

The aim of this section is to prove next proposition.

Proposition 4.1. *For every $h \geq 0$, let $E_f^{\leq}(h) = \{x : h_f(x) \leq h\}$. Then for every $h \in [0, 1]$,*

$$\dim E_f^{\leq}(h) \leq g \circ f(h).$$

In particular, $d_f(h) \leq g \circ f(h)$.

The proof is based on the following technical proposition.

Proposition 4.2. *Let $0 < h < 1$, $x \in [0, 1]$, and $r > 0$. Assume that $n \geq 2$ is so large that*

$$(12) \quad \alpha_{n,n-2} > h, \quad \max\left(\frac{2\pi^2}{3}, 9j_n\right) < 2^{\frac{j_n}{n}}.$$

Let n be the unique integer such that

$$(13) \quad 2^{-j_{n+1}} \leq r < 2^{-j_n}.$$

Let $i_{n,h} \in \{1, \dots, n-1\}$ be the unique integer such that

$$(14) \quad \alpha_{n,i_{n,h}-1} < h \leq \alpha_{n,i_{n,h}}.$$

Assume that

$$(15) \quad f(x+r) - f(x) > r^h.$$

Then, necessarily,

$$(16) \quad 2^{-j_n/(\alpha_{n,1} \cdot (1-\alpha_{n,n-1}))} < r < 2^{-j_n/\alpha_{n,i_{n,h}+1}}$$

and

$$(17) \quad x \in \bigcup_{i:i \leq i_{n,h}+1} \bigcup_{k \in \mathcal{A}_{n,i}} I_{j_n,k}(i, r),$$

where

$$I_{j_n,k}(i, r) = I_{j_n,k}(\alpha_{n,i}) + B(0, r).$$

Proof. Observe that our assumptions imply that $i_{n,h} \leq n-2$, hence $h + 1/n \leq 1 - 1/n$.

If (15) holds, then

$$\sum_{m \geq 1} (\Phi_m(x+r) - \Phi_m(x)) \geq r^h \text{ since } f(x) = \sum_{m \geq 1} \Phi_m(x).$$

Each Φ_n being an increasing function, all the contributions $\Phi_m(x+r) - \Phi_m(x)$ are positive.

• **Case $m < n$:** By assumption, Φ_m is continuous, increasing, piecewise affine. The greatest slope appearing in the formulas giving Φ_m is given in item 2. of Definition 3.6, and it has the form

$$\frac{\xi_m((k+1)2^{-j_m}) - \xi_m(x_{j_m,k}(\alpha_{m,l}))}{2^{-j_m/\alpha_{m,l}}}, \quad l \in \{1, \dots, m-1\}$$

Using Lemma 2.6, $\frac{1}{m} = \alpha_{m,1} \leq \alpha_{m,l}$ and equations (8) and (12), one gets

$$(18) \quad \begin{aligned} \frac{\xi_m((k+1)2^{-j_m}) - \xi_m(x_{j_m,k}(\alpha_{m,l}))}{2^{-j_m/\alpha_{m,l}}} &\leq \frac{9 \cdot 2^{-j_m-m}}{2^{-j_m/\alpha_{m,1}}} \leq 9 \cdot 2^{mj_m} \leq 9 \frac{j_{m+1}}{(m+1)^2} \\ &\leq 9 \frac{j_n}{(m+1)^2} \leq \frac{r^{-1/n}}{(m+1)^2}. \end{aligned}$$

The maximal value of $\Phi_m(x+r) - \Phi_m(x)$ is reached when the whole interval $[x, x+r]$ is contained in a part of the interval of $[0, 1]$ where the slope of Φ_m is maximal, and in this case

$$\Phi_m(x+r) - \Phi_m(x) \leq \frac{r^{1-1/n}}{(m+1)^2}.$$

Summing over all $m \in \{1, \dots, n-1\}$, one obtains

$$\sum_{m=1}^{n-1} \Phi_m(x+r) - \Phi_m(x) \leq \frac{\pi^2}{6} r^{1-1/n} < \frac{r^{-1/n}}{4} r^{1-1/n} < \frac{r^h}{4},$$

where we have used (12) and $h \leq 1 - \frac{2}{n}$.

