RANDOM SPARSE SAMPLING IN A GIBBS WEIGHTED TREE

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Abstract. Let μ be the geometric realization on $[0,1]$ of a Gibbs measure on $\Sigma = \{0,1\}^{\mathbb{N}}$ associated with a Hölder potential. The thermodynamic and multifractal properties of μ are well known to be linked via the multifractal formalism. In this article, the impact of a random sampling procedure on this structure is studied.

More precisely, let $\{I_w\}_{w\in\Sigma^*}$ stand for the collection of dyadic subintervals of $[0,1]$ naturally indexed by the set of finite dyadic words Σ^*. Fix $\eta\in(0,1)$, and a sequence $(p_w)_{w\in\Sigma^*}$ of independent Bernoulli variables of parameters $2^{-|w|(1-\eta)}$ ($|w|$ is the length of w). We consider the (very sparse) remaining values $\widetilde{\mu} = \{\mu(I_w) : w\in\Sigma^*, p_w = 1\}$.

We prove that when $\eta < 1/2$, it is possible to entirely reconstruct μ from the sole knowledge of $\widetilde{\mu}$, while it is not possible when $\eta > 1/2$, hence a first phase transition phenomenon.

We show that, for all $\eta\in(0,1)$, it is possible to reconstruct a large part of the initial multifractal structure of μ, via the fine study of $\widetilde{\mu}$. After reorganization, these coefficients give rise to a random capacity with new remarkable scaling and multifractal properties: its L^q-spectrum exhibits two phase transitions, and has a rich thermodynamic and geometric structure.

1. Introduction

Statistical mechanics and multifractals are well known to be closely related. Typical situations are provided by the energy model associated with a Gibbs measure on the boundary Σ of the dyadic tree Σ^* in the context of the thermodynamic formalism [34, 15, 33], or the random energy model associated with a branching random walk on Σ^*, namely directed polymers on disordered trees [16, 14, 24, 31, 2, 3]. The purpose of this paper is to investigate the thermodynamic and geometric impact of a random sparse sampling on such structures.

Let us start by describing the interplay between thermodynamics and multifractals.

1.1. **Free energy and singularity spectrum as a Legendre pair.** For the sake of generality, we work on the d-dimensional dyadic tree and on $[0,1]^d$, $d\geq 1$. Let Σ_j be the set of words of length $j\geq 1$ over the alphabet $\{0,1\}^d$, i.e.

$$\Sigma_j = \Big\{(w_1w_2\cdots w_j) : \forall\, k\in\{1,...,j\},\ w_k = (w_k^{(1)}, w_k^{(2)},,...,w_k^{(d)})\in\{0,1\}^d\Big\}.$$

If $w\in\Sigma_j$, we denote by $|w| = j$ its length (or its generation). Then, $\Sigma^* = \bigcup_{j\geq 1}\Sigma_j$ and $\Sigma = (\{0,1\}^d)^{\mathbb{N}_+}$ denote the set of finite words and infinite words over $\{0,1\}^d$ respectively.

Date: Received: date / Revised version: date.

2010 *Mathematics Subject Classification.* 28A78, 28A80, 37D35, 60F10, 60K35.

Key words and phrases. Hausdorff dimension, Random sparse sampling, Thermodynamic formalism, Phase transitions, Ubiquity theory, Multifractals, Large deviations.

The set Σ is endowed with the standard ultra-metric distance, and $\Sigma^*\cup\Sigma$ is endowed with the shift operation denoted σ.

If $w\in\Sigma^*\cup\Sigma$ and $1\le j\le |w|$ is finite, $w_{|j}$ stands for the prefix of length j of w. If $W\in\Sigma^*$, $[W]$ is the *cylinder* of those words $w\in\Sigma$ such that $w_{||W|}=W$.

With each $w=w_1...w_j\in\Sigma_j$ is naturally associated the dyadic point

$$x_w=\left(\sum_{k=1}^j w_k^{(i)}2^{-k}\right)_{1\le i\le d}, \tag{1}$$

of $[0,1]^d$, and the dyadic subcube $I_w=\prod_{i=1}^d\left[x_w^{(i)},x_w^{(i)}+2^{-j}\right]$ of $[0,1]^d$.

If $x=(x^{(1)},x^{(2)},,...,x^{(d)})\in[0,1]^d$ has no dyadic component, then x is encoded by a unique $w=w^{(1)}w^{(2)}...w^{(d)}\in\Sigma$, and $I_j(x)$ stands for $I_{w_{|j}}$. When $x^{(i)}$ is dyadic, we choose $w^{(i)}$ as the largest element of $\{0,1\}^{\mathbb{N}_+}$ in lexicographical order which encodes $x^{(i)}$. In both cases, $w_{|j}$ is also denoted $x_{|j}$.

Definition 1. *We call* capacity *a non-negative and non-decreasing function μ of the dyadic subcubes of $[0,1]^d$, i.e. for every $W,w\in\Sigma^*$ such that $I_w\subset I_W$, $0\le\mu(I_w)\le\mu(I_W)$.*

The set of capacities is denoted by $\mathrm{Cap}([0,1]^d)$.

The support of $\mu\in\mathrm{Cap}([0,1]^d)$ is the set $\mathrm{supp}(\mu)=\bigcap_{j\ge1}\ \bigcup_{w\in\Sigma_j:\mu(I_w)>0} I_w$.

We focus on two quantities especially relevant in the thermodynamic and geometric measure theoretic contexts.

• The *free energy* of a capacity $\mu\in\mathrm{Cap}([0,1]^d)$ with a non empty support is defined as the thermodynamic (lower) limit given for $q\in\mathbb{R}$ by

$$\tau_\mu(q)=\liminf_{j\to\infty}\tau_{\mu,j}(q),\qquad \text{where }\ \tau_{\mu,j}(q):=\frac{-1}{j}\log_2\sum_{w\in\Sigma_j:\mu(I_w)>0}\mu(I_w)^q, \tag{2}$$

and q is interpreted as the inverse of a temperature when it is positive (the precise connection with statistical mechanics terminology is that in finite volume j, $\tau_{\mu,j}(q)$ is the free energy associated with the potential $V(w)=-\log(\mu(I_w))$, $w\in\Sigma_j$).

When the free energy $\tau_\mu(q)$ is a limit (not only a liminf) and is differentiable, the value $\tau_\mu(q)$ allows one to describe the asymptotical distribution properties of μ over Σ_j thanks to large deviations theory, which roughly gives the approximation:

$$\forall H\in\mathbb{R},\ \#\left\{w\in\Sigma^*:|w|=j,\ \mu(I_w)\approx 2^{-jH}\right\}\approx 2^{j\tau_\mu^*(H)}\quad\text{as } j\to+\infty,$$

where τ_μ^* is the Legendre transform of τ_μ, i.e.

$$\tau_\mu^*(H):=\inf_{q\in\mathbb{R}}\big(Hq-\tau_\mu(q)\big). \tag{3}$$

• The *singularity*, or *multifractal*, *spectrum* of μ is defined as

$$D_\mu:H\mapsto\dim\underline{E}_\mu(H),\quad H\in\mathbb{R},$$

where

$$\underline{E}_\mu(H)=\left\{x\in\mathrm{supp}(\mu):\liminf_{j\to\infty}\frac{\log_2\left(\mu(I_{x_{|j}})\right)}{-j}=H\right\}.$$

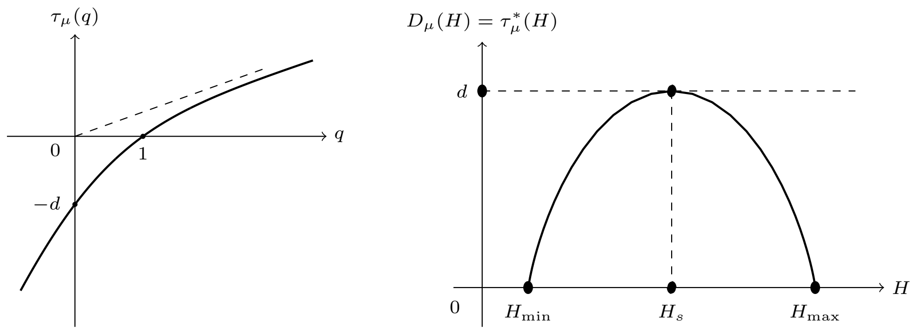

FIGURE 1. **Left:** Free energy function of a Gibbs measure μ on $[0,1]^d$. **Right:** The singularity spectrum of μ.

The Hausdorff dimension in $\mathbb{R}^d$ is denoted by dim, and by convention, $\dim \emptyset = -\infty$. The singularity spectrum provides a fine geometric description of the energy distribution at small scales by giving the Hausdorff dimension of the iso-Hölder sets $\underline{E}_\mu(H)$ of μ.

It turns out that when μ possesses nice scaling properties, one has

$$\forall\, H \in \mathbb{R}, \quad D_\mu(H) = \tau_\mu^*(H).$$

Definition 2. *When the above formula is satisfied, τ_μ and D_μ are said to form a Legendre pair (see Figure 1). In this situation, one says that μ obeys the multifractal formalism at any $H \in \mathbb{R}$.*

Forming a Legendre pair implies that the geometric description of μ provided by its singularity spectrum D_μ matches with the asymptotic statistical description of the energy distribution μ provided by the free energy τ_μ and its Legendre transform. Our goal is to investigate the impact on such well-organized structures of a natural sampling procedure.

1.2. **Random sparse sampling operation on capacities.** We perform on any capacity μ the random sampling process consisting in acting independently on the vertices of Σ^* by letting a vertex of generation j survive with probability $2^{-jd(1-\eta)}$, where $\eta \in (0,1)$ (it is also a special case of decimation rule used in percolation theory on Σ^*). More formally:

Definition 3. *Fix a real parameter $0 < \eta < 1$, called the* sampling index. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(p_w)_{w\in\Sigma^*}$ a sequence of independent Bernoulli random variables so that $p_w \sim B(2^{-d(1-\eta)|w|})$, i.e.*

$$\mathbb{P}(p_w = 1) = 1 - \mathbb{P}(p_w = 0) = 2^{-d(1-\eta)|w|}. \tag{4}$$

When $p_w = 1$, w is said to be a surviving vertex (or a survivor).
For every $j \geq 1$, denote by $\mathcal{S}_j(\eta)$ the (random) set of surviving vertices in Σ_j:

$$\mathcal{S}_j(\eta) := \{w \in \Sigma_j : p_w = 1\}.$$

Let $\mu \in \mathrm{Cap}([0,1]^d)$. We denote by $\widetilde{\mu} : \Sigma^* \to \mathbb{R}^+$ the function defined by

$$\forall\, w \in \Sigma^*, \quad \widetilde{\mu}(I_w) = \mu(I_w) \cdot p_w.$$

See Figure 2 for an illustration. The problems we address are the following.

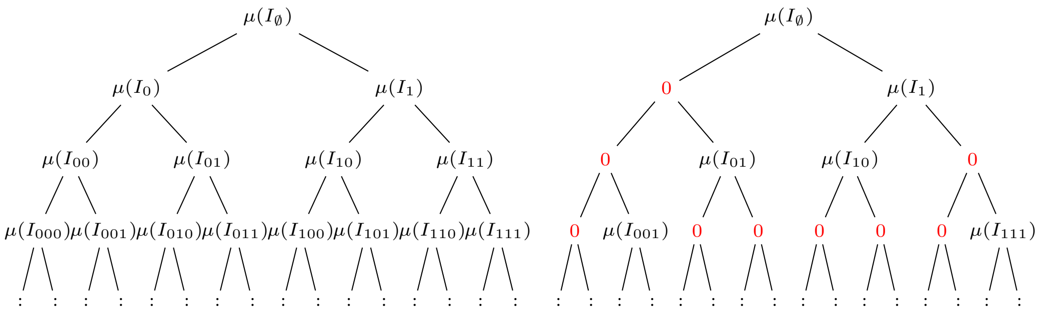

FIGURE 2. **Left:** Capacity μ on dyadic cubess. **Right:** Function $\widetilde{\mu}$ and surviving vertices after sampling.

- **Recovering from sparse information:** The set of surviving vertices $\mathcal{S}_j(\eta)$ has a cardinality of expectation $2^{dj\eta}$ (which is exponentially less than the 2^{dj} initial coefficients), and is very sparse. The first question concerns the information remaining after the sampling operation. Can one recover the initial Gibbs measure μ (i.e. all the values $(\mu(I_w))_{w\in\Sigma^*}$) from the sole knowledge of $\widetilde{\mu}$? If not, what about recovering the free energy and multifractal spectrum of μ?
- **Structure of $\widetilde{\mu}$:** The new object $\widetilde{\mu}$ is not a capacity any more. Does it have a well-organized structure though?

The last two above questions are of course related to each other.

Concerning the reconstruction problematics, recovering the scaling behavior from sparse information is a very natural issue in signal processing (this is one issue in compressive sensing). This allows one to evaluate the "incompressible" information represented by the initial capacity. We bring an answer when μ is the geometric realization on $[0,1]$ of a Gibbs measure associated with a Hölder continuous potential on Σ, and more generally a non trivial Gibbs capacity. Specifically, the capacity μ satisfies that there exists a Gibbs measure ν, $K>0$ and $(\alpha,\beta)\in\mathbb{R}_+\times\mathbb{R}_+$ such that

$$\mu(I_w)=K\nu(I_w)^{\alpha}2^{-\beta|w|},\quad \forall\, w\in\Sigma^*,$$

and μ is not constant, so $(\alpha,\beta)\neq(0,0)$ (see Section 2.1 for a precise definition of Gibbs measures and capacities).

Theorem 1. *Suppose that μ is a Gibbs capacity. With probability one, when $\eta<1/2$, one can reconstruct, up to some multiplicative constant depending only on μ, all the values $\{\mu(I_w):w\in\Sigma^*\}$, while when $\eta>1/2$, it is impossible provided μ is not built from a potential on Σ depending on only finitely many letters.*

See Section 3, Theorem 4 for a more precise statement. This constitutes a first phase transition phenomenon at $\eta=1/2$.

Regarding recovering of statistical and geometrical properties of μ, we first reorganize the surviving information in a suitable and exploitable way, as follows. If $w,v\in\Sigma^*$, wv stands for the concatenation of w and v.

Definition 4. *Let $\mu\in\mathrm{Cap}([0,1]^d)$. We consider the random capacity $\mathsf{M}_\mu\in\mathrm{Cap}([0,1]^d)$ associated with μ and the sequence $(p_w)_{w\in\Sigma^*}$ defined by*

$$\mathsf{M}_\mu(I_w)=\max\Big\{\mu(I_{wv}):v\in\Sigma^*\ \text{ and } p_{wv}=1\Big\}. \tag{5}$$

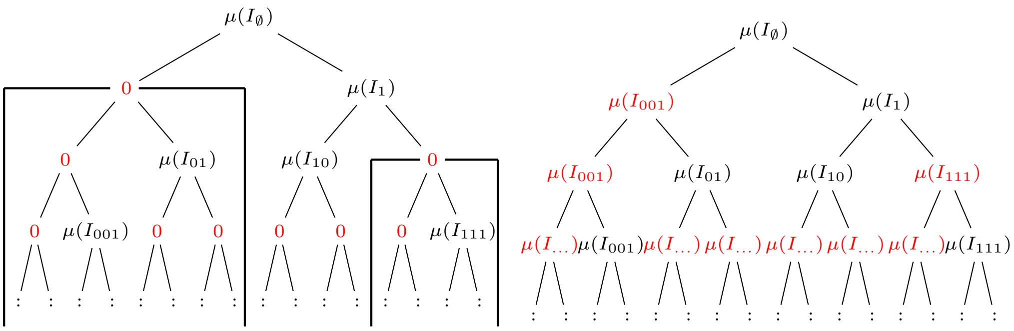

FIGURE 3. **Left:** surviving vertices after sampling, and the coefficients used to compute $\mathsf{M}_\mu(I_0)$ and $\mathsf{M}_\mu(I_{11})$. **Right:** The capacity M_μ.

See Figure 3 for the construction of M_μ. By construction, any capacity $\mu \in \mathrm{Cap}([0,1]^d)$ satisfies $\mu(I_w) = \max\left\{\mu(I_{wv}) : v \in \Sigma^*\right\}$, hence (5) is the most natural formula to be used to build a capacity from $\widetilde{\mu}$.

It is not difficult to see that with probability 1, for every $w \in \Sigma^*$, the set $\{v \in \Sigma^* \text{ and } p_{wv} = 1\}$ is non-empty, so that M_μ is well defined. Observe that by our choice (4), most of the coefficients $\widetilde{\mu}(I_w)$ equal 0, hence typically one has $\mathsf{M}_\mu(I_w) \ll \mu(I_w)$ when $\lim_{j\to+\infty} \max\{\mu(I_w) : w \in \Sigma_j\} = 0$.

The definition of M_μ can be rephrased as

$$\mathsf{M}_\mu(I_w) = \max\{\mu(I_v) : v \text{ survives}, [v] \subset [w]\} = \max\{\widetilde{\mu}(I_{wv}) : v \in \Sigma^*\}.$$

We notice that M_μ and $\widetilde{\mu}$ are equivalent objects in the following sense. If μ is strictly positive, $\widetilde{\mu}$ can be recovered from M_μ since $\widetilde{\mu}(I_w) \neq 0$ if and only if $\mathsf{M}_\mu(I_w) > \mathsf{M}_\mu(I_{wv})$ for all $v \in \Sigma^*$ such that $|v| \geq 1$. From now on, we work with the capacitiy M_μ only.

Starting from a positive capacity μ whose free energy τ_μ and singularity spectrum D_μ form a Legendre pair, we consider the following questions in order to estimate the structural perturbations induced by the sampling process:

- Do the free energies in finite volume $\tau_{\mathsf{M}_\mu,j}$ converge to a thermodynamic limit τ_{M_μ} as $j \to +\infty$?
- Is it possible to conduct a fine analysis of the local behavior of M_μ so that D_{M_μ} is computable? If so, do τ_{M_μ} and D_{M_μ} form a Legendre pair?
- Are there explicit relations between the new pair $(\tau_{\mathsf{M}_\mu}, D_{\mathsf{M}_\mu})$ and the original one (τ_μ, D_μ), so that one can recover the initial information (before sampling)?

When μ is a Gibbs Capacity, we are going to prove that the free energy τ_{M_μ} exists as a limit, and that it forms a Legendre pair with the singularity spectrum of M_μ. Nevertheless, we will see that the sampling deeply modifies and complexifies the initial structure, creating several phenomenological differences between μ and M_μ, both from thermodynamic and geometric viewpoints.

1.3. **Statement of the main result for the random capacity M_μ when μ is Gibbsian.** We only consider capacities with full support, i.e. $\mu(I_w) > 0$ for all $w \in \Sigma^*$.

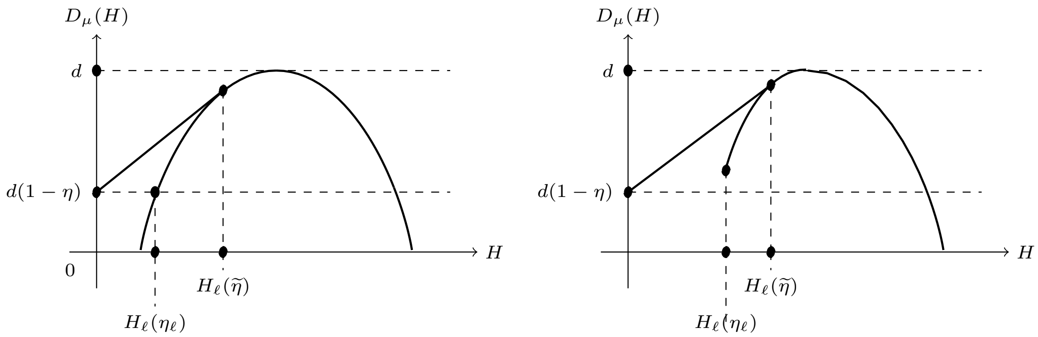

FIGURE 4. Values of $H_\ell(\eta_\ell)$ and $H_\ell(\widetilde\eta)$ depending on D_μ and η: **Left:** when $D_\mu(H_{\min}) \le d(1-\eta)$. **Right:** when $D_\mu(H_{\min}) > d(1-\eta)$.

Definition 5. *Let $\mu \in \mathcal{C}([0,1]^d)$ with full support. For $x \in [0,1]^d$, the lower and upper local dimensions of μ at x are respectively defined as*

$$\underline{\dim}(\mu,x) = \liminf_{r\to 0^+} \frac{\log_2 \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{\dim}(\mu,x) = \limsup_{r\to 0^+} \frac{\log_2 \mu(B(x,r))}{\log r}.$$

When $\underline{\dim}(\mu,x) = \overline{\dim}(\mu,x)$, their common value is denoted by $\dim(\mu,x)$.
For $H \in \mathbb{R}$, set

$$\underline{E}_\mu(H) = \left\{ x \in [0,1]^d : \underline{\dim}(\mu,x) = H \right\},$$
$$\overline{E}_\mu(H) = \left\{ x \in [0,1]^d : \overline{\dim}(\mu,x) = H \right\},$$
$$E_\mu(h) = \underline{E}_\mu(H) \cap \overline{E}_\mu(H).$$

Recall that the singularity spectrum of μ is the mapping

$$D_\mu : \ H \in \mathbb{R} \longmapsto \dim \underline{E}_\mu(H).$$

The lower local dimension is distinguished with respect to $\overline{\dim}(\mu,x)$ or $\dim(\mu,x)$, because it provides at any x the best local control of the capacity μ. Since μ is bounded, one has $\dim(\mu,x) \ge 0$ at any x, hence $\underline{E}_\mu(H) = \emptyset = \overline{E}_\mu(H)$ for all $H < 0$.

The multifractal formalism states that for every capacity $\mu \in \mathcal{C}([0,1]^d)$ with full support,

$$\dim \underline{E}_\mu(H) \le \tau_\mu^*(H) := \inf_{q\in\mathbb{R}} \big(Hq - \tau_\mu(q)\big), \quad \forall\, H \in \mathbb{R}, \tag{6}$$

see for instance [11, 32], which deal with measures, but easily extend to capacities. Recall that the multifractal formalism holds for μ at $H \in \mathbb{R}$ when there is equality in (6).

We consider a non-homogeneous Gibbs capacity μ, i.e. associated with a Hölder potential non cohomologous to a constant (see Definition 8 in Section 2.1 for a precise description). For such an object, the following statement gathers information deduced from the study of Gibbs measures and almost-additive potentials [34, 15, 33, 11, 23, 21, 20]: Let $H_{\min} = \tau_\mu'(+\infty) \le H_s := \tau_\mu'(0) \le H_{\max} = \tau_\mu'(-\infty)$.

(1) The free energy function τ_μ is the limit of $(\tau_{\mu,j})_{j\ge 1}$ as $j \to +\infty$. The function τ_μ is analytic, increasing, and strictly concave on $\mathbb{R}$.

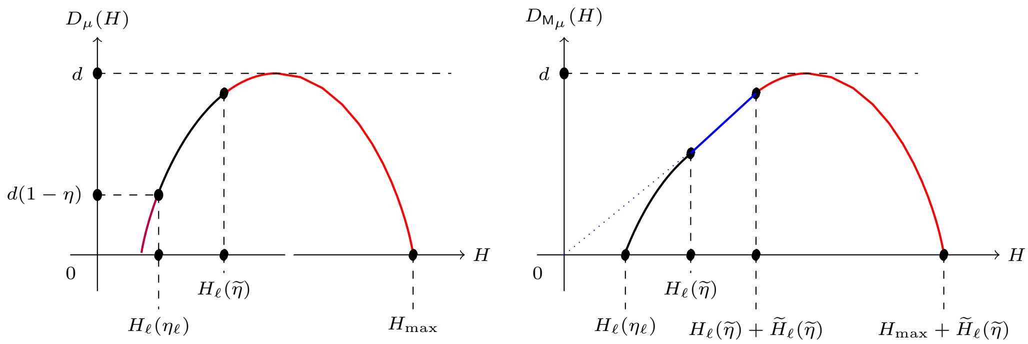

FIGURE 5. **Case** $D_\mu(H_{\min}) \le d(1-\eta)$: **Left:** singularity spectrum of μ. **Right:** Almost sure singularity spectrum of M_μ. The parts drawn with same color are translated copies of each other. One sees that the left part $H \le H_\ell(\widetilde{\eta})$ of the spectrum of μ (drawn in purple) does not appear in the singularity spectrum of M_μ, and a linear part appears in D_{M_μ} which was not present in D_μ. Observe that the slope of D_{M_μ} at $H_\ell(\eta_\ell)$ is finite.

(2) The stricly concave function τ_μ^* is non-negative on its domain of definition, namely $[H_{\min}, H_{\max}] \subset \mathbb{R}_+^*$, and analytic on $(H_{\min}, H_{\max})$. It reaches its maximum at H_s, and $\tau_\mu^*(H_s) = d$.

(3) For all $H \ge 0$, we have $D_\mu(H) = \dim E_\mu(H) = \dim \overline{E}_\mu(H) = \tau_\mu^*(H)$. The multifractal formalism holds for μ, and (τ_μ, D_μ) forms a Legendre pair.

Let us describe our result on the random capacity M_μ obtained after the sampling of μ. For this, let us introduce some notations.

Definition 6. *Let μ be a non-homogeneous Gibbs capacity. Given $\eta \in (0,1)$, one introduces three exponents $H_\ell(\eta_\ell)$, $H_\ell(\widetilde{\eta})$ and $\widetilde{H}_\ell(\widetilde{\eta})$, which depend on μ and η only, by the following formulas:*

- *$H_\ell(\eta_\ell)$ is defined as*

  $H_\ell(\eta_\ell) = \min\{H \ge 0 : D_\mu(H) \ge d(1-\eta)\}$. *Then we set* $q_{\eta_\ell} = D'_\mu(H_\ell(\eta_\ell))$.

- *$H_\ell(\widetilde{\eta})$ is the (unique) real number such that the tangent to the graph of D_μ at $(H_\ell(\widetilde{\eta}), D_\mu(H_\ell(\widetilde{\eta}))$ passes through $(0, d(1-\eta))$. Also let $q_{\widetilde{\eta}} = D'_\mu(H_\ell(\widetilde{\eta}))$.*

- *Finally, $\widetilde{H}_\ell(\widetilde{\eta}) = -\frac{\tau_\mu(q_{\widetilde{\eta}})}{q_{\widetilde{\eta}}}$.*

See Figure 4 for an illustration. The origin and roles of the three exponents $H_\ell(\eta_\ell)$, $H_\ell(\widetilde{\eta})$ and $\widetilde{H}_\ell(\widetilde{\eta})$, as well as the notations themselves, will be explained in Sections 4 and next. Observe that these exponents depend continuously on D_μ and η.

Theorem 2. *Let μ be a non-homogeneous Gibbs capacity on $[0,1]^d$. Let $0 < \eta < 1$ be a sampling parameter. With probability 1:*

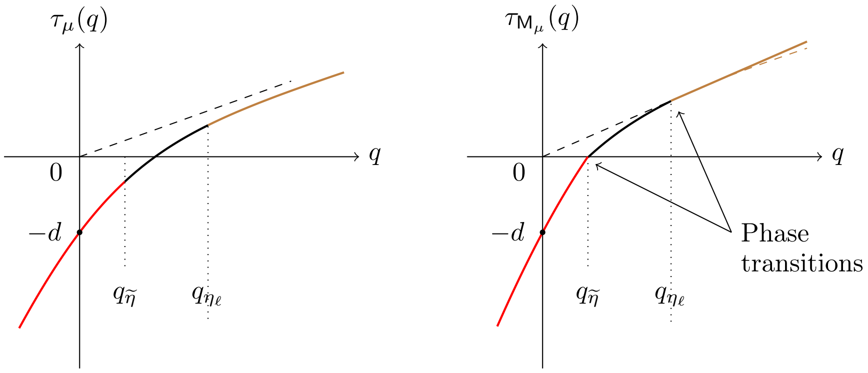

FIGURE 6. **Case** $D_\mu(H_{\min}) \le d(1-\eta)$: **Left:** Free energy τ_μ of μ; **Right:** Free energy τ_{M_μ} of M_μ.

(1) *The singularity spectrum of* M_μ *reads:*

$$D_{\mathsf{M}_\mu}(H) = \begin{cases} D_\mu(H) - d(1-\eta) & \text{when} \quad H_\ell(\eta_\ell) \le H \le H_\ell(\widetilde{\eta}), \\ q_{\widetilde{\eta}} \cdot H & \text{when} \quad H_\ell(\widetilde{\eta}) \le H \le H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), \\ D_\mu\big(H - \widetilde{H}_\ell(\widetilde{\eta})\big) & \text{when} \quad H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}) \le H \le H_{\max} + \widetilde{H}_\ell(\widetilde{\eta}), \\ -\infty & \text{otherwise.} \end{cases}$$

(2) *The free energy function of* M_μ *is is the limit of* $(\tau_{M_\mu,j})_{j\ge1}$ *as* $j\to\infty$, *and* $(\mathsf{M}_\mu, \tau_{\mathsf{M}_\mu})$ *forms a Legendre pair. One has*

$$\tau_{\mathsf{M}_\mu}(q) = \begin{cases} \tau_\mu(q) + \widetilde{H}_\ell(\widetilde{\eta}) \cdot q & \text{when} \quad q \le q_{\widetilde{\eta}}, \\ \tau_\mu(q) + d(1-\eta) & \text{when} \quad q_{\widetilde{\eta}} < q < q_{\eta_\ell}, \\ H_\ell(\eta_\ell) \cdot q & \text{when} \quad q_{\eta_\ell} < +\infty \text{ and } q \ge q_{\eta_\ell}. \end{cases}$$

(3) *For all* $H \ge H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})$,

$$\dim E_{\mathsf{M}_\mu}(H) = \dim \overline{E}_{\mathsf{M}_\mu}(H) = \begin{cases} D_\mu\big(H - \widetilde{H}_\ell(\widetilde{\eta})\big) & \text{if } H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}) \le H \le H_{\max} + \widetilde{H}_\ell(\widetilde{\eta}), \\ -\infty & \text{if } H > H_{\max} + \widetilde{H}_\ell(\widetilde{\eta}). \end{cases}$$

1.4. **Comments.** • It is quite easy to see that the lower local dimension of M_μ at any x must be greater than $H_\ell(\eta_\ell)$ (see Lemma 4). It is much more involved to define and to understand the role of the other parameters.

• From the free energy function τ_{M_μ}, one recovers the initial free energy τ_μ, except for $q \ge q_{\eta_\ell}$. Similarly, one recovers D_μ from D_{M_μ} for $H \ge \widetilde{H}_\ell(\eta_\ell)$. In this sense, the sampling procedure implies a loss of information on the local dimensions, since the values of the singularity spectrum $D_\mu(H)$ are "lost" when $H < H_\ell(\eta_\ell)$.

• The singularity spectra associated with the level sets $E_{\mathsf{M}_\mu}(H)$ or $\overline{E}_{\mathsf{M}_\mu}(H)$ are just translated from D_μ over $[H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$. In fact, $D_\mu(\cdot - \widetilde{H}_\ell(\widetilde{\eta}))$ is still a lower bound for these spectra over $[H_{\min} + \widetilde{H}_\ell(\widetilde{\eta}), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}))$, but, whether $D_\mu(\cdot - \widetilde{H}_\ell(\widetilde{\eta}))$ is a sharp upper bound in this case or not, remains an open question (see Remark 5).

• The thermodynamic and geometric phase transitions mentioned earlier can now be made more precise. The free energy τ_{M_μ} is not differentiable at $q_{\widetilde{\eta}}$, and it differentiable but not twice differentiable at q_{η_ℓ} when $d(1-\eta) > D_\mu(H_{\min})$. Moreover, τ_{M_μ} is analytic outside these singularities. In the thermodynamics language, τ_{M_μ} presents a first order phase transition at the inverse temperature $q_{\widetilde{\eta}}$, and a second order phase transition at the inverse temperature q_{η_ℓ} whenever $d(1-\eta) > D_\mu(H_{\min})$.

Let us mention that the study of phase transitions for weak Gibbs measures associated with continuous potentials, started with [34, 25], is still an active domain of research [35, 26, 12, 13, 19, 22].

• In most of the usual situations, upper bounds for dimensions of "fractal sets" are easily deduced from covering arguments, and lower bounds are more difficult to derive. The structure of M_μ, combining random and dynamical phenomena, makes both the derivation of the sharp upper bound *and* lower bound for D_{M_μ} delicate.

It is too soon in the paper to give an intuition of the proofs. Let us only say that they follow from a careful analysis of the distribution and the scaling behavior (with respect to μ) of the surviving vertices. Also, results on large deviations for Gibbs measures, heterogeneous mass transference principles (which combines ergodic and approximation theories) and percolation theory, are involved.

• One may also want to describe the asymptotical statistical distribution of M_μ through the notion of large deviations, as is often the case in statistical physics.

