MULTIFRACTAL PROPERTIES OF TYPICAL CONVEX FUNCTIONS

ZOLTÁN BUCZOLICH AND STÉPHANE SEURET

ABSTRACT. We study the singularity (multifractal) spectrum of continuous convex functions defined on $[0,1]^d$. Let $E_f(h)$ be the set of points at which f has a pointwise exponent equal to h. We first obtain general upper bounds for the Hausdorff dimension of these sets $E_f(h)$, for all convex functions f and all $h \geq 0$. We prove that for typical/generic (in the sense of Baire) continuous convex functions $f : [0,1]^d \to \mathbb{R}$, one has $\dim E_f(h) = d - 2 + h$ for all $h \in [1,2]$, and in addition, we obtain that the set $E_f(h)$ is empty if $h \in (0,1) \cup (1,+\infty)$. Also, when f is typical, the boundary of $[0,1]^d$ belongs to $E_f(0)$.

1. INTRODUCTION AND MAIN RESULTS

In this paper we investigate the multifractal properties of the continuous convex functions defined on $[0,1]^d$. This paper is a quite natural continuation of our papers [3, 4] where generic multifractal properties of measures and of functions monotone increasing in several variables (in short: MISV) were studied. It is interesting that all these natural objects supported on $[0,1]^d$ have very different typical multifractal behaviors.

Let us first recall that the pointwise Hölder exponent and the singularity spectrum for a locally bounded function are defined as follows.

Definition 1. Let $f \in L^\infty([0,1]^d)$. For $h \geq 0$ and $\mathbf{x} \in [0,1]^d$, the function f belongs to $C^h(\mathbf{x})$ if there are a polynomial P of degree strictly less than $[h]$ and a constant $C > 0$ such that, for all $\mathbf{x}'$ close to $\mathbf{x}$,

$$|f(\mathbf{x}') - P(\mathbf{x}' - \mathbf{x})| \leq C|\mathbf{x}' - \mathbf{x}|^h. \tag{1}$$

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The pointwise Hölder exponent of f at $\mathbf{x}$ is

$$h_f(\mathbf{x}) = \sup\{h \geq 0 : \ f \in C^h(\mathbf{x})\}.$$

In the following, $\dim_H$ denotes the Hausdorff dimension.

Definition 2. The singularity spectrum of f is the mapping

$$d_f(h) = \dim_H E_f(h), \quad \text{where } E_f(h) = \{\mathbf{x} : h_f(\mathbf{x}) = h\}.$$

By convention $\dim \emptyset = -\infty$. We will also use the sets

$$E_f^{\leq}(h) = \{\mathbf{x} : h_f(\mathbf{x}) \leq h\} \supset E_f(h). \tag{2}$$

We denote by $\mathcal{CC}^d$ the set of continuous convex functions $f : [0,1]^d \to \mathbb{R}$. Equipped with the supremum norm $\|\cdot\|$, $\mathcal{CC}^d$ is a separable complete metric space. An open ball in $\mathcal{CC}^d$ of center $f \in \mathcal{CC}^d$ of radius $r \geq 0$ is written as $B_{\|\cdot\|}(f,r)$, and a closed ball is $\overline{B}_{\|\cdot\|}(f,r)$

In this paper, we first prove an upper bound for the multifractal spectrum of all functions in $\mathcal{CC}^d$.

Theorem 1. *For any function $f \in \mathcal{CC}^d$, one has*

$$d_f(h) \leq \begin{cases} d-1 & \text{if } h \in [0,1) \\ d+h-2 & \text{if } h \in [1,2] \\ 2 & \text{if } h > 2. \end{cases}$$

Then we compute the multifractal spectrum of typical functions in $\mathcal{CC}^d$. Recall that a property is typical, or generic in a complete metric space E, when it holds on a residual set, i.e. a set with a complement of first Baire category (a set is of first Baire category if it is the union of countably many nowhere dense sets).