• **Case $m > n$:** In this case, $r \geq 2^{-j_m}$. The interval $[x, x+r]$ intersects at most $r2^{j_m} + 2$ dyadic intervals of generation j_m . The oscillation of Φ_m on each dyadic interval $I_{j_m, k}$ is the same as the one of the function ξ_m , which is estimated as follows. Using Lemma 2.6, one has

$$\Phi_m(x+r) - \Phi_m(x) \leq \sum_{i=1}^{r2^{j_m}+2} 9 \cdot 2^{-j_m-m} \leq 3 \cdot 9r2^{-m} \leq \frac{r^{1-1/n}}{2^m},$$

where we used $r \geq 2^{-j_m}$ and (12) to get the final step.

Summing all contributions for $m \geq n+1$, we get (recalling that $n \geq 4$ and $h < 1 - \frac{1}{n}$)

$$\sum_{m \geq n+1} \Phi_m(x+r) - \Phi_m(x) \leq 2^{-n} r^{1-1/n} < \frac{r^h}{4},$$

• **Case $m = n$:** Finally, from the above we deduce that, in order to have (15) realized, it is necessary that

$$(19) \quad \Phi_n(x+r) - \Phi_n(x) \geq \frac{r^h}{2}.$$

If $\frac{2^{-j_n}}{2} \leq r < 2^{-j_n}$, then by construction $\Phi_n(x+r) - \Phi_n(x)$ is less than the oscillation of Φ_n on two consecutive dyadic intervals of generation j_n , which is at most

$$\Phi_n(x+r) - \Phi_n(x) \leq 2 \cdot 9 \cdot 2^{-j_n-n} \leq 9r2^{-n+2} \leq \frac{r^{1-\frac{1}{n}}}{4} \leq \frac{r^h}{4},$$

where we used (10), $n \geq 4$, and (12). Hence (15) cannot be realized.

Moreover, if the interval $[x, x+r]$ does not meet any interval $I_{j_n, k}(\alpha_{n, i})$ with $k \in \mathcal{A}_{n, i}$, a similar computation yields that (15) cannot be realized either.

Hence, we assume that there exists a unique interval $I_{j_n, k}(\alpha_{n, i})$, i.e. a unique $i \in \{1, \dots, n-1\}$ and a unique $k \in \mathcal{A}_{n, i}$ such that $I_{j_n, k}(\alpha_{n, i}) \cap [x, x+r] \neq \emptyset$. Recall the definition (14) of the integer $i_{n, h}$.

$$\text{First case: } 2^{\frac{-j_n}{\alpha_{n, (i_{n, h}+1)}}} \leq r < \frac{2^{-j_n}}{2}.$$

In this case, $\Phi_n(x+r) - \Phi_n(x)$ is less than the oscillation of Φ_n on two consecutive intervals. Using (12), $h \leq \alpha_{n,i_n,h}$ and $\alpha_{n,(i_n,h+1)} = \alpha_{n,i_n,h} + \frac{1}{n}$, one has

$$\begin{aligned} \Phi_n(x+r) - \Phi_n(x) &\leq 2 \cdot 9 \cdot 2^{-j_n-n} \leq \frac{2 \cdot 9}{2^n} r^{\alpha_{n,(i_n,h+1)}} \leq \frac{r^{-\frac{1}{n}}}{4} r^{\alpha_{n,(i_n,h+1)}} \\ &< \frac{r^{-\frac{1}{n}}}{4} r^{h+1/n} < \frac{r^h}{4}. \end{aligned}$$

Hence (15) cannot be realized.

Second case: $r < 2^{-j_n/\alpha_{n,i_n,h+1}}$.

First subcase: $r < 2^{-j_n/(\alpha_{n,1} \cdot (1-\alpha_{n,n-1}))} = 2^{-j_n n^2}$. As was shown in (18), the slope in the affine part corresponding to $I_{j_n,k}(\alpha_{n,i})$ is bounded above. One deduces that

$$\Phi_n(x+r) - \Phi_n(x) \leq 9 \cdot 2^{j_n} r \leq 9r^{-1/n} r < \frac{r^h}{4}.$$

where the fact that $r < 2^{-j_n n^2}$ has been used. Again, this makes (15) impossible.