Definition 7. *Let $\mu \in \mathcal{C}([0,1]^d)$ with full support. For every set $I \subset \mathbb{R}^+$, and every integer $j \geq 1$, set*

$$\mathcal{E}_\mu(j, I) = \left\{ w \in \Sigma_j : \frac{\log_2 \mu(I_w)}{-j} \in I \right\}.$$

If $H \geq 0$ and $\varepsilon > 0$, we introduce the notation

$$\mathcal{E}_\mu(j, H \pm \varepsilon) = \left\{ w \in \Sigma_j : \frac{\log_2 \mu(I_w)}{-j} \in [H-\varepsilon, H+\varepsilon] \right\}.$$

Then, the lower and upper large deviations spectra of μ are respectively

$$\begin{aligned} \underline{f_\mu}(H) &= \lim_{\varepsilon\to 0} \liminf_{j\to+\infty} \frac{\log_2 \#\mathcal{E}_\mu(j, H\pm\varepsilon)}{j} \\ \text{and} \quad \overline{f_\mu}(H) &= \lim_{\varepsilon\to 0} \limsup_{j\to+\infty} \frac{\log_2 \#\mathcal{E}_\mu(j, H\pm\varepsilon)}{j}. \end{aligned}$$

Heuristically, one should have in mind that the number of words of length j satisfying $\mu(I_w) \sim 2^{-jH}$ is between $2^{j\underline{f_\mu}(H)}$ and $2^{j\overline{f_\mu}(H)}$. Next theorem states that M_μ behaves nicely with respect to the large deviations theory, as the Gibbs capacity μ does.

Theorem 3. *Under the same assumptions as in Theorem 2, with probability 1, we have*

$$\text{for all } H \geq 0, \qquad \underline{f}_{\mathsf{M}_\mu}(H) = \overline{f}_{\mathsf{M}_\mu}(H) = D_{\mathsf{M}_\mu}(H).$$

1.5. **Conclusion and further perspectives.** The hierarchical structure of the initial capacity μ is so robust that, although we greatly sample it, the remaining coefficients still possess a rich structure, especially in terms of scaling properties and multifractal formalism. For instance, one consequence of Theorem 2 is that no matter how close to 0 η is (i.e. even if only a very small logarithmic proportion of vertices are kept), it is always possible to reconstruct from the knowledge of τ_{M_μ} all the dimensions of the set of points with local

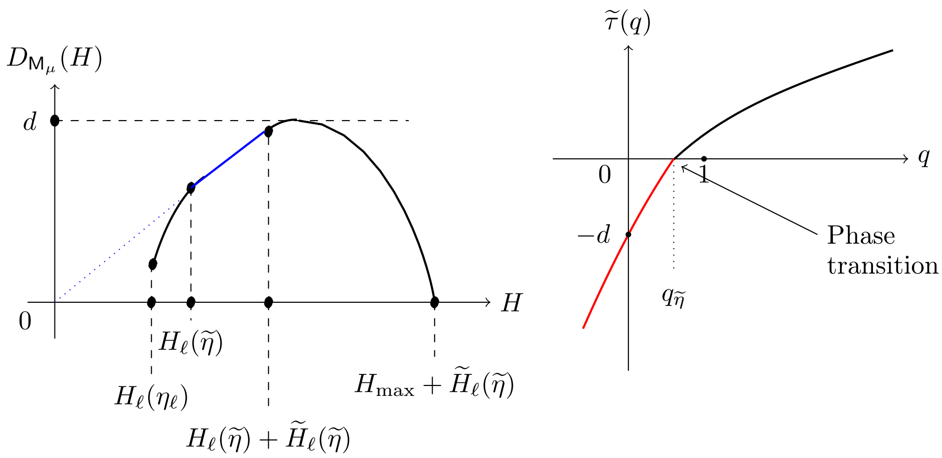

FIGURE 7. **Case** $D_\mu(H_{\min}) > d(1-\eta)$: Observe that D_μ and D_{M_μ} have an infinite slope at $H_\ell(\eta_\ell) = H_{\min}$.

dimension greater than H_s (one can even show that when $\eta = 0$, one has $\underline{E}_{\mathsf{M}_\mu}(H) = E_{\mathsf{M}_\mu}(H) = \overline{E}_{\mathsf{M}_\mu}(H) = \emptyset$ if $H < H_s$, $\dim \underline{E}_{\mathsf{M}_\mu}(H) = \dim E_{\mathsf{M}_\mu}(H) = \dim \overline{E}_{\mathsf{M}_\mu}(H) = 0$ if $H \geq H_s$, and $\dim E_{\mathsf{M}_\mu}(+\infty) = d$).

This phenomenon is remarkable, since at the same time, most of the information on the dimensions of the set of points with local dimension smaller than H_s is lost. This asymmetry was, at least from our point of view, unexpected.

Let us finish with some perspectives:

- A remaining question concerns the possible reconstruction of the Gibbs tree at the critical parameter $1/2$ (see Section 3).
- It is natural to expect our result to extend to capacities obtained after sampling of branching random walks.
- Instead of starting by assigning the value $\mu(I_w)$ at every node $w \in \Sigma_j$, one could give the value $\mu(I_{w_{|\lfloor j\rho \rfloor}})$ with $\rho < 1$. This creates redundancy in the dyadic tree, which may balance the sparsity associated with the sampling process and provide different behaviors than those exhibited in Theorem 2.
- Other sampling procedures can be investigated. In particular, one would like to allow correlations between the p_w, or make η depend on the vertex w. One may also multiply $\mu(I_w)$ by some positive random variable when $p_w = 1$. Other interesting phase transitions phenomena will certainly occur.
- It is tempting to iterate the sampling process by applying it to M_μ. Unfortunately our analysis does not apply to M_μ any more, since M_μ is not a Gibbs capacity in the sense considered in this paper. An interesting related question is whether the capacity M_μ could be made equivalent, after a natural renormalization procedure, to a measure, as it is the case for Gibbs capacities.

The paper is organized as follows.

Section 2 provides the reader with details on Gibbs measures and capacities, and gathers some information about large deviations and multifractal analysis.

Section 3 focuses on the reconstruction of the original capacity μ from its sample $\widetilde{\mu}$.

The rest of the paper is devoted to the investigation of the structure of M_μ.

We first need to introduce new definitions to explain the origin of the parameters introduced in Theorem 2. This is achieved in Section 4. There, we first explain that we will work with a slight, and natural, modification of M_μ possessing the same statistical and geometric properties as M_μ, but necessary to get an application of our result to Gibbs weighted wavelets series.

In Section 5, we investigate the scaling and distribution properties of the surviving vertices. A key decomposition of the value of $\mu(I_w)$ when w survives, is proved (see Proposition 5).

Sections 6 and 7 respectively establish the sharp upper bound and lower bound for the singularity spectrum D_{M_μ}, while Section 9 is devoted to the dimensions of the sets $E_{\mathsf{M}_\mu}(H)$ and $\overline{E}_{\mathsf{M}_\mu}(H)$. The studies achieved in the Sections 6 and 7 are used in Section 8 to get the free energy τ_{M_μ} as the limit of $(\tau_{\mathsf{M}_\mu,j})_{j\geq 1}$, as well as the large deviations spectra $\underline{f}_{\mathsf{M}_\mu}$ and $\overline{f}_{\mathsf{M}_\mu}$. The case of homogeneous Gibbs capacities (i.e. when the associated Gibbs measure is the Lebesgue measure) is dealt with in Section 10.

Notational conventions:

- We always use:
  - **capital letters** ($E_{\mathsf{M}_\mu}(H)$, F_μ, ...) to characterize sets of points $x \in [0,1]^d$ enjoying some properties,
  - **curved letters** for sets of finite words having specific properties ($\mathcal{S}_j(\eta, W)$ for some surviving coefficients, $\mathcal{R}_\mu(j,\eta',\alpha\pm\varepsilon)$ or $\mathcal{T}_\mu(j,\eta',\varepsilon)$ for words with specific properties, see next Definition 17).
  - **calligraphic letters** ($\mathscr{A}$, $\mathscr{B}$,...) to denote probabilistic events.

- For every finite word $W \in \Sigma_J$, $\mathcal{N}(W)$ stands for the set of 3^d-1 words of length J corresponding to the 3^d-1 neighboring cubes at generation J of I_W. Sometimes we will write $\mathcal{N}_J(W)$ when the length J of W is specified.

2. Complements on Gibbs measures and capacities structure

2.1. **Formal definition of Gibbs measures and capacities.** Let $\psi : \Sigma \to \mathbb{R}$ be a Hölder continuous mapping. Then, the function Ψ defined as

$$\Psi([w]) = \sup_{t\in[w]} \sum_{i=0}^{|w|-1} \psi(\sigma^i t), \quad \forall\, w \in \Sigma^*$$

is almost additive: there exists $C_1 \in \mathbb{R}$ such that for all $u, v \in \Sigma^*$,

$$|\Psi([u]) + \Psi([v]) - \Psi([uv])| \leq C_1$$

(see [34]). This almost additivity property implies that the topological pressure

$$P(\sigma,\phi) = \lim_{j\to\infty} \frac{1}{j} \log \sum_{w\in\Sigma_j} \exp(\Psi([w]))$$

exists in $\mathbb{R}$, and there exists a fully supported Gibbs measure ν on Σ such that for another constant $C_2 > 0$ one has

$$C_2^{-1} \exp(\Psi([w]) - nP(\sigma,\psi)) \leq \nu([w]) \leq C_2 \exp(\Psi([w]) - nP(\sigma,\psi)), \quad \forall\, w \in \Sigma^*.$$

Also, there is a unique choice of such a ν so that ν is ergodic. Moreover, the mapping $q \in \mathbb{R} \mapsto P(\sigma, q\psi)$ is convex, analytic, and it is strictly convex if and only if ψ is not cohomologous to a constant, i.e. there is no continuous function φ on Σ and constant $c \in \mathbb{R}$ such that $\psi = c + \varphi - \varphi \circ \sigma$. These are important facts from thermodynamic formalism (see e.g. [34]).

Definition 8. *A capacity $\mu \in \mathrm{Cap}([0,1]^d)$ is a Gibbs capacity if*

$$\mu(I_w) = K\nu([w])^\alpha e^{-|w|\beta}, \quad \forall\, w \in \Sigma^*, \tag{7}$$

where $K > 0$, $(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(0,0)\}$, and ν is a Gibbs measure associated with a Hölder continuous potential ψ has above.

Equivalently, one says that μ is associated with the Hölder potential

$$\phi = \alpha\psi - \alpha P(\sigma, \psi) - \beta.$$

The capacity μ is said to be homogeneous *when ψ is cohomologous to a constant or $\alpha = 0$, i.e. when ϕ is cohomologous to a constant, and* non-homogeneous *otherwise.*

Observe that if $(\alpha, \beta) = (1, 0)$, μ reduces to the Gibbs measure associated with ψ, and that

$$\tau_\mu(q) = \frac{1}{\log(2)}\Big((\beta + \alpha P(\sigma, \psi))q - P(\sigma, \alpha q\psi)\Big), \quad \forall\, q \in \mathbb{R}.$$

The following fact is key: the capacity μ possesses self-similarity properties expressed through the following almost multiplicative property (easy to check): there exists a constant $C > 0$ such that

$$\text{for all words } v \text{ and } w, \quad C^{-1}\mu(I_w)\mu(I_v) \le \mu(I_{wv}) \le C\mu(I_w)\mu(I_v). \tag{8}$$

2.2. **Large deviations and multifractal properties.** Let $\mu \in \mathcal{C}([0,1]^d)$ with non empty support. The concave function τ_μ^* is called Legendre spectrum of μ (recall that the Legendre transform τ_μ^* is given by (3)). For a non-homogeneous Gibbs capacity μ, one always has:

- τ_μ is strictly concave and analytic, and D_μ is strictly concave, and real analytic over $(H_{\min}, H_{\max})$. Also, $D_\mu = \tau_\mu^*$ and $(D_\mu^*)^* = D_\mu$.
- If $H = \tau_\mu'(q)$, then $\tau_\mu(q) = D_\mu^*(q) = qH - D_\mu(H) = q\tau_\mu'(q) - D_\mu(\tau_\mu'(q))$.
- If $q = {D_\mu}'(H)$, then $D_\mu(H) = \tau_\mu^*(H) = Hq - \tau_\mu(q) = HD_\mu'(H) - \tau_\mu(D_\mu'(H))$.

These relationships will be used repeatedly in the following.

Definition 9. *For any fully supported capacity $\mu \in \mathcal{C}([0,1]^d)$, define the level sets*

$$E_\mu^{\le}(H) = \{x \in [0,1]^d : \dim(\mu, x) \le H\} \quad \text{and} \quad E_\mu^{\ge}(H) = \{x \in [0,1]^d : \dim(\mu, x) \ge H\}.$$

The sets $\underline{E}_\mu^{\le}(H)$, $\underline{E}_\mu^{\ge}(H)$, $\overline{E}_\mu^{\le}(H)$, $\overline{E}_\mu^{\ge}(H)$ are defined similarly using the lower and upper local dimensions, respectively.

If $j \ge 1$ and $w \in \Sigma_j$, denote by $\mathcal{N}_j(w)$ the set of at most 3^d elements $v \in \Sigma_{|w|}$ such that I_v is a neighbor of I_w in $\mathbb{R}^d$. Also, for $x \in [0,1]^d$ and $j \ge 1$, set $\mathcal{N}_j(x) = \mathcal{N}(x_{|j})$. One defines the set

$$\widetilde{E}_\mu(H) = \left\{x \in [0,1]^d : \lim_{j\to+\infty} \frac{\log_2 \max_{w\in\mathcal{N}_j(x)} \mu(I_w)}{j} = \lim_{j\to+\infty} \frac{\log_2 \min_{w\in\mathcal{N}_j(x)} \mu(I_w)}{j} = H\right\}.$$

Obviously $\widetilde{E}_\mu(H) \subset E_\mu(H)$. This refinement of $E_\mu(H)$ is needed when looking for the lower bound of the Hausdorff dimensions of some sets in Section 7.

A direct consequence of large deviations theory (see e.g. [11, 32]) is a property valid for all capacities.

Proposition 1. *Let $\mu \in \mathcal{C}([0,1]^d)$ with full support. For all $H \leq \tau_\mu'(0^+)$, one has*

$$\limsup_{j\to\infty} \frac{1}{j} \log_2 \#\mathcal{E}_\mu(j,[0,H]) \leq \tau_\mu^*(H).$$

Next proposition gathers information about upper bounds for the singularity spectrum of D_μ in terms of Legendre and large deviation spectra, when μ is a Gibbs capacity.

Proposition 2. *Let μ be a non-homogeneous Gibbs capacity. Recall that $H_{\min} = \tau_\mu'(+\infty) < H_s := \tau_\mu'(0) < H_{\max} = \tau_\mu'(-\infty)$.*

(1) *For every $H \geq 0$, one has*
$$\dim \underline{E}_\mu(H) = \dim E_\mu(H) = \dim \overline{E}_\mu(H) = \underline{D_\mu}(H) = \overline{D_\mu}(H) = \tau_\mu^*(H) = D_\mu(H),$$
with $\underline{E}_\mu(H) = \emptyset$ if and only if $D_\mu(H) = -\infty$.

(2) *For every $H \in [H_{\min}, H_s]$ (i.e., in the increasing part of the singularity spectrum D_μ), one has*
$$\dim E_\mu^{\leq}(H) = \dim \underline{E}_\mu^{\leq}(H) = \dim \overline{E}_\mu^{\leq}(H) = D_\mu(H).$$

(3) *For every $H \in [H_s, H_{\max}]$ (i.e. in the decreasing part of D_μ), one has*
$$\dim E_\mu^{\geq}(H) = \dim \underline{E}_\mu^{\geq}(H) = \dim \overline{E}_\mu^{\geq}(H) = D_\mu(H).$$

(4) *For every possible local dimension $H \in (H_{\min}, H_{\max})$, there exists a unique $q \in \mathbb{R}$ such that $H = \tau_\mu'(q)$. The Gibbs measure μ_H associated with the potential $q\phi$ is exact dimensional with dimension $D_\mu(H)$, and $\mu_H\big(E_\mu(H)\big) = \mu\big(\widetilde{E}_\mu(H)\big) = 1$.*

(5) *For every $\varepsilon > 0$ and every interval $I \subset \mathbb{R}_+$, there exists an integer J_I such that for every $j \geq J_I$,*
$$\left| \frac{\log_2 \mathcal{E}_\mu(j,I)}{j} - \sup_{h\in I} D_\mu(h) \right| \leq \varepsilon.$$

(6) *There exists a constant $K > 0$ such that for every finite word $w \in \Sigma^*$,*
$$\left| \frac{\log_2 \mu(I_w)}{-|w|} \right| \leq K.$$

This is deduced from [11, 32, 29, 10].

Items (1) and (3) of the last proposition say in particular that the Hausdorff dimension of the sets of points at which $\dim(\mu,x) = H$ is the same as the Hausdorff dimension of the set of points at which $\underline{\dim}(\mu,x) = H$. This will be of particular importance.

We often use item (5) under the following form. Recall the formula for $\mathcal{E}_\mu(j, H \pm \varepsilon)$ in Definition 7: heuristically, $\mathcal{E}_\mu(j, H \pm \varepsilon)$ contains those words of length j such that $\mu(I_w) \sim 2^{-j(H\pm\varepsilon)}$. For every $H_{\min} \leq H \leq H_{\max}$ and $\varepsilon, \widetilde{\varepsilon} > 0$, there exists a generation J such that $j \geq J$ implies

$$\left| \frac{\log_2 \#\mathcal{E}_\mu(j, H\pm\varepsilon)}{j} - \sup_{h\in[H-\varepsilon,H+\varepsilon]} D_\mu(h) \right| \leq \widetilde{\varepsilon}. \tag{9}$$

One needs to keep in mind that $\#\mathcal{E}_\mu(j, H\pm\varepsilon) \approx 2^{jD_\mu(H)}$.

3. Reconstruction of the initial capacity μ

Fix a Gibbs capacity μ. We investigate the possibility to reconstitute the whole Gibbs tree $(\mu(I_w))_{w\in\Sigma^*}$ from the sole knowledge of $\tilde{\mu}$ (or equivalently, from M_μ).

Assume first that the capacity μ is associated with a Bernoulli measure, i.e. there exists $q_0, q_1 > 0$ such that for any word $w \in \Sigma_*$ one has $\mu(I_{w1}) = q_1\mu(I_w)$ and $\mu(I_{w0}) = q_0\mu(I_w)$. Hence, in order to reconstitute μ, it is enough to find q_0 and q_1. Assume that two surviving vertices w and w' have different proportions of zeros and ones in their dyadic decomposition. It is easy to check that this event has probability one. Then the knowledge of $\mu(I_w)$ and $\mu(I_{w'})$ leads to two linearly independent equations with unknowns q_0 and q_1, hence to their values.

This idea generalizes to the case where μ is constructed from a Markov measure, i.e. there exist an integer $k \geq 0$ and $((q_{v0}, q_{v_1}))_{v\in\Sigma_k} \in (0,\infty)^{2^{k+1}}$ such that for all $w \in \Sigma_*$ and $v \in \Sigma_k$ one has $\mu(I_{wv0}) = q_{v0}\mu(I_{wv})$ and $\mu(I_{wv1}) = q_{v1}\mu(I_{wv})$.

When μ is associated with a general Gibbs measure the situation is not that simple. The answer we propose uses the basic tools we have at our disposal, namely concatenation of words and quasi-Bernoulli property (8); it depends on the value of η, and there is a phase transition at $\eta = 1/2$.

Definition 10. *Let $k \in \mathbb{N}^*$. A word $u \in \Sigma^*$ is* k-reconstructible *when there is a finite sequence of words $(w_1, u_1, w_2, u_2, ...w_k, u_k)$ in Σ^* such that*

- *for every $i \in \{1, ..., k\}$, $p_{w_i} = p_{w_iu_i} = 1$,*
- *$u = u_1u_2\cdots u_k$.*

One says that $S \subset \Sigma^*$ is k-reconstructible when every word $u \in S$ is k-reconstructible.

This definition follows from the idea that when u is k-reconstructible, after sampling of the initial tree one has access to the value of the weights $\mu(I_{w_i})$ and $\mu(I_{w_iu_i})$ for every i. Hence, by the quasi-Bernoulli property (8), one estimates, up to the constant $C > 1$, the value of $\mu(I_{u_i})$, and by concatenation of the words u_1, ..., u_k and (8) again, one reconstructs the value of $\mu(I_{u_i})$ up to the constant C^{k+1}. Next Theorem completes Theorem 1 in the introduction.

Theorem 4. *When $\eta < 1/2$, Σ^* is 1-reconstructible.*

When $\eta > 1/2$, Σ^* is not k-reconstructible, for any integer $k \geq 1$.

Proof. • Assume first that $\eta < 1/2$.

Fix a generation $\ell \geq 1$, and a word $u \in \Sigma_\ell$. By construction, for any word $w \in \Sigma_j$,

$$\mathbb{P}(p_wp_{wu} = 1) = 2^{-j(1-\eta)}2^{-(j+\ell)(1-\eta)} = 2^{-\ell(1-\eta)}2^{-j2(1-\eta)}. \tag{10}$$

Consider the random variable $Z_j = \#\{w \in \Sigma_j :\ p_wp_{wu} = 1\}$ and the event $\mathcal{Z}_j = \{Z_j = 0\}$. By independence, $\mathbb{P}(\mathcal{Z}_j) = \left(1 - 2^{-\ell(1-\eta)}2^{-j2(1-\eta)}\right)^{2^j} = e^{-2^{-\ell(1-\eta)+j(1-2(1-\eta))+o(j)}}$, which tends exponentially fast to zero. By the Borel-Cantelli Lemma, there exists almost surely an (infinite number of) words $w \in \Sigma^*$ such that $p_wp_{wu} = 1$, i.e. u is 1-reconstructible.

• Assume now that $\eta > 1/2$.

For every $u \in \Sigma^*$, denote by r_u the random variable equal to 1 if u is 1-reconstructible, and 0 otherwise. Hence, r_u is a Bernoulli variable, with parameter $\tilde{p}_{|u|}$, the probability that there exists $w \in \Sigma^*$ such that $p_w p_{wu} = 1$ (which depends only on $|u|$). By (10),

$$\forall\, j \geq 1, \quad \tilde{p}_j \leq \sum_{w \in \Sigma^*} \mathbb{P}(p_w p_{wu} = 1) \leq \tilde{C} 2^{-j(1-\eta)},$$

for some constant $\tilde{C}$ independent on w.

Fix $\varepsilon > 0$ so small that $\eta + \varepsilon < 1$ and $(\varepsilon_j)_{j\geq 1}$ a positive sequence converging to zero, such that $0 < \varepsilon_j \leq \varepsilon$ and $\sum_{j\geq 1} 2^{-j\varepsilon_j} < +\infty$.

Let us introduce $\widetilde{Z}^1_j = \sum_{u \in \Sigma_j} r_u$, the number of 1-reconstructible words at generation j. The random variable $\widetilde{Z}^1_j$ is a sum of non-independent random variables with common law the Bernoulli law with parameter $\tilde{p}_j$. Markov's inequality yields $\mathbb{P}(\widetilde{Z}^1_j \geq 2^{j\varepsilon_j} 2^j \tilde{p}_j) \leq 2^{-j\varepsilon_j}$, and Borel-Cantelli's lemma implies that almost surely, for j large enough, we have

$$\widetilde{Z}^1_j \leq \tilde{C} 2^{j\varepsilon_j} 2^j \tilde{p}_j \leq C_1 2^{j(\eta+\varepsilon_j)}, \tag{11}$$

for some other constant C_1. This implies that Σ^* is not 1-reconstructible, since at most $C_1 2^{j(\eta+\varepsilon_j)} \ll 2^j$ words can be reconstructed.

Assume that for $k \geq 2$, the number $\widetilde{Z}^k_j$ of k-reconstructible words at any generation j is bounded from above by $C_k j^k 2^{j(\eta+\varepsilon)}$ for some constant C_k. Let $J \geq k+1$. Any $(k+1)$-reconstructible word u in Σ_J is the concatenation of a k-reconstructible word and a 1-reconstructible word. Hence, by (11), for the constant $C_{k+1} = C_1 C_k$, one has

$$\widetilde{Z}^{k+1}_J \leq \sum_{i=1}^{J-k} \widetilde{Z}^1_i \widetilde{Z}^k_{J-i} \leq C_1 C_k \sum_{i=1}^{J-k} 2^{i(\eta+\varepsilon_i)} (J-i)^k 2^{(J-i)(\eta+\varepsilon)} \leq C_{k+1} J^k 2^{J(\eta+\varepsilon)}.$$

One concludes that Σ^* is not k-reconstructible, for any k, since $\widetilde{Z}^k_J \ll 2^J$. □

The situation at the critical sampling index $\eta = 1/2$ must still be investigated.

4. New parameters, alternative definitions for the parameters $H_\ell(\eta_\ell)$, $H_\ell(\widetilde{\eta})$ and $\widetilde{H}(\widetilde{\eta})$

From now on, we consider a non-homogeneous Gibbs capacity μ. The homogeneous case will be dealt with at the all end of the paper (Section 10).

We work with the $\| \ \|_\infty$ over $\mathbb{R}^d$, so that balls are Euclidean cubes.

4.1. Modified version of the capacity M_μ.

We will study a slight modification of M_μ.

Definition 11. *Let $\mu \in \mathrm{Cap}([0,1]^d)$. We set*

$$\widetilde{\mathsf{M}}_\mu(I_w) = \max_{u \in \mathcal{N}_j(w)} \mathsf{M}_\mu(I_u) = \max\left\{\mu(I_{uv}) : u \in \mathcal{N}_j(w), v \in \Sigma^*, p_{uv} = 1\right\}. \tag{12}$$

Thus, the difference between the capacities M_μ and $\widetilde{\mathsf{M}}_\mu$ is that $\widetilde{\mathsf{M}}_\mu(I_j(x))$ carries information about the behavior of μ in the neighborhood of x, not only in the dyadic cube $I_j(x)$ of generation j containing x. We consider $\widetilde{\mathsf{M}}_\mu$ for the following reasons. First it is natural to extend M_μ to balls: for $x \in [0,1]^d$ and $r > 0$ one denotes $B(x,r)$ the closed ball of radius r centered at x, and defines $\mathsf{M}_\mu(B(x,r)) = \max\{\mathsf{M}_\mu(I_w) : I_w \subset B(x,r)\}$. Then the multifractal analysis of M_μ using the more intrinsic logarithmic density $\frac{\log(\mu(B(x,r)))}{\log(r)}$ to define the local dimensions of M_μ is given by the multifractal analysis of $\widetilde{\mathsf{M}}_\mu$.

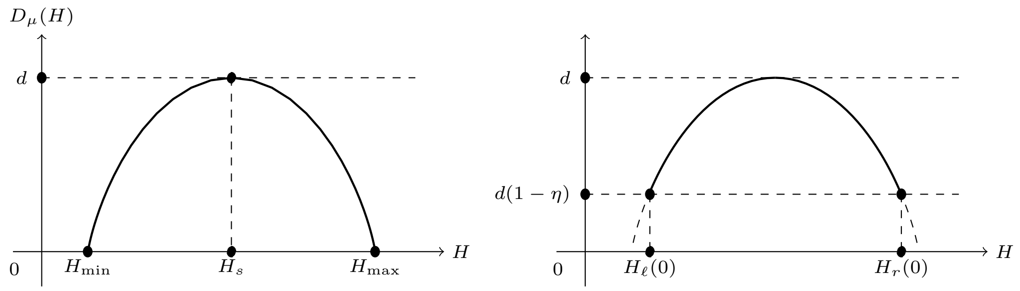

FIGURE 8. **Left:** Typical singularity spectrum of a Gibbs measure. **Right:** Parameters $H_\ell(0)$ and $H_r(0)$.

The second reason is that knowing the mutifractal nature of $\widetilde{\mathsf{M}}_\mu$ yields that of the sparse wavelets series weighted by using the random sample $\widetilde{\mu}$ of μ (see [28] for an account of multifractal analysis of functions).

From now on, only $\widetilde{\mathsf{M}}_\mu$ will be considered, and we denote it as M_μ.

The reader will check that our proofs to study the capacity defined by (12) are easily adapted to the case where the capacity is defined by (5). In fact, the case we study is a little bit more complicated, since it involves a control of all the immediate neighbors.

4.2. New parameters.

Definition 12. *The real number $\eta_\ell \in [0,\eta]$ is defined as*

$$\eta_\ell = \begin{cases} 0 & \text{if } 0 \le D_\mu(H_{\min}) \le d(1-\eta) \\ 1 - \frac{d(1-\eta)}{D_\mu(H_{\min})} & \text{otherwise.} \end{cases}$$

For all $\eta' \in [\eta_\ell, \eta]$, there exists a unique $H_\ell(\eta') \in [H_{\min}, H_s]$ such that

$$D_\mu\big(H_\ell(\eta')\big) = \frac{d(1-\eta)}{1-\eta'}. \tag{13}$$

See Figures 8 and 9 for a geometrical interpretation of $H_\ell(\eta')$, which makes it easier to understand. By construction one has:

- $H_\ell(\eta) = H_s$,
- if $D_\mu(H_{\min}) \le d(1-\eta)$, $\eta_\ell = 0$ and $H_\ell(\eta_\ell)$ is the unique solution of $D_\mu(H) = d(1-\eta)$ in $[H_{\min}, H_s]$,
- if $D_\mu(H_{\min}) > d(1-\eta)$, $\eta_\ell > 0$ and $H_\ell(\eta_\ell) = H_{\min}$.

Definition 13. *For $\eta' \in [\eta_\ell, \eta] \setminus \{0\}$, let*

$$\begin{aligned} \widetilde{H}_\ell(\eta') &= \left(\frac{1}{\eta'} - 1\right) H_\ell(\eta') \\ \textit{and} \quad \widetilde{\eta} &= \operatorname{argmin}_{\eta' \in [\eta_\ell, \eta]\setminus\{0\}} \widetilde{H}_\ell(\eta'). \end{aligned} \tag{14}$$

Again, see Figure 9 for a geometrical interpretation of these parameters (in the case we discard at the moment, i.e. when μ is homogeneous, the function τ_μ is linear so that $H_{\min} = H_s = H_{\max}$ and $\widetilde{\eta} = \eta_\ell = \eta$). It is easily seen that by definition the value $\widetilde{\eta}$ is so that the straight line passing through the points $(0, d(1-\eta))$ and $\left(H_\ell(\widetilde{\eta}), \frac{d(1-\eta)}{1-\widetilde{\eta}}\right)$ is

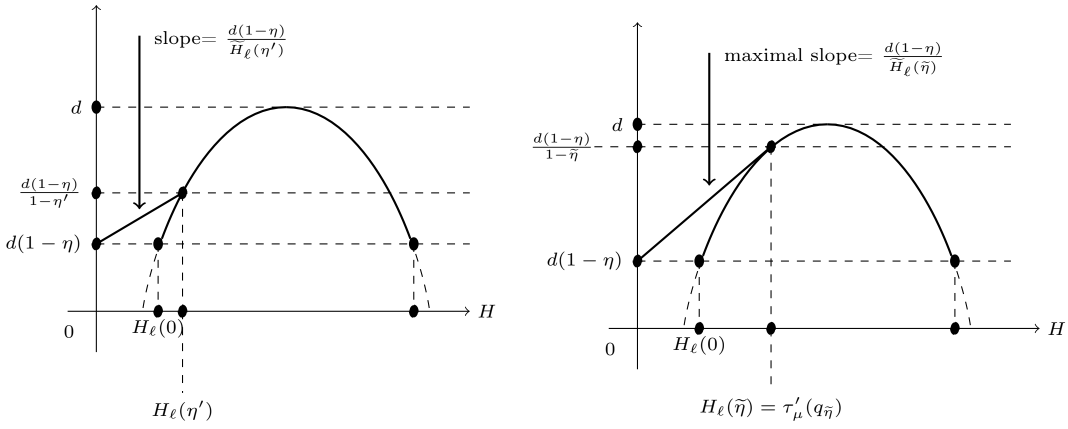

FIGURE 9. **Left:** Parameters $H_\ell(\eta')$ and $\widetilde{H}_\ell(\eta')$. **Right:** Optimal parameter $\widetilde{\eta}$.

tangent to the singularity spectrum of μ. This value always exists and is unique. Since D_μ is strictly concave, $D_\mu{}'(H_{\min}+) = \infty$ and $D_\mu{}'(H_s) = 0$, one has $\widetilde{\eta} \in (\eta_\ell, \eta)$.

Definition 14. *Let $q_{\widetilde{\eta}}$ be the unique solution to the equation*

$$H_\ell(\widetilde{\eta}) = \tau'_\mu(q_{\widetilde{\eta}}) \tag{15}$$

$$\text{and} \quad q_{\eta_\ell} = \sup\left\{q \geq 0 : \tau^*_\mu(\tau'_\mu(q)) \geq d(1-\eta)\right\}. \tag{16}$$

Let $\widetilde{\tau} : \mathbb{R} \to \mathbb{R}$ be the mapping defined as

$$\widetilde{\tau}(q) = \begin{cases} \tau_\mu(q) + \widetilde{H}_\ell(\widetilde{\eta})q & \text{if } q \leq q_{\widetilde{\eta}}, \\ \tau_\mu(q) + d(1-\eta) & \text{if } q_{\widetilde{\eta}} < q < q_{\eta_\ell}, \\ H_\ell(0)q & \text{if } q_{\eta_\ell} < \infty \text{ and } q \geq q_{\eta_\ell}. \end{cases} \tag{17}$$

See Figure 6 for a representation of $\widetilde{\tau}$.

Observe that $q_{\eta_\ell} < +\infty$ if and only if $\eta_\ell = 0$, and in this case $\tau'_\mu(q_{\eta_\ell}) = H_\ell(0)$.