Theorem 2. *For typical functions $f \in \mathcal{CC}^d$, one has*

$$d_f(h) = \begin{cases} d-1 & \text{if } h = 0 \\ d+h-2 & \text{if } h \in [1,2] \\ -\infty & \text{otherwise.} \end{cases}$$

More precisely, one has $E_f(0) = \partial([0,1]^d)$.

It is interesting to compare Theorem 2 with the regularity of other typical objects naturally defined on the cube $[0,1]^d$.

When $d = 1$, generic properties of continuous functions on $[0,1]$ have been studied for a long time (see for instance [5, 6], and the other references we mention in the present paper). It was proved that typical continuous functions belong to $C^0(x)$, for every $x \in [0,1]$. In [2], typical monotone continuous functions on the interval $[0,1]$ were

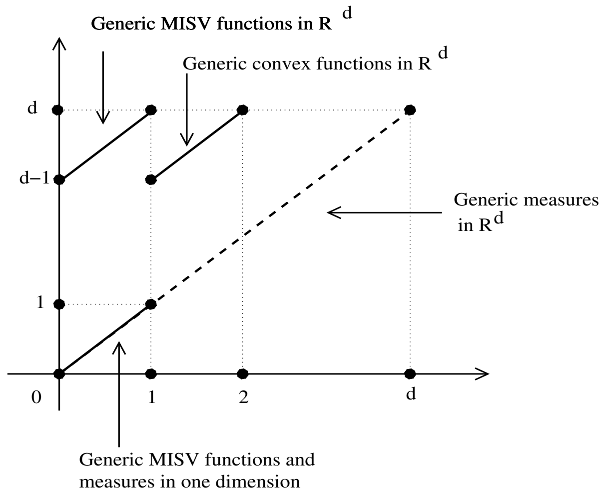

FIGURE 1. Typical spectra for measures, for MISV and for convex functions

proved to have a much more interesting multifractal behavior: such functions $f : [0,1] \to \mathbb{R}$ satisfy

$$d_f(h) = \dim E_f^{\leq}(h) = \begin{cases} h & \text{if } h \in [0,1] \\ -\infty & \text{otherwise.} \end{cases} \tag{3}$$

The same holds true for typical monotone functions (not necessarily continuous). We remark that it also follows, for example from results in [2], that for arbitrary monotone functions there is an upper estimate

$$\dim E_f^{\leq}(h) \leq h \text{ for } h \in [0,1]. \tag{4}$$

It is interesting to extend these results to higher dimensions.

The first natural way is to consider Borel measures on the cube. In [3] (see also [1] for a nice generalization to all compact sets in $\mathbb{R}^d$), it is proved that typical measures μ supported on $[0,1]^d$ satisfy a multifractal formalism, and

$$d_\mu(h) = \begin{cases} h & \text{if } h \in [0,d] \\ -\infty & \text{otherwise.} \end{cases}$$

Another interesting class is constituted of the continuous monotone increasing in several variables (in short: MISV) functions. These functions extend to higher dimensions in a different direction the one-dimensional monotone functions. A function $f : [0,1]^d \to \mathbb{R}$ is MISV when for all $i \in \{1, ..., d\}$, the functions

$$f^{(i)}(t) = f(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_d) \tag{5}$$

are continuous monotone increasing. We use the notation

$$\mathcal{M}^d = \{f \in C([0,1]^d) : f \text{ MISV}\}.$$

The space $\mathcal{M}^d$ is a separable complete metric space when equipped with the supremum L^∞ norm for functions. Typical MISV functions satisfy (see [2, 4])

$$d_f(h) = \begin{cases} d-1+h & \text{if } h \in [0,1] \\ -\infty & \text{otherwise.} \end{cases} \tag{6}$$

On Figure 1 we compare our new results about generic continuous convex functions with the earlier results we mentioned.