Second subcase: $2^{-j_n/(\alpha_{n,1} \cdot (1-\alpha_{n,n-1}))} < r < 2^{-j_n/\alpha_{n,i_n,h+1}}$ and $i \geq i_{n,h} + 2$. The slope in the affine part corresponding to $I_{j_n,k}(\alpha_{n,i})$ is bounded above by $\frac{9 \cdot 2^{-j_n-n}}{2^{-j_n/\alpha_{n,i}}}$ (as proved in the case $m < n$). One deduces that

$$\begin{aligned} \Phi_n(x+r) - \Phi_n(x) &\leq \frac{9 \cdot 2^{-j_n-n}}{2^{-j_n/\alpha_{n,i}}} r \leq 9 \cdot 2^{-n} 2^{j_n(\frac{1}{\alpha_{n,i}}-1)} r \\ &\leq 9 \cdot 2^{-n} r^{1-\alpha_{n,i_n,h+1}(\frac{1}{\alpha_{n,i}}-1)} \leq 9 \cdot 2^{-n} r^{\alpha_{n,i_n,h+1}+(1-\frac{\alpha_{n,i_n,h+1}}{\alpha_{n,i}})} \\ &\leq 9 \cdot 2^{-n} r^{h+\frac{1}{n}+(1-\frac{i_{n,h+1}}{i})} < \frac{r^h}{4}. \end{aligned}$$

Once more, in this case, (15) is not realized.

Third subcase: $2^{-j_n/(\alpha_{n,1} \cdot (1-\alpha_{n,n-1}))} < r < 2^{-j_n/\alpha_{n,i_n,h+1}}$ and $i \leq i_{n,h} + 1$. It is enough for our purpose to notice that (15) is possible only if r and x satisfy these assumptions, i.e. $B(x,r) \cap I_{j_n,k}(\alpha_{n,i}) \neq \emptyset$ with r satisfying (16) and $i \leq i_{n,h} + 1$. This yields the result. \square

Now that Proposition 4.2 is proved, we use it to prove Proposition 4.1.

Proof. Let $\varepsilon > 0$. If $h_f(x) \leq h \leq 1$, then for an infinite number of integers j_n , one has $f(x+r) - f(x) > r^{h+\varepsilon}$ with $2^{-j_n+1} \leq r < 2^{-j_n}$. By Proposition 4.2, one has $2^{-j_n/(\alpha_{n,1} \cdot (1-\alpha_{n,n-1}))} < r < 2^{-j_n/\alpha_{n,i_n,h+\varepsilon+1}}$, and

$$x \in \bigcup_{i:i \leq i_{n,h+\varepsilon+1}} \bigcup_{k \in \mathcal{A}_{n,i}} I_{j_n,k}(i,r),$$

where $i_{n,h+\varepsilon} \in \{1, \dots, n-1\}$ is the unique integer such that $\alpha_{n,i_{n,h+\varepsilon}-1} < h \leq \alpha_{n,i_{n,h+\varepsilon}}$. In particular, necessarily,

$$x \in \bigcap_{N \geq 1} \bigcup_{n \geq N} \bigcup_{i: i \leq i_{n,h+\varepsilon}+1} \bigcup_{k \in \mathcal{A}_{n,i}} I_{j_n,k}(i, 2^{-j_n/\alpha_{n,i_{n,h+\varepsilon}+1}}).$$

Let $1 > \delta > g \circ f(h + 3\varepsilon)$, and let $\eta > 0$ be small. We can choose $N \geq \frac{1}{\varepsilon}$ so that for $n \geq N$, $2^{-j_n/\alpha_{n,i_{n,h+\varepsilon}+1}} < \eta$. For such an integer N , using the above inclusion, one deduces that the set $\bigcup_{n \geq N} \bigcup_{i: i \leq i_{n,h+\varepsilon}+1} \bigcup_{k \in \mathcal{A}_{n,i}} I_{j_n,k}(i, 2^{-j_n/\alpha_{n,i_{n,h+\varepsilon}+1}})$ forms a covering of $E_f^{\leq}(h)$ by sets of diameter less than η . One can then estimate the \mathcal{H}_η^δ -Hausdorff pre-measure of $E_f^{\leq}(h)$ computed using sets of diameters less than η as follows:

$$\mathcal{H}_\eta^\delta(E_f^{\leq}(h)) \leq \sum_{n \geq N} \sum_{i: i \leq i_{n,h+\varepsilon}+1} \sum_{k \in \mathcal{A}_{n,i}} \left| I_{j_n,k}(i, 2^{-j_n/\alpha_{n,i_{n,h+\varepsilon}+1}}) \right|^\delta.$$