Definition 15. *The real number $\eta_r \in [0, \eta]$ is defined as*

$$\eta_r = \begin{cases} 0 & \text{if } 0 \leq D_\mu(H_{\max}) \leq d(1-\eta) \\ 1 - \frac{d(1-\eta)}{D_\mu(H_{\max})} & \text{otherwise.} \end{cases}$$

For all $\eta' \in [\eta_r, \eta]$, there exists a unique $H_r(\eta') \in [H_s, H_{\max}]$ such that

$$D_\mu\big(H_r(\eta')\big) = \frac{d(1-\eta)}{1-\eta'}.$$

By construction one has:

- $H_r(\eta) = H_s$,
- if $D_\mu(H_{\max}) \leq d(1-\eta)$, $\eta_r = 0$ and $H_r(\eta_r)$ is the unique solution of $D_\mu(H) = d(1-\eta)$ in $[H_s, H_{\max}]$,
- if $D_\mu(H_{\max}) > d(1-\eta)$, $\eta_r > 0$ and $H_r(\eta_r) = H_{\max}$.

The existence of H_ℓ and H_r is ensured by the continuity of the Legendre spectrum D_μ on its support.

As $\widetilde{H}_\ell(\eta')$ was defined in Definition 13, we can also define a parameter $\widetilde{H}_r(\eta')$ as follows: for every $\eta' \in [\eta_r, \eta] \setminus \{0\}$, let

$$\widetilde{H}_r(\eta') = \left(\frac{1}{\eta'} - 1\right) H_r(\eta').$$

The geometrical interpretation is the same as the one for $\widetilde{H}_\ell(\eta')$ (see Figure 9), except now that everything is done on the decreasing part of the spectrum.

Finally, the following lemma provides us with another interpretation of the exponent $H_\ell(\widetilde{\eta})$ (see (14)), which is useful to simplify some formulas and to understand its role.

Lemma 1. *One has*

$$H_\ell(\widetilde{\eta}) = \mathrm{argmax}_H \left(\frac{D_\mu(H)}{H + \widetilde{H}_\ell(\widetilde{\eta})}\right). \tag{18}$$

Proof. Due to the unimodal character of D_μ, the maximum we seek for is reached at $H \in [H_{\min}, H_s]$. A rapid calculation shows that since D_μ is strictly concave and differentiable over $(H_{\min}, H_s]$ with ${D_\mu}'(H_{\min}+) = +\infty$ and ${D_\mu}'(H_s) = 0$, then for any $\gamma > 0$, $H \mapsto \frac{D_\mu(H)}{H+\gamma}$ reaches its maximum at a unique point of $(H_{\min}, H_s)$. Notice that from its definition the function $\eta' \mapsto H_\ell(\eta')$ is differentiable.

Let us introduce the function $\varphi(\eta') = \eta' \widetilde{H}_\ell(\eta') = (1-\eta')H_\ell(\eta')$. Recall that by (13), $D_\mu\big(H_\ell(\eta')\big) = \dfrac{d(1-\eta)}{1-\eta'}$. So ${D_\mu}'\big(H_\ell(\eta')\big){H_\ell}'(\eta') = \dfrac{d(1-\eta)}{(1-\eta')^2} = \dfrac{D_\mu\big(H_\ell(\eta')\big)}{1-\eta'}$. One deduces that

$$\varphi'(\eta') = -H_\ell(\eta') + (1-\eta'){H_\ell}'(\eta') = -H_\ell(\eta') + \frac{D_\mu\big(H_\ell(\eta')\big)}{{D_\mu}'\big(H_\ell(\eta')\big)} = -\frac{{D_\mu}^*\big({D_\mu}'\big(H_\ell(\eta')\big)\big)}{{D_\mu}'\big(H_\ell(\eta')\big)},$$

since ${D_\mu}^*(H) = H{D_\mu}'(H) - D_\mu(H)$.

On the other hand, the derivative of $H \mapsto \dfrac{D_\mu(H)}{H + \widetilde{H}_\ell(\widetilde{\eta})}$ vanishes at $\alpha = \mathrm{argmax}_H \left(\frac{D_\mu(H)}{H+\widetilde{H}_\ell(\widetilde{\eta})}\right)$. This yields

$${D_\mu}'(\alpha)(\alpha + \widetilde{H}_\ell(\widetilde{\eta})) - D_\mu(\alpha) = 0,$$

i.e.

$$\widetilde{H}_\ell(\widetilde{\eta}) = -\frac{{D_\mu}^*({D_\mu}'(\alpha))}{{D_\mu}'(\alpha)}.$$

Since $\widetilde{\eta}$ is chosen so that $\widetilde{H}_\ell(\eta')$ is minimal at $\widetilde{\eta}$, we have $\widetilde{H}'_\ell(\widetilde{\eta}) = 0$. This implies that $\varphi'(\widetilde{\eta}) = \widetilde{H}_\ell(\widetilde{\eta})$, so finally

$$-\frac{{D_\mu}^*\big({D_\mu}'(\alpha)\big)}{{D_\mu}'(\alpha)} = -\frac{{D_\mu}^*\big({D_\mu}'\big(H_\ell(\widetilde{\eta})\big)\big)}{{D_\mu}'\big(H_\ell(\widetilde{\eta})\big)}. \tag{19}$$

Recalling that D_μ is the Legendre transform of τ_μ, we know that $q \in \mathbb{R}_+^* \mapsto \tau'_\mu(q)$ is a bijection onto $(H_{\min}, H_s)$. Hence, since the mapping $q > 0 \mapsto -\dfrac{\tau_\mu(q)}{q}$ is injective (τ_μ being strictly concave), the identification $\Big(H, q, {D_\mu}^*\big({D_\mu}'(H)\big)\Big) = \Big(\tau'_\mu(q), {D_\mu}'(H), \tau_\mu(q)\Big)$

implies that $H \in (H_{\min}, H_s) \mapsto -\dfrac{{D_\mu}^*({D_\mu}'(H))}{{D_\mu}'(H)}$ is injective as well. Equation (19) yields finally $\alpha = H_\ell(\widetilde{\eta})$. □

The previous definitions and discussion clarify the origin of the parameters introduced to state Theorem 2. The rest of the paper is devoted to the proof of the multifractal properties of M_μ definied by (11).

5. Analysis of the surviving vertices

5.1. **Basic properties of the distribution of the surviving vertices.** Recall the Definition 3 in which $\mathcal{S}_j(\eta)$ is defined, and recall that x_w, defined by (1), is the dyadic point corresponding to the projection of the finite word $w \in \Sigma_j$ to $[0,1]^d$. The first question concerns the distribution of the points x_w, for $w \in \mathcal{S}_j(\eta)$.

Definition 16. *For every $j \geq 1$, and every finite word $W \in \Sigma^*$, we define*

$$\mathcal{S}_j(\eta, W) = \{w \in \mathcal{S}_j(\eta) : I_w \subset I_W\}.$$

The set $\mathcal{S}_j(\eta, W)$ describes the *surviving* coefficients at generation j included in I_W. Obviously, for every $J \leq j$,

$$\mathcal{S}_j(\eta) = \bigcup_{W \in \Sigma_J} \mathcal{S}_j(\eta, W).$$

Lemma 2. *There exists a positive sequence $(\varepsilon_j)_{j\geq 1}$ converging to 0 such that, with probability 1, for every j large enough, for every $W \in \Sigma_{\lfloor j(\eta-\varepsilon_j)\rfloor}$, $\mathcal{S}_j(\eta, W) \neq \emptyset$.*

In other words, every cylinder of generation $\lfloor j(\eta-\varepsilon_j)\rfloor$ contains a surviving vertex w of generation j.

Proof. Fix a positive sequence $(\varepsilon_j)_{j\geq 1}$ converging to 0. For each $j \geq 1$ and $W \in \Sigma_{\lfloor j(\eta-\varepsilon_j)\rfloor}$, the cylinder $[W]$ contains exactly $2^{j-\lfloor j(\eta-\varepsilon_j)\rfloor}$ distinct cylinders $[w]$, with $w \in \Sigma_j$. Denote this set by $S(W)$. The probability of the event $\mathscr{E}(W) = \{\forall\, w \in S(W),\ p_w = 0\}$ is given by $(1-2^{-j(1-\eta)})^{2^{j-\lfloor j(\eta-\varepsilon_j)\rfloor}}$. Thus,

$$\begin{aligned}\mathbb{P}\Big(\bigcup_{W \in \Sigma_{\lfloor j(\eta-\varepsilon_j)\rfloor}} \mathscr{E}(W)\Big) &\leq 2^{\lfloor j(\eta-\varepsilon_j)\rfloor}(1-2^{-j(1-\eta)})^{2^{j-\lfloor j(\eta-\varepsilon_j)\rfloor}} \\ &\leq 2^{\lfloor j(\eta-\varepsilon_j)\rfloor}\exp(-2\cdot 2^{j\varepsilon_j}).\end{aligned}$$

If we choose $\varepsilon_j = (\log^2(j))/j$, we get $\sum_{j\geq 1}\mathbb{P}\Big(\bigcup_{v\in\Sigma_{\lfloor j(\eta-\varepsilon_j)\rfloor}} \mathscr{E}(W)\Big) < \infty$. So the Borel-Cantelli lemma yields that, with probability 1, for j large enough, for all $W \in \Sigma_{\lfloor j(\eta-\varepsilon_j)\rfloor}$, there exists $w \in \Sigma_j$ such that $I_w \subset I_W$ and $p_w = 1$, i.e. $w \in \mathcal{S}_j(\eta, W)$. □

The sequence $(\varepsilon_j)_{j\geq 1}$ is now fixed.

Lemma 2 has the following consequence: Almost surely, the set of points belonging to an infinite number of balls of the form $B(x_w, 2^{-\lfloor |w|(\eta-\varepsilon_{|w|})\rfloor})$ with $p_w = 1$ is exactly the whole cube $[0,1]^d$, i.e.

$$[0,1]^d = \limsup_{j\to+\infty} \bigcup_{w\in\mathcal{S}_j(\eta)} B(x_w, 2^{-\lfloor |w|(\eta-\varepsilon_{|w|})\rfloor}). \tag{20}$$

Next we obtain an upper bound for the cardinality of $\mathcal{S}_j(\eta, W)$ when $W \in \Sigma_{\lfloor \eta j\rfloor}$.

Lemma 3. *With probability one, for every large j, for every $W \in \Sigma_{\lfloor \eta j\rfloor}$, $\#\mathcal{S}_j(\eta, W) \leq j$.*

Proof. This is standard computations. Denote for every $j \geq 1$ and every word $W \in \Sigma_{\lfloor \eta j\rfloor}$, the random variable

$$B_W = \sum_{w\in\Sigma_j:\, I_w\subset I_W} p_w$$

is equal to the cardinality of $\mathcal{S}_j(\eta, W)$.

With this formulation, the $(B_W)_{W\in\Sigma_{\lfloor \eta j\rfloor}}$ are i.i.d. random variables with common law the binomial law $B(n_j, \rho_j)$ of parameters $n_j = 2^{d(j-\lfloor \eta j\rfloor)}$ and $\rho_j = 2^{-dj(1-\eta)}$. We have

$$\mathbb{P}\big(B(n_j,\rho_j)\geq j\big) = \sum_{l=j}^{n_j} \binom{n_j}{l} (\rho_j)^l (1-\rho_j)^{n_j-l} =$$

$$\sum_{l=j}^{n_j} \frac{2^{d(j-\lfloor \eta j\rfloor)}(2^{d(j-\lfloor \eta j\rfloor)}-1)\dots(2^{d(j-\lfloor \eta j\rfloor)}-(l-1))}{l!} 2^{-djl(1-\eta)}(1-2^{-dj(1-\eta)})^{2^{d(j-\lfloor \eta j\rfloor)}-l}$$

Observing that

$$\frac{2^{d(j-\lfloor \eta j\rfloor)}\dots(2^{d(j-\lfloor \eta j\rfloor)}-(l-1))}{2^{djl(1-\eta)}} = (2^{d(\eta j-[\eta j])})\dots(2^{d(\eta j-[\eta j])}-(l-1)2^{-dj(1-\eta)}),$$

this quantity is upper bounded by 2^{dl}. Finally,

$$\mathbb{P}\big(B(n_j,\rho_j)\geq j\big) \leq \sum_{l=j}^{n_j} \frac{2^{dl}}{l!}(1-2^{-dj(1-\eta)})^{2^{d(j-\lfloor \eta j\rfloor)}-l} \leq \sum_{l=j}^{+\infty}\frac{2^{dl}}{l!} \leq 2^{-dj}$$

for j large enough. We deduce that $\sum_{j\geq 1} 2^{d\lfloor \eta j\rfloor}\mathbb{P}\big(B(n_j,\rho_j)\geq j\big) < +\infty$. Then the Borel-Cantelli lemma yields that almost surely there exists $J \geq 1$ such that for all $j \geq J$, for all $W \in \Sigma_{\lfloor \eta j\rfloor}$, one has $B_W < j$. □

As a conclusion, one keeps in mind the intuition that every cylinder $W \in \Sigma_{\lfloor \eta j\rfloor}$ contains at least one, but not much more than one surviving vertex $w \in \mathcal{S}_j(\eta)$.

5.2. **Analysis of the values of μ at the surviving vertices.** The above lemmas give some hints about the possible values for $\mu(I_w)$ for $w \in \mathcal{S}_j(\eta)$. Indeed, observe that any word w can be written as the concatenation $w = w_{|\lfloor \eta j\rfloor}\cdot \sigma^{\lfloor \eta j\rfloor} w$. Further, by the almost multiplicativity property of μ, one has

$$\mu(I_w) \approx \mu(I_{w_{|\lfloor \eta j\rfloor}})\mu(I_{\sigma^{\lfloor \eta j\rfloor} w}).$$

Lemmas 2 and 3 assert that all the possible values for $\mu(I_{w_{|\lfloor \eta j\rfloor}})$ are reached. Hence, in order to describe the values of $\mu(I_w)$, it is necessary to investigate the possible values for $\mu(I_{\sigma^{\lfloor \eta j\rfloor} w})$ when $w \in \mathcal{S}_j(\eta)$. A quick analysis could lead to the intuition that since most of the coefficients are put to zero, only the most frequent local dimension H_s survive, i.e. $\mu(I_{\sigma^{\lfloor \eta j\rfloor} w}) \approx 2^{-\lfloor j\eta\rfloor H_s}$.

The goal of this section is to prove that this intuition is neither true, nor absolutely false. In fact, we are going to explain that in order to investigate the values of $\mu(I_w)$ for $w \in \mathcal{S}_j(\eta)$, one needs to look at all the decompositions

$$w = w_{|\lfloor \eta' j\rfloor}\cdot \sigma^{\lfloor \eta' j\rfloor} w, \tag{21}$$

and to use that

$$\mu(I_w) \approx \mu(I_{w_{|\lfloor \eta' j\rfloor}})\mu(I_{\sigma^{\lfloor \eta' j\rfloor} w}),$$

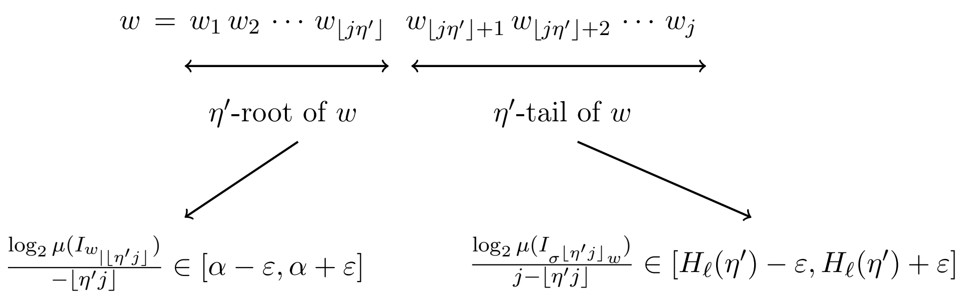

FIGURE 10. Decomposition of a word $w \in \mathcal{R}_\mu(j, \eta', \alpha \pm \varepsilon) \cap \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon)$ into its η'-tail and its η'-root.

for all possible values of $\eta' \in [\eta_\ell, \eta] \cup [\eta_r, \eta]$, and that the most frequent behaviors for $\mu(I_{\sigma^{\lfloor \eta' j \rfloor} w})$ are related to the local dimensions $H_\ell(\eta')$ and $H_r(\eta')$, H_s corresponding to $H_\ell(\eta) = H_r(\eta)$.

These considerations lead to the following definition.

Definition 17. *Let $\alpha, \varepsilon \geq 0$ be two real numbers, and let $\eta' \in [0, \eta]$.*

*When $w \in \Sigma_j$, the prefix $w_{|\lfloor \eta' j \rfloor}$ is referred to as the η'-*root *of w, and the suffix $\sigma^{\lfloor \eta' j \rfloor} w$ is the η'-*tail *of w.*

We introduce the following sets:

- *$\mathcal{R}_\mu(j, \eta', \alpha \pm \varepsilon)$ is the set of those finite words $w \in \Sigma_j$ whose η'-root $w_{|\lfloor \eta' j \rfloor}$ belongs to $\mathcal{E}_\mu(\lfloor \eta' j \rfloor, \alpha \pm \varepsilon)$, i.e.*
$$\frac{\log_2 \mu(I_{w_{|\lfloor \eta' j \rfloor}})}{-\lfloor \eta' j \rfloor} \in [\alpha - \varepsilon, \alpha + \varepsilon].$$
- *When $W \in \Sigma^*$, $\mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon, W)$ is the set of those finite words $w \in \Sigma_j$ satisfying $I_w \subset I_W$ and whose η'-tail $\sigma^{\lfloor \eta' j \rfloor} w$ belongs to $\mathcal{E}_\mu(j - \lfloor \eta' j \rfloor, H_\ell(\eta') \pm \varepsilon)$, i.e.*
$$\frac{\log_2 \mu(I_{\sigma^{\lfloor \eta' j \rfloor} w})}{j - \lfloor \eta' j \rfloor} \in [H_\ell(\eta') - \varepsilon, H_\ell(\eta') + \varepsilon]. \tag{22}$$
- *the sets $\mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon)$ is the set of all finite words $w \in \Sigma_j$ satisfying (22), so for every $J \leq j$,*
$$\mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon) = \bigcup_{W \in \Sigma_J} \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon, W).$$
- *$\mathcal{T}_{\mu,r}(j, \eta', \varepsilon, W)$ and $\mathcal{T}_{\mu,r}(j, \eta', \varepsilon)$ are defined as $\mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon, W)$ and $\mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon)$ by replacing $H_\ell(\eta')$ by $H_r(\eta')$.*
- *$\mathcal{T}_\mu(j, \eta', \varepsilon) = \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon) \cup \mathcal{T}_{\mu,r}(j, \eta', \varepsilon)$.*

Recall the decomposition (21) of any finite word w. The idea, illustrated by Figure 10, is that the sets $\mathcal{R}_\mu(j, \eta', \alpha \pm \varepsilon)$ describe the scaling behavior of the η'-root $w_{|\lfloor \eta' j \rfloor}$ of the word $w \in \Sigma_j$, while $\mathcal{T}_\mu(j, \eta', \varepsilon)$ describe the scaling behavior of the η'-tail $\sigma^{\lfloor \eta' j \rfloor} w$ of w. Observe that we focus on the cases where the η'-tail of w behaves with a local dimension close to some $H_\ell(\eta')$ or $H_r(\eta')$. Indeed, these specific behaviors of the η'-tail will turn out to be central to explain the structure of the local dimensions of M_μ, and Propositions 3, and 5 to 7 will exhibit essential properties related to them.

Observe that the knowledge of which sets $\mathcal{R}_\mu(j,\eta',\alpha\pm\varepsilon)$ and $\mathcal{T}_\mu(j,\eta',\varepsilon)$ a given word w belongs to, yields $\mu(I_w)$ up to a multiplicative factor of order $2^{\pm\varepsilon j}$.

The first proposition gives upper and lower bounds for the possible values of $\mu(I_w)$, when w survives after sampling.

Proposition 3. *Almost surely, there exists a positive sequence $(\varepsilon^1_j)_{j\geq 1}$ converging to 0 such that for j large enough, for all $w\in\mathcal{S}_j(\eta)$, one has*

$$j(H_\ell(\eta_\ell)-\varepsilon^1_j)\leq -\log_2\mu(I_w)\leq j(H_r(\eta_r)+\varepsilon^1_j).$$

Proof. This is a consequence of the large deviations properties of Gibbs measures. Fix an integer $p\geq 1$. Consider the interval $I_p=[0,H_\ell(\eta_\ell)-2^{-p}]\cup[H_r(\eta_r)+2^{-p},+\infty)$. By definition of H_ℓ and H_r, one has $\sup\{D_\mu(h):h\in I_p\}<d(1-\eta)$. Let us call $\xi_p=d(1-\eta)-\sup\{D_\mu(h):h\in I_p\}$.

By item (5) of Proposition 2, there exists a generation J_p such that $j\geq J_p$ implies

$$\left|\frac{\log\#\mathcal{E}_\mu(j,I_p)}{-j}\log 2^j-\sup_{h\in I_p}D_\mu(h)\right|\leq\xi_p/2.$$

Using the definition of ξ_p, this rephrases as

$$\#\mathcal{E}_\mu(j,I_p)\leq 2^{j(\sup_{h\in I_p}D_\mu(h)+\xi_p/2)}\leq 2^{j(d(1-\eta)-\xi_p/2)}.$$

Let us compute the probability of the event $\mathscr{A}^p_j=\{\mathcal{S}_j(\eta)\cap\mathcal{E}_\mu(j,I_p)\neq\emptyset\}$. One has

$$\begin{aligned}\forall\, j\geq J_p,\quad \mathbb{P}(\mathscr{A}^p_j)&\leq 1-(1-2^{-dj(1-\eta)})^{\#\mathcal{E}_\mu(j,I_p)}\\&\leq 1-(1-2^{-dj(1-\eta)})^{2^{j(d(1-\eta)-\xi_p/2)}}\\&\leq 2^{-j\xi_p/4}.\end{aligned}$$

The Borel-Cantelli lemma implies that, almost surely, $\mathscr{A}^p_j$ is not realized when j becomes greater than some integer $J'_p\geq J_p$. In the construction, one can ensure that J_{p+1} is always strictly greater than J_p, for all integers p.

Choosing now $\varepsilon^1_j=2^{-p}$ for $j\in[J_p,J_{p+1})$ yields the result. □

The next proposition is complementary to the previous one: it precisely estimates the number of surviving vertices with a given μ-local dimension. We state it for the sake of completeness but it will not be used.

Proposition 4. *Almost surely, for every $\varepsilon>0$ and every $H\in[H_\ell(\eta_\ell),H_r(\eta_r)]$, there exists $\widetilde{\varepsilon}>0$ with $\widetilde{\varepsilon}\to 0$ as $\varepsilon\to 0$ such that for j large enough one has*

$$2^{j\left(D_\mu(H)-d(1-\eta)-\widetilde{\varepsilon}\right)}\leq\#\left(\mathcal{E}_\mu(j,H\pm\varepsilon)\cap\mathcal{S}_j(\eta)\right)\leq 2^{j\left(D_\mu(H)-d(1-\eta)+\widetilde{\varepsilon}\right)}.$$

Proof. The proof is close to that of Proposition 3 and is omitted. □

Next proposition is crucial. It shows that the parameters η', $H_\ell(\eta')$ and $H_r(\eta')$ play a special role in our problem. The underlying idea is the following: The almost multiplicativity property implies that for every word $w\in\Sigma$, for every $\eta'\in[\eta_\ell,\eta]\cup[\eta_r,\eta]$, one has

$$\mu(I_w)\approx\mu(I_{w_{|\lfloor\eta' j\rfloor}})\mu(I_{w_{\sigma^{\lfloor\eta' j\rfloor}w}}).$$

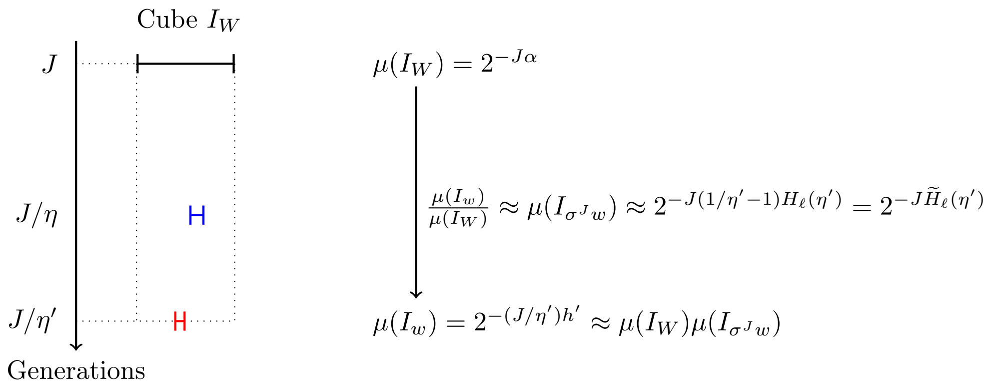

FIGURE 11. Behavior of the surviving vertices inside a cube I_W

But if a vertex w survives after sampling, i.e. if $w \in \Sigma_j(\eta)$, then we are going to prove that $\mu(I_w)$ can be decomposed as

$$\mu(I_w) \approx \mu(I_{w_{|\lfloor \eta' j \rfloor}}) 2^{-(j-\lfloor \eta' j \rfloor)H_\ell(\eta')} \quad \text{or} \quad \mu(I_w) \approx \mu(I_{w_{|\lfloor \eta' j \rfloor}}) 2^{-(j-\lfloor \eta' j \rfloor)H_r(\eta')},$$

for some suitable choice of η' (depending on w). So we have an explicit formula for its η'-tail. We will then establish a complementary information (Proposition 6): η' being fixed, with probability one for j large enough, for $W \in \Sigma_{\lfloor \eta' j \rfloor}$, there is necessarily at least one word $w \in \mathcal{S}_j(\eta, W)$ such that the above decomposition holds.

Proposition 5. *With probability one, there exists a positive sequence $(\varepsilon_j^2)_{j \geq 1}$ converging to 0 such that for all $w \in \mathcal{S}_j(\eta)$, there exists $\eta' \in [\eta_\ell, \eta] \cup [\eta_r, \eta]$ such that $w \in \mathcal{T}_\mu(j, \eta', \varepsilon_j^2)$.*

Proof. We fix $w \in \mathcal{S}_j(\eta)$, and we look for a suitable η'. See Figure 11.

Let us denote, for all $j \geq 1$, $\alpha_j := -\dfrac{\log_2 \mu(I_w)}{j}$, and for all $\eta' \in [0, \eta]$, $\alpha_j(\eta') = -\dfrac{\log_2 \mu(I_{w_{|\lfloor \eta' j \rfloor}})}{\lfloor \eta' j \rfloor}$ and $H_j(\eta') = \dfrac{-\log_2 \mu(I_{\sigma^{\lfloor \eta' j \rfloor} w})}{j - \lfloor \eta' j \rfloor}$. By the almost multiplicativity property of μ, we have

$$\alpha_j j = \alpha_j(\eta') \lfloor \eta' j \rfloor + H_j(\eta')(j - \lfloor \eta' j \rfloor) + O(1), \tag{23}$$

where $O(1)$ is bounded independently on w, j and η' (it depends only on the constant C involved in (8)).

On the other hand, for $\eta,' \eta'' \in [0, \eta]$ we have

$$H_j(\eta'')(j - \lfloor \eta'' j \rfloor) - H_j(\eta')(j - \lfloor \eta' j \rfloor) = -\log_2 \mu(I_{\sigma^{\lfloor \eta' j \rfloor} w}) + \log_2 \mu(I_{\sigma^{\lfloor \eta'' j \rfloor} w}),$$

which is bounded above by $c|\lfloor \eta' j \rfloor - \lfloor \eta'' j \rfloor|$ for some constant $c > 0$ by (8). Also, by item (6) of Proposition 2, $H_j(\eta')$ and $H_j(\eta'')$ are bounded by a constant $K > 0$ independently of j, w and η'. Subsequently,

$$\begin{aligned} |H_j(\eta'') - H_j(\eta')| &\leq \left| H_j(\eta'') - H_j(\eta') \frac{j - \lfloor \eta' j \rfloor}{j - \lfloor \eta'' j \rfloor} \right| + H_j(\eta') \left| 1 - \frac{j - \lfloor \eta' j \rfloor}{j - \lfloor \eta'' j \rfloor} \right| \\ &\leq (c + K) \frac{|\lfloor \eta' j \rfloor - \lfloor \eta'' j \rfloor|}{j - \lfloor \eta'' j \rfloor} \leq (c + K) \frac{|\eta'' - \eta'| + 1/j}{1 - \eta}. \end{aligned}$$

From this inequality, one deduces that there exists a continuous function $\widetilde{H}_j : [0,\eta] \to \mathbb{R}^+$ such that
$$s_j = \sup\{|H_j(\eta') - \widetilde{H}_j(\eta')| : \eta' \in [0,\eta]\} = O(1/j)$$
independently of w as $j \to \infty$, and (23) holds with $\widetilde{H}_j$ instead of H_j.

• Suppose that $\widetilde{H}_j(\eta) = H_s$. Since $H_\ell(\eta) = H_s$, Proposition 5 is proved with $\eta' = \eta$.

• Suppose now that $\widetilde{H}_j(\eta) < H_s = H_\ell(\eta)$.

- Suppose first that $\eta_\ell = 0$. Recall that $H_\ell(0) = H_{\min}$.

If $\widetilde{H}_j(0) \le H_\ell(0)$, then we see that $j\alpha_j = j\widetilde{H}_j(0) + O(1) \le j(H_\ell(0) + O(1/j))$, which due to Proposition 3 implies that $H_\ell(0) - \varepsilon^1_j \le \widetilde{H}_j(0) + O(1/j) \le H_\ell(0) + O(1/j)$. Hence (24) is satisfied with $\eta' = 0$.

If $\widetilde{H}_j(0) > H_\ell(0)$, observe that the mapping $\eta' \mapsto (\widetilde{H}_j - H_\ell)(\eta')$ is continuous, positive at $\eta' = 0$, negative at $\eta' = \eta$. The continuity ensures the existence of $\eta' \in (0,\eta)$ such that $\widetilde{H}_j(\eta') = H_\ell(\eta')$, and (24) is satisfied with this η'.

- Suppose now that $\eta_\ell > 0$ and recall that $H_\ell(\eta')$ ranges in $[H_\ell(\eta_\ell), H_s]$. Notice that for any η', $j - \lfloor \eta' j\rfloor \ge j - \lfloor \eta j \rfloor$ which tends to $+\infty$ when $j \to +\infty$. Hence, by Proposition 3, for j large enough we have $H_j(\eta') \ge H_\ell(\eta_\ell) - \varepsilon^1_{j-\lfloor \eta' j\rfloor}$, so that for all $\eta' \in [\eta_\ell, \eta]$,
$$\widetilde{H}_j(\eta') \ge H_\ell(\eta_\ell) - \varepsilon^1_{j-\lfloor \eta' j\rfloor} - s_j.$$
By continuity of $\eta' \mapsto (\widetilde{H}_j - H_\ell)(\eta')$, there exists $\eta' \in [\eta_\ell, \eta]$ such that $|\widetilde{H}_j(\eta') - H_\ell(\eta')| \le \varepsilon^1_{j-\lfloor \eta' j\rfloor} + s_j$. In all cases, we found $\eta' \in [\eta_\ell, \eta]$ such that $|\widetilde{H}_j(\eta') - H_\ell(\eta')| \le \varepsilon^1_{j-\lfloor \eta' j\rfloor} + 2s_j + O(1/j)$. Since H_j and $\widetilde{H}_j$ differ by $o(1)$, the result follows.

• Finally suppose that $\widetilde{H}_j(\eta) > H_s$. Similar arguments as above yield $\eta' \in [\eta_r, \eta]$ such that $|H_j(\eta') - H_r(\eta')| \le \varepsilon^1_{j-\lfloor \eta' j\rfloor} + 2s_j + O(1/j)$. We let the reader check the details.

Since the bound $\varepsilon^1_{j-\lfloor \eta' j\rfloor} + 2s_j + O(1/j)$ tends to 0 uniformly in $\eta' \in [0,\eta]$ as $j \to +\infty$, the sequence $(\varepsilon^2_j := \varepsilon^1_{j-\eta j} + 2s_j + O(1/j))_{j\ge 1}$ fulfills the conditions of Proposition 5. □

The previous proposition tells us that every surviving vertex $w \in \mathcal{S}_j(\eta)$ is such that its η'-tail satisfies either for some $\eta' \in [\eta_\ell, \eta]$ (depending on w),

$$(j - \lfloor \eta' j\rfloor)(H_\ell(\eta') - \varepsilon^2_j) \le -\log_2 \mu(I_{\sigma^{\lfloor \eta' j\rfloor} w}) \le (j - \lfloor \eta' j\rfloor)(H_\ell(\eta') + \varepsilon^2_j), \tag{24}$$

or for some $\eta' \in [\eta_r, \eta]$ (also depending on w) that
$$(j - \lfloor \eta' j\rfloor)(H_r(\eta') - \varepsilon^2_j) \le -\log_2 \mu(I_{\sigma^{\lfloor \eta' j\rfloor} w}) \le (j - \lfloor \eta' j\rfloor)(H_r(\eta') + \varepsilon^2_j).$$
Next proposition claims that η' being fixed in $[\eta_\ell, \eta]$, almost surely, for j large enough, for all $W \in \Sigma_{\lfloor \eta' j\rfloor}$, there is a surviving vertex $w \in \mathcal{S}_j(\eta, W)$ such that $\dfrac{-\log_2 \mu(I_{\sigma^{\lfloor \eta' j\rfloor} w})}{j - \lfloor \eta' j\rfloor} \approx H_\ell(\eta')$. This property shall be understood as a renewal property for the local dimensions $H_\ell(\eta')$. See Figure 10 for an illustration of this decomposition.