Remark 1. One cannot directly infer Theorems 1 and 2 by integrating MISV functions or measures on $[0,1]^d$. For instance, letting $f(x_1,x_2) = 10(x_1^2+x_2^2)+x_1^2x_2^2$, its second differential $d^2f(x_1,x_2,h_1,h_2) = (20+2x_2^2)h_1^2+(20+2x_1^2)h_2^2+4x_1x_2h_1h_2$ is positive definite for any $(x_1,x_2) \in [-1,1]^2$. Hence this function f is strictly convex on $[-1,1]^2$, but $\partial_1 f = 20x_1+2x_1x_2^2$ is monotone only in x_1 and $\partial_2 f = 20x_2+2x_2x_1^2$ is monotone only in x_2.

2. Preliminary Results

2.1. **Basic notation.** If it is not stated otherwise we work in $\mathbb{R}^d$. Points in the space are denoted by $\mathbf{x} = (x_1,...,x_d)$. The j'th unitvector is denoted by

$$\mathbf{e}_j = (0,...,0,\underset{\substack{\uparrow \\ j}}{1},0,...,0).$$

Open balls in $\mathbb{R}^d$ are denoted by $B(\mathbf{x},r)$ (to be distinguished from the balls $B_{\|\cdot\|}(f,\varepsilon)$ in $\mathcal{CC}^d$).

2.2. **Local regularity results of continuous convex functions.** First, recall that C^∞ functions are dense in $\mathcal{CC}^d$.

Remark 2. To see this, take $\psi \geq 0$ a C^∞ function, which is 0 outside $\mathbf{B}(\mathbf{0},1)$, such that $\displaystyle\int_{\mathbb{R}^d}\psi = 1$. Put $\psi_\lambda = \lambda^d\psi(\frac{1}{\lambda}\mathbf{x})$. If f is convex and continuous on $[-1,1]^d$, then the convolution $\overline{f}_\lambda(\mathbf{x}) = \int_{[0,1]^d} f(\mathbf{x})\psi_\lambda(\mathbf{y}-\mathbf{x})d\mathbf{y}$ is convex on $[-1+\lambda,1-\lambda]^d$ and $f_\lambda(\mathbf{x}) = \overline{f}_\lambda(\frac{1}{1-\lambda}\mathbf{x})$ is convex on $[-1,1]^d$. Using the uniform continuity of f on $[-1,1]^d$, one easily sees that $||f-f_\lambda|| \to 0$ as $\lambda \to 0$. One concludes by using an obvious linear transformation.

A first lemma allows one to control the left and right partial derivatives of all functions in a neighborhood of a convex differentiable function in $\mathcal{CC}^d$. The notations $\partial_{j,+}f$ and $\partial_{j,-}f$ are used for the right and left partial j-th derivatives of a convex function f.

Lemma 3. *Suppose $f \in \mathcal{CC}^d \cap C^1([0,1]^d)$ and $\varepsilon > 0$. There exists $\varrho_{f,\varepsilon} > 0$, such that for all $j \in \{1,...,d\}$, if $g \in B_{\|\cdot\|}(f, \varrho_{f,\varepsilon})$, then for every $x_j \in [\varepsilon, 1-\varepsilon]$, $x_i \in [0,1]$, $i \in \{1,...,d\} \setminus \{j\}$ we have*

$$|\partial_{j,\pm}g(x_1,...,x_d) - \partial_j f(x_1,...,x_d)| < \varepsilon. \tag{7}$$

Proof. It is enough to fix one $j \in \{1,...,d\}$. Recall that $f \in \mathcal{CC}^d \cap C^1([0,1]^d)$ implies that $\partial_j f$ is non-decreasing in x_j and is uniformly continuous on $[0,1]^d$.