Since

$$\begin{aligned} \left| I_{j_n,k}(i, 2^{-j_n/\alpha_{n,i_{n,h+\varepsilon}+1}}) \right|^\delta &= \left| I_{j_n,k}(\alpha_{n,i}) + B(0, 2^{-j_n/\alpha_{n,i_{n,h+\varepsilon}+1}}) \right|^\delta \\ &\leq (2^{-j_n/\alpha_{n,i}} + 2 \cdot 2^{-j_n/\alpha_{n,i_{n,h+\varepsilon}+1}})^\delta \\ &\leq 4^\delta 2^{-\delta j_n/\alpha_{n,i_{n,h+\varepsilon}+1}}, \end{aligned}$$

one has

$$\begin{aligned} \mathcal{H}_\eta^\delta(E_f^{\leq}(h)) &\leq \sum_{n \geq N} \sum_{i: i \leq i_{n,h+\varepsilon}+1} \sum_{k \in \mathcal{A}_{n,i}} 4^\delta 2^{-\delta j_n/\alpha_{n,i_{n,h+\varepsilon}+1}} \\ &\leq 4^\delta \sum_{n \geq N} \sum_{i: i \leq i_{n,h+\varepsilon}+1} 2^{\left[j_n \frac{g \circ f_{n-1}(\alpha_{n,i})}{\alpha_{n,i}} \right]} 2^{-\delta j_n/\alpha_{n,i_{n,h+\varepsilon}+1}}, \end{aligned}$$

where we used Definition 3.5.

We can choose N so large that $\alpha_{n,i_{n,h+\varepsilon}+1} \leq h + 2\varepsilon$. By the properties of the sequence $(g \circ f_n)$ (increasing visibility and monotonicity), one has :

$$\frac{g \circ f_{n-1}(\alpha_{n,i})}{\alpha_{n,i}} \leq \frac{g \circ f(\alpha_{n,i})}{\alpha_{n,i}} \leq \frac{g \circ f(\alpha_{n,i_{n,h+\varepsilon}+1})}{\alpha_{n,i_{n,h+\varepsilon}+1}} \leq \frac{g \circ f(h+2\varepsilon)}{h+2\varepsilon}.$$

Combining this with the fact that $\delta > g \circ f(h + 3\varepsilon)$, one obtains

$$\begin{aligned}
\mathcal{H}_\eta^\delta(E_f^{\leq}(h)) &\leq 4^\delta \sum_{n \geq N} \sum_{i: i \leq i_{n, h+\varepsilon}+1} 2^{j_n \left(\frac{g \circ f(\alpha_n, i_{n, h+\varepsilon}+1)}{\alpha_n, i_{n, h+\varepsilon}+1} - \frac{g \circ f(h+3\varepsilon)}{\alpha_n, i_{n, h+\varepsilon}+1} \right)} \\
&\leq 4^\delta \sum_{n \geq N} \sum_{i: i \leq i_{n, h+\varepsilon}+1} 2^{j_n \left(\frac{g \circ f(h+2\varepsilon)}{h+2\varepsilon} - \frac{g \circ f(h+3\varepsilon)}{h+2\varepsilon} \right)} \\
&\leq 4^\delta \sum_{n \geq N} n 2^{j_n \left(\frac{g \circ f(h+2\varepsilon)}{h+2\varepsilon} - \frac{g \circ f(h+3\varepsilon)}{h+2\varepsilon} \right)}.
\end{aligned}$$

Since $g \circ f$ is increasing, the exponent after j_n is strictly negative. Subsequently, the above series converges. Letting η go to zero (hence N to infinity), one deduces that $\mathcal{H}^\delta(E_f^{\leq}(h)) = 0$, i.e. $\dim(E_f^{\leq}(h)) \leq \delta$. Since this holds for any $\delta > g \circ f(h + 3\varepsilon)$ and any $\varepsilon > 0$, we get the result by right continuity of $g \circ f$. \square

We took care of the exponents less than 1. In order to complete the upper bound, one needs to prove the following proposition.