Proposition 6. *Given $\eta' \in [\eta_\ell, \eta)$, there exists a positive sequence $(\varepsilon^3_j)_{j\ge 1}$ converging to 0 such that, with probability 1, for j large enough, for all $W \in \Sigma_{\lfloor \eta' j\rfloor}$, $\mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon^3_j) \neq \emptyset$.*

Of course, the same holds true for $\mathcal{T}_{\mu,r}(j, \eta', \varepsilon^3_j)$, but we do not need this second property.

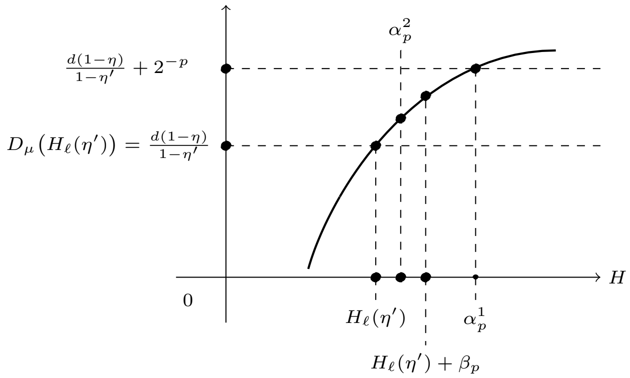

FIGURE 12. Relative positions of $H_\ell(\eta')$, $H_\ell(\eta') + \beta_p$, α_p^1, α_p^2.

Proof. Fix $\eta' \in [\eta_\ell, \eta)$, which implies that $D_\mu(H_\ell(\eta')) < d = \|D_\mu\|_\infty$. For every integer $p \geq 1$ so large that $D_\mu\big(H_\ell(\eta')\big) + 2^{-p} < d$, let $\alpha_p^1, \alpha_p^2, \beta_p$ be such that

- α_p^1 is the unique real number in $[H_\ell(\eta_\ell), H_s]$ such that $D_\mu(\alpha_p^1) = D_\mu(H_\ell(\eta')) + 2^{-p}$,
- $\beta_p = (\alpha_p^1 - H_\ell(\eta'))/2$,
- α_p^2 is such that $H_\ell(\eta') < \alpha_p^2 < H_\ell(\eta') + \beta_p = \alpha_p^1 - \beta_p$.

Observe that $D_\mu(H_\ell(\eta')) < D_\mu(\alpha_p^2) < D_\mu(\alpha_p^1 - \beta_p) < D_\mu(\alpha_p^1)$ (see Figure 12).

For every integer $p \geq 1$, due to the large deviations properties of μ (part (5) of Proposition 2 and equation (9)), we can fix an integer j_p such that for all $j \geq j_p$,

$$\#\mathcal{E}_\mu(j, \alpha_p^1 \pm \beta_p) \geq 2^{jD_\mu(\alpha_p^2)}.$$

Using the definition of our parameters, this implies that

$$\#\mathcal{E}_\mu(j, H_\ell(\eta') \pm \widetilde{\varepsilon}_p) \geq 2^{j\left(D_\mu(H_\ell(\eta'))+\widehat{\varepsilon}_p\right)},$$

where $\widetilde{\varepsilon}_p = 3\beta_p$ and $\widehat{\varepsilon}_p = D_\mu(\alpha_p^2) - D_\mu(H_\ell(\eta')) > 0$.

It is clear from the continuity and monotonicity of D_μ that $(\widetilde{\varepsilon}_p)_{p\geq 1}$ and $(\widehat{\varepsilon}_p)_{p\geq 1}$ are two positive decreasing sequences, and that $\lim_{p\to+\infty} \widetilde{\varepsilon}_p = \lim_{p\to+\infty} \widehat{\varepsilon}_p = 0$.

For $j \geq j_p/(1-\eta')$ (hence so that $j - \lfloor \eta' j \rfloor \geq j_p$) and $W \in \Sigma_{\lfloor \eta' j \rfloor}$, consider the event

$$\mathscr{A}(\eta', \widetilde{\varepsilon}_p, W) = \Big\{ \forall\, w' \in \mathcal{E}_\mu\big(j - \lfloor \eta' j \rfloor, H_\ell(\eta') \pm \widetilde{\varepsilon}_p\big),\ p_{Ww'} = 0 \Big\}.$$

One has

$$\begin{aligned}
\mathbb{P}\Big(\mathscr{A}(\eta', \widetilde{\varepsilon}_p, W)\Big) &= (1 - 2^{-d(1-\eta)j})^{\#\mathcal{E}_\mu\left(j-\lfloor \eta' j\rfloor, H_\ell(\eta')\pm\widetilde{\varepsilon}_p\right)} \\
&\leq \exp\left(-2^{-d(1-\eta)j} \#\mathcal{E}_\mu(j - \lfloor \eta' j \rfloor, H_\ell(\eta') \pm \widetilde{\varepsilon}_p)\right) \\
&\leq \exp\left(-2^{-d(1-\eta)j + (j-\lfloor \eta' j\rfloor)\left(D_\mu(H_\ell(\eta'))+\widehat{\varepsilon}_p\right)}\right).
\end{aligned}$$

Recalling that $D_\mu\big(H_\ell(\eta')\big) = \frac{d(1-\eta)}{1-\eta'}$, we get

$$\mathbb{P}\Big(A(\eta', \widetilde{\varepsilon}_p, W)\Big) \leq \exp(-2^{(j-\lfloor \eta' j \rfloor)\widehat{\varepsilon}_p} + O(1)) \leq C \exp(-2^{(1-\eta')j\widehat{\varepsilon}_p}).$$

We choose the sequence $(\varepsilon_j^3)_{j\geq 1}$ as follows: we first build some sequences of integers by induction. Pick an integer p_0 so large that the previous inequality holds true for

$j \geq j_{p_0}/(1-\eta')$. Also, choose $\widetilde{j}_{p_0} > j_{p_0}$ so large that for all $j \geq \widetilde{j}_{p_0}/(1-\eta')$, one has $C\exp(-2^{(1-\eta')j\widehat{\varepsilon}_{p_0}}) \leq 2^{-dj}$.

Then, assume that integers and $\widetilde{j}_{p_0}, ..., \widetilde{j}_{p_0+m}$ are found such that for $n = 1, ..., m$:

- $\widetilde{j}_{p_0+n} > \max(j_{p_0+n}, \widetilde{j}_{p_0+n-1})$,
- for $j(1-\eta') \geq \widetilde{j}_{p_0+n}$ one has $C\exp(-2^{(1-\eta')j\widehat{\varepsilon}_{p_0+n}}) \leq 2^{-dj}$.

Then we choose $\widetilde{j}_{p_0+m+1} > \max(j_{p_0+m+1}, \widetilde{j}_{p_0+m})$ so large that for all $j \geq \widetilde{j}_{p_0+m+1}/(1-\eta')$, one has $C\exp(-2^{(1-\eta')j\widehat{\varepsilon}_{p_0+m+1}}) \leq 2^{-dj}$.

Finally, for every $j \geq \widetilde{j}_{p_0}/(1-\eta')$, there is a unique integer m_j such that

$$\widetilde{j}_{p_0+m_j}/(1-\eta') \leq j < \widetilde{j}_{p_0+m_j+1}/(1-\eta'), \tag{25}$$

and we set $\varepsilon_j^3 = \widetilde{\varepsilon}_{p_0+m_j}$. By construction we obtain

$$\mathbb{P}\Big(\mathscr{A}(\eta', \varepsilon_j^3, W)\Big) \leq C\exp(-2^{(1-\eta')j\widehat{\varepsilon}_{p_0+m_j}}) \leq 2^{-dj}. \tag{26}$$

Subsequently,

$$\begin{aligned} & \mathbb{P}\Big(\Big\{\exists\, j \geq \widetilde{j}_{p_0}/(1-\eta') \text{ and } \exists\, W \in \Sigma_{\lfloor \eta' j\rfloor} : \mathscr{A}(\eta', \varepsilon_j^3, W) \text{ holds}\Big\}\Big) \\ \leq & \sum_{j \geq \widetilde{j}_{p_0}/(1-\eta')} \ \sum_{W \in \Sigma_{\lfloor \eta' j\rfloor}} \mathbb{P}\Big(\mathscr{A}(\eta', \varepsilon_j^3, W)\Big) \\ \leq & \sum_{j \geq \widetilde{j}_{p_0}/(1-\eta')} 2^{d\lfloor \eta' j\rfloor} 2^{-dj} < +\infty. \end{aligned}$$

We conclude thanks to the Borel-Cantelli lemma. □

Last proposition can be realized simultaneously on several $\eta' \in [\eta_\ell, \eta]$.

Corollary 1. *For all integers $N \geq 1$ and $0 \leq k \leq N-1$, let $\eta_{N,k} = \eta_\ell + \frac{k}{N}(\eta - \eta_\ell)$. There exists a positive sequence $(\varepsilon_j^{4,N})_{j\geq 1}$ converging to 0 when j tends to infinity, such that with probability 1, for $N \geq 2$ and j large enough, for all $0 \leq k \leq N-1$ and all $W \in \Sigma_{\lfloor \eta_{N,k} j\rfloor}$, $\mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \eta_{N,k}, \varepsilon_j^{4,N}) \neq \emptyset$.*

Proof. Fix $N \geq 2$. For each $k \in \{0, ..., N-1\}$, we apply Proposition 5, so that we get a sequence $(\varepsilon_j^3(k))_{j\geq 1}$ and a sequence $(\widetilde{j}_p(k))_{p \geq p_0(k)}$, such that (26) holds true, i.e. for every $j \geq \widetilde{j}_{p_0(k)}/(1-\eta')$ and $W \in \Sigma_{\lfloor \eta_{N,k} j\rfloor}$,

$$\mathbb{P}\Big(A(\eta_{N,k}, \varepsilon_j^3(k), W)\Big) \leq 2^{-dj}. \tag{27}$$

Observe that if $0 < \varepsilon < \varepsilon'$, $\mathscr{A}(\eta', \varepsilon', W) \subset \mathscr{A}(\eta', \varepsilon, W)$. Hence, we can choose the integer $p = \max(p_0(0), ..., p_0(N-1))$, and the sequences $\varepsilon_j^{4,N} := \max(\varepsilon_j^3(0), ..., \varepsilon_j^3(N-1))$ and $\widetilde{j}_p := \max(\widetilde{j}_p(0), ..., \widetilde{j}_p(N-1))$, so that we have the following property: for all $0 \leq k \leq N-1$, for all $j \geq \widetilde{j}_{p_0}/(1-\eta)$, for all $W \in \Sigma_{\lfloor \eta_{N,k} j\rfloor}$, (27) holds true with $\varepsilon_j^{4,N}$ instead of $\varepsilon_j^3(k)$.

Thus,

$$\mathbb{P}\Big(\Big\{\exists\, j \ge \widetilde{j}_{p_0}/(1-\eta),\ \exists\, k\in\{0,...,N-1\},\ \exists\, W\in\Sigma_{\lfloor \eta_{N,k} j\rfloor} : A(\eta_{N,k},\varepsilon_j^{4,N},W) \text{ holds}\Big\}\Big)$$

$$\le \sum_{k=0}^{N-1}\mathbb{P}\Big(\Big\{\exists\, j\ge \widetilde{j}_{p_0}/(1-\eta) \text{ and } \exists\, W\in\Sigma_{\lfloor \eta_{N,k} j\rfloor} : A(\eta_{N,k},\varepsilon_j^{3}(k),W) \text{ holds}\Big\}\Big)$$

$$\le \sum_{k=0}^{N-1}\sum_{j\ge \widetilde{j}_{p_0}/(1-\eta)}\sum_{W\in\Sigma_{\lfloor \eta_{N,k} j\rfloor}} \mathbb{P}\Big(A(\eta_{N,k},\varepsilon_j^{3}(k),W)\Big)$$

$$\le \sum_{k=0}^{N-1} 2^{-(d-\eta_{N,k})\widetilde{j}_{p_0}/(1-\eta)} < +\infty.$$

The result follows again by the Borel-Cantelli lemma. □

Next proposition completes the previous corollary by showing (roughly speaking), that for a fixed $W\in\Sigma_J$ with J large enough, for η' in some interval $[\eta_0,\eta]$ fixed in advance, the probability to find $w\in\bigcup_{J/\eta\le j\le J/\eta_0}\mathcal{S}_j(\eta,W)$ with a η'-tail having a local dimension smaller than $H_\ell(\eta')$ decreases exponentially with J.

Proposition 7. *Let* $\eta_0 = \begin{cases}\frac{H_{\min}}{H_{\min}+\widetilde{H}_\ell(\widetilde{\eta})} & \text{if } \eta_\ell = 0\\ \eta_0=\eta_\ell & \text{if } \eta_\ell>0.\end{cases}$

For all integers $N\ge 1$ and $k\in\{-1,0...,N-1\}$, set $\widetilde{\eta}_{N,k}=\eta-(\eta-\eta_0)\frac{k}{N}$.
For $J\ge 1$ and $W\in\Sigma_J$, consider the event $\mathscr{C}(N,J,W)$ defined as

$$\mathscr{C}(N,J,W)=\left\{\begin{array}{l}\exists\, k\in\{-1,0,...,N-1\},\ \exists\, j\in[J/\widetilde{\eta}_{N,k},J/\widetilde{\eta}_{N,k+1}],\\ \quad \exists\, w\in\mathcal{S}_j(\widetilde{\eta}_{N,k},W) \text{ such that } \ \mu(I_{\sigma^J w})>2^{-J(\widetilde{H}_\ell(\widetilde{\eta}_{N,k})+\varepsilon_N)}\end{array}\right\},$$

with the convention that $\widetilde{H}_\ell(\widetilde{\eta}_{N,-1})=\widetilde{H}_\ell(\widetilde{\eta}_{N,0})=\widetilde{H}_\ell(\eta)$.

With probability one, there exists a positive sequence $(\varepsilon_N)_{N\ge1}$ converging to 0 such that for all $N\ge1$, $J\ge1$ and $W\in\Sigma_J$, we have $\mathbb{P}\big(\mathscr{C}(N,J,W)\big)\le 2^{-J\varepsilon_N}$.

The proof uses arguments similar to those developed earlier, and is left to the reader.

Proposition 6 asserts that for all $W\in\Sigma_{\lfloor\eta' j\rfloor}$, $\mathcal{S}_j(\eta,W)\cap\mathcal{T}_{\mu,\ell}(j,\eta',\varepsilon_j^3)$ is not empty when j becomes large. The last proposition of this section shows that its cardinality cannot be very large. This fact will be interpreted geometrically as a *weak redundancy* property from the viewpoint of ubiquity theory [6, 9] and has nice geometric consequences for our study.

Proposition 8. (1) *For all $\eta'\in[\eta_\ell,\eta]\setminus\{0\}$, for all $\varepsilon\in(0,1)$, there exists $\beta>0$ such that with probability 1, for every j large enough and all $W\in\Sigma_{\lfloor\eta' j\rfloor}$,*

$$1 \le \#\big(\mathcal{S}_j(\eta,W)\cap\mathcal{T}_{\mu,\ell}(j,\eta',\beta)\big) \le 2^{\eta' j\varepsilon}. \tag{28}$$

(2) *The same holds true for $\eta'\in[\eta_r,\eta]\setminus\{0\}$ and the sets $\mathcal{S}_j(\eta,W)\cap\mathcal{T}_{\mu,r}(j,\eta',\beta)$.*

Proof. (1) It is clear that it is enough to get the conclusion for ε small enough. Fix $\varepsilon\in(0,1)$ and $\eta'\in\eta'\in[\eta_\ell,\eta]\setminus\{0\}$. Due to the almost multiplicativity property of μ, and equation (9), there exists $\beta>0$ and J_0 such that for $j\ge J_0$, for each $W\in\Sigma_{\lfloor\eta' j\rfloor}$,

$$\#\mathcal{T}_{\mu,\ell}(j,\eta',\beta,W)\le 2^{\big(D_\mu(H_\ell(\eta'))+d\varepsilon^2\big)(j-\lfloor\eta' j\rfloor)}. \tag{29}$$

Notice that the cardinality $n_j = \#\mathcal{T}_{\mu,\ell}(j,\eta',\beta,W)$ is independent of W. Since $D_\mu(H_\ell(\eta')) = d(1-\eta)/(1-\eta') \le d$, $\varepsilon < 1$ and $\eta' \le \eta < 1$, for $j \ge J_0$ we have

$$n_j \le 2^{\left(D_\mu(H_\ell(\eta'))+d\varepsilon^2\right)(j-\lfloor \eta' j\rfloor)} \le 2^{d(1-\eta)j}2^{d\varepsilon^2 j+d}.$$

By definition, we have

$$\#\big(\mathcal{S}_j(\eta,W)\cap\mathcal{T}_{\mu,\ell}(j,\eta',\varepsilon_j^3)\big) = \sum_{w\in\mathcal{T}_{\mu,\ell}(j,\eta',\beta,W)} p_w.$$

Denote this random variable by $B(j,\eta',\beta,W)$. Its law is a binomial law of parameters $(n_j, 2^{-d(1-\eta)j})$. Thus

$$\begin{aligned}\mathbb{P}\Big(B(j,\eta',\beta,W)\ge 2^{\varepsilon\eta' j}\Big) &\le \sum_{2^{\varepsilon\eta' j}\le l\le n_j}\binom{n_j}{l}(2^{-d(1-\eta)j})^l \le \sum_{2^{\varepsilon\eta' j}\le l\le n_j}\frac{(n_j 2^{-d(1-\eta)j})^l}{l!}\\ &\le \sum_{2^{\varepsilon\eta' j}\le l\le n_j}\frac{2^{dj\varepsilon^2 l+dl}}{l!} \le \sum_{l\ge 2^{\varepsilon\eta' j}}\left(\frac{e2^{dj\varepsilon^2+d}}{l}\right)^l\end{aligned}$$

for j large enough by Stirling's formula. Then, if $\varepsilon \le \eta'/(4d)$, there is another integer J_0' such that for $j \ge J_0'$, for all $l \ge 2^{\varepsilon\eta' j}$, we have $\dfrac{e2^{dj\varepsilon^2+d}}{l} \le 2^{-\varepsilon\eta' j/2} \le 1/2$, hence

$$\mathbb{P}\Big(B(j,\eta',\beta,W)\ge 2^{\varepsilon\eta' j}\Big) \le 2\cdot 2^{-\lfloor 2^{\varepsilon\eta' j}\rfloor \varepsilon\eta' j/2},$$

and

$$\sum_{j\ge J_0'}\sum_{W\in\Sigma_{\lfloor\eta' j\rfloor}}\mathbb{P}\Big(B(j,\eta',\beta,W)\ge 2^{\varepsilon\eta' j}\Big) \le \sum_{j\ge J_0'} 2^{d\lfloor\eta' j\rfloor}2\cdot 2^{-\lfloor 2^{\varepsilon\eta' j}\rfloor\varepsilon\eta' j/2} < \infty.$$

The desired conclusion follows from the Borel-Cantelli lemma.

(2) The computations are identical for $\eta' \in [\eta_r,\eta]\setminus\{0\}$ and $\#\big(\mathcal{S}_j(\eta,W)\cap\mathcal{T}_{\mu,\ell}(j,\eta',\beta)\big)$. ☐

6. Upper bound for the singularity spectrum of M_μ

Section 6.1 derives the sharp upper bound provided by Theorem 2 for the decreasing part of D_{M_μ}; this bound comes rather directly after the preparation achieved in Section 5.2. Next, Section 6.2 examines in detail the possible local behaviors of the surviving coefficients $\mu(I_w)$ which contribute to a given set $\underline{E}_{\mathsf{M}_\mu}(H)$ and provides a first expression for the upper bound of the increasing part of D_{M_μ}. This bound is then simplified in Section 6.3 into the formula given by Theorem 2. Also, precious information are pointed out in preparation of next Section 7, which deals with the lower bound for D_{M_μ}.

Remark 1. *This section is rather long and technical, but it is key to understand what phenomena rule the local behavior of M_μ at a point $x\in[0,1]^d$. Let us mention that there is a slightly faster way to obtain the upper bound for the multifractal spectrum of M_μ, using the multifractal formalism and a lower bound for the L^q-spectrum τ_{M_μ} obtained in Section 8.2. Nevertheless, we choose to keep up with the first method, mainly for two reasons:*

- *The second method is "blind", since it does not give any clue on how to obtain the lower bound for the spectrum. Indeed, it will appear soon that for every possible local dimension H, there is a favorite scenario which leads a point x to satisfy*

$\underline{\dim}(\mathsf{M}_\mu, x) = H$. *This can absolutely not be guessed without the precise study achieved in the following pages.*

- *Using the multifractal formalism to get an upper bound for the multifractal spectrum is efficient only when the multifractal formalism is satisfied by the object under consideration. Fortunately, this is the case for M_μ, and the bound is sharp. But there are closely related sampling processes (not developed in this paper) not satisfying the multifractal formalism, and in this case this method is useless.*

6.1. **Upper bound for the decreasing part of D_{M_μ}.** It turns out that finding an upper bound in the decreasing part of the singularity spectrum D_{M_μ}, i.e. for $H \geq H_s + \widetilde{H}_\ell(\widetilde{\eta})$, is much easier than in the increasing one.

We start with quite a direct upper bound for all the local dimensions of M_μ.

Proposition 9. *Almost surely, for every $x \in [0,1]^d$,*

$$\underline{\dim}(\mathsf{M}_\mu, x) \leq \overline{\dim}(\mathsf{M}_\mu, x) \leq \overline{\dim}(\mu, x) + \widetilde{H}_\ell(\widetilde{\eta}).$$

As a consequence, for every $x \in [0,1]^d$, $\underline{\dim}(\mathsf{M}_\mu, x) \leq H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})$.

Proof. Let $x \in [0,1]^d$. Due to Proposition 6 applied with $\eta' = \widetilde{\eta}$, for each j large enough we have

$$\mathsf{M}_\mu(I_{\lfloor j\widetilde{\eta}\rfloor}(x)) \geq C^{-1}\mu(I_{\lfloor j\widetilde{\eta}\rfloor}(x))2^{-(j-\lfloor j\widetilde{\eta}\rfloor)(H_\ell(\widetilde{\eta})+\varepsilon_j^3)}.$$

Taking logarithm on both sides, dividing by $-\lfloor j\widetilde{\eta}\rfloor \log(2)$, and taking the lim inf as $j \to \infty$ yields the desired conclusion.

Since for every $x \in [0,1]^d$, $\overline{\dim}(\mu, x) \leq H_{\max}$, the result follows. □

Using this upper bound for the local dimensions, and by anticipation the lower bound given by Lemma 4 below, one deduces that the domain of D_{M_μ} is included in $[H_\ell(\eta_\ell), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$. Also, we get an upper bound for the decreasing part of the singularity spectrum.

Proposition 10. *For all $H \in [H_s + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$, one has*

$$D_{\mathsf{M}_\mu}(H) \leq D_\mu\big(H - \widetilde{H}_\ell(\widetilde{\eta})\big),$$

and for all $H > H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})$,

$$\dim \underline{E}^{\geq}_{\mathsf{M}_\mu}(H') = -\infty.$$

Proof. Let $H \geq H_s + \widetilde{H}_\ell(\widetilde{\eta})$. By Proposition 9, if $x \in \underline{E}_{\mathsf{M}_\mu}(H)$, then $\overline{\dim}(\mu, x) \geq H - \widetilde{H}_\ell(\widetilde{\eta})$, hence $\underline{E}_{\mathsf{M}_\mu}(H) \subset \overline{E}^{\geq}_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))$. Using Part (3) of Proposition 2, one deduces that

$$\dim \underline{E}_{\mathsf{M}_\mu}(H) \leq D_\mu\big(H - \widetilde{H}_\ell(\widetilde{\eta})\big),$$

since $H - \widetilde{H}_\ell(\widetilde{\eta}) \geq H_s$ (this corresponds to the decreasing part of D_μ). □

6.2. **Upper bound for the increasing part of D_{M_μ}.** Let us start with the lower bound for the left end-point of the support of the singularity spectrum of M_μ.

Lemma 4. *With probability 1, for every $x \in [0,1]^d$, $\underline{\dim}(\mathsf{M}_\mu, x) \geq H_\ell(\eta_\ell)$.*

Proof. By Proposition 3, with probability 1, for j large enough, the surviving vertices $w \in \mathcal{S}_j(\eta)$ all satisfy $\mu(I_w) \leq 2^{-j(H_\ell(\eta_\ell)-\varepsilon_j^1)}$. Hence, for every large integer J and every word $W \in \Sigma_J$, $\mathsf{M}_\mu(I_W) \leq 2^{-J(H_\ell(\eta_\ell)-\varepsilon_J^1)}$, since $\mathsf{M}_\mu(I_W)$ is the maximum of $\mu(I_w)$ over all surviving words w such that $I_w \subset I_W$. Subsequently, for every x, $\underline{\dim}(\mathsf{M}_\mu, x) \geq H_\ell(\eta_\ell)$. □

Further, we are going to provide a first expression for the sharp upper bound of $\dim \underline{E}_{\mathsf{M}_\mu}$ when $H_\ell(\eta_\ell) \leq H \leq H_s + \widetilde{H}_\ell(\widetilde{\eta})$ in Proposition 12. It is based on the following definition and Proposition 11, which describe the possible scenarii leading to the property $\underline{\dim}(\mathsf{M}_\mu, x) \leq H$.

Definition 18. *For each $j \geq 1$, $k \geq 1$, $\alpha \in \mathbb{R}_+$, $\eta' \in [\eta_\ell, \eta] \setminus \{0\}$ and $\delta \geq 1$, let*

$$F_{\mu,\ell}(j, \alpha, \eta', \delta, k) = \left\{ x \in [0,1]^d : \left\{ \begin{array}{l} \exists\, w \in \mathcal{S}_j(\eta) \cap \mathcal{R}_\mu(j, \eta', \alpha \pm 1/k) \cap \mathcal{T}_{\mu,\ell}(j, \eta', 1/k) \\ \text{such that } \max(2^{-j}, d(x, I_w)) \leq 2^{-\eta' j \delta} \end{array} \right. \right\}.$$

Let $\mathcal{P}_\ell$ be a countable set of parameters (α, η', δ) dense in $[H_{\min}, H_{\max}] \times (\eta_\ell, \eta] \times [1, +\infty)$. Then, for $H \geq 0$, let

$$F_{\mu,\ell}(H) = \bigcap_{\varepsilon \in (0,1)} \bigcap_{k \geq 1} \bigcup_{\substack{(\alpha, \eta', \delta) \in \mathcal{P}_\ell: \\ \delta \in [1, 1/\eta'],\ \frac{\alpha + \widetilde{H}_\ell(\eta')}{\delta} \leq H + \varepsilon}} \limsup_{j \to +\infty} F_{\mu,\ell}(j, \alpha, \eta', \delta - \varepsilon, k). \tag{30}$$

The sets $F_{\mu,r}(j, \alpha, \eta', \delta, k)$, $\mathcal{P}_r$ and $F_{\mu,r}(H)$ are similarly defined.

The definition of the sets $F_{\mu,\ell}(j, \alpha, \eta', \delta, k)$ and $F_{\mu,r}(j, \alpha, \eta', \delta, k)$ is rather long and difficult to handle with at first sight. Nevertheless, it is the mathematical counterpart of the following intuition: $F_{\mu,\ell}(j, \alpha, \eta', \delta, k)$ contains those points x which are $2^{-\eta' j \delta}$ close to some cube I_w (i.e. $\max(2^{-j}, d(x, I_w)) \leq 2^{-\eta' j \delta}$) associated with a surviving vertex w whose η'-root and η'-tail have a prescribed behavior with respect to the capacity μ.

The fact that these sets play a key role is illustrated by the following proposition, which is the main result of this section.

Proposition 11. *With probability 1, for all $H \geq 0$,*

$$\underline{E}_{\mathsf{M}_\mu}(H) \subset F_{\mu,\ell}(H) \cup F_{\mu,r}(H).$$

Proof. Fix $H \geq 0$ and $x \in \underline{E}_{\mathsf{M}_\mu}(H)$. We denote by $\mathsf{M}_j(x)$ the coefficient $\mathsf{M}_\mu(I_{x_{|j}})$, and $\mathcal{N}_j(x)$ the set $\mathcal{N}(x_{|j})$ (recall Definition 9). Let $\varepsilon \in (0,1)$.

By definition, since $\underline{\dim}(\mathsf{M}_\mu, x) = H$ there is an infinite number of integers J_n such that $\frac{-\log_2 \mathsf{M}_{J_n}(x)}{J_n} \leq H + \varepsilon/2$. We write $W_n = x_{|J_n}$. By definition of the quantity $\mathsf{M}_{J_n}(x) = \mathsf{M}_\mu(I_{W_n})$, there exists a surviving vertex $w_n \in \mathcal{S}_{j_n}(\eta)$, $j_n \geq J_n$, such that $w_n \in \mathcal{N}_{J_n}(x)$ and $\mathsf{M}_{J_n}(x) = \mu(I_{w_n})$. One can assume that $I_{w_n} \subset I_{W_n}$, the other cases, i.e. when I_{w_n} is included in a neighboring cube $I_{w'}$ of I_{W_n} of generation J_n, are absolutely similar by switching W_n and w').

By Proposition 5, there are $\eta_{j_n} \in [\eta_\ell, \eta] \cup [\eta_r, \eta]$ and $\alpha_{\lfloor j_n \eta_{j_n} \rfloor} \in [H_{\min}, H_{\max}]$ such that

(1) either $\eta_{j_n} \in [\eta_\ell, \eta]$, and

$$-\log_2 \mu(I_{w_n}) = \alpha_{\lfloor j_n \eta_{j_n} \rfloor} \lfloor j_n \eta_{j_n} \rfloor + (j_n - \lfloor j_n \eta_{j_n} \rfloor) H_\ell(\eta_{j_n}) + o(j_n), \tag{31}$$

$$-\log_2 \mu(I_{\sigma^{\lfloor j_n \eta_{j_n} \rfloor} w_n}) = (j_n - \lfloor j_n \eta_{j_n} \rfloor) H_\ell(\eta_{j_n}) + o(j_n), \tag{32}$$

(2) or $\eta_{j_n} \in [\eta_r, \eta]$, and the same holds with $H_r(\eta_{j_n})$ instead of $H_\ell(\eta_{j_n})$.

Obviously, $\max(2^{-j_n}, d(x, I_{w_n})) \leq 2 \cdot 2^{-J_n}$ (recall that we work with the $\|.\|_\infty$ norm).

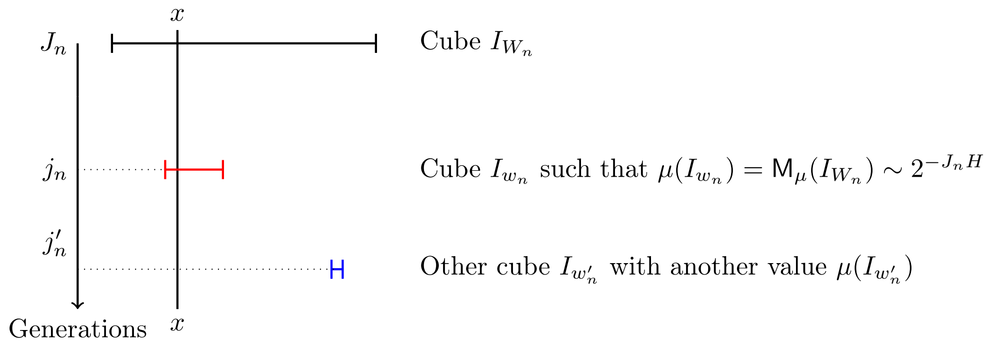

FIGURE 13. Competition between surviving vertices

- Assume that the subsequence $(\eta_{j_n})_{n\geq 1}$ has an accumulating point in $(0,\eta]$.

Without loss of generality we can assume that the whole sequence $(\eta_{j_n})_{n\geq 1}$ converges to some $\eta' \in [\eta_\ell, \eta] \setminus \{0\}$ and that we are in the above situation above (1) for infinitely many values of j_n, along which $H_\ell(\eta_{j_n})$ converges to $H_\ell(\eta')$. The other situation, absolutely symmetric, is that the sequence $(\eta_{j_n})_{n\geq 1}$ converges to some $\eta' \in [\eta_r, \eta] \setminus \{0\}$, the situation (2) holds for n large enough, and $H_r(\eta_{j_n})$ converges to $H_r(\eta')$.

Lemma 5. *One has* $\lim_{n\to+\infty} j_n \eta_{j_n} = +\infty$*, and for* n *large enough, one has*

$$j_n \eta_{j_n} \leq J_n(1+\varepsilon/6). \tag{33}$$

Proof. The first part $\lim_{n\to\infty} j_n\eta_{j_n} = +\infty$ is obvious since $\lim_{n\to\infty} \eta_{j_n} = \eta' > 0$.