Using this uniform continuity, there exists a partition $0 = x_{j,0} < x_{j,1} < ... < x_{j,K} = 1$ such that $x_{j,1} < \varepsilon$ and for every $x_i \in [0,1]$ with $i \in \{1,...,d\} \setminus \{j\}$, one has for every $l = 1,...,K$,

$$\partial_j f(x_1,...,x_{j-1},x_{j,l},x_{j+1},...,x_d) - \partial_j f(x_1,...,x_{j-1},x_{j,l-1},x_{j+1},...,x_d) < \frac{\varepsilon}{4}. \tag{8}$$

Set

$$\varrho_{f,\varepsilon,j} = \frac{\varepsilon}{4} \min_{l=2,...,K} (x_{j,l} - x_{j,l-1}). \tag{9}$$

Consider any function $g \in B_{\|\cdot\|}(f, \varrho_{f,\varepsilon,j})$, and fix $\mathbf{x} = (x_1,...,x_d) \in [0,1]^d$, so that $x_j \in [\varepsilon, 1-\varepsilon]$.

There exists an integer $l \in \{1,...,K-1\}$ such that $x_j \in [x_{j,l}, x_{j,l+1}]$.

Set $\mathbf{x}(l') = (x_1,...,x_{j-1},x_{j,l'},x_{j+1},...,x_d)$.

Since the two functions $t \mapsto g(x_1,...,x_{j-1},t,x_{j+1},...,x_d)$ and $t \mapsto f(x_1,...,x_{j-1},t,x_{j+1},...,x_d)$ are real convex, one has

$$\begin{aligned}
\partial_{j,-}g(\mathbf{x}) &\geq \partial_{j,-}g(\mathbf{x}(l)) \geq \frac{g(\mathbf{x}(l)) - g(\mathbf{x}(l-1))}{x_{j,l} - x_{j,l-1}} \\
&\geq \frac{f(\mathbf{x}(l)) - f(\mathbf{x}(l-1)) - 2\varrho_{f,\varepsilon,j}}{x_{j,l} - x_{j,l-1}} \\
&\geq \frac{f(\mathbf{x}(l)) - f(\mathbf{x}(l-1))}{x_{j,l} - x_{j,l-1}} - \frac{\varepsilon}{2} \\
&\geq \partial_j f(\mathbf{x}(l-1)) - \frac{\varepsilon}{2}.
\end{aligned} \tag{10}$$

Thanks to the convexity of $t \mapsto f(x_1,...,x_{j-1},t,x_{j+1},...,x_d)$, equation (8) gives

$$\partial_j f(\mathbf{x}) \leq \partial_j f(\mathbf{x}(l+1)) < \partial_j f(\mathbf{x}(l-1)) + 2\frac{\varepsilon}{4}.$$

Hence, one can continue (10) to obtain

$$\partial_{j,-}g(\mathbf{x}) > \partial_j f(\mathbf{x}) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = \partial_j f(\mathbf{x}) - \varepsilon.$$

Moreover, a similar argument can show that

$$\partial_{j,+}g(\mathbf{x}) < \partial_j f(\mathbf{x}) + \varepsilon.$$

This ends the proof, since $\partial_{j,-}g(\mathbf{x}) \leq \partial_{j,+}g(\mathbf{x})$.

The conclusion follows by taking $\varrho_{f,\varepsilon} = \min(\varrho_{f,\varepsilon,j} : j = 1, ..., d)$. □

The one-dimensional version of Lemma 3 is stated as follows.

Lemma 4. *Suppose $f \in \mathcal{CC}^1 \cap C^1([0,1])$. For $\varepsilon > 0$ there exist $\varepsilon > 0$ and $\varrho_{\varepsilon,f} > 0$ such that for any $g \in B_{\|\cdot\|}(f, \varrho_{\varepsilon,f})$ and $x \in [\varepsilon, 1-\varepsilon]$ we have*

$$|g'_{\pm}(x) - f'(x)| < \varepsilon. \tag{11}$$

Next, one compares the pointwise exponents of a differentiable convex function f and its derivative f'. It is a general property that $h_{f'}(x) \leq h_f(x) - 1$, for every differentiable f. A surprising property is that equality necessarily holds when f is convex and $h_f(x) \in [1,2)$,