Proposition 4.3. *For every $h > 1$, $E_f(h) = \emptyset$.*

Proof. It suffices to show that for every $x \in [0, 1]$ and every $\varepsilon > 0$, there exists a sequence r_n converging to zero such that

$$f(x + r_n) - f(x) > r_n^{1+\varepsilon}.$$

Observe that the smallest slope appearing for the function Φ_n has the form :

$$\frac{\Phi_n(x_{j_n, k}(\alpha_{n, l})) - \Phi_n(k2^{-j_n})}{x_{j_n, k}(\alpha_{n, l}) - k2^{-j_n}}, \quad l \in \{1, \dots, n-1\}.$$

and one has :

$$\begin{aligned}
\frac{\Phi_n(x_{j_n, k}(\alpha_{n, l})) - \Phi_n(k2^{-j_n})}{x_{j_n, k}(\alpha_{n, l}) - k2^{-j_n}} &= \frac{\xi_n(x_{j_n, k}(\alpha_{n, l})) - \xi_n(k2^{-j_n})}{x_{j_n, k}(\alpha_{n, l}) - k2^{-j_n}} \\
&= \frac{\widehat{\xi}(x_{j_n, k}(\alpha_{n, l})2^{j_n}) - \widehat{\xi}(k)}{\widehat{\xi}(2^{j_n})2^n(x_{j_n, k}(\alpha_{n, l}) - k2^{-j_n})} \\
&= \frac{\frac{\widehat{\xi}(k)}{k}x_{j_n, k}(\alpha_{n, l})2^{j_n} - \widehat{\xi}(k)}{\widehat{\xi}(2^{j_n})2^n(x_{j_n, k}(\alpha_{n, l}) - k2^{-j_n})} \\
&= \frac{\widehat{\xi}(k)2^{j_n}(x_{j_n, k}(\alpha_{n, l}) - k2^{-j_n})}{k\widehat{\xi}(2^{j_n})2^n(x_{j_n, k}(\alpha_{n, l}) - k2^{-j_n})} \\
&= \frac{\widehat{\xi}(k)2^{j_n}}{k\widehat{\xi}(2^{j_n})2^n} \geq \frac{2^{j_n}}{\widehat{\xi}(2^{j_n})2^n} \\
&\geq \frac{2^{j_n}}{2^{j_n+n}(j_n \ln 2 + 1)} \geq \frac{1}{2^n(1 + j_n)},
\end{aligned}$$

where increasing visibility and Lemma 2.5 were used.

Let N be such that $\forall n \geq N$, $1 + \epsilon > 1 + \frac{1}{n}$ and $\frac{1}{2^n(1+j_n)} \geq 2^{-j_n}$. We put $r_n = 2^{-j_n n}$, and finally we get

$$f(x + r_n) - f(x) \geq \Phi_n(x + r_n) - \Phi_n(x) \geq \frac{r_n}{2^n(1+j_n)} \geq \frac{r_n}{2^{j_n}} \geq r_n^{1+1/n} \geq r_n^{1+\epsilon}.$$

□

5. LOWER BOUND FOR THE SPECTRUM OF f .

Now that we know that $d_f(h) \leq g \circ f(h)$ for every $h \in [0, 1]$, in order to complete the proof it is enough to establish the converse inequality. This is achieved thanks to the mass transference principle by Beresnevich and Velani [4, 5].