Recall Corollary 1 and the notations therein. For $N \geq 1$, let us write $\eta_{j_n}(N)$ for the unique $\eta_{N,k}$ such that

$$\eta_{j_n} \in \big[\eta_{N,k}, \eta_{N,k} + (\eta - \eta_\ell)/N\big). \tag{34}$$

We apply Corollary 1 with an integer $j'_n > J_n$ such that $\lfloor j'_n \eta_{j_n}(N)\rfloor = J_n$, to the word $W_n \in \Sigma_{J_n}$: there exists $w'_n \in \mathcal{S}_{j'_n}(\eta)$ such that $I_{w'_n} \subset I_{W_n}$ and $w'_n \in \mathcal{T}_{\mu,\ell}(j'_n, \eta_{j_n}(N), \varepsilon^{4,N}_{j'_N})$, i.e.

$$2^{-(j'_n - J_n)\left(H_\ell(\eta_{j_n}(N)) + \varepsilon^{4,N}_{j'_n}\right)} \leq \mu(I_{\sigma^{\lfloor j'_n \eta_{j_n}(N)\rfloor} w'_n}) \leq 2^{-(j'_n - J_n)\left(H_\ell(\eta_{j_n}(N)) - \varepsilon^{4,N}_{j'_n}\right)},$$

see Figure 13. Observe that $I_{w'_n} \subset I_{W_n}$ (hence $(w'_n)_{|\lfloor j'_n \eta_{j_n}(N)\rfloor} = (w'_n)_{|J_n} = W_n$), so

$$\mu(I_{w'_n}) \quad\geq\quad C^{-1}\mu(I_{w'_{n|J_n}})\mu(I_{\sigma^{J_n} w'_n}) \geq C^{-1}\mu(I_{W_n}) 2^{-(j'_n - J_n)\left(H_\ell(\eta_{j_n}(N)) + \varepsilon^{4,N}_{j'_n}\right)}. \tag{35}$$

Assume towards contradiction that (33) is not true. We are going to prove that $\mu(I_{w'_n}) > \mu(I_{w_n})$, contradicting the maximality of $\mu(I_{w_n})$ and the fact that $\mathsf{M}_{J_n}(x) = \mu(I_{w_n})$. When (33) does not hold, one has

$$j'_n - J_n \quad\leq \frac{J_n + 1}{\eta_{j_n}(N)} - J_n = J_n\left(\frac{1 + 1/J_n}{\eta_{j_n}(N)} - 1\right) < \frac{j_n \eta_{j_n}}{1+\varepsilon/6} \frac{1 + 1/J_n - \eta_{j_n}(N)}{\eta_{j_n}(N)}.$$

Moreover, since η_{j_n} tends to η' and $|\eta_{j_n}(N) - \eta_{j_n}| \leq 1/N$, we choose N so large that that when n becomes large we have

$$j'_n - J_n \leq \frac{j_n - \lfloor j_n \eta_{j_n}\rfloor}{1 + \varepsilon/12}.$$

Observe that $j_n\eta_{j_n} \geq J_n(1+\varepsilon/6)$ also yields $I_{w_{n|\lfloor j_n\eta_{j_n}\rfloor}} \subset I_{W_n}$. This, together with the last inequality and (35) yields

$$\mu(I_{w'_n}) \quad \geq \quad C^{-1}\mu(I_{w_{n|\lfloor j_n\eta_{j_n}\rfloor}})2^{-\frac{j_n-\lfloor j_n\eta_{j_n}\rfloor}{1+\varepsilon/12}\left(H_\ell(\eta_{j_n}(N))+\varepsilon^{4,N}_{j'_n}\right)}.$$

One chooses now the integers N and n so large that:

- $|H_\ell(\eta_{j'_n}) - H_\ell(\eta')| < H_\ell(\eta_{j_n})\varepsilon/96$,
- $|H_\ell(\eta_{j_n}(N)) - H_\ell(\eta_{j_n})| < H_\ell(\eta_{j_n})\varepsilon/96$,
- $\varepsilon^{4,N}_{j'_n} < H_\ell(\eta_{j_n})\varepsilon/96$.

This is possible since $\eta_{j'_n} \to \eta' > 0$, $H_\ell(\eta') > 0$, and H_ℓ is differentiable as function of $\eta' \in (\eta_\ell, \eta)$. With these choices, one sees that

$$\frac{j_n - \lfloor j_n\eta_{j_n}\rfloor}{1+\varepsilon/12}\left(H_\ell(\eta_{j_n}(N)) + \varepsilon^{4,N}_{j'_n}\right) \quad \leq \quad (j_n - \lfloor j_n\eta_{j_n}\rfloor)H_\ell(\eta_{j_n})(1-\varepsilon/48).$$

Finally, we use (32) to get, for n large enough

$$\begin{aligned}
\mu(I_{w'_n}) \quad &\geq \quad C^{-1}\mu(I_{w_{n|\lfloor j_n\eta_{j_n}\rfloor}})2^{-(j_n-\lfloor j_n\eta_{j_n}\rfloor)H_\ell(\eta_{j_n})(1-\varepsilon/48)}\\
&\geq \quad C^{-2}\mu(I_{w_{n|\lfloor j_n\eta_{j_n}\rfloor}})\mu(I_{\sigma^{\lfloor j_n\eta_{j_n}\rfloor}w_n})2^{(j_n-\lfloor j_n\eta_{j_n}\rfloor)H_\ell(\eta_{j_n})\varepsilon/48+o(j_n)}\\
&\geq \quad C'\mu(I_{w_n})2^{(j_n-\lfloor j_n\eta_{j_n}\rfloor)H_\ell(\eta_{j_n})\varepsilon/96}.
\end{aligned}$$

When n becomes large, we conclude that $\mu(I_{w'_n}) > \mu(I_{w_n})$, hence a contradiction. □

Now we prove that x is well-approximated by some cubes I_w whose η'-tail and η'-root have a controlled local behavior with respect to μ.

By construction and the last lemma, since η_{j_n} tends to η' when $n \to +\infty$, we have

$$2^{-j_n} \leq \max(2^{-j_n}, d(x, I_{w_n})) \leq 2\cdot 2^{-J_n} \leq 2\cdot 2^{-\lfloor j_n\eta_{j_n}\rfloor/(1+\varepsilon/6)} \leq 2^{-j_n\eta'(1-\varepsilon/3)}.$$

Consequently, for n large enough we can write $\max(2^{-j_n}, d(x, I_{w_n})) = 2^{-j_n\eta'(\delta_{j_n}-\varepsilon/2)}$ for some $\delta_{j_n} \in [1, 1/\eta']$.

Up to extraction of a subsequence, the sequences $(\delta_{j_n})_{n\geq 1}$ and $(\alpha_{\lfloor j_n\eta_{j_n}\rfloor})_{n\geq 1}$ can be assumed to converge to some $\delta \in [1, 1/\eta']$ and $\alpha \in [H_{\min}, H_{\max}]$. In addition, by construction we have $\mathsf{M}_{\lfloor j_n\eta'\delta_{j_n}\rfloor}(x) = \mathsf{M}_{J_n}(x) = \mu(I_{w_n})$ and $J_n \leq \lfloor j_n\eta'\delta_{j_n} + 1\rfloor$. Thus, recalling (31), one finally gets

$$\frac{\alpha_{\lfloor j_n\eta_{j_n}\rfloor}\lfloor j_n\eta_{j_n}\rfloor + (j_n - \lfloor j_n\eta_{j_n}\rfloor)H_\ell(\eta_{j_n}) + o(j_n)}{\eta_{j_n}j_n\delta_{j_n}} \leq \frac{-\log_2 \mathsf{M}_{J_n}(x)}{J_n} \leq H + \varepsilon/2,$$

which can also be written

$$\alpha_{\lfloor j_n\eta_{j_n}\rfloor}\frac{\lfloor j_n\eta_{j_n}\rfloor}{\eta_{j_n}j_n\delta_{j_n}} + \frac{(j_n - \lfloor j_n\eta_{j_n}\rfloor)}{\eta_{j_n}j_n\delta_{j_n}}(H_\ell(\eta_{j_n}) + o(1)) \leq H + \varepsilon/2.$$

Finally, fix an integer $k \geq 1$. Choosing ε such that $\varepsilon < 1/(4k)$ and a triplet $(\widetilde{\alpha}, \widetilde{\delta}, \widetilde{\eta}')$ in the dense set $\mathcal{P}_\ell$ so that for n large $\|(\alpha_{j_n}, \delta_{j_n}, \eta_{j_n}) - (\widetilde{\alpha}, \widetilde{\delta}, \widetilde{\eta}')\|_\infty \leq \varepsilon/4$, we see that when n becomes large, we ensured that:

- $\widetilde{\delta} \in [1, 1/\eta']$,
- $\frac{\widetilde{\alpha}+\widetilde{H}_\ell(\widetilde{\eta}')}{\widetilde{\delta}} = \frac{\widetilde{\alpha}+(1/\widetilde{\eta}'-1)H_\ell(\widetilde{\eta}')}{\widetilde{\delta}} \leq H + \varepsilon$,
- $\max(2^{-j_n}, d(x, I_{w_n})) = 2^{-j_n\widetilde{\eta}'(\widetilde{\delta}-\varepsilon)}$,
-

- $\left|\dfrac{-\log_2\mu(I_{(w_n)_{|\lfloor j_n\widetilde{\eta}'\rfloor}})}{\lfloor j_n\widetilde{\eta}'\rfloor}-\widetilde{\alpha}\right|\le 1/k$, hence $w_n\in\mathcal{R}_\mu(j,\widetilde{\eta}',\widetilde{\alpha}\pm 1/k)$,
- $\left|\dfrac{-\log_2\mu\left(I_{\sigma^{\lfloor j_n\widetilde{\eta}'\rfloor}w_n}\right)}{j_n-\lfloor j_n\widetilde{\eta}'\rfloor}-H_\ell(\widetilde{\eta}')\right|\le 1/k$, hence $w_n\in\mathcal{T}_{\mu,\ell}(j,\widetilde{\eta}',1/k)$.

Recalling Definition 18 and equation (30), this precisely shows that x belongs to the set $F_{\mu,\ell}(j_n,\widetilde{\alpha},\widetilde{\eta}',\widetilde{\delta}-\varepsilon,k)$ with $\widetilde{\delta}\in[1,1/\eta']$ and $\frac{\widetilde{\alpha}+\widetilde{H}_\ell(\widetilde{\eta}')}{\widetilde{\delta}}\le H+\varepsilon$, as claimed in the initial statement.

• Assume now that $(\eta_{j_n})_{n\ge1}$ converges to 0 (this implies that $\eta_\ell=0$). Fix a small $\eta'>0$, and rewrite (31) and (32) as

$$-\log_2\mu(I_{w_n})=\alpha_{j_n}\lfloor j_n\eta'\rfloor+(j_n-\lfloor j_n\eta'\rfloor)H_\ell(\eta')+o(j_n)+j_n\xi_1(j_n,\eta'),$$
$$-\log_2\mu(I_{\sigma^{\lfloor j_n\eta'\rfloor}w_n})=(j_n-\lfloor j_n\eta'\rfloor)H_\ell(\eta')+o(j_n)+j_n\xi_2(j_n,\eta'),$$

where both $\xi_1(j_n,\eta')$ and $\xi_2(j_n,\eta')$ are $O\left(|H_\ell(\eta')-H_\ell(\eta_{j_n})|+|\eta'|\right)$. If one chooses η' so small that $\limsup_{n\to+\infty}|H_\ell(\eta')-H_\ell(\eta_{j_n})|+\eta'\le|H_\ell(\eta')-H_\ell(0)|+\eta'\le\varepsilon^2$, we are back to the previous situation. □

Using Proposition 11, we are now able to find an upper bound for the Hausdorff dimension of any $\underline{E}_{\mathsf{M}_\mu}(H)$.

Proposition 12. *For $H>0$ and $\varepsilon\ge0$ let*

$$D(H,\varepsilon)=\sup\left\{\frac{D_\mu(\alpha)}{\delta}:\begin{cases}\alpha\in[H_{\min},H_{\max}],\\ \exists i\in\{\ell,r\},\quad \eta'\in[\eta_i,\eta]\setminus\{0\},\\ 1\le\delta\le 1/\eta',\quad \dfrac{\alpha+\widetilde{H}_i(\eta')}{\delta}\le H+\varepsilon\end{cases}\right\}, \tag{36}$$

with the convention $\sup\emptyset=-\infty$. For all $H>0$, we have

$$\dim\underline{E}_{\mathsf{M}_\mu}(H)\le D(H):=\lim_{\varepsilon\to0^+}D(H,\varepsilon). \tag{37}$$

Proof. Recalling Proposition 11, it is enough to find an upper bound for the Hausdorff dimensions of $F_{\mu,\ell}(H)$ and $F_{\mu,r}(H)$. But for $F_{\mu,\ell}(H)$ (the same hold true for $F_{\mu,r}(H)$), one needs only to focus on $\limsup_{j\to\infty}F_{\mu,\ell}(j,\alpha,\eta',\delta-\varepsilon,k)$, for the suitable values of the parameters $\alpha,\delta,\eta',\varepsilon$ described in (30). We are going to prove that the weak redundancy (described in Proposition 8) implies that for $\varepsilon\in(0,1)$, for k large enough, uniformly in (α,η',δ) (under the constraint $\delta\in[1,1/\eta']$), one has

$$\dim\limsup_{j\to\infty}F_{\mu,\ell}(j,\alpha,\eta',\delta-\varepsilon,k)\le\frac{D_\mu(\alpha)}{\delta}+\theta(\varepsilon), \tag{38}$$

where $\lim_{\varepsilon\to0}\theta(\varepsilon)=0$. Then, the result follows by taking the supremum over (α,η',δ) in $\mathcal{P}_\ell$ and letting ε tend to 0.

We prove (38). Fix $\varepsilon>0$.

For every $\alpha\in[H_{\min},H_{\max}]$, using Proposition 2 and the large deviations properties (9) of μ, there exists $\beta_\alpha>0$ and $j_\alpha\in\mathbb{N}$ such that for $j\ge j_\alpha$,

$$\#\mathcal{E}_\mu(j,\beta_\alpha,\alpha)\le 2^{j(D_\mu(\alpha)+\varepsilon/2)}. \tag{39}$$

Moreover, D_μ being continuous over $[H_{\min},H_{\max}]$, one can choose β_α so small that $D_\mu(\alpha)\le D_\mu(\alpha')+\varepsilon/2$ for all $\alpha'\in[\alpha-\beta_\alpha,\alpha+\beta_\alpha]$. Using the compactness of $[H_{\min},H_{\max}]$, there exists finitely many real numbers $\alpha_1, \dots, \alpha_p \in [H_{\min}, H_{\max}]$, such that for the associated numbers $\beta_1, ..., \beta_p > 0$ and integers $j_1, ..., j_p$, one has $[H_{\min}, H_{\max}] \subset \bigcup_{i=1}^p [\alpha_i - \beta_i, \alpha_i + \beta_i]$ and (39) holds for every $j \geq j_i$.

Pick an integer $k \geq 1$ such that $1/k \leq \min\{\beta_i : 1 \leq i \leq p\}$. For all $j \geq J_\varepsilon^1 = \max\{j_i : 1 \leq i \leq p\}$, for every $\alpha \in [H_{\min}, H_{\max}]$, there exists $1 \leq i \leq p$ such that $\alpha \in [\alpha_i - \beta_i, \alpha_i + \beta_i]$, hence because of (39), one has

$$\#\mathcal{E}_\mu(j, 1/k, \alpha) \leq 2^{j(D_\mu(\alpha_i)+\varepsilon/2)} \leq 2^{j(D_\mu(\alpha)+\varepsilon)}. \tag{40}$$

Consider one triplet $(\alpha, \eta', \delta) \in \mathcal{P}_\ell$ with $\delta \in [1, 1/\eta']$. We choose the integer k such that $1/k \leq \beta$. Observe that with these parameters,

$$F_{\mu,\ell}(j, \alpha, \eta', \delta - \varepsilon, k) \subset \bigcup_{W \in \Sigma_{\lfloor j\eta' \rfloor}} T_{\mu,\ell}(j, \eta', \beta, W).$$

We use Proposition 8 to find an upper bound for the cardinality $T_{\mu,\ell}(j, \eta', \beta, W)$. Applying part (1) of this proposition, one can find $\beta > 0$ and an integer J_ε^2 such that for all $j \geq J_\varepsilon^2$, equation (28) holds true.

For each $j \geq J_\varepsilon := \max(J_\varepsilon^1/\eta', J_\varepsilon^2)$, every point x belonging to $F_{\mu,\ell}(j, \alpha, \eta', \delta - \varepsilon, k)$ is close at distance at most $2^{-\lfloor \eta' j \rfloor(\delta-\varepsilon)}$ from some cube I_w such that $w \in \mathcal{S}_j(\eta) \cap \mathcal{R}_\mu(j, \eta', \alpha \pm 1/k) \cap \mathcal{T}_{\mu,\ell}(j, \eta', 1/k)$, with $\delta \in [1, 1/\eta']$.

One combines now two facts:

- By (40), the cardinality of those words $W \in \Sigma_{\lfloor \eta' j \rfloor}$ satisfying
$$\left| \frac{-\log_2 \mu(I_W)}{\lfloor j\eta' \rfloor} - \alpha \right| \leq 1/k$$
is bounded from above by
$$\#\mathcal{E}_\mu(\lfloor \eta' j \rfloor, \alpha, 1/k) \leq 2^{\lfloor j\eta' \rfloor(D_\mu(\alpha)+\varepsilon)}.$$
This applies to the words $W = w_{\lfloor \eta' j \rfloor}$ when $w \in \mathcal{R}_\mu(j, \eta', \alpha \pm 1/k)$
- If $W \in \Sigma_{\lfloor \eta' j \rfloor}$ is fixed, we know by (28) that the cardinality the words $w \in \mathcal{T}_{\mu,\ell}(j, \eta', 1/k)$ is bounded from above by $2^{\eta' j \varepsilon}$.

The first item allows us to control the number of possible η'-roots $w_{\lfloor \eta' j \rfloor}$ of w, and the second item the number of possible η'-tails $w_{\sigma^{j-\lfloor \eta' j \rfloor} w}$.

We deduce that for all $J \geq J_\varepsilon$, the set $F_{\mu,\ell}(j, \alpha, \eta', \delta - \varepsilon, k)$ is covered by at most $2^{\lfloor j\eta' \rfloor(D_\mu(\alpha)+\varepsilon)}$ times $2^{\eta' j \varepsilon}$ cubes of diameter $2^{\lfloor \eta' j \rfloor(\delta-\varepsilon)}$.

Fix a real number $s > \dfrac{D_\mu(\alpha) + 2\varepsilon}{\delta}$. The s-Hausdorff measure $\mathcal{H}^s$ of the limsup set $F := \limsup_{j \to +\infty} F_{\mu,\ell}(j, \alpha, \eta', \delta - \varepsilon, k)$ satisfies for every integer $J \geq J_\varepsilon$

$$\mathcal{H}^s(F) \quad \leq \quad \sum_{j \geq J} 2^{\lfloor j\eta' \rfloor(D_\mu(\alpha)+\varepsilon)} 2^{\eta' j \varepsilon} 2^{-\delta \lfloor \eta' j \rfloor(\delta-\varepsilon)} < +\infty.$$

Hence $\dim F \leq \dfrac{D_\mu(\alpha) + 2\varepsilon}{\delta}$. Since $\delta \geq 1$, $\dim F \leq \dfrac{D_\mu(\alpha)}{\delta} + 2\varepsilon$, and (39) is proved. □

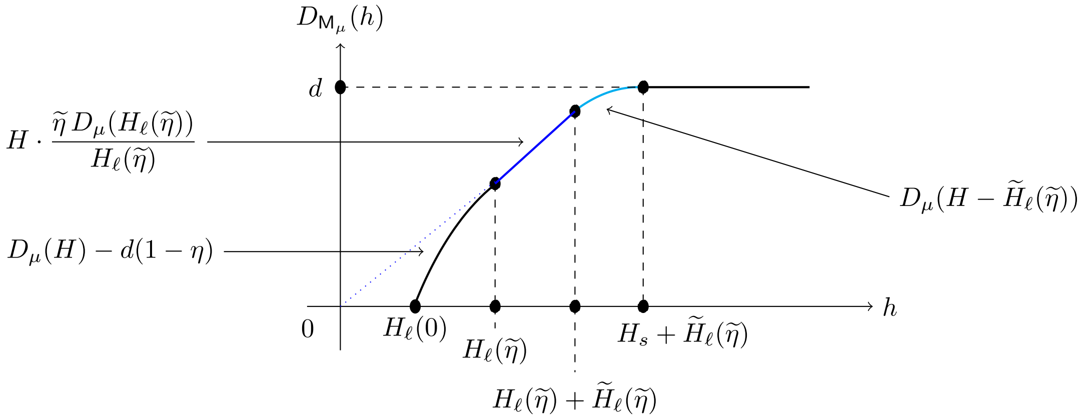

FIGURE 14. The mapping $H \mapsto D(H)$

6.3. **Simplification of the upper bound formula (37) of Section 6.2.** Summarizing the previous information, we already established that, with probability 1, the domain of the singularity spectrum D_{M_μ} of M_μ is included in $[H_\ell(\eta_\ell), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$ (Lemma 4 and Section 6.1), and for all $H \in [H_\ell(\eta_\ell), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$ we found the upper bound $D(H)$ for $D_{\mathsf{M}_\mu}(H)$ in Proposition 12.

Formula (37) defining $D(H)$ is useless for the moment for at least two reasons: it is not tractable as it it is written, and we ignore yet whether this upper bound is optimal. In this section we investigate the optimization problem raised by the expression of $D(H)$ and prove the following proposition, which yields the upper bound announced in Theorem 2 for the increasing part of D_{M_μ}.

Proposition 13. *For every $H \in [H_\ell(\eta_\ell), H_s + \widetilde{H}_\ell(\widetilde{\eta})]$, one has*

$$D(H) = \begin{cases} D_\mu(H) - d(1-\eta) & \text{if } \quad H_\ell(\eta_\ell) \le H \le H_\ell(\widetilde{\eta}), \\ H \cdot \dfrac{\widetilde{\eta}\, D_\mu(H_\ell(\widetilde{\eta}))}{H_\ell(\widetilde{\eta})} & \text{if } \quad H_\ell(\widetilde{\eta}) \le H \le H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), \\ D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta})) & \text{if } \quad H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}) \le H \le H_s + \widetilde{H}_\ell(\widetilde{\eta}). \end{cases}$$

In particular $H \longmapsto D(H)$ is strictly increasing on the interval $[H_\ell(\eta_\ell), H_s + \widetilde{H}_\ell(\widetilde{\eta}))$, $D(H_\ell(\eta_\ell)) = D_\mu(H_\ell(\eta_\ell)) - d(1-\eta)$ and $D(H_s + \widetilde{H}_\ell(\widetilde{\eta})) = d$.

Remark 2. *For every $H \in (H_\ell(\eta_\ell), H_s + \widetilde{H}_\ell(\widetilde{\eta})]$, we are going to show that $D(H) = D(H,0)$ and there exists a unique triplet $(\alpha_H, \eta_H, \delta_H)$ which realizes the supremum in $D(H,0)$. Lemma 6 is particularly important, since it describes the precise scenario leading to the optimal upper and lower bounds for the dimensions of the level sets. In particular, Lemma 6 distinguishes this triplet, which yields the simplified formula and is key for the next Section 7, where we prove that the upper bound $D(H)$ is a lower bound for $D_{\mathsf{M}_\mu}(H)$ as well, when $H \in [H_\ell(\eta_\ell), H_s + \widetilde{H}_\ell(\widetilde{\eta})]$.*

The rest of the section is less essential and more technical, since it involves only computations based on Legendre transforms.

For the proof of Proposition 13, we will use the following facts:

Fact 1: By Lemma 1, the mapping $\alpha \mapsto \frac{D_\mu(\alpha)}{\alpha+\widetilde{H}_\ell(\widetilde{\eta})}$ is increasing when $\alpha \le H_\ell(\widetilde{\eta})$, decreasing when $\alpha \ge H_\ell(\widetilde{\eta})$.

Fact 2: For all $\eta' \in [\eta_\ell, \eta] \setminus \{0\}$, one has $\eta' = \frac{H_\ell(\eta')+\widetilde{H}_\ell(\eta')}{H_\ell(\eta')}$.

Fact 3: For all $\eta' \in [\eta_\ell, \eta]$, one has $D_\mu(H_\ell(\eta')) = \dfrac{d(1-\eta)}{1-\eta'}$.

Proof of Proposition 13. We start with some direct observations.

Observe first that $D(H,\varepsilon)$ and $D(H)$ are non-decreasing mappings with respect to the variable H. Hence we only need to deal with $H > H_\ell(\eta_\ell)$, since it will follow from our computations that $D(H_\ell(\eta_\ell)) = \lim_{H\to H_\ell(\eta_\ell)^+} D(H) = D_\mu(H_\ell(\eta_\ell)) - d(1-\eta)$.

In addition, for all $\eta' \in [\eta_r, \eta]\setminus\{0\}$, one has $\widetilde{H}_r(\eta') \ge \widetilde{H}_\ell(\eta) \ge \widetilde{H}_\ell(\widetilde{\eta})$. Hence it is enough to consider $\widetilde{H}_\ell(\eta')$ and $\eta' \in [\eta_\ell, \eta] \setminus \{0\}$ to obtain the greatest upper bound in $D(H,\varepsilon)$. One deduces that (36) reduces to

$$D(H,\varepsilon) = \sup\left\{ \frac{D_\mu(\alpha)}{\delta} : \begin{cases} \alpha \in [H_{\min}, H_{\max}], \quad \eta' \in [\eta_\ell, \eta] \setminus \{0\}, \\ 1 \le \delta \le 1/\eta', \quad \dfrac{\alpha + \widetilde{H}_\ell(\eta')}{\delta} \le H + \varepsilon \end{cases} \right\}.$$

We call $\mathcal{P}_{H,\varepsilon}$ the domain of admissible values appearing in the right hand-side above, so that

$$D(H,\varepsilon) = \sup\left\{ \frac{D_\mu(\alpha)}{\delta} : (\alpha, \eta', \delta) \in \mathcal{P}_{H,\varepsilon} \right\}.$$

One starts by finding the expected lower bounds for $D(H)$.

Lemma 6. (1) *If $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_s + \widetilde{H}_\ell(\widetilde{\eta}))$, the triplet*

$$\big(\alpha_H := H - \widetilde{H}_\ell(\widetilde{\eta}), \widetilde{\eta}, 1\big) \tag{41}$$

belongs to the domain $\mathcal{P}_{H,0}$, and

$$D(H) \ge \frac{D_\mu(\alpha_H)}{1} = D_\mu\big(H - \widetilde{H}_\ell(\widetilde{\eta})\big). \tag{42}$$

(2) *If $H \in [H_\ell(\widetilde{\eta}), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}))$, the triplet*

$$\left(H_\ell(\widetilde{\eta}), \widetilde{\eta}, \frac{H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})}{H_\ell(\widetilde{\eta})} \right) \tag{43}$$

belongs to the domain $\mathcal{P}_{H,0}$ and

$$D(H) \ge H \frac{D_\mu(H_\ell(\widetilde{\eta}))}{H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})} = H \cdot \frac{\widetilde{\eta} \cdot D_\mu(H_\ell(\widetilde{\eta}))}{H_\ell(\widetilde{\eta})}. \tag{44}$$

(3) *If $H \in (H_\ell(\eta_\ell), H_\ell(\widetilde{\eta}))$, let $\eta_H \in (\eta_\ell, \widetilde{\eta})$ be the unique real number such that*

$$H = H_\ell(\eta_H). \tag{45}$$

The triplet

$$(H_\ell(\eta_H), \eta_H, 1/\eta_H) \tag{46}$$

belongs to $\mathcal{P}_{H,0}$ *and*

$$D(H) \geq \frac{D_\mu(H_\ell(\eta_H))}{1/\eta_H} = D_\mu(H) - d(1-\eta). \tag{47}$$

Proof. Observe that $\mathcal{P}_{H,0} \subset \bigcap_{\varepsilon>0} \mathcal{P}_{H,\varepsilon}$. So, if a triplet (α, η', δ) belongs to $\mathcal{P}_{H,0}$, $D(H,\varepsilon) \geq \frac{D_\mu(\alpha)}{\delta}$ for every ε, and one necessarily has $D(H) \geq \frac{D_\mu(\alpha)}{\delta}$.

The fact that the three triplets belong to the associated domains $\mathcal{P}_{H,0}$ is a simple calculation.

Part (1) is immediate.

Part (2) follows from Fact 2.

Concerning Part (3), one observes that

$$\begin{aligned} \frac{D_\mu(H_\ell(\eta_H))}{1/\eta_H} &= \eta_H \frac{d(1-\eta)}{1-\eta_H} = \frac{d(1-\eta)}{1-\eta_H} - d(1-\eta) \\ &= D_\mu(H_\ell(\eta_H)) - d(1-\eta) = D_\mu(H) - d(1-\eta). \end{aligned}$$

□

From the above lower bounds, one deduces that $D(H) = \lim_{\varepsilon\to 0^+} D(H,\varepsilon) = D(H,0)$. Indeed, take any $H \in (H_\ell(\eta_\ell), H_s + \widetilde{H}_\ell(\widetilde{\eta})]$. Obviously $d \geq D(H,\varepsilon) \geq D(H) > 0$, for every $\varepsilon \geq 0$. Recalling that $D(H,\varepsilon)$ is a supremum of quantities $\frac{D_\mu(\alpha)}{\delta}$ and $D_\mu(\alpha) \leq d$, it is enough to consider triplets of the form (α, η', δ) with $\delta \leq d/D(H,\varepsilon)$. As a consequence, in this range of triplets, recalling the constraint $\dfrac{\alpha + \widetilde{H}_\ell(\eta')}{\delta} \leq H + \varepsilon$, the quantity $\widetilde{H}_\ell(\eta')$ must be bounded to contribute to the value of $D(H,\varepsilon)$. This means that η' is bounded from below by some positive constant depending on H only.

We know now that the domain of suitable parameters leading to $D(H,\varepsilon)$ can be chosen to be compact, and independent of $\varepsilon > 0$. The continuity of all the functions involved in the domain $\mathcal{P}_{H,\varepsilon}$, $\varepsilon \geq 0$, allows us to conclude that $D(H,0) = \lim_{\varepsilon\to 0^+} D(H,\varepsilon)$, and

$$D(H) = D(H,0) = \max \left\{ \frac{D_\mu(\alpha)}{\delta} : \begin{cases} \exists\, \alpha \in [H_{\min}, H_{\max}],\ \exists\, \eta' \in [\eta_\ell, \eta] \setminus \{0\}, \\ 1 \leq \delta \leq 1/\eta', \quad \dfrac{\alpha + \widetilde{H}_\ell(\eta')}{\delta} \leq H \end{cases} \right\}, \tag{48}$$

where we know that the maximum is effectively reached (it is not only a supremum as in formula (36)).

Now let us make some general remarks on $D(H,0)$.

Suppose the maximum in (48) is realized at some triplet (α, η', δ). Observe that if α were strictly greater than H_s, one could improve the bound $D_\mu(\alpha)/\delta$ by replacing α by H_s and not changing the value of the other parameters. This contradicts the maximality of $D_\mu(\alpha)/\delta$. As a conclusion, $\alpha \leq H_s$.

Suppose now that $\dfrac{\alpha + \widetilde{H}_\ell(\eta')}{\delta} < H$. Then necessarily $\delta = 1$, otherwise one could improve the upper bound $\frac{D_\mu(\alpha)}{\delta}$ by slightly decreasing δ. When $\delta = 1$, one has $\alpha + \widetilde{H}_\ell(\eta') < H \leq H_s + \widetilde{H}_\ell(\widetilde{\eta})$. Observe that it is necessary to have $\alpha < H_s$, since $\widetilde{H}_\ell(\eta') \geq \widetilde{H}_\ell(\widetilde{\eta})$. By taking $\alpha' \in (\alpha, H_s)$ still satisfying $\alpha' + \widetilde{H}_\ell(\eta') < H$, one gets a larger value for $\frac{D_\mu(\alpha')}{\delta}$, which contradicts again the maximality of $\frac{D_\mu(\alpha)}{\delta}$. So, for the optimal triplet, one has necessarily the equality

$$\frac{\alpha+\widetilde{H}_\ell(\eta')}{\delta}=H. \tag{49}$$

Finally, observe that (49) implies that

$$D(H,0)\le H\cdot\max\left\{\frac{D_\mu(\alpha)}{\alpha+\widetilde{H}_\ell(\eta')}:\begin{cases}\alpha\in[H_{\min},H_{\max}],\ \eta'\in[\eta_\ell,\eta]\setminus\{0\},\\ \exists\ 1\le\delta\le 1/\eta',\ \dfrac{\alpha+\widetilde{H}_\ell(\eta')}{\delta}=H\end{cases}\right\}. \tag{50}$$

We now distinguish three cases.

- **First case:** $H\in(H_\ell(\widetilde{\eta})+\widetilde{H}_\ell(\widetilde{\eta}),H_s+\widetilde{H}_\ell(\widetilde{\eta}))$.

  Recall the triplet (41), which satisfies (49).
  If $\alpha>\alpha_H$, Fact 1 yields

$$H\frac{D_\mu(\alpha)}{\alpha+\widetilde{H}_\ell(\eta')}\le H\frac{D_\mu(\alpha)}{\alpha+\widetilde{H}_\ell(\widetilde{\eta})}\le H\frac{D_\mu(\alpha_H)}{\alpha_H+\widetilde{H}_\ell(\widetilde{\eta})}=D_\mu(\alpha_H)$$

  If $\alpha<\alpha_H$, we have $D_\mu(\alpha)/\delta<D_\mu(\alpha_H)/1=D_\mu(\alpha_H)$.
  Consequently, the maximum is reached necessarily at $(\alpha_H,\widetilde{\eta},1)$, and it equals (42).