Lemma 5. *If f is convex and differentiable on (a,b) and $h_f(x) \in [1,2)$ for some $x \in (a,b)$, then $h_f(x) = h_{f'}(x) + 1$.*

Proof. It is enough to prove that $h_f(x) \leq h_{f'}(x) + 1$. Since $h_f(x) \in [1,2)$, necessarily $h = h_{f'}(x) < 1$. Hence, there exists a sequence $(x_n)_{n\geq 1}$ converging to x such that $|f'(x_n) - f'(x)| > |x_n - x|^{h+\frac{1}{n}}$. Without limiting generality, suppose that $x_n > x$. By the monotonicity of f', one has

$$f'(x_n) > f'(x) + (x_n - x)^{h+\frac{1}{n}}.$$

By convexity of f,

$$f(x_n) \geq f(x) + f'(x)(x_n - x).$$

Setting $x'_n = x + 2(x_n - x)$, and using again the convexity of f at x_n, one gets

$$\begin{aligned}
f(x'_n) &\geq f(x_n) + f'(x_n)(x'_n - x_n) \\
&\geq f(x) + f'(x)(x_n - x) + (f'(x) + (x_n - x)^{h+\frac{1}{n}})(x'_n - x_n) \\
&\geq f(x) + f'(x)(x'_n - x) + (x_n - x)^{h+\frac{1}{n}} \frac{1}{2}(x'_n - x) \\
&= f(x) + f'(x)(x'_n - x) + \frac{1}{2^{h+1+\frac{1}{n}}}(x'_n - x)^{h+1+\frac{1}{n}}.
\end{aligned}$$

This implies $h_f(x) \leq h + 1$, hence the result. □

One investigates what happens for non-differentiable convex functions.

Lemma 6. *If f is convex on $(a, b) \subset \mathbb{R}$ and $h_f(x) \in [1, 2)$ for some $x \in (a, b)$, then* $\min(h_{f'_+}(x), h_{f'_-}(x)) \leq h_f(x) - 1$*.*

Proof. When $h_f(x) = 1$ the lemma is obvious. Set $h = h_f(x) > 1$, and let $\varepsilon > 0$ so that $h - \varepsilon > 0$. By definition, there exists $M \in \mathbb{R}$ such that one has

$$|f(x + y) - f(x) - My| \leq |y|^{h-\varepsilon}$$

for every small y, and there exists a sequence $(y_n)_{n\geq 1}$ converging to zero such that

$$|f(x + y_n) - f(x) - My_n| \geq |y_n|^{h+\varepsilon}.$$

Hence,

$$|y_n|^{h+\varepsilon-1} \leq \left| \frac{f(x + y_n) - f(x)}{y_n} - M \right| \leq |y_n|^{h-\varepsilon-1}.$$

Since the left and right derivatives $f'_+(x)$ and $f'_-(x)$ both exist, they both equal $M = f'(x)$.

Assume, without loss of generality, that there are infinitely many positive y_n's. For every y_n,

$$f'_+(x + y_n) - f'_+(x) \geq \frac{f(x + y_n) - f(x)}{y_n} - f'_+(x) \geq |y_n|^{h+\varepsilon-1},$$

thus $h_{f'_+}(x) \leq h + \varepsilon - 1$, which gives the result. □

We also prove the following proposition, which somehow asserts that a convex function cannot have exceptional isolated directional pointwise regularity.

For this, consider the d-dimensional unit sphere $S_d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\}$. Then, we select a finite set of pairwise distinct points $(\mathbf{z}_1, \mathbf{z}_2, ...\mathbf{z}_N) \in (S_d)^N$ for some integer $N \geq 1$ such that the convex hull of $\{\mathbf{z}_1, \mathbf{z}_2, ...\mathbf{z}_N\}$ contains the d-dimensional ball $B(\mathbf{0}, 1/2)$.