Let $h \in (0, 1)$. For every $n \geq 1$, recall that $i_{n,h}$ is the integer satisfying (14). Obviously, due to our construction, if a point x belongs to an interval $I_{j_n,k}(\alpha_{n,i_{n,h}})$, with $k \in \mathcal{A}_{n,i_{n,h}}$, then for $r_n = \pm \frac{2^{-j_n/\alpha_{n,i_{n,h}}}}{2}$:

One has, for n so large that $\frac{2^{\alpha_{n,i_{n,h}}}}{2^{n+1}(j_n+1)} > |r|^{1/n}$,

$$\begin{aligned} |f(x + r_n) - f(x)| &> |\Phi_n(x + r_n) - \Phi_n(x)| > |r_n| \frac{\Phi_n((k+1)2^{-j_n}) - \Phi_n(x_{j_n,k}(\alpha_{n,i_{n,h}}))}{2^{-j_n/\alpha_{n,i_{n,h}}}} \\ &> \frac{|r_n|}{2^{-j_n/\alpha_{n,i_{n,h}}}} \cdot \frac{1}{2^{j_n+n}(j_n+1)} > \frac{1}{2} \cdot \frac{2^{-j_n}}{2^n(j_n+1)} > \frac{|r_n|^{\alpha_{n,i_{n,h}}} 2^{\alpha_{n,i_{n,h}}}}{2^{n+1}(j_n+1)}. \end{aligned}$$

Now, if n is large enough, one has :

$$|f(x + r_n) - f(x)| > |r_n|^{\alpha_{n,i_{n,h}}} |r_n|^{1/n}.$$

Hence, if x belongs an infinite number of intervals $I_{j_n,k}(\alpha_{n,i_{n,h}})$, the convergence of the sequence $(\alpha_{n,i_{n,h}})_{n \geq 1}$ towards h yields $h_f(x) \leq h$.

From these considerations, it is natural to investigate the Hausdorff dimension of the set of those points belonging an infinite number of intervals $I_{j_n,k}(\alpha_{n,i_{n,h}})$. This is where the mass transference principle intervenes.

Let us denote by $([x_j - l_j, x_j + l_j] = B(x_j, l_j))_{j \geq 1}$ the sequence of the re-ordered intervals $(I_{j_n,k}(\alpha_{n,i_{n,h}}))_{n \geq 1, k \in \mathcal{A}_{n,i_{n,h}}}$, where the sequence is re-arranged so that l_j is decreasing and goes to zero. Every l_j can be written $1/2 \cdot 2^{-j_n/\alpha_{n,i_{n,h}}}$ for some integer n . From the above remarks one deduces that

$$\limsup_{j \rightarrow +\infty} B(x_j, l_j) \subset E_f^{\leq}(h).$$

This remains true if one modifies a little bit the radii of the balls: for every positive non-increasing sequence $(\tilde{\epsilon}_j)_{j \geq 1}$ converging to zero,

$$(20) \quad \limsup_{j \rightarrow +\infty} B(x_j, l_j^{1-\tilde{\epsilon}_j}) \subset E_f^{\leq}(h).$$

Observe that for every fixed integer $n \geq 1$, the dyadic numbers $\{(k+1)2^{-jn} : k \in \mathcal{A}_{n,i_n,h}\}$ is very well distributed in $[0, 1]$. Indeed, recalling Definition 3.5, one clearly has

$$[0, 1] \subset \bigcup_{k \in \mathcal{A}_{n,i_n,h}} B \left((k+1)2^{-jn}, 2^{-jn \frac{g \circ f_{n-1}(\alpha_{n,i_n,h})}{\alpha_{n,i_n,h}}} \right).$$

This holds true for every integer n , hence every real number $x \in [0, 1]$ belongs to an infinite number of balls of the form

$$B \left((k+1)2^{-jn}, 2^{-jn \frac{g \circ f_{n-1}(\alpha_{n,i_n,h})}{\alpha_{n,i_n,h}}} \right).$$

This implies in turn, since every center x_j is very close to a dyadic $(k+1)2^{-jn}$ such that $k \in \mathcal{A}_{n,i_n,h}$, that every real number $x \in [0, 1]$ belongs to an infinite number of balls of the form $B \left(x_j, (2l_j)^{g \circ f_{n-1}(\alpha_{n,i_n,h})} \right)$, where n is the unique integer so that $l_j = 1/2 \cdot 2^{-jn/\alpha_{n,i_n,h}}$. By our choice for $i_{n,h}$, there exists a sequence $(\varepsilon_j)_{j \geq 1}$ converging to zero such that

$$(2l_j)^{g \circ f_{n-1}(\alpha_{n,i_n,h})} \leq \tilde{l}_j := (l_j)^{g \circ f(h) - \varepsilon_j}.$$