- **Second case:** $H\in[H_\ell(\widetilde{\eta}),H_\ell(\widetilde{\eta})+\widetilde{H}_\ell(\widetilde{\eta}))$.

  For any $\alpha\in[H_{\min},H_{\max}]$ and any η', by Fact 1 we have

$$H\frac{D_\mu(\alpha)}{\alpha+\widetilde{H}_\ell(\eta')}\le H\frac{D_\mu(\alpha)}{\alpha+\widetilde{H}_\ell(\widetilde{\eta})}\le H\frac{D_\mu(H_\ell(\widetilde{\eta}))}{H_\ell(\widetilde{\eta})+\widetilde{H}_\ell(\widetilde{\eta})},$$

which is the value obtained in (44) with the triplet (43), which satisfies (49).

- **Third case:** $H\in(H_\ell(\eta_\ell),H_\ell(\widetilde{\eta}))$.

The optimization process here is more intricate. Indeed, in the first two cases, only one amongst the three parameters involved in the optimal triplet depends on H. In this range of local dimensions, the three parameters of the optimal triplet are functions of H.

Lemma 7. *When $H\in(H_\ell(\eta_\ell),H_\ell(\widetilde{\eta}))$, the value of $D(H,0)$ is reached at some triplet $(\alpha,\eta',1/\eta')$, for some $\eta'\in[\eta_\ell,\widetilde{\eta})\setminus\{0\}$.*

Proof. Suppose that $D(H,0)$ is reached at some triplet (α,η',δ), with $1<\delta<1/\eta'$.

If $\eta'\neq\widetilde{\eta}$, we can perturb slightly (α,η',δ) into a new triplet $(\alpha'',\eta'',\delta'')$ such that $\alpha''\ge\alpha$, $\widetilde{H}_\ell(\eta'')<\widetilde{H}_\ell(\eta')$, $1<\delta''<\delta'$ and $\delta''<1/\eta''$, and such that (49) still holds for the new triplet. This contradicts the maximality of $D_\mu(\alpha)/\delta$. One deduces that $\eta'=\widetilde{\eta}$, and the optimal triplet is in fact $(\alpha,\widetilde{\eta},\delta)$ with $1<\delta<1/\widetilde{\eta}$.

Now, the constraints (49) and $H<H_\ell(\widetilde{\eta})$ imply that $\alpha<H_\ell(\widetilde{\eta})$, since $H_\ell(\widetilde{\eta})=\widetilde{\eta}(H_\ell(\widetilde{\eta})+\widetilde{H}_\ell(\widetilde{\eta}))$. Then, optimizing in α the ratio $D_\mu(\alpha)/\delta$ under the constraint (49) amounts to studying the mapping

$$\alpha\mapsto H\frac{D_\mu(\alpha)}{\alpha+\widetilde{H}_\ell(\widetilde{\eta})}$$

over $[\alpha_{\min}, H_\ell(\widetilde{\eta}))$. Fact 1 ensures that it is increasing. If the optimal triplet is $(\alpha, \widetilde{\eta}, \delta)$ with $1 < \delta < 1/\widetilde{\eta}$, we can slightly increase the values of α and δ in α' and δ', preserve (49), $\alpha' < H_\ell(\widetilde{\eta})$ and $1 < \delta' < 1/\widetilde{\eta}$, and get the contradiction $H\frac{D_\mu(\alpha')}{\alpha'+\widetilde{H}_\ell(\widetilde{\eta})} > D(H,0) = \frac{D_\mu(\alpha)}{\alpha+\widetilde{H}_\ell(\widetilde{\eta})}$.

Next suppose that $D(H,0)$ is reached at some triplet $(\alpha, \eta', 1)$, for some $\eta' \in [\eta_\ell, \eta] \setminus \{0\}$. This imposes $H = \alpha + \widetilde{H}_\ell(\eta')$, and one looks for the maximum of $D_\mu(\alpha)$. Necessarily $\alpha \in I = [H_\ell(\eta_\ell) - \widetilde{H}_\ell(\eta'), H_\ell(\widetilde{\eta}) - \widetilde{H}_\ell(\eta'))$, which is an interval included in the increasing part of the spectrum D_μ. However D_μ does not reach a maximum over I, hence a new contradiction.

The previous cases leading to a contradiction, we deduce that necessarily $D(H,0)$ is reached at some triplet $(\alpha, \eta', 1/\eta')$, for some $\eta' \in [\eta_\ell, \eta] \setminus \{0\}$. Hence we have

$$H = \eta'(\alpha + \widetilde{H}_\ell(\eta')) \text{ and } D(H,0) = \eta' D_\mu(\alpha), \tag{51}$$

with $\eta' \in [\eta_\ell, \eta] \setminus \{0\}$.

Further, we prove that it is enough to consider $\eta' \in [\eta_\ell, \widetilde{\eta})$.

Suppose that (51) holds for some $\eta' \in [\widetilde{\eta}, \eta]$. Since $H < \widetilde{\eta}(H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}))$ and $\widetilde{H}_\ell(\eta') \geq \widetilde{H}_\ell(\widetilde{\eta})$, we have $\alpha < H_\ell(\widetilde{\eta})$. Consequently, there exists $\alpha' \in (\alpha, H_\ell(\widetilde{\eta}))$ such that $H = \widetilde{\eta}(\alpha' + \widetilde{H}_\ell(\widetilde{\eta}))$. For the triplet $(\alpha', \widetilde{\eta}, 1/\widetilde{\eta})$, one has

$$\widetilde{\eta} D_\mu(\alpha') = H\frac{D_\mu(\alpha')}{\alpha' + \widetilde{H}_\ell(\widetilde{\eta})} > H\frac{D_\mu(\alpha)}{\alpha + \widetilde{H}_\ell(\widetilde{\eta})},$$

where we used Fact 1 and $H_\ell(\widetilde{\eta}) > \alpha' > \alpha$. Finally, since $\widetilde{H}_\ell(\widetilde{\eta}) \leq \widetilde{H}_\ell(\eta')$, one sees that

$$\widetilde{\eta} D_\mu(\alpha') > H\frac{D_\mu(\alpha)}{\alpha + \widetilde{H}_\ell(\eta')} = D(H,0),$$

which is a (last) contradiction. □

From last Lemma, the maximum $D(H,0)$ reduces to

$$D(H,0) = \max\Big\{\eta' D_\mu(\alpha) : \alpha \in [H_{\min}, H_s],\ \eta' \in [\eta_\ell, \widetilde{\eta}) \setminus \{0\},\ \eta'(\alpha + \widetilde{H}_\ell(\eta')) = H\Big\}.$$

This is standard optimization under constraints. The maximum is reached when

$$D_\mu{}'(\alpha)(\alpha + \widetilde{H}'_\ell(\eta')) + D_\mu(\alpha) = 0,$$

or equivalently when

$$\widetilde{H}'_\ell(\eta') = -\frac{D^*_\mu(D_\mu{}'(\alpha))}{D'_\mu(\alpha)}.$$

When η' is fixed, this happens if and only if $\alpha = H_\ell(\eta')$. This means that $H = \eta'(H_\ell(\eta') + \widetilde{H}_\ell(\eta'))$, so $H = H_\ell(\eta')$. This leads to the choice (45) for η', and to the triplet (46), which gives the value (47) for $D(H)$. □

Remark 3. *A key observation for the following is that actually we proved a little more than what we announced. Indeed, as a direct by-product of the proof, not only we know that when* $H \leq H_s + \widetilde{H}_\ell(\widetilde{\eta})$, $D_{\mathsf{M}_\mu}(H)(= \dim \underline{E}_{\mathsf{M}_\mu}(H)) \leq D(H)$, *we also have that*

$$\dim \underline{E}^{\leq}_{\mathsf{M}_\mu}(H) \leq D(H). \tag{52}$$

This inequality is useful in the following section.

Remark 4. *There is no chance for* $D(H)$ *to be an optimal bound in the decreasing part of the singularity spectrum of* M_μ, *since the mapping* $H \mapsto D(H)$ *is non-decreasing.*

7. Lower bound for the singularity spectrum

For each admissible local dimension H, we are going to exhibit an auxiliary probability measure ν (which depends on H) such that $\nu\big(\underline{E}_{\mathsf{M}_\mu}(H)\big) = 1$, and such that the dimension of ν equals the announced value for $D_{\mathsf{M}_\mu}(H)$: i.e. $D(H)$ when $H \leq H_s + \widetilde{H}_\ell(\widetilde{\eta})$, and $D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))$ when $H > H_s + \widetilde{H}_\ell(\widetilde{\eta})$.

These auxiliary measures do not always have the same nature, depending on H. They can be taken as a Gibbs measures when $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta}))$, but not for the other values of H.

We introduce two families of measures in Section 7.1, whose properties are established in Section 7.5. Then we obtain the sharp lower bound for D_{M_μ} in Sections 7.2 to 7.4.

7.1. Two families of measures.

The first family will be used to obtain a sharp lower bound for $D_{\mathsf{M}_\mu}(H)$ when $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$. It is based on the following result.

Recall Proposition 7 in which the event $\mathscr{C}(N, J, W)$ is defined.

Theorem 5. *With probability 1, for all $\alpha \in [H_{\min}, H_{\max}]$, there exists an exact dimensional Borel probability measure ν_α of Hausdorff dimension $D_\mu(\alpha)$ supported on $\widetilde{E}_\mu(\alpha)$ (i.e. $\nu_\alpha(\widetilde{E}_\mu(\alpha)) = 1$), such that:*

(1) *for all $\delta > 1$ we have*

$$\nu_\alpha\left(\bigcap_{J\geq 1}\ \bigcup_{j\geq J}\ \bigcup_{w\in\mathcal{S}_j(\eta)} B\big(x_w, (2\cdot 2^{-\lfloor \eta j\rfloor})^{\delta}\big)\right) = 0.$$

(2) *for all $N > 1/\eta$, for ν_α-almost every x, there exists an integer $J_{N,\alpha,x} \geq 1$ such that for all $J \geq J_{N,\alpha,x}$, the event $\mathscr{C}(N, J, x_{|J})$ is not realized.*

Theorem 5 is proved at the end of this Section (Section 7.5). Observe that the result holds simultaneously for all $\alpha \in [H_{\min}, H_{\max}]$.

In the first item, the limsup set contains those points $x \in [0,1]^d$ that are very close to the surviving coefficients, i.e. those x satisfying for some $\delta > 1$

$$|x - x_w| < 2\cdot 2^{-\lfloor |w|\eta\rfloor \delta}$$

for infinitely many surviving words w. We know by the covering Lemma 2 that when $\delta < 1$, every $x \in [0,1]^d$ satisfies the last inequality infinitely many times. Part (1) of Theorem 5 states that this is no longer true when $\delta > 1$, in the sense that the ν_α-measure of these sets of points is always 0.

The second part of the Theorem is technical, and used in the proofs below.

The second family of measures allows us to compute the value of $D_{\mathsf{M}_\mu}(H)$ when $H \in [H_\ell(\eta_\ell), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})]$. These measures are built thanks to the theory of heterogeneous ubiquity theory, developed in [5, 6, 9, 18], whose main results can be resumed as follows.

Theorem 6. *Let $\mathcal{F} = ((x_n, r_n))_{n\geq 1}$ be a sequence of couples such that $(x_n)_{n\geq 1}$ is a sequence of points in $[0,1]^d$, and $(r_n)_{n\geq 1}$ is a positive sequence converging to zero. Assume that*

$$(0,1)^d \subset \limsup_{n\to+\infty} B(x_n, r_n). \tag{53}$$

Let $\alpha \in (H_{\min}, H_{\max})$. Recall that the Gibbs measure μ_α was defined in Proposition 2(4). For every $\delta \geq 1$ and any positive sequence $\widetilde{\beta} := (\widetilde{\beta}_n)_{n\geq 1}$ converging to zero, define

$$U_\mu\big(\alpha, \delta, \mathcal{F}, \widetilde{\beta}\big) := \bigcap_{N\geq 1} \bigcup_{\substack{n\geq N: \\ (r_n)^{\alpha+\widetilde{\beta}_n} \leq \mu_\alpha(B(x_n,r_n)) \leq (r_n)^{\alpha-\widetilde{\beta}_n}}} B\Big(x_n, (r_n)^\delta\Big). \tag{54}$$

For every $\delta \geq 1$, there exists a Borel probability measure $\nu_{\alpha,\delta}$ and a positive sequence $\widetilde{\beta} := (\widetilde{\beta}_n)_{n\geq 1}$ converging to zero such that

$$\nu_{\alpha,\delta}\left(U_\mu\big(\alpha, \delta, \mathcal{F}, \widetilde{\beta}\big)\right) = 1,$$

and $\nu_{\alpha,\delta}(E) = 0$ for every set E such that $\dim E < D_\mu(\alpha)/\delta$.

In particular, one has

$$\dim U_\mu\big(\alpha, \delta, \mathcal{F}, \widetilde{\beta}\big) \geq \dim \nu_{\alpha,\delta} \geq \frac{D_\mu(\alpha)}{\delta}.$$

Moreover, if $(\alpha^{(p)}, \mathcal{F}^{(p)}, \delta^{(p)})_{p\geq 1}$ stands for a sequence of parameters satisfying the above conditions, there exists a measure $\widetilde{\nu}$ and sequences $\widetilde{\beta}^{(p)} := (\widetilde{\beta}^{(p)}_n)_{p\geq 1, n\geq 1}$ converging to zero satisfying

$$\widetilde{\nu}\left(\bigcap_{p\geq 1} U_\mu\big(\alpha^{(p)}, \delta^{(p)}, \mathcal{F}^{(p)}, \widetilde{\beta}^{(p)}\big)\right) = 1,$$

and $\widetilde{\nu}(E) = 0$ for every set E such that $\dim E < \inf_{p\geq 1} \dfrac{D_\mu(\alpha^{(p)})}{\delta^{(p)}}$.

In particular,

$$\dim \bigcap_{p\geq 1} U_\mu\big(\alpha^{(p)}, \delta^{(p)}, \mathcal{F}^{(p)}, \widetilde{\beta}^{(p)}\big) \geq \inf_{p\geq 1} \frac{D_\mu(\alpha^{(p)})}{\delta^{(p)}}.$$

The last property is due to the fact that the sets $U_\mu\big(\alpha, \delta, \mathcal{F}, \widetilde{\beta}\big)$ enjoy the large intersection property, i.e. when intersecting a countable number of them, the Hausdorff dimension of the resulting set is at least the infimum of all the dimensions, see [9, 18].

We are going to apply Theorem 6 with specific families $(x_n, r_n)_{n\geq 1}$:

Let $\mathcal{D}_\ell$ be a dense countable subset of $[\eta_\ell, \eta] \setminus \{0\}$, such that $\widetilde{\eta} \in \mathcal{D}_\ell$. With probability 1, for all $\eta' \in \mathcal{D}_\ell$, Proposition 6 proves the existence of words $w \in \mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon_j^3)$, for j large enough, for all $W \in \Sigma_{\lfloor \eta' j \rfloor}$. For such a surviving word w, we set $r_w = 2 \cdot 2^{-\lfloor \eta' j \rfloor}$. The sequence of couples (x_w, r_w) obtained in this way is denoted

$$\mathcal{F}_{\eta'} := \big(x_n(\eta'), r_n(\eta')\big)_{n\geq 1}$$

after being re-ordered so that the sequence of radii $(r_n(\eta'))_{n\geq 1}$ is non-increasing. By construction, the covering property (53) is satisfied for the family $\mathcal{F}'_\eta$, so that the second part of Theorem 6 can be applied with the countable number of families $\big(\mathcal{F}_{\eta'}\big)_{\eta' \in \mathcal{D}_\ell}$.

7.2. **The right part of the spectrum D_{M_μ}.**

Recall that for $H \in \big[H_\ell(\widetilde\eta)+\widetilde H_\ell(\widetilde\eta), H_{\max}+\widetilde H_\ell(\widetilde\eta)\big]$ we defined $\alpha_H = H-\widetilde H_\ell(\widetilde\eta)$. Also recall Theorem 5 in which the measure ν_{α_H} is defined.

Lemma 8. *With probability 1, for all $H \in [H_\ell(\widetilde\eta)+\widetilde H_\ell(\widetilde\eta), H_{\max}+\widetilde H_\ell(\widetilde\eta)]$, there exists a set $G_H \subset \widetilde E_\mu(\alpha_H)$ such that:*

- $\nu_{\alpha_H}(G_H)=1$
- *for all $x\in G_H$, for all integers $N>1/\eta$, there exists $J_N(x)\ge 1$ such that for all $J\ge J_N(x)$, for all $J\le j< J/(\widetilde\eta_{N,-1})$, one has*

$$\bigcup_{W\in\mathcal N_J(x)} \mathcal S_j(\eta,W)=\emptyset.$$

Proof. Notice that for all $N\ge 1$ we have $1/\widetilde\eta_{N,-1}<1/\eta$. Assume towards contradiction that with positive probability, there exists $H\in[H_\ell(\widetilde\eta)+\widetilde H_\ell(\widetilde\eta), H_{\max}+\widetilde H_\ell(\widetilde\eta)]$, a set F_H of positive ν_{α_H}-measure, and $N\ge 1$ such that

$$F_H\subset \bigcap_{J\ge 1}\bigcup_{j\ge J}\bigcup_{w\in\mathcal S_j(\eta)} B\big(x_w,(2\cdot 2^{-\lfloor \eta j\rfloor})^{\delta}\big)$$

for all $\delta\in(1,\widetilde\eta^{-1}\widetilde\eta_{N,-1})$. This contradicts Theorem 5.

Consequently we get that, with probability 1, for all $H\in[H_\ell(\widetilde\eta)+\widetilde H_\ell(\widetilde\eta), H_{\max}+\widetilde H_\ell(\widetilde\eta)]$ a set G_H such that the two items of the statement hold. Moreover, G_H can be taken a subset of $\widetilde E_\mu(\alpha_H)$ since $\nu_{\alpha_H}\big(\widetilde E_\mu(\alpha_H)\big)=1$. □

Now, we prove that $D_{\mathsf{M}_\mu}(H)\ge D_\mu\big(H-\widetilde H(\widetilde\eta)\big)$.

Consider a set Ω' of probability 1 over which the conclusions of Theorem 5 and Lemma 8 hold true.

Lemma 9. *For all $\omega\in\Omega'$, for all $H\in[H_\ell(\widetilde\eta)+\widetilde H_\ell(\widetilde\eta), H_{\max}+\widetilde H_\ell(\widetilde\eta)]$, one has $G_H\subset \underline{E}_{\mathsf{M}_\mu}(H)$.*

Proof. Take $\omega\in\Omega'$, and fix an integer $N>1/\eta$. Fix $x\in G_H$. We focus on the values of $M_\mu(I_J(x))$. We analyse the values of $\mu(I_w)$ when $w\in\mathcal S_j(\eta)$ is a surviving vertex such that I_w is included in the neighborhood $\mathcal N_J(x)$ of x. For this, we apply the quasi-Bernoulli property (8) to get

$$\mu(I_w)\approx \mu(I_{w_{|J}})\mu(I_{\sigma^J w}),$$

and we use some of the inequalities we proved above.

First, combining part (2) of Theorem 5 and Lemma 8, for all J large enough, for all $W\in\mathcal N_J(x)$, one has:

- for all $J\le j\le J/\widetilde\eta_{N,-1}$, $\bigcup_{W\in\mathcal N_J(x)}\mathcal S_j(\eta,W)=\emptyset$;
- for all $-1\le k\le N-1$, for all $J/\widetilde\eta_{N,k}\le j\le J/\widetilde\eta_{N,k+1}$, for all $w\in\mathcal S_j(\eta,W)$,
$$\mu(I_{\sigma^J w})\le 2^{-J(\widetilde H_\ell(\eta_{N,k})-\varepsilon_N)}\le 2^{-J(\eta_{N,k}^{-1}-1)\varepsilon_N}2^{-J\widetilde H_\ell(\widetilde\eta)};$$
- if $j>J/\eta_0$, for all $w\in\Sigma_j$ such that $I_w\subset I_W$, we have

$$\mu(I_{\sigma^J w})\le 2^{-J(\eta_0^{-1}-1)(H_{\min}-\epsilon_N)}\le \begin{cases} 2^{-J(\eta_0^{-1}-1)(H_{\min}-\epsilon_N)}\le 2^{-J\left(\widetilde H_\ell(\widetilde\eta)-(\eta_0^{-1}-1)\epsilon_N\right)} & \text{if } \eta_\ell=0\\ 2^{-j(\eta_\ell^{-1}-1)(H_\ell(\eta_\ell)-\epsilon_N)}\le 2^{-J\left(\widetilde H_\ell(\widetilde\eta)-(\eta_\ell^{-1}-1)\epsilon_N\right)} & \text{if } \eta_\ell>0.\end{cases}$$

Secondly, since $x \in G_H \subset \widetilde{E}_\mu(\alpha_H)$, there exists a sequence $(\widetilde{\varepsilon}_J)_{J\geq 1}$ (depending on x) tending to 0 as $J \to +\infty$ such that $x_{|J} \in \mathcal{E}_\mu(J, \alpha_H \pm \widetilde{\varepsilon}_J)$. In particular, one has for $I_W \in \mathcal{N}_J(x)$,

$$\mu(I_W) \leq 2^{-J(\alpha_H - \widetilde{\varepsilon}_J)}.$$

When $\eta_\ell = 0$, combining the previous inequalities, we get

$$\begin{aligned} \mathsf{M}_\mu(I_J(x)) &= \max\{\mu(I_w) : w \in \mathcal{S}_j(\eta, W),\ W \in \mathcal{N}_J(x)\} \\ &\leq C \cdot \max\{\mu(I_W) : I_W \in \mathcal{N}_J(x)\} \cdot \max\{\mu(I_{\sigma^J w}) : w \in \mathcal{S}_j(\eta, W),\ W\mathcal{N}_J(x))\} \\ &\leq C 2^{-J\left(\alpha_H - \widetilde{\varepsilon}_J + \widetilde{H}_\ell(\widetilde{\eta}) - (\eta_0^{-1} - 1)\varepsilon_N\right)}. \end{aligned}$$

Consequently,

$$\underline{\dim}(\mathsf{M}_\mu, x) \geq \alpha_H + \widetilde{H}_\ell(\widetilde{\eta}) - (\eta_0^{-1} - 1)\varepsilon_N = H - (\eta_0^{-1} - 1)\varepsilon_N.$$

This holds for all $N > 1/\eta$ hence $\underline{\dim}(\mathsf{M}_\mu, x) \geq H$, for every $x \in G_H$. The same estimate are true when $\eta_\ell > 0$ by replacing η_0 by η_ℓ.

On the other side, by Proposition 9 we know that $\overline{\dim}(\mathsf{M}_\mu, x) \leq \overline{\dim}(\mu, x) + \widetilde{H}_\ell(\widetilde{\eta}) = \alpha_H + \widetilde{H}_\ell(\widetilde{\eta}) = H$, hence $\underline{\dim}(\mathsf{M}_\mu, x) = H$ (in fact we obtained that $G_H \subset E_{\mathsf{M}_\mu}(H)$). □

We can now conclude. Recall that with probability 1, simultaneously for all $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$, we have $\nu_{\alpha_H}(G_H) = 1$, so one has $\dim G_H \geq D_\mu(\alpha_H) = D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))$. Finally, since $G_H \subset \underline{E}_{\mathsf{M}_\mu}(H)$, one has

$$\dim\, \underline{E}_{\mathsf{M}_\mu}(H) \geq \dim\, G_H \geq D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta})).$$

Remark 5. *Observe that the previous arguments give also a lower bound for the Hausdorff dimension of the level sets of the limit local dimension: for any $H \in [H_{\min} + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$, $\dim\, E_{\mathsf{M}_\mu}(H) \geq D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))$.*

7.3. The middle part of the spectrum D_{M_μ}.

Let $H \in [H_\ell(\widetilde{\eta}), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})]$. We apply Theorem 6 with the parameters:

- $\eta' = \widetilde{\eta}$,
- the family $\mathcal{F}_{\widetilde{\eta}} = (x_n(\widetilde{\eta}), r_n(\widetilde{\eta}))_{n\geq 1}$,
- $\alpha = H_\ell(\widetilde{\eta})$,
- $\delta = H_\ell(\widetilde{\eta})/(\widetilde{\eta} H)$ (which does belong to $[1, 1/\widetilde{\eta}]$).

There exists a sequence $\widetilde{\beta} := (\widetilde{\beta}_n)_{n\geq 1}$ and a Borel probability measure $\nu_{\alpha,\delta}$ supported on the set $U_\mu\big(H_\ell(\widetilde{\eta}), \delta, \mathcal{F}_{\widetilde{\eta}}, \widetilde{\beta}\big)$ and such that

$$\dim\, \nu_{\alpha,\delta} \geq \frac{\dim \mu_{H_\ell(\widetilde{\eta})}}{\delta} = \widetilde{\eta} H \frac{D_\mu(H_\ell(\widetilde{\eta}))}{H_\ell(\widetilde{\eta})} = D(H).$$

Lemma 10. *One has $U_\mu\big(H_\ell(\widetilde{\eta}), \delta, \mathcal{F}_{\widetilde{\eta}}, \widetilde{\beta}\big) \subset \underline{E}^{\leq}_{\mathsf{M}_\mu}(H)$.*

Proof. Let $x \in U_\mu\big(H_\ell(\widetilde{\eta}), \delta, \mathcal{F}_{\widetilde{\eta}}, \widetilde{\beta}\big)$. By definition of this limsup set, there is an increasing sequence of integers $(j_k)_{k\geq 1}$ and words $w_k \in \mathcal{S}_{j_k}(\eta) \cap \mathcal{R}_\mu(j_k, \widetilde{\eta}, H_\ell(\widetilde{\eta}) \pm \widetilde{\beta}_{j_k}) \cap \mathcal{T}_{\mu,\ell}(j_k, \widetilde{\eta}, \varepsilon_{j_k}^3)$ such that for each $k \geq 1$, $x \in B\big(x_{w_k}, (2 \cdot 2^{-\lfloor \widetilde{\eta} j_k \rfloor})^\delta\big)$. In other words, w_k satisfies

$$\left\{ \begin{array}{lllll} 2^{-\lfloor \widetilde{\eta} j_k \rfloor (H_\ell(\widetilde{\eta}) + \widetilde{\beta}_{\lfloor \widetilde{\eta} j_k \rfloor})} & \leq & \mu(I_{w_{k|\lfloor \widetilde{\eta} j_k \rfloor}}) & \leq & 2^{-\lfloor \widetilde{\eta} j_k \rfloor (H_\ell(\widetilde{\eta}) - \widetilde{\beta}_{\lfloor \widetilde{\eta} j_k \rfloor})} \\ 2^{-(j_k - \lfloor \widetilde{\eta} j_k \rfloor)(H_\ell(\widetilde{\eta}) + \varepsilon_{j_k}^3)} & \leq & \mu(I_{\sigma^{\lfloor \widetilde{\eta} j_k \rfloor} w_k}) & \leq & 2^{-(j_k - \lfloor \widetilde{\eta} j_k \rfloor)(H_\ell(\widetilde{\eta}) - \varepsilon_{j_k}^3)}. \end{array} \right.$$

Consider for each $k \geq 1$ the largest integer J_k such that $2^{-J_k} \geq (2 \cdot 2^{-\lfloor \widetilde{\eta} j_k \rfloor})^\delta$. With such a choice, one has $I_{w_k} \subset \mathcal{N}_{J_k}(x)$, so that $\mathsf{M}_{J_k}(x) \geq \mu(I_{w_k})$. Since $J_k = \delta \widetilde{\eta} j_k + o(1/k)$, one concludes that

$$\mathsf{M}_{J_k}(x) \geq \mu(I_{w_k}) \geq C^{-1} 2^{-\lfloor \widetilde{\eta} j_k \rfloor (H_\ell(\widetilde{\eta}) + \widetilde{\beta}_{\lfloor \widetilde{\eta} j_k \rfloor})} 2^{-(j_k - \lfloor \widetilde{\eta} j_k \rfloor)(H_\ell(\widetilde{\eta}) + \varepsilon^3_{j_k})} \geq 2^{-\frac{J_k}{\delta}(H_\ell(\widetilde{\eta}) + \widehat{\beta}_k)},$$

for some sequence $\widehat{\beta}_k$ converging to 0 as $k \to +\infty$. Taking the liminf as $k \to +\infty$ on both sides yields $\underline{\dim}(\mathsf{M}_\mu, x) \leq H$. □

From the previous lemma, we deduce that

$$\nu_{\alpha,\delta}\Big(E^{\leq}_{\mathsf{M}_\mu}(H)\Big) \geq \nu_{\alpha,\delta}\big(U_\mu\big(H_\ell(\widetilde{\eta}), \delta, \mathcal{F}_{\widetilde{\eta}}, \widetilde{\beta}\big) > 0.$$

But for any $H' < H$, applying formula (52) of Remark 3, one knows that $\dim E^{\leq}_{\mathsf{M}_\mu}(H') \leq D(H') < D(H)$. In addition, Theorem 6 asserts that $\nu_{\alpha,\delta}\big(E^{\leq}_{\mathsf{M}_\mu}(H')\big) = 0$. We deduce that

$$\nu_{\alpha,\delta}\left(E^{\leq}_{\mathsf{M}_\mu}(H) \setminus \bigcup_{n\geq 1} \underline{E}^{\leq}_{\mathsf{M}_\mu}(H - 1/n)\right) = 1.$$

Since $\underline{E}_{\mathsf{M}_\mu}(H) = E^{\leq}_{\mathsf{M}_\mu}(H) \setminus \bigcup_{n\geq 1} \underline{E}^{\leq}_{\mathsf{M}_\mu}(H-1/n)$, we conclude that $\nu_{\alpha,\delta}\big(\underline{E}_{\mathsf{M}_\mu}(H)\big) = 1$, i.e. $D_{\mathsf{M}_\mu}(H) \geq D(H)$. Since we already proved the converse inequality, equality holds.

7.4. **The left part of the spectrum D_{M_μ}.**

Let $H \in \big[H_\ell(\eta_\ell), H_\ell(\widetilde{\eta})\big)$. Recall the definition (45) of η_H when $H \in (H_\ell(\eta_\ell), H_\ell(\widetilde{\eta}))$. When $H = H_\ell(\eta_\ell)$, we set $\eta_{H_\ell(\eta_\ell)} = \eta_\ell$. Let $(H^{(p)})_{p\geq 1}$ be a decreasing sequence of real numbers in the interval $\big(H_\ell(\eta_\ell), H_\ell(\widetilde{\eta})\big)$ converging to H, with the constraint that $\eta_{H^{(p)}} \in \mathcal{D}_\ell$. For each $p \geq 1$, we consider any sequence $(\delta^{(p)})_{p\geq 1}$ converging to $1/\eta_H$ as $p \to +\infty$, and such that the sequence of real numbers $\left(\dfrac{D_\mu(H^{(p)})}{\delta^{(p)}}\right)_{p\geq 1}$ is non increasing.

We apply the second part of Theorem 6: there exist a collection of positive sequences $\big(\widetilde{\beta}^{(p)} := (\widetilde{\beta}^{(p)}_n)_{n\geq 1}\big)_{p\geq 1}$ converging to 0, such that the set $\bigcap_{p\geq 1} U_\mu\big(H^{(p)}, \delta^{(p)}, \mathcal{F}_{\eta_{H^{(p)}}}, \widetilde{\beta}^{(p)}\big)$ supports a measure $\widetilde{\nu}_H$, whose dimension is greater than or equal to

$$\inf_{p\geq 1} \frac{D_\mu(H^{(p)})}{\delta^{(p)}} = \eta_H D_\mu(H) = D(H).$$

Also, similarly to what was done in Section 7.3, we can get

$$\bigcap_{p\geq 1} U_\mu\big(H^{(p)}, \delta^{(p)}, \mathcal{F}_{\eta_{H^{(p)}}}, \widetilde{\beta}^{(p)}\big) \ \subset\ \underline{E}^{\leq}_{\mathsf{M}_\mu}(H)$$

and $\widetilde{\nu}_H\left(\underline{E}^{\leq}_{\mathsf{M}_\mu}(H - 1/n)\right) = 0$ for all $n \geq 1$. This yields

$$\widetilde{\nu}_H\left(E^{\leq}_{\mathsf{M}_\mu}(H) \setminus \bigcup_{n\geq 1} \underline{E}^{\leq}_{\mathsf{M}_\mu}(H - 1/n)\right) = 1,$$

and finally that $D_{\mathsf{M}_\mu}(H) = D(H)$.

7.5. **Proof of Part (1) of Theorem 5.** If $\alpha \in (H_{\min}, H_{\max})$, we can choose for ν_α the Gibbs measure μ_α of item (4) of Proposition 2 and it is not too difficult to prove the desired property by using natural coverings because the Hausdorff dimension of μ_α is positive.

We give a construction of a measure ν_α that works for $\alpha \in \{H_{\min}, H_{\max}\}$, based on a concatenation method. It is also possible to adapt this method to get another choice for ν_α when $\alpha \in (H_{\min}, H_{\max})$ (as explained at the end of the proof).

Due to Lemma 3 we can fix a positive sequence $(\varepsilon_j)_{j\geq 1}$ converging to 0, such that, with probability 1, for j large enough, for all $W \in \Sigma_{\lfloor \eta j \rfloor}$,

$$\#\mathcal{S}_j(\eta, W) \leq 2^{\eta j \varepsilon_j}. \tag{55}$$

Without loss of generality we can assume that $(\varepsilon_j)_{j\geq 1}$ is non-increasing, $1/j \leq \varepsilon_j \leq d$ for all $j \geq 1$, and $\varepsilon_{j+1}/\varepsilon_j$ converges to 1 as $j \to +\infty$.