Let us choose $\varepsilon_c > 0$ so small that :

- $0 < \varepsilon_c \leq \frac{1}{1000} \min(\|\mathbf{z}_i - \mathbf{z}_j\|,\ i \neq j,\ i, j \in \{1, ..., N\})$.
- Setting for every $i \in \{1, ..., N\}$

(12) $$C_i = S_d \cap B(\mathbf{z}_i, \varepsilon_c),$$

  then for any choice of $\mathbf{z}'_i \in C_i$, the convex hull of $\{\mathbf{z}'_1, \mathbf{z}'_2, ...\mathbf{z}'_N\}$ contains the ball $B(\mathbf{0}, 1/4)$.

Proposition 7. *If $h_f(\mathbf{x}) = h$, then there exists $i \in \{1, ..., N\}$ such that for every $\mathbf{z}'_i \in C_i$ (see (12)), the restriction of f to the straight line passing through $\mathbf{x}$ parallel to the vector $\mathbf{z}'_i$ has a pointwise Hölder exponent equal to h.*

Proof. Let n be such that $n \leq h < n+1$. We assume without loss of generality that $\mathbf{x} = \mathbf{0}$, $f(\mathbf{0}) = 0$, $D^k f(\mathbf{0}, ..., \mathbf{0}) = 0$ for every $k \in \{1, ..., n\}$. Let $\varepsilon > 0$. By definition, for every $\mathbf{x}$ close to $\mathbf{0}$, $|f(\mathbf{x})| \leq |\mathbf{x}|^{h-\varepsilon}$, and there exists a sequence $(\mathbf{x}_n = (x_{n,1}, ..., x_{n,d}))_{n\geq 1}$ of elements in $\mathbb{R}^d$, converging to $\mathbf{0}$, such that

$$|\mathbf{x}_n|^{h+\varepsilon} \leq |f(\mathbf{x}_n)| \leq |\mathbf{x}_n|^{h-\varepsilon}. \tag{13}$$

Consider such an element $\mathbf{x}_n$, and the sets $(C_{n,i} := 4\|\mathbf{x}_n\| \cdot C_i)_{i=1,...,N}$. Let us prove that there exists $i_n \in \{1, ..., N\}$ such that for every $n \in \mathbb{N}$ and $\mathbf{z}'_{i_n} \in C_{n,i_n}$, $|f(\mathbf{z}'_{i_n})| \geq (|\mathbf{z}'_{i_n}|/4)^{h+\varepsilon}$.

Assume first that for every $i \in \{1, ..., N\}$, there exists $\mathbf{z}'_i \in C_{n,i}$ such that $|f(\mathbf{z}'_i)| < |f(\mathbf{x}_n)|$. By construction, the ball $B(\mathbf{0}, \|\mathbf{x}_n\|)$ is included in the convex hull of these points $(\mathbf{z}'_i)_{i=1,...,N}$. By convexity, this would imply that $|f(\mathbf{x}_n)| \leq \max(|f(\mathbf{z}'_i)| : i = 1, ..., N)$, hence a contradiction.

Hence, there exists an $i_n \in \{1, ..., N\}$ such that for every $\mathbf{z}'_{i_n} \in C_{n,i_n}$, $|f(\mathbf{z}'_{i_n})| \geq |f(\mathbf{x}_n)| \geq |\mathbf{x}_n|^{h+\varepsilon} = (|\mathbf{z}'_{i_n}|/4)^{h+\varepsilon}$.

Turning to a subsequence, one can assume that i_n is constant, equal to $i \in \{1, ..., N\}$.