To see that, observe that since $g \in C^\eta(\mathbb{R})$, for some constant depending on g only, one has

$$g \circ f(h) \leq g(f_{n-1}(\alpha_{n,i_n,h})) + C |f_{n-1}(\alpha_{n,i_n,h}) - f(h)|^\eta.$$

Moreover,

$$\begin{aligned} f(h) &\leq f(\alpha_{n,i_n,h}) = f_{n-1}(\alpha_{n,i_n,h}) + \sum_{k=n}^{\infty} \Phi_k(\alpha_{n,i_n,h}) \\ &\leq f_{n-1}(\alpha_{n,i_n,h}) + \sum_{k=n}^{\infty} \frac{\alpha_{n,i_n,h}}{2^k} \leq f_{n-1}(\alpha_{n,i_n,h}) + 2^{-(n-1)}. \end{aligned}$$

Hence, for some constant C independent of j and n , one has

$$g \circ f(h) \leq g(f_{n-1}(\alpha_{n,i_n,h})) + C2^{-n\eta}.$$

This yields, for some other constant \tilde{C} ,

$$(2l_j)^{g \circ f_{n-1}(\alpha_{n,i_n,h})} \leq \tilde{C} (l_j)^{g \circ f(h) - C2^{-n\eta}} = (l_j)^{g \circ f(h) - \varepsilon_j},$$

where ε_j tends to 0 when j tends to infinity (since the integer n associated with j tends to infinity). Finally, we deduce that every $x \in [0, 1]$ belongs to an infinite number of balls $B(x_j, \tilde{l}_j)$, i.e.

$$(21) \quad [0, 1] \subset \limsup_{j \rightarrow +\infty} B(x_j, \tilde{l}_j).$$

We use now the mass transference principle proved by Beresnevich and Velani [4, 5].

Theorem 5.1 (Mass transference principle). *Let $(y_j)_{j \geq 1}$ be a sequence of points in $[0, 1]$, and let $(m_j)_{j \geq 1}$ be a sequence of positive real numbers, non-increasing and converging to zero. If the Lebesgue measure of*

$$\limsup_{j \rightarrow +\infty} B(y_j, m_j)$$

is 1, then for any $\delta > 1$, the $1/\delta$ -Hausdorff measure of

$$\limsup_{j \rightarrow +\infty} B(y_j, (m_j)^\delta)$$

is infinite, i.e. $\mathcal{H}^{1/\delta}(\limsup_{j \rightarrow +\infty} B(y_j, (m_j)^\delta) = +\infty$.

Equation (21) guarantees us that the condition in Theorem 5.1 is realized for our sequence $(x_j, \tilde{l}_j)_{j \geq 1}$. Choosing $\delta = 1/g \circ f(h)$, one deduces that

$$\mathcal{H}^{g \circ f(h)} \left(\limsup_{j \rightarrow +\infty} B(x_j, l_j^{1-\varepsilon_j/g \circ f(h)}) \right) = +\infty.$$

Recalling (20) applied with $\tilde{\varepsilon}_j := \varepsilon_j/g \circ f(h)$, this implies

$$\mathcal{H}^{g \circ f(h)}(E_f^\leq(h)) = +\infty.$$

The conclusion follows now from two facts. Obviously,

$$E_f^\leq(h) = E_f(h) \cup \bigcup_{i \geq 1} E_f^\leq(h - 1/i).$$

Since we proved that $\dim_{\mathcal{H}} E_f^\leq(h - 1/i) \leq g \circ f(h - 1/i) < g \circ f(h)$, one has

$$\forall i \geq 1, \quad \mathcal{H}^{g \circ f(h)}(E_f^\leq(h - 1/i)) = 0.$$

The last three equations imply that $\mathcal{H}^{g \circ f(h)}(E_f(h)) = +\infty$, i.e. $\dim_{\mathcal{H}} E_f(h) \geq g \circ f(h)$. Hence the result.

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