We treat the case $H_{\min}$, the case $H_{\max}$ is identical.

7.5.1. *Construction of the measure $\nu_{H_{\min}}$ and an associated Cantor set $\mathcal{C}_{H_{\min}}$.* Let $(q_k)_{k\geq 1}$ be an increasing sequence of real numbers, and let $\alpha_k := \tau'_\mu(q_k)$. We assume that :

- If $D_\mu(H_{\min}) = 0$, we choose q_k such that $D_\mu(\alpha_k) = \sqrt{\varepsilon_k}$, for every $k \geq 1$. Hence (q_k) satisfies $\lim_{k\to+\infty} \alpha_k = H_{\min}$ and $\lim_{k\to\infty} D_\mu(\alpha_k) = 0$.
- If $D_\mu(H_{\min}) > 0$, we choose q_k such that $\lim_{k\to+\infty} \alpha_k = H_{\min}$ and one also has $\lim_{k\to+\infty} D_\mu(\alpha_k) = D_\mu(H_{\min})$.

In all cases, by construction we have

$$|D_\mu(\alpha_{k+1}) - D_\mu(\alpha_k)| = \theta_k \sqrt{\varepsilon_k} \leq \theta_k D_\mu(\alpha_k), \tag{56}$$

with $\lim_{k\to+\infty} \theta_k = 0$.

We start by selecting some intervals words at which μ and μ_{α_k} have the desired scaling properties.

Recall that by item (4) of Proposition 2, the measure Gibbs μ_{α_k} satisfies

$$\mu_{\alpha_k}\big(\widetilde{E}_\mu(\alpha_k)\big) = \mu_{\alpha_k}\big(\widetilde{E}\mu_{\alpha_k}(D_\mu(\alpha_k))\big) = 1.$$

Hence, for all $k \geq 1$, the sets

$$\begin{aligned} \mathcal{A}_J^k &= \big\{W \in \Sigma_J : \forall W' \in \mathcal{N}(W),\ W' \in \mathcal{E}_\mu(J, \alpha_k \pm \varepsilon_k)\big\} \\ \text{and} \quad \mathcal{B}_J^k &= \big\{W \in \Sigma_J : \forall W' \in \mathcal{N}(W),\ W' \in \mathcal{E}_{\mu_{\alpha_k}}(J, D_\mu(\alpha_k) \pm \varepsilon_k)\big\} \end{aligned}$$

satisfy $\lim_{J\to+\infty} \mu_{\alpha_k}(\mathcal{A}_J^k) = \lim_{J\to+\infty} \mu_{\alpha_k}(\mathcal{B}_J^k) = 1$. Up to extraction of a subsequence, one deduces that there exists an integer $J_k \in \mathbb{N}_+$ and a collection $\mathcal{W}_k$ of words of generation J_k such that the cubes I_W, $W \in \mathcal{W}_k$, are pairwise disjoint, $\sum_{W\in\mathcal{W}_k} \mu_{\alpha_k}(I_W) \geq e^{-\varepsilon_k}$, and

$$\forall W \in \mathcal{W}_k, \forall W' \in \mathcal{N}(W),\ \ W' \in \mathcal{E}_\mu(J, \alpha_k \pm \varepsilon_k) \cap \mathcal{E}_{\mu_{\alpha_k}}(J, D_\mu(\alpha_k) \pm \varepsilon_k). \tag{57}$$

Now, let $(N_k)_{k\geq 1}$ be an increasing sequence of integers such that for all $k \geq 1$,

$$\begin{aligned} \sum_{p=1}^{k-1} N_p J_p \max\big(1, \alpha_p + 2\varepsilon_p, D_\mu(\alpha_p) + 2\varepsilon_p\big) &\leq \varepsilon_k N_k J_k, \\ \frac{J_{k+1}}{N_k J_k} \max\big(1, \alpha_{k+1} + 2\varepsilon_{k+1}, D_\mu(\alpha_{k+1}) + 2\varepsilon_{k+1}\big) &\leq \varepsilon_{k+1}\alpha_k. \end{aligned} \tag{58}$$

We also define $\widetilde{J}_k = \sum_{p=1}^k N_p J_p$, which satisfies

$$N_k J_k \leq \widetilde{J}_k \leq N_k J_k(1+\varepsilon_k).$$

Then we define recursively a Cantor-like set $\mathcal{C}_{H_{\min}}$ and simultaneously a Borel probability measure $\nu_{H_{\min}}$ on $[0,1]^d$ supported on $\mathcal{C}_{H_{\min}}$. To do so, we use a construction by concatenation: the measure $\nu_{H_{\min}}$ behaves like μ_{α_k} between the generations $\widetilde{J}_{k-1}+1$ and $\widetilde{J}_k$. More precisely:

- We set $I_\emptyset = [0,1]^d$ and $\nu_{H_{\min}}([0,1]^d) = 1$,
- For every $k \geq 1$, we write $\widetilde{W}_k \in \mathcal{W}_k^{N_k}$ as $\widetilde{W}_k = W_{k,1}\cdots W_{k,N_k}$ where $W_{k,i} \in \mathcal{W}_k \subset \Sigma_{J_k}$,
- The Cantor set is

$$\mathcal{C}_{H_{\min}} = \bigcap_{k\geq 1} \bigcup_{(\widetilde{W}_1,\dots,\widetilde{W}_k)\in \mathcal{W}_1^{N_1}\times\cdots\times\mathcal{W}_k^{N_k}} I_{\widetilde{W}_1\cdots\widetilde{W}_k}$$

- The measure $\nu_{H_{\min}}$ is defined recursively as follows: for every $k \geq 1$, for every $(\widetilde{W}_1,\dots,\widetilde{W}_k) \in \mathcal{W}_1^{N_1}\times\cdots\times\mathcal{W}_k^{N_k}$, we set for every $i\in\{1,\dots,N_k\}$

$$\nu_{H_{\min}}\left(I_{\widetilde{W}_1\cdots\widetilde{W}_{k-1}W_{k,1}\cdots W_{k,i-1}W_{k,i}}\right) = \nu_{H_{\min}}\left(I_{\widetilde{W}_1\cdots\widetilde{W}_{k-1}W_{k,1}\cdots W_{k,i-1}}\right)\frac{\mu_{\alpha_k}(I_{W_{k,i}})}{\sum_{W'_k\in\mathcal{W}_k}\mu_{\alpha_k}(I_{W'_k})}.$$

It is clear that this measure $\nu_{H_{\min}}$, defined only on the cubes appearing in the Cantor's construction, uniquely extends to a Borel probability measure on the cube $[0,1]^d$.

7.5.2. *Properties of the measure $\nu_{H_{\min}}$.* We first prove that the Cantor set lies on the elements $x\in[0,1]^d$ satisfying simultaneously $\underline{\dim}(\mu,x) = H_{\min}$ and $\underline{\dim}(\nu_{H_{\min}},x) = D_\mu(H_{\min})$.

Lemma 11. *One has $\mathcal{C}_{H_{\min}} \subset \widetilde{E}_\mu(H_{\min}) \cap \widetilde{E}_{\nu_{H_{\min}}}(D_\mu(H_{\min}))$.*

Proof. If $k\geq 2$ and $\widetilde{J}_k < J \leq \widetilde{J}_{k+1}$, we set $k_J = k$.

Fix $x\in\mathcal{C}_{H_{\min}}$. Using the quasi-Bernoulli property (8) of μ and the inequalities (57), one gets that for every $k\geq 2$ and $\widetilde{J}_k < J \leq \widetilde{J}_{k+1}$, and for every cube $W\in\mathcal{N}_J(x)$,

$$2^{-\left(N_1J_1(\alpha_1+\varepsilon_1)+\dots+N_kJ_k(\alpha_k+\varepsilon_k)+(J-\widetilde{J}_k)(\alpha_{k+1}+\varepsilon_{k+1})\right)}$$
$$\leq \mu(I_W) \leq 2^{-\left(N_1J_1(\alpha_1-\varepsilon_1)+\dots+N_kJ_k(\alpha_k-\varepsilon_k)+(J-\widetilde{J}_k)(\alpha_{k+1}-\varepsilon_{k+1})\right)}.$$

Using the equations (58), we deduce that

$$2^{-\left((\alpha_k+2\varepsilon_k)N_kJ_k+(J-\widetilde{J}_k)(\alpha_{k+1}+\varepsilon_{k+1})\right)} \leq \mu(I_W) \leq 2^{-\left((\alpha_k-2\varepsilon_k)N_kJ_k+(J-\widetilde{J}_k)(\alpha_{k+1}-\varepsilon_{k+1})\right)},$$

which yields

$$2^{-\left((\alpha_k+2\varepsilon_k)(1+\varepsilon_k)\widetilde{J}_k+(J-\widetilde{J}_k)(\alpha_{k+1}+\varepsilon_{k+1})\right)} \leq \mu(I_W) \leq 2^{-\left((\alpha_k-2\varepsilon_k)(1+\varepsilon_k)\widetilde{J}_k+(J-\widetilde{J}_k)(\alpha_{k+1}-\varepsilon_{k+1})\right)}.$$

Since $\alpha_k \to H_{\min}$ when $k\to+\infty$, we deduce that $\lim_{J\to+\infty}\frac{\log_2\mu(I_W)}{-J} = H_{\min}$, where $W\in\mathcal{N}_J(x)$. This proves that $x\in\widetilde{E}_\mu(H_{\min})$.

Similarly, the same arguments show that for every $k\geq 2$ and $\widetilde{J}_k < J \leq \widetilde{J}_{k+1}$, and for every cube $W\in\mathcal{N}_J(x)$,

$$2^{-\left((D_\mu(\alpha_k)+2\varepsilon_k)(1+\varepsilon_k)\widetilde{J}_k+(J-\widetilde{J}_k)(D_\mu(\alpha_{k+1})+\varepsilon_{k+1})\right)}$$
$$\leq \nu_{H_{\min}}(I_W) \leq 2^{-\left((D_\mu(\alpha_k)-2\varepsilon_k)(1+\varepsilon_k)\widetilde{J}_k+(J-\widetilde{J}_k)(D_\mu(\alpha_{k+1})-\varepsilon_{k+1})\right)}.$$

The equation (56) then yields

$$2^{-JD_\mu(\alpha_k)(1+\widetilde{\theta}_k)} \leq \nu_{H_{\min}}(I_W) \leq 2^{-JD_\mu(\alpha_k)(1-\widetilde{\theta}_k)}, \tag{59}$$

for some decreasing sequence $\widetilde{\theta}_k$, tending to 0 when $k \to +\infty$. This yields that $x \in \widetilde{E}_{\nu_{H_{\min}}}(D_\mu(H_{\min}))$, since $D_\mu(\alpha_k) \to D_\mu(H_{\min})$ when $k \to +\infty$. □

Observe also that (59) implies that for each j large enough, we have

$$\#\{W \in \Sigma_j : I_W \cap \mathcal{C}_{H_{\min}} \neq \emptyset\} \leq 2^{jD_\mu(\alpha_{k_j})(1+\widetilde{\theta}_{k_j})}. \tag{60}$$

7.5.3. *Proof that well-approximated points have $\nu_{H_{\min}}$-measure 0.* Fix an approximation rate $\delta > 1$. To get the result, one focuses first on the value of

$$\nu_{H_{\min}}\Big(\bigcup_{w\in\mathcal{S}_j(\eta)} B(x_w,(2\cdot 2^{-\lfloor \eta j\rfloor})^\delta)\Big) = \nu_{H_{\min}}\Big(\bigcup_{W\in\Sigma_{\lfloor\eta j\rfloor}}\ \bigcup_{w\in\mathcal{S}_j(\eta,W)} B(x_w,(2\cdot 2^{-\lfloor \eta j\rfloor})^\delta)\Big).$$

For each j large enough, consider $W \in \Sigma_{\lfloor j\eta\rfloor}$ such that $I_W \cap \mathcal{C}_{H_{\min}} \neq \emptyset$. One looks for points $x \in I_W \cap \mathcal{C}_{H_{\min}}$ such that $x \in B(x_w,(2\cdot 2^{-\lfloor \eta' j\rfloor})^\delta)$ for some surviving word $w \in \mathcal{S}_j(\eta, W)$. Hence, one sees that

$$\begin{aligned}
&\nu_{H_{\min}}\Big(\bigcup_{W\in\Sigma_{\lfloor\eta j\rfloor}}\ \bigcup_{w\in\mathcal{S}_j(\eta,W)} B(x_w,(2\cdot 2^{-\lfloor \eta j\rfloor})^\delta)\Big)\\
\leq &\sum_{W\in\Sigma_{\lfloor\eta j\rfloor}: I_W\cap\mathcal{C}_{H_{\min}}\neq\emptyset}\ \sum_{\substack{w\in\mathcal{S}_j(\eta,W),\\ I_w\cap\mathcal{C}_{H_{\min}}\neq\emptyset}} \nu_{H_{\min}}\left(B(x_w,(2\cdot 2^{-\lfloor \eta j\rfloor})^\delta)\right).
\end{aligned}$$

Recall that by (55), the number of such possible surviving vertices w (in the second sum above) is bounded from above by $2^{\eta j\varepsilon_j}$. Applying (60) to $B(x_w,(2\cdot 2^{-\lfloor \eta j\rfloor})^\delta)$ for the generation $J = \lfloor \eta j\delta\rfloor$, we get

$$\begin{aligned}
&\nu_{H_{\min}}\Big(\bigcup_{W\in\Sigma_{\lfloor\eta j\rfloor}}\ \bigcup_{w\in\mathcal{S}_j(\eta,W)} B(x_w,(2\cdot 2^{-\lfloor \eta j\rfloor})^\delta)\Big)\\
\leq\ &\Big(\#\{W\in\Sigma_{\lfloor\eta j\rfloor} : I_W\cap\mathcal{C}_{H_{\min}}\neq\emptyset\}\Big)\cdot 2^{\eta j\varepsilon_j}\cdot 2^{-\lfloor\eta j\delta\rfloor D_\mu(\alpha_{k_{\lfloor\eta j\delta\rfloor}})(1-\widetilde{\theta}_{k_{\lfloor\eta j\delta\rfloor}})}\\
\leq\ &2^{\eta j\varepsilon_j+\lfloor\eta j\rfloor D_\mu(\alpha_{k_{\lfloor\eta j\rfloor}})(1+\widetilde{\theta}_{k_{\lfloor\eta j\rfloor}})-\lfloor\eta j\delta\rfloor(D_\mu(\alpha_{k_{\lfloor\eta j\delta\rfloor}})(1-\widetilde{\theta}_{k_{\lfloor\eta j\delta\rfloor}})}.
\end{aligned}$$

It follows now from the properties imposed to the sequences $(\varepsilon_k)_{k\geq1}$ and $(\alpha_k)_{k\geq1}$ that

$$\xi_j := \nu_{H_{\min}}\Big(\bigcup_{w\in\mathcal{S}_j(\eta)} B(x_w,(2\cdot 2^{-\lfloor \eta j\rfloor})^\delta)\Big) \leq C' 2^{\eta j(1-\delta)D_\mu(\alpha_{k_{\lfloor\eta j\rfloor}})(1+o(1))},$$

where C is another constant coming from the fact that we dropped some integer parts.

- When $D_\mu(H_{\min}) > 0$, it is direct that the series $\sum_j \xi_j$ converges.
- When $D_\mu(H_{\min}) = 0$, for large values of j one has by construction $j > k_{\lfloor\eta j\rfloor}$, so $j^{-1} \leq \varepsilon_j \leq \varepsilon_{k_{\lfloor\eta j\rfloor}}$, and $D_\mu(\alpha_{k_{\lfloor\eta j\rfloor}}) = \sqrt{\varepsilon_{k_{\lfloor\eta j\rfloor}}}$. Thus, for j large enough we get
$$2^{\eta j(1-\delta)D_\mu(\alpha_{k_{\lfloor\eta j\rfloor}})(1+o(1))} \leq 2^{-\sqrt{j}\eta(1-\delta)(1+o(1))},$$

hence the series $\sum_j \xi_j$ still converges.

Finally, the Borel-Cantelli lemma proves Part (1) of Theorem 5.

Observe that everything works similarly if we replace $H_{\min}$ with $H_{\max}$ and change the sequence α_k accordingly. When $\alpha \in (H_{\min}, H_{\max})$, we can even take the sequence $(\alpha_k)_{k\geq 1}$ to be constant (if not, this process gives other measures sitting on $\widetilde{E}_\mu(\alpha)$).

7.6. **Proof of Part (2) of Theorem 5.** Recall Proposition 7. Applying the Borel-Cantelli lemma, it is enough to prove that for all integers $N \geq 1$ and $p > 2(H_{\max} - H_{\min})^{-1}$,

$$\mathbb{E}\Big(\sup_{\alpha \in \{H_{\min}, H_{\max}\} \cup I_p} \sum_{J \geq 1} \sum_{W \in \Sigma_J} \nu_\alpha(I_W) \mathbf{1}_{\mathscr{C}(N,J,W)} \Big) < +\infty. \tag{61}$$

where $I_p = [H_{\min} + 1/p, H_{\max} - 1/p]$.

At first, notice that for any Borel probability measure ν on $[0,1]$ we have

$$\begin{aligned} \mathbb{E}\Big(\sum_{J \geq 1} \sum_{W \in \Sigma_J} \nu(I_W) \mathbf{1}_{\mathscr{C}(N,J,W)} \Big) &= \sum_{J \geq 1} \sum_{W \in \Sigma_J} \nu(I_W) \mathbb{P}\big(\mathscr{C}(N,J,W)\big) \\ &\leq \sum_{J \geq 1} 2^{-J\varepsilon_N} \sum_{W \in \Sigma_J} \nu(I_W) < \infty. \end{aligned}$$

Applying this to $\nu_{H_{\min}}$ and $\nu_{H_{\max}}$ constructed above, it remains us to prove (61) only with the interval I_p.

Recall that when $\alpha \in I_p \subset (H_{\min}, H_{\max})$, one can take $\nu_\alpha = \mu_\alpha$ (where μ_α is the Gibbs measure ofProposition 2) .

Let us write the interval I_p as $I_p = \tau'_\mu([q'_p, q_p])$, for some real numbers $q_p > q'_p$. Recall that the Gibbs capacity μ is associated with a Hölder potential ϕ which belongs to the C^β Hölder class, for some $\beta > 0$. Standard arguments based on the bounded distorsion property give that for $\kappa = 2\|\phi\|_\infty / \log(2)$ and $C_{q,q'} = e^{\frac{(|q|+|q'|)C}{(1-2^{-\beta})}}$, for all $q, q' \in [q'_p, q_p]$ and $W \in \Sigma^*$, setting $\alpha_q = \tau'_\mu(q)$, we have

$$\mu_{\alpha_{q'}}(I_W) \leq C_{q,q'} 2^{\kappa |q-q'| \cdot |W|} \mu_{\alpha_q}(I_W).$$

The interval I_p being compact, one can extract a finite collection of intervals $[\alpha_{\widetilde{q}_{k-1}}, \alpha_{\widetilde{q}_k}]$, $1 \leq k \leq K$, such that $q_p = \widetilde{q}_0 > \ldots > \widetilde{q}_K = q'_p$ and $|\widetilde{q}_k - \widetilde{q}_{k-1}| \leq \epsilon_N/(2\kappa)$. Setting $C_k = \sup_{q' \in [\widetilde{q}_k, \widetilde{q}_{k-1}]} C_{q', \widetilde{q}_k}$, one rewrites the above properties as follows: for all $W \in \Sigma^*$, for all $1 \leq k \leq K$,

$$\sup_{q' \in [\widetilde{q}_k, \widetilde{q}_{k-1}]} \mu_{\alpha_{q'}}(I_W) \leq C_k 2^{|W| \varepsilon_N / 2} \mu_{\alpha_{q_k}}(I_W).$$

From these considerations, we get for $1 \leq k \leq K$,

$$\begin{aligned} &\mathbb{E}\Big(\sup_{\alpha \in [\alpha_{\widetilde{q}_{k-1}}, \alpha_{\widetilde{q}_k}]} \sum_{J \geq 1} \sum_{W \in \Sigma_J} \nu_\alpha(I_W) \mathbf{1}_{\mathscr{C}(N,J,W)} \Big) \\ \leq\ &\mathbb{E}\Big(\sum_{J \geq 1} \sum_{W \in \Sigma_J} \sup_{\alpha \in [\alpha_{\widetilde{q}_{k-1}}, \alpha_{\widetilde{q}_k}]} \nu_\alpha(I_W) \mathbf{1}_{\mathscr{C}(N,J,W)} \Big) \\ \leq\ &C_k \sum_{J \geq 1} \sum_{W \in \Sigma_J} 2^{J\varepsilon_N/2} \nu_{\alpha_{q_k}}(I_W) \mathbb{P}\big(\mathscr{C}(N,J,W)\big) \\ \leq\ &C_k \sum_{J \geq 1} 2^{-J\varepsilon_N/2} < +\infty. \end{aligned}$$

It follows that

$$\mathbb{E}\Big(\sup_{\alpha\in I_p} \Big(\sum_{J\geq 1} \sum_{W\in\Sigma_J} \nu_\alpha(I_W) \mathbf{1}_{\mathscr{C}(N,J,W)} \Big)\Big) \leq \sum_{k=1}^{K} C_k \sum_{J\geq 1} 2^{-J\varepsilon_N/2} < +\infty,$$

i.e. (61) holds.

8. Free energy and large deviations for M_μ

Recall the definitions (15) and (16) for $q_{\widetilde{\eta}}$ and q_{η_ℓ}, and also formula (17) for $\widetilde{\tau}(q)$ that we reproduce for convenience:

$$\widetilde{\tau}(q) = \begin{cases} \tau_\mu(q) + \widetilde{H}_\ell(\widetilde{\eta})q & \text{if } q \leq q_{\widetilde{\eta}}, \\ \tau_\mu(q) + d(1-\eta) & \text{if } q_{\widetilde{\eta}} < q < q_{\eta_\ell}, \\ H_\ell(0)q & \text{if } q_{\eta_\ell} < \infty \text{ and } q \geq q_{\eta_\ell}. \end{cases}$$

In this section we prove that, with probability 1:

- (Section 8.1) the Legendre transform of $\tilde{\tau}$ equals D_{M_μ},
- (Section 8.2) the lower L^q-spectrum τ_{M_μ} is bounded below by $\tilde{\tau}$,
- (Section 8.3) the lower large deviation spectrum satisfies

$$\underline{f}_{\mathsf{M}_\mu}(H) \geq \tau^*_{\mathsf{M}_\mu}(H). \tag{62}$$

Using that $D_{\mathsf{M}_\mu}(H) \leq \tau^*_{\mathsf{M}_\mu}(H)$ holds true for every H, the first result yields $\widetilde{\tau} \geq \tau_{\mathsf{M}_\mu}$. The second result ensures that there is in fact equality: $\widetilde{\tau} = \tau_{\mathsf{M}_\mu}$.
Since one always has $\underline{f}_{\mathsf{M}_\mu}(H) \leq \overline{f}_{\mathsf{M}_\mu}(H) \leq \tau^*_{\mathsf{M}_\mu}(H)$, we conclude that $D_{\mathsf{M}_\mu}(H) = \tau^*_{\mathsf{M}_\mu}(H) = \underline{f}_{\mathsf{M}_\mu}(H) = \overline{f}_{\mathsf{M}_\mu}(H) = \widetilde{\tau}(H)$. In particular, M_μ obeys the multifractal formalism.
Finally, by Varadhan's lemma (or in our situation very simple estimates), the free energy $\tau_{\mathsf{M}_\mu}(q)$ exists as a limit, not only as a liminf, for all $q \in \mathbb{R}$.

This completes Part (2) of Theorem 2 and Theorem 3.

8.1. Equality between $\widetilde{\tau}^*$ and the singularity spectrum of M_μ.

First, we prove that $D_{\mathsf{M}_\mu}(H)$ (given by part (2) of Theorem 2) is indeed the Legendre transform of $\widetilde{\tau}$.

Lemma 12. *With probability one, one has* $\widetilde{\tau}^* = D_{M_\mu}$.

Proof. Let us start with a few observations. By definition of $H_\ell(\widetilde{\eta})$ (see Figure 9), $q_{\widetilde{\eta}}$ is the slope of the tangent line to the graph of τ^*_μ at $H_\ell(\widetilde{\eta})$, and this tangent line passes through the point $(0, d(1-\eta))$. Hence $\tau^*_\mu(H_\ell(\widetilde{\eta})) - d(1-\eta) = q_{\widetilde{\eta}} H_\ell(\widetilde{\eta})$. Recalling that $\tau^*_\mu(H_\ell(\widetilde{\eta})) = d\frac{1-\eta}{1-\widetilde{\eta}}$, one deduces that

$$\widetilde{\eta}\tau^*_\mu(H_\ell(\widetilde{\eta})) = q_{\widetilde{\eta}}\tau'_\mu(q_{\widetilde{\eta}}).$$

But by the definition of the Legendre transform, one has $\tau^*_\mu(H_\ell(\widetilde{\eta})) = \tau^*_\mu(\tau'_\mu(q_{\widetilde{\eta}})) = q_{\widetilde{\eta}}\tau'_\mu(q_{\widetilde{\eta}}) - \tau_\mu(q_{\widetilde{\eta}})$. We deduce that

$$\tau_\mu(q_{\widetilde{\eta}}) = \tau^*_\mu(H_\ell(\widetilde{\eta}))(\widetilde{\eta} - 1) = -d(1-\eta),$$

i.e. $\widetilde{\tau}(q_{\widetilde{\eta}}) = 0$. Notice that $\widetilde{\tau}$ is continuous on $\mathbb{R}$, and that there is a first order phase transition at $q_{\widetilde{\eta}}$, since $H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}) = \widetilde{\tau}'(q_{\widetilde{\eta}}{}^-) > \widetilde{\tau}'(q_{\widetilde{\eta}}{}^+) = H_\ell(\widetilde{\eta})$.

• When $H \geq H_\ell(\widetilde{\eta})$: Since $\widetilde{\tau}$ and τ_μ differ by a linear term of slope $\widetilde{H}_\ell(\widetilde{\eta})$ over $(-\infty, q_{\widetilde{\eta}}]$, their Legendre transform are translated versions of each other by $\widetilde{H}_\ell(\widetilde{\eta})$ over the interval

$[\widetilde{\tau}'(q_{\widetilde{\eta}}), +\infty) = [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), +\infty)$. Hence, for $H \geq H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})$, one has $\widetilde{\tau}^*(H) = \tau_\mu^*(H - \widetilde{H}_\ell(\widetilde{\eta})) = D_{\mathsf{M}_\mu}(H)$.

• When $H \in [H_\ell(\widetilde{\eta}), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})]$: The discontinuity of $(\widetilde{\tau})'$ at $q_{\widetilde{\eta}}$ implies that for H in the interval $[\widetilde{\tau}'(q_{\widetilde{\eta}}{}^+), \widetilde{\tau}'(q_{\widetilde{\eta}}{}^-)] = [H_\ell(\widetilde{\eta}), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})]$, one has

$$\widetilde{\tau}^*(H) = \inf_{q\in\mathbb{R}} (qH - \widetilde{\tau}(q)) = q_{\widetilde{\eta}} H - \widetilde{\tau}(q_{\widetilde{\eta}}) = q_{\widetilde{\eta}} H = D_{\mathsf{M}_\mu}(H).$$

• When $\eta_\ell = 0$ and $H \leq H_\ell(\widetilde{\eta})$: In this case we have $q_{\eta_\ell} < +\infty$. Since $\widetilde{\tau}$ and τ_μ differ by the constant $d(1-\eta)$ over $[q_{\widetilde{\eta}}, q_{\eta_\ell}]$, for $H \in \big[\widetilde{\tau}'(q_{\eta_\ell}), \widetilde{\tau}'(q_{\widetilde{\eta}})\big] = \big[H_\ell(0), H_\ell(\widetilde{\eta})\big]$, one has $\widetilde{\tau}^*(H) = \tau_\mu^*(H) - d(1-\eta)$. Then, when $q \geq q_{\eta_\ell}$, $\widetilde{\tau}$ is linear with slope $\widetilde{\tau}'(q_{\eta_\ell}^-)$, so $\widetilde{\tau}^*(H) = -\infty$ for all $H < H_\ell(0)$. In all cases, $\widetilde{\tau}^*(H) = D_{\mathsf{M}_\mu}(H)$.

• When $\eta_\ell > 0$ and $H \leq H_\ell(\widetilde{\eta})$: Here $q_{\eta_\ell} = +\infty$ and $H_\ell(\eta_\ell) = H_{\min}$. The same argument as above yields $\widetilde{\tau}^*(H) = \tau_\mu^*(H) - d(1-\eta)$ for all $H \leq H_\ell(\widetilde{\eta})$. □

We know now that $D_{\mathsf{M}_\mu}(H) = \widetilde{\tau}^*(H)$, for all $H \in \mathbb{R}$. Since the multifractal formalism states that $D_{\mathsf{M}_\mu}(H) \leq \tau_{\mathsf{M}_\mu}^*(H)$ for all $H \in \mathbb{R}$, one deduces that $\widetilde{\tau}^* \leq \tau_{\mathsf{M}_\mu}^*$. By inverse Legendre transform, one gets

$$\text{for all } q \in \mathbb{R}\ , \quad \widetilde{\tau}(q) \geq \tau_{\mathsf{M}_\mu}(q).$$

The next section establishes that $\widetilde{\tau} \leq \tau_{\mathsf{M}_\mu}$, so that equality indeed holds almost surely.

8.2. **Lower bound for τ_{M_μ}.**

8.2.1. *When $q_{\widetilde{\eta}} < q < q_{\eta_\ell}$:* The sub-multiplicativity property of μ gives for $j \geq 1$

$$\begin{aligned}\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q &= \sum_{W\in\Sigma_J} \left(\max_{W'\in\mathcal{N}_J(W)} \max_{w\in\mathcal{S}_j(\eta,W')} \mu(I_w)\right)^q \\ &\leq 3^d \sum_{W\in\Sigma_J} \max_{w\in\mathcal{S}_j(\eta,W)} \mu(I_w)^q \\ &\leq 3^d C^q \sum_{W\in\Sigma_J} \mu(I_W)^q \sum_{w\in\Sigma^*,\, p_{Ww}=1} \mu(I_w)^q \\ &= 3^d C^q \sum_{W\in\Sigma_J} \mu(I_W)^q \sum_{k\geq 0} \sum_{w\in\Sigma_k} \mu(I_w)^q p_{Ww}.\end{aligned}$$

The random variables p_{Ww} being independent, with law $B(2^{-d(J+k)(1-\eta)})$, this yields

$$\mathbb{E}\left(\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q\right) \leq 3^d C^q \left(\sum_{W\in\Sigma_J} \mu(I_W)^q\right) \sum_{k\geq 0} 2^{-(J+k)d(1-\eta)} \left(\sum_{w\in\Sigma_k} \mu(I_w)^q\right).$$

Observe that a direct consequence of (8) is that for some positive constant $C_q > 0$,

$$\sup_{k\geq 1} 2^{k\tau_\mu(q)} \sum_{w\in\Sigma_k} \mu(I_w)^q \leq C_q. \tag{63}$$

Consequently,

$$\mathbb{E}\left(\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q\right) \leq 3^d C_q C^q \left(2^{-Jd(1-\eta)} \sum_{W\in\Sigma_J} \mu(I_W)^q\right) \sum_{k\geq 0} 2^{-k(\tau_\mu(q)+d(1-\eta))}.$$

Since $q > q_{\widetilde{\eta}}$, we have $\tau_\mu(q) + d(1-\eta) > 0$. Hence for some constant C'_q depending on q only,

$$\mathbb{E}\left(\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q\right) \le 3^d C'_q \left(2^{-Jd(1-\eta)} \sum_{W\in\Sigma_J} \mu(I_W)^q\right).$$

Finally, for every $\varepsilon > 0$, applying (63) we get

$$\mathbb{E}\left(\sum_{J\ge1} 2^{J(\tau_\mu(q)+d(1-\eta)-\varepsilon)} \sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q\right) \le 3^d C'_q C_q \sum_{J\ge1} 2^{-J\varepsilon},$$

which is finite. We conclude that with probability 1, we have

$$\liminf_{J\to+\infty} \frac{1}{J}\log_2 \sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q \le -\tau_\mu(q) - d(1-\eta),$$

i.e. $\tau_{\mathsf{M}_\mu}(q) \ge \tau_\mu(q) + d(1-\eta) = \tilde{\tau}(q)$.

This holds for each $q_{\widetilde{\eta}} < q < q_{\eta_\ell}$ almost surely, and by concavity (hence continuity) of τ_{M_μ} and $\widetilde{\tau}$, this holds almost surely for all $q_{\widetilde{\eta}} \le q \le q_{\eta_\ell}$.

8.2.2. *When $q \in (0, q_{\widetilde{\eta}})$:* • Suppose for a while that both η_ℓ and η_r are positive.

Fix $0 < \varepsilon < H_\ell(\eta_\ell)$, and two integers $N_\ell, N_r > 2/\varepsilon$. Due to Proposition 5 and the continuity of the mappings H_ℓ and H_r, there exists $j_0 \ge 1$, and two sets of parameters $\eta_\ell = \eta_{\ell,1} < \ldots < \eta_{\ell,N_\ell} = \eta$ and $\eta_r = \eta_{r,1} < \ldots < \eta_{r,N_r}$ such that for $j \ge j_0$, if $w \in \mathcal{S}_j(\eta)$, then $w \in \mathcal{T}_\mu(j, \eta_{i,k}, \varepsilon)$ for some $i \in \{\ell, r\}$ and $1 \le k \le N_i$, i.e.

$$(j - \lfloor \eta_{i,k} j\rfloor)(H_i(\eta_{i,k}) - \varepsilon) \le -\log_2 \mu(I_{\sigma^{\lfloor \eta_{i,k} j\rfloor} w}) \le (j - \lfloor \eta_{i,k} j\rfloor)(H_i(\eta_{i,k}) + \varepsilon). \tag{64}$$

For J large enough, given $W \in \Sigma_J$, assume that $\mathsf{M}_\mu(I_W)$ is realized at w, i.e. $\mathsf{M}_\mu(I_W) = \mu(I_w)$, for some word of length $|w| = j \ge J$, $I_w \subset I_{W'}$ and $W' \in \mathcal{N}_J(W)$. In this case, there exists $\eta_{i,k}$ such that (64) holds. We have to distinguish the two following possibilities for the parameters $\{\eta_{i,k}\}_{i\in\{\ell,r\},k\in\{1,\ldots,\max(N_\ell,N_r)\}}$:

- if $\lfloor \eta_{i,k} j\rfloor \le J$, then $I_W \subset \bigcup_{u\in\mathcal{N}_{\lfloor \eta_{i,k} j\rfloor}(w_{|\lfloor \eta_{i,k} j\rfloor})} I_u$.
- if $\lfloor \eta_{i,k} j\rfloor > J$, then
$$\mathsf{M}_\mu(I_W) \le C\mu(I_w)\mu(I_{\sigma^{j-\lfloor \eta_{i,k} j\rfloor} w}) \le C\mu(I_W) 2^{(j-\lfloor j\eta_{i,k}\rfloor)(H_i(\eta_{i,k})-\varepsilon)},$$
where (64) has been used.

In the second case, some information is lost between the generations J and $\lfloor \eta' j\rfloor$. We deduce from these observations and the quasi-Bernoulli property of μ that

$$\begin{aligned}\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q \le& \sum_{i\in\{\ell,r\}} \sum_{\substack{k=1\\ \lfloor \eta_{i,k} j\rfloor \le J}}^{N_i} \sum_{w\in\mathcal{T}_\mu(j,\eta_{i,k},\varepsilon)} \mu(I_w)^q + \sum_{i\in\{\ell,r\}} \sum_{\substack{k=1\\ \lfloor \eta_{i,k} j\rfloor > J}}^{N_i} \sum_{w\in\mathcal{T}_\mu(j,\eta_{i,k},\varepsilon)} \mu(I_w)^q \\ \le& \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} 3^d \sum_{J\le j\le J/\eta_{i,k}} \sum_{u\in\Sigma_{\lfloor j\eta_{i,k}\rfloor}} C^q \mu(I_u)^q 2^{-q(j-\lfloor j\eta_{i,k}\rfloor)(H_i(\eta_{i,k})-\varepsilon)} \\ &+ \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} 3^d \sum_{W\in\Sigma_J} \sum_{j>J/\eta_{i,k}} C^q \mu(I_W)^q 2^{-q(j-\lfloor j\eta_{i,k}\rfloor)(H_i(\eta_{i,k})-\varepsilon)}.\end{aligned}$$

Recalling (63) and the fact that $\widetilde{H}_i(\eta') = H_i(\eta')(\eta'^{-1} - 1)$ for every η', the first term in the last sum is bounded from above by

$$3^d C^q C_q \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} \sum_{J\leq j\leq J/\eta_{i,k}} 2^{-\lfloor j\eta_{i,k}\rfloor \tau_\mu(q)} 2^{-q(j-\lfloor j\eta_{i,k}\rfloor)(H_i(\eta_{i,k})-\varepsilon)}$$
$$\leq C'_q \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} 2^{qJ\varepsilon/\eta_{i,k}} \sum_{J\leq j\leq J/\eta'_{i,k}} 2^{-j\eta_{i,k}(\tau_\mu(q)+q\widetilde{H}_i(\eta_{i,k}))}$$

and the second by

$$3^d C^q C_q \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} 2^{-J\tau_\mu(q)} \sum_{j>J/\eta_{i,k}} 2^{-q(j-\lfloor j\eta_{i,k}\rfloor)(H_i(\eta_{i,k})-\varepsilon)}$$
$$\leq C'_q \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} 2^{-J(\tau_\mu(q)+q\widetilde{H}_i(\eta_{i,k}))} 2^{qJ(\eta_{i,k}{}^{-1}-1)\varepsilon}$$

for some other constant C'_q. Since $q\geq 0$, $\tau_\mu(q) + q\widetilde{H}_i(\eta_{i,k})$ is bounded from below for all $p\in\{1,...,\max(N_\ell,N_r)\}$ by $\tau_\mu(q)+q\widetilde{H}_\ell(\widetilde{\eta})$, which is negative. Consequently,

$$\sum_{J\leq j\leq J/\eta_{i,k}} 2^{-j\eta_{i,k}(\tau_\mu(q)+qHt_i(\eta_{i,k}))} = O(2^{-J(\tau_\mu(q)+q\widetilde{H}_\ell(\widetilde{\eta}))}).$$

In addition, one always has $\eta_{i,k}\geq\eta_i$, hence

$$2^{-J(\tau_\mu(q)+q\widetilde{H}_i(\eta_{i,k}))} 2^{qJ(\eta_{i,k}{}^{-1}-1)\varepsilon} \leq 2^{-J(\tau_\mu(q)+q\widetilde{H}(\widetilde{\eta}))} 2^{qJ(\eta_i{}^{-1}-1)\varepsilon}.$$

Putting everything together we get for some $C''_q>0$

$$\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q$$
$$\leq C''_q \Big(N_\ell(2^{qJ\varepsilon/\eta_\ell} + 2^{qJ(\eta_\ell{}^{-1}-1)\varepsilon}) + N_r(2^{qJ\varepsilon/\eta_r} + 2^{qJ(\eta_r{}^{-1}-1)\varepsilon})\Big) 2^{-J(\tau_\mu(q)+q\widetilde{H}_\ell(\widetilde{\eta}))}.$$

This yields $\tau_{\mathsf{M}_\mu}(q) \geq \tau_\mu(q) + q\widetilde{H}_\ell(\widetilde{\eta}) + O(\varepsilon)$, and letting ε tend to 0 gives the desired conclusion.

- Now we deal with the case where at least one of parameters η_ℓ and η_r equals zero.

According to the value of η_i, we construct a subset $\Sigma_J^{(i)}$ of words of length J having specific properties:

First case: $\eta_i > 0$**:** set $\Sigma_J^{(i)} = \emptyset$.

Second case: $\eta_i = 0$**:** in this case, $D_\mu(H_i(\eta_i)) = d(1-\eta)$. Heuristically, $\Sigma_J^{(i)}$ contains those words W such that $\mathsf{M}_\mu(I_W) = \mu(I_w)$ for some surviving vertex $w\in\mathcal{S}_j(\eta)$ having an "extreme" behavior, i.e. $\mu(I_w)\sim 2^{-jH_i(\eta_i)}$. We proceed as follows:

At first, let $K\geq H_{\max}$ be as in Proposition 2(6). For $\eta'_i\in[\eta_i,\eta]$ close to η_i, we denote by $\widehat{\eta}_i$ the unique real number in $[\eta_i,\eta]$ such that $H_i(\widehat{\eta}_i) = H_i(\eta'_i) + K\eta'_i$ if $i=\ell$ and $H_i(\widehat{\eta}') = H_i(\eta'_i) - K\eta'_i$ if $i=r$. Notice that $\widehat{\eta}_i > \eta'_i$.

Now, fix $\varepsilon = qH_i(\eta_i)/4$, and choose η_i' small enough so that $(1-\eta_i')(H_i(\widehat{\eta}_i) - K\widehat{\eta}_i) > H_i(\eta_i)/2$ if $i=\ell$ and $H_i(\eta_i') - 2K\widehat{\eta}_i > H_i(\eta_i)/2$ if $i=r$, and

$$\begin{cases} H_i(\eta_i') + 2K\widehat{\eta}_i < H_s \text{ and } D_\mu(H_i(\eta_i') + 2K\widehat{\eta}_i) \leq D_\mu(H_i(\eta_i) + \varepsilon/2 & \text{if } i=\ell, \\ H_i(\eta_i') - 2K\widehat{\eta}_i > H_s \text{ and } D_\mu(H_i(\eta_i') - 2K\widehat{\eta}_i) \leq D_\mu(H_i(\eta_i)) + \varepsilon/2 & \text{if } i=r \end{cases}.$$

By item (5) of Proposition 2, there exists an integer J_i such that for $j \geq J_i$,

$$\begin{array}{ll} \#\mathcal{E}_\mu(j,[0,H_\ell(\eta_i')+2K\widehat{\eta}_i]) \leq 2^{j(D_\mu(H_i(\eta_i')+2K\eta_i')+\varepsilon/2)} \leq 2^{j(D_\mu(H_i(\eta_i))+\varepsilon)} & \text{if } i=\ell, \\ \#\mathcal{E}_\mu(j,[H_i(\eta_i')-2K\widehat{\eta}_i,+\infty)) \leq 2^{j(D_\mu(H_r(\eta_i')+2K\eta_i')+\varepsilon/2)} \leq 2^{j(D_\mu(H_i(\eta_i))+\varepsilon)} & \text{if } i=r \end{array}. \tag{65}$$

We can also choose J_i such that $\varepsilon_j^2 \leq K\eta_i'/2 \leq K\widehat{\eta}_i/2$ for $j \geq J_i$, where$(\varepsilon_j^2)_{j\geq 1}$ is the sequence introduced in Proposition 5.

For $J \geq J_i$, we take $\Sigma_J^{(i)}$ as the set of those words $W \in \Sigma_J$ such that $\mathsf{M}_\mu(I_W) = \mu(I_w)$, where $w \in \mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,i}(j,\eta',\varepsilon_j^2, W)$ for some η' satisfying

$$\begin{cases} H_\ell(\eta') + \varepsilon_j^2 \leq H_\ell(\eta_\ell') + K\eta_\ell' & \text{if } i=\ell, \\ H_r(\eta') - \varepsilon_j^2 \geq H_r(\eta_r') - K\eta_r' & \text{if } i=r \end{cases}.$$

In particular, we have $\eta' \leq \widehat{\eta}_i$. The words $W \in \Sigma_J^{(i)}$ are the ones that may cause problems when compared to the case where $\eta_\ell, \eta_r > 0$. The other words W are such that $\mathsf{M}_\mu(I_W)$ is reached at some w associated with η' satisfying $H_\ell(\eta') \in [H_\ell(\eta_\ell') + K\eta_\ell'/2, H_r(\eta_r') - K\eta_r'/2]$, i.e. η' stays bounded away from 0.

When $J \geq J_i$ and $W \in \Sigma_J^{(i)}$, for the associated word $w \in \mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,i}(j,\eta',\varepsilon_j^2, W)$ (according to the previous notations), we have, using (8) and the definition of K:

$$C^{-1}2^{-\lfloor j\eta'\rfloor K}\mu(I_{\sigma^{\lfloor \eta' j\rfloor}w}) \leq \mathsf{M}_\mu(I_W) = \mu(I_w) \leq C\mu(I_{\sigma^{\lfloor \eta' j\rfloor}w}),$$

which yields, due to the property of (W,w) and the fact that $\eta' \leq \widehat{\eta}_i$:

$$C^{-1}2^{-j\widehat{\eta}_i K}2^{-j(H_i(\eta')+\varepsilon_j^2)} \leq \mathsf{M}_\mu(I_W) = \mu(I_w) \leq C2^{-(j-\lfloor \widehat{\eta}_i j\rfloor)(H_i(\eta')-\varepsilon_j^2)}.$$

This yields, for J large enough,

$$\begin{cases} 2^{-j(H_i(\eta_i')+2K\widehat{\eta}_i)} \leq \mathsf{M}_\mu(I_W) = \mu(I_w) \leq 2^{-(j(1-\eta_i')(H_i(\widehat{\eta}_i)-K\widehat{\eta}_i)} & \text{if } i=\ell, \\ \mathsf{M}_\mu(I_W) = \mu(I_w) \leq 2^{-j(H_i(\eta_i')-2K\widehat{\eta}_i)} & \text{if } i=r \end{cases}.$$

Hence, each such word W is associated with one surviving word $w \in \mathcal{E}_\mu(j,[0,H_i(\eta_i')+2K\widehat{\eta}_i])$, for some $j \geq J$ if $i=\ell$, and one surviving word $w \in \mathcal{E}_\mu(j,[H_i(\eta_i')-2K\widehat{\eta}_i,+\infty))$, for some $j \geq J$ if $i=r$.

Then, if $i=\ell$, writing $j=J+k$ one gets :

$$\sum_{W\in\Sigma_J^{(i)}} \mathsf{M}_\mu(I_W)^q \leq \sum_{k=0}^{+\infty} \sum_{w\in\mathcal{E}_\mu(J+k,[0,H_i(\eta_i')+2K\widehat{\eta}_i])} p_w C^q 2^{-(J+k)q(1-\eta_i')(H_i(\widehat{\eta}_i)-K\widehat{\eta}_i)}.$$

Taking expectations and recalling (65), one gets

$$\begin{aligned} \mathbb{E}\Big(\sum_{W\in\Sigma_J^{(i)}} \mathsf{M}_\mu(I_W)^q\Big) &\leq 3^dC^q\sum_{k\geq 0} 2^{-(J+k)d(1-\eta)}2^{(J+k)(d(1-\eta)+\varepsilon)}2^{-q(J+k)(1-\eta_i')(H_i(\widehat{\eta}_i)-K\widehat{\eta}_i)} \\ &= 3^dC^q\sum_{k\geq 0} 2^{(J+k)(\varepsilon-q(1-\eta_i')(H_i(\widehat{\eta}_i)-K\widehat{\eta}_i))}. \end{aligned}$$

Now by our choice for ε and η_i', we have $\varepsilon - q(1-\eta_i')(H_i(\widehat{\eta}_i) - K\widehat{\eta}_i) \leq -\varepsilon$. We deduce that

$$\mathbb{E}\Big(\sum_{W\in\Sigma_J^{(i)}} \mathsf{M}_\mu(I_W)^q\Big) \leq C_{q,\varepsilon} 2^{-J\varepsilon}$$

for some constant $C_{q,\varepsilon} > 0$. Finally, applying the Borel-Cantelli lemma, we deduce that with probability 1, for J large enough we have

$$\sum_{W\in\Sigma_J^{(i)}} \mathsf{M}_\mu(I_W)^q \leq 1. \tag{66}$$

Observe that (66) holds true even if $\eta_i > 0$ (in which case $\Sigma_J^{(i)}$ is empty). If $i = r$, similar computations yield

$$\mathbb{E}\Big(\sum_{W\in\Sigma_J^{(i)}} \mathsf{M}_\mu(I_W)^q\Big) \leq 3^d C^q \sum_{k\geq 0} 2^{(J+k)(\varepsilon - q(H_i(\eta_i') - 2K\widehat{\eta}_i))},$$

with a similar conclusion.

Finally, the same estimates as when both η_ℓ and η_r are strictly positive yield

$$\liminf_{J\to+\infty} \frac{-1}{J} \log_2 \sum_{W\in\Sigma_J\setminus(\Sigma_J^{(\ell)}\cup\Sigma_J^{(r)})} \mathsf{M}_\mu(I_W)^q \geq \tau_\mu(q) + q\widetilde{H}_\ell(\widetilde{\eta}) = \tilde{\tau}(q).$$

Since $\tilde{\tau}(q) < 0$ and (66) holds for J large enough, we conclude that $\tau_{\mathsf{M}_\mu}(q) \geq \tilde{\tau}(q)$.

8.2.3. *When $q < 0$:* Applying Proposition 6 with $\eta' = \widetilde{\eta}$, there exists a positive sequence $(\varepsilon_j^3)_{j\geq 1}$ converging to 0 such that with probability 1, for j large enough, for all $W \in \Sigma_{\lfloor\widetilde{\eta}j\rfloor}$, there exists $w \in \mathcal{S}_j(\eta, W)$ such that the $\widetilde{\eta}$-tail of w satisfies

$$2^{-(j-\lfloor\widetilde{\eta}j\rfloor)(H_\ell(\widetilde{\eta})+\varepsilon_j^3)} \leq \mu(I_{\sigma^{\lfloor\widetilde{\eta}j\rfloor}w}).$$

The quasi-Bernoulli property implies that $\mathsf{M}_\mu(I_{Ww}) \geq C^{-1}\mu(I_W)2^{-j(1-\widetilde{\eta})(H_\ell(\widetilde{\eta})+\varepsilon_j)}$, which for $q < 0$ yields

$$\begin{aligned}\sum_{W\in\Sigma_{\lfloor\widetilde{\eta}j\rfloor}} \mathsf{M}_\mu(I_W)^q &\leq C^q 2^{-jq(1-\widetilde{\eta})(H_\ell(\widetilde{\eta})+\varepsilon_j)} \sum_{W\in\Sigma_{\lfloor\widetilde{\eta}j\rfloor}} \mu(I_W)^q \\ &\leq C^{q+1} 2^{-\lfloor\widetilde{\eta}j\rfloor q(\widetilde{H}_\ell(\widetilde{\eta})+\varepsilon_j/\widetilde{\eta})} \sum_{W\in\Sigma_{\lfloor\widetilde{\eta}j\rfloor}} \mu(I_W)^q.\end{aligned}$$

One concludes that $\tau_{\mathsf{M}_\mu}(q) \geq \tau_\mu(q) + \widetilde{H}_\ell(\widetilde{\eta})q = \tilde{\tau}(q)$.

8.2.4. *When $q_{\eta_\ell} < +\infty$ and $q > q_{\eta_\ell}$.* Recall that this implies $\eta_\ell = 0$. We have already shown that $\tau_{\mathsf{M}_\mu}(q) \geq \tau_\mu(q) + d(1-\eta)$ when $q \in [q_{\widetilde{\eta}}, q_{\eta_\ell}]$.

The tangent to the graph of $q \mapsto \tau_{\mathsf{M}_\mu}(q)$ at $(q_{\eta_\ell}, \tau_{\mathsf{M}_\mu}(q_{\eta_\ell}))$ is the affine line passing through $(0,0)$, whose slope is $\tau'_{\mathsf{M}_\mu}(q_{\eta_\ell}) = H_\ell(0)$. Consequently, the concavity of τ_{M_μ} implies that $\tau_{\mathsf{M}_\mu}(q) \leq qH_\ell(0)$ for all $q \geq q_{\eta_\ell}$. On the other hand, if $q \geq q_{\eta_\ell}$, for all integers $J \geq 1$ we have

$$\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q \leq \Big(\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^{q_{\eta_\ell}}\Big)^{q/q_{\eta_\ell}},$$

from which it follows that $\tau_{\mathsf{M}_\mu}(q) \geq \frac{q}{q_{\eta_\ell}} \tau_{\mathsf{M}_\mu}(q_{\eta_\ell}) = qH_\ell(0)$.

8.3. **Lower bound for the lower large deviations spectrum $\underline{f}_{\mathsf{M}_\mu}(H)$.** Let us check that (62) holds. It is enough to deal with a dense countable subset of the support $[H_\ell(\eta_\ell), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$ of τ_{M_μ}.

• Suppose first that $H \in [H_\ell(\eta_\ell), H_\ell(\widetilde{\eta})]$. Recall the definition (45) of η_H: $H = H_\ell(\eta_H)$. By item (4) of Proposition 2, for every $\varepsilon > 0$, there exists $\beta(\varepsilon) > 0$ such that when j becomes large,

$$\#\mathcal{E}_\mu(\lfloor \eta_H j \rfloor, [0, H_\ell(\eta_H) + \varepsilon]) \geq 2^{\lfloor \eta_H j \rfloor (D_\mu(H_\ell(\eta_H)) - \beta(\varepsilon))}.$$

One also knows that $\beta(\varepsilon)$ can be taken so that $\beta(\varepsilon) \to 0$ when $\varepsilon \to 0$.

In addition, by Proposition 6, there is a positive sequence $(\varepsilon_j^3)_{j\geq 1}$ converging to 0 such that, with probability 1, for j large enough, each cube I_W (with $W \in \mathcal{E}_\mu(\lfloor \eta_H j \rfloor, [0, H_\ell(\eta_H) + \varepsilon])$) contains a smaller cube I_w, with $w \in \mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \eta_H, \varepsilon_j^3)$.

By the quasi-Bernoulli property of μ,

$$\mathsf{M}_\mu(I_w) \geq \mu(I_w) \geq C^{-1} 2^{-\lfloor \eta_H j \rfloor (H_\ell(\eta_H) + \varepsilon)} 2^{-(j - \lfloor \eta_H j \rfloor)(H_\ell(\eta') + \varepsilon_j^3)} = 2^{-j(H_\ell(\eta_H) + 2\varepsilon)}$$

when j becomes large. Thus,

$$\liminf_{j\to+\infty} \frac{1}{j} \log_2 \#\mathcal{E}_{\mathsf{M}_\mu}(j, [0, H_\ell(\eta_H) + 2\varepsilon]) \geq \eta_H \Big(D_\mu(H_\ell(\eta_H)) - \beta(\varepsilon) \Big),$$

Since by construction $H = H_\ell(\eta_H)$ and $\eta_H D_\mu(H_\ell(\eta_H)) = D_\mu(H) - d(1-\eta)$, letting ε go to zero gives

$$\lim_{\varepsilon\to 0^+} \liminf_{j\to+\infty} \frac{1}{j} \log_2 \#\mathcal{E}_{\mathsf{M}_\mu}(j, [0, H + 2\varepsilon]) \geq D_\mu(H) - d(1-\eta) = \tau^*_{\mathsf{M}_\mu}(H).$$

We conclude that $\underline{f}_{\mathsf{M}_\mu}(H) \geq \widetilde{\tau}^*(H)$, for otherwise there would exist $H' < H$ such that

$$\limsup_{j\to+\infty} \frac{1}{j} \log_2 \#\mathcal{E}_{\mathsf{M}_\mu}(j, [0, H']) \geq \tau^*_{\mathsf{M}_\mu}(H) > \tau^*_{\mathsf{M}_\mu}(H'),$$

which contradicts the fact that for all $H' \leq H_s + \widetilde{H}_\ell(\widetilde{\eta}) = \tau'_{\mathsf{M}_\mu}(0)$, Proposition 1 yields

$$\limsup_{j\to+\infty} \frac{1}{j} \log_2 \#\mathcal{E}_{\mathsf{M}_\mu}(j, [0, H']) \leq \tau^*_{\mathsf{M}_\mu}(H').$$

• For $H \in [H_\ell(\widetilde{\eta}), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\eta_\ell)]$ we use the same idea: There exist a positive sequence $(\beta_j)_{j\geq 1}$ converging to 0 such that, for j is large enough, at generation $\lfloor j\widetilde{\eta} \rfloor$ there are at least $2^{\lfloor j\widetilde{\eta} \rfloor (D_\mu(H_\ell(\widetilde{\eta})) - \varepsilon_j))}$ elements W in $\mathcal{E}_\mu(\lfloor j\widetilde{\eta} \rfloor, [0, H_\ell(\widetilde{\eta}) + \beta_j])$.

In addition, by Proposition 6, with probability 1, for j large enough, each of these I_W contains a smaller cube I_w, with $w \in \mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \widetilde{\eta}, \varepsilon_j^3)$.

Then, let w' be the word of generation $j' = \lfloor jH_\ell(\widetilde{\eta})/H \rfloor$ such that $I_w \subset I_{w'} \subset I_W$. We have $-\log_2 \mathsf{M}_\mu(I_{w'}) \geq -\log \mathsf{M}_\mu(I_W) \sim jH_\ell(\eta') \sim j'H$. It follows that for any $\varepsilon > 0$,

$$\liminf_{j'\to+\infty} \frac{1}{j'} \log_2 \#\mathcal{E}_{\mathsf{M}_\mu}(j', [0, H + \varepsilon]) \geq \frac{\widetilde{\eta} D_\mu(H_\ell(\widetilde{\eta})}{H_\ell(\widetilde{\eta})} H = \widetilde{\tau}^*(H) = \tau^*_{\mathsf{M}_\mu}(H).$$

We conclude as in the previous case.

• For $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$, we can use Section 7.2 which directly yields the conclusion.

9. Dimension of the level sets $E_{\mathsf{M}_\mu}(H)$ and $\overline{E}_{\mathsf{M}_\mu}(H)$ for $H \geq H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})$

Due to Remark 5 we only have to prove the sharp upper bound for $\dim \overline{E}_{\mathsf{M}_\mu}(H)$.

Proposition 9 yields $\overline{\dim}(\mu, x) \geq \overline{\dim}(\mathsf{M}_\mu, x) - \widetilde{H}_\ell(\widetilde{\eta})$ for all $x \in [0,1]^d$. Hence for all $H \geq 0$ we have $\overline{E}_{\mathsf{M}_\mu}(H) \subset \overline{E}_\mu^{\geq}(H - \widetilde{H}_\ell(\widetilde{\eta}))$. By item (3) of Proposition 2, we deduce that $\dim \overline{E}_{\mathsf{M}_\mu}(H) \leq D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))$ for $H \geq H_s + \widetilde{H}_\ell(\widetilde{\eta})$.

It remains us to treat the case $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_s + \widetilde{H}_\ell(\widetilde{\eta}))$. We already know by item (2) of Proposition 2 that $\dim \left(\overline{E}_{\mathsf{M}_\mu}(H) \cap \underline{E}_\mu^{\leq}(H - \widetilde{H}_\ell(\widetilde{\eta}))\right) \leq D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))$.

In order to complete the proof, it is enough to prove that $\dim \left(\overline{E}_{\mathsf{M}_\mu}(H) \cap \underline{E}_\mu^{\geq}(H - \widetilde{H}_\ell(\widetilde{\eta}))\right) \leq D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))$. For this, consider $x \in \overline{E}_{\mathsf{M}_\mu}(H) \cap \underline{E}_\mu^{\geq}(H - \widetilde{H}_\ell(\widetilde{\eta}))$. Based on the discussion achieved in the proof of Proposition 11, we know that $x \in \widetilde{F}_\mu(H)$, where

$$\widetilde{F}_\mu(H) = \bigcup_{i \in \{\ell, r\}} \bigcap_{\varepsilon \in (0,1)} \bigcap_{k \geq 1} \bigcup_{\substack{(\alpha, \eta', \delta) \in \mathcal{P}_i(H): \\ \delta \in [1, 1/\eta'],\ \frac{\alpha + \widetilde{H}_i(\eta')}{\delta} \leq H + \varepsilon}} \limsup_{j \to +\infty} F_{\mu,\ell}(j, \alpha, \eta', \delta - \varepsilon, k),$$

and $\mathcal{P}_i(H)$ is a countable set of parameters (α, η', δ) dense in $[H - \widetilde{H}_\ell(\widetilde{\eta}), H_{\max}] \times (\eta_i, \eta] \times [1, +\infty)$. Also, if $\alpha \geq H - \widetilde{H}_\ell(\widetilde{\eta})$ and $\frac{\alpha + \widetilde{H}_i(\eta')}{\delta} \leq H + \varepsilon$, then

$$\frac{D_\mu(\alpha)}{\delta} \leq (H + \varepsilon) \frac{D_\mu(\alpha)}{\alpha + \widetilde{H}_i(\eta')} \leq (H + \varepsilon) \frac{D_\mu(\alpha)}{\alpha + \widetilde{H}_\ell(\widetilde{\eta})} \leq (H + \varepsilon) \frac{D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))}{H},$$

since $\alpha \mapsto \frac{D_\mu(\alpha)}{\alpha + \widetilde{H}_\ell(\widetilde{\eta})}$ is decreasing over $[H_\ell(\widetilde{\eta}), H_{\max}]$ and $H \geq H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})$. Then, using the same estimates as in the proof of Proposition 12, we get $\dim \widetilde{F}_\mu(H) \leq D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))$, hence $\dim \left(\overline{E}_{\mathsf{M}_\mu}(H) \cap \underline{E}_\mu^{\geq}(H - \widetilde{H}_\ell(\widetilde{\eta}))\right) \leq D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))$. Hence the conclusion.

Remark 6. *We could conclude for all $H \in [H_{\min} + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$ if we were able to prove the property $\overline{\dim}(\mathsf{M}_\mu, x) \geq \underline{\dim}(\mu, x) + \widetilde{H}_\ell(\widetilde{\eta})$ for all $x \in [0,1]^d$ to hold.*

10. Case of a homogeneous Gibbs measure

We rapidly explain the case of a homogeneous capacity that we denote λ. We assume without loss of generality that for some $\beta > 0$, for every finite word $w \in \Sigma^*$, $\lambda(I_w) \sim 2^{-\beta|w|}$.

In this situation, $H_{\min} = H_s = H_{\max} = \beta$, so $\eta_\ell = \eta_r = \widetilde{\eta} = \eta$, and $H_\ell(\eta_\ell) = H_r(\eta_r) = \beta$. One also has $\widetilde{H}_\ell(\widetilde{\eta}) = \beta(1/\eta - 1)$. Moreover, $q_{\eta_\ell} = +\infty$.

The free energy function $\tau_\lambda(q) = \beta q - d$ is linear, and $q_{\widetilde{\eta}}$ is the solution to $\tau_\lambda(q) = -d(1-\eta)$, i.e. $q_{\widetilde{\eta}} = d\eta/\beta$.

The proof follows exactly the same lines as in the previous sections, except that most of the arguments are trivial. Indeed, all the survivors at a given generation j satisfy $\lambda(I_w) \sim 2^{-j\beta}$ (there is no dependence of the value $\lambda(I_w)$ on the location of w). The sets $\mathcal{R}_\lambda$, $\mathcal{T}_\lambda$ are similarly defined, but are also trivial.

The obtained energy function is

$$\tau_{\mathsf{M}_\lambda}(q) = \begin{cases} \tau_\lambda(q) + \beta(1/\eta - 1)q = q\beta/\eta - d & \text{if } q \leq d\eta/\beta, \\ \tau_\lambda(q) + d(1-\eta) = q\beta - d(1-\eta) & \text{if } q > d\eta/\beta, \end{cases}$$

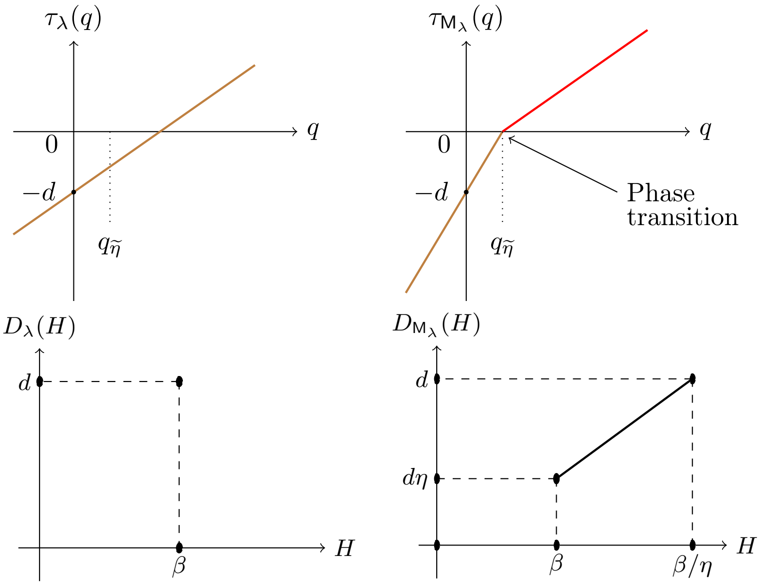

FIGURE 15. **Left:** Free energy function (top) and associated multifractal spectrum (bottom) of a homogeneous capacity λ. **Right:** The almost sure free energy (top) and the multifractal spectrum (bottom) of M_λ.

and the associated multifractal spectrum is

$$D_{\mathsf{M}_\lambda}(H) = \begin{cases} \dfrac{d\eta}{\beta} H & \text{if } H \in [\beta, \beta/\eta], \\ -\infty & \text{otherwise.} \end{cases}$$

Actually, this case was already studied by Jaffard in the context of "lacunary wavelet series" and multifractal analysis of functions [27]. More precisely, Jaffard computes the singularity spectrum of wavelet series whose wavelet coefficients are defined as follows: Fix $(\psi_{j,k})_{j,k\in\mathbb{Z}}$, a wavelet basis of $L^2(\mathbb{R})$ associated with a smooth mother wavelet and normalized so that all its elements have the same L^∞ norm. Fix $\beta > 0$, and for each $j \geq 1$ select uniformly and independently $2^{\lfloor \eta j \rfloor}$ intervals among the 2^j dyadic subintervals of $[0, 1]$ of generation j. Then assign the coefficient $2^{-\beta j}$ to $\psi_{j,k}$ if $[k2^{-j}, (k + 1)2^{-j}]$ has been selected; otherwise assign the coefficient 0. Though different, this sparse collection of coefficients is close to that obtained by sampling the homogeneous capacity λ as above in the special situation where $\lambda(I_w) = 2^{-\beta|w|}$ for all $w \in \Sigma^*$. It turns out that the multifractal analysis of the resulting sparse wavelet series is essentially reducible to that of M_λ, which in this case follows from quite a direct application of homogeneous ubiquity theory [17, 27]. Of course, Jaffard obtained the same multifractal spectrum, although he did not compute the free energy function.

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