Now, as a consequence of what precedes, for every vector $\mathbf{z}'_i \in C_i$, there exists an infinite number of values $(r_n = |\mathbf{x}_n|)_{n\geq 1}$ such that $|f(r_n \mathbf{z}'_i)| \geq (|r_n \mathbf{z}'_i|/4)^{h+\varepsilon}$. Hence, the restriction of f to the straight line passing through $\mathbf{0}$ parallel to $\mathbf{z}'_i$ has a pointwise Hölder exponent less than $h + \varepsilon$. Since this holds for every $\varepsilon > 0$, and obviously this exponent is bounded below by h (i.e. the exponent of f), one concludes that this restriction has exactly exponent h. □

3. First typical properties of continuous convex functions

Based on Lemma 3 it is very easy to see that the typical function in $\mathcal{CC}^d$ is continuously differentiable on $(0,1)^d$. This was proved in [5, 6] for instance. We give another proof for completeness.

Proposition 8. *There is a dense G_δ set $\mathcal{G}$ in $\mathcal{CC}^d$ such that every $f \in \mathcal{G}$ is continuously differentiable on $(0,1)^d$.*

Proof. By convexity, the partial derivatives $\partial_{j,\pm} f(\mathbf{x})$ exist for any $f \in \mathcal{CC}^d$, $\mathbf{x} \in (0,1)^d$ and $j \in \{1, ..., d\}$.

Since $\mathcal{CC}^d$ is separable, one can choose a sequence of convex functions $\{f_m : m = 1, ...\}$ dense in $\mathcal{CC}^d$. In addition, by Remark 2, one can assume that all these functions f_m are $C^\infty([0,1]^d)$ functions.

By uniform continuity of all the partial derivatives of f_m, there is a $\delta_{n,m} > 0$ such that for every j, for every $\mathbf{x}, \mathbf{x}' \in [0,1]^d$,

$$|\partial_j f_m(\mathbf{x}) - \partial_j f_m(\mathbf{x}')| < \frac{1}{n}, \quad \text{when } |\mathbf{x} - \mathbf{x}'| < \delta_{n,m}. \tag{14}$$

Applying Lemma 3, it is possible to choose $0 < \varrho_{n,m} < \frac{1}{n+m}$ such that if $f \in B_{\|\cdot\|}(f_m, \varrho_{n,m})$ then for every $j \in \{1, ..., d\}$, if $\mathbf{x} = (x_1, ..., x_d) \in [0,1]^d$ with $x_j \in [\frac{1}{n}, 1 - \frac{1}{n}]$, then

$$|\partial_{j,\pm} f(\mathbf{x}) - \partial_j f_m(\mathbf{x})| < \frac{1}{n}. \tag{15}$$

Let us introduce the sets

$$\mathcal{G}_n = \bigcup_{m=1}^{\infty} B_{\|\cdot\|}(f_m, \varrho_{n,m}) \quad \text{and} \quad \mathcal{G} = \bigcap_{n=1}^{\infty} \mathcal{G}_n.$$

It is clear that $\mathcal{G}$ is a dense G_δ set in $\mathcal{CC}^d$.

We prove that any $f \in \mathcal{G}$ is continuously differentiable.

By definition, there exists an infinite sequence of integers $(m_n)_{n\geq 1}$ such that $f \in B_{\|\cdot\|}(f_{m_n}, \varrho_{n,m_n})$.

Fix $j \in \{1, .., d\}$, and focus on the j-th partial derivatives. Combining inequalities (14) and (15), if $\mathbf{x}, \mathbf{x}' \in [0,1]^d$, with $x_j, x'_j \in [\frac{1}{n}, 1 - \frac{1}{n}]$ and $||\mathbf{x} - \mathbf{x}'|| < \varrho_{n,m_n}$, then

$$|\partial_{j,\pm} f(\mathbf{x}) - \partial_{j,\pm} f(\mathbf{x}')| < |\partial_j f_{m_n}(\mathbf{x}) - \partial_j f_{m_n}(\mathbf{x}')| + \frac{2}{n} < \frac{3}{n}.$$

The $\pm$ is the above inequality means that any choice of left or right derivative can be made.

From this, letting $n \to \infty$, it follows easily that $\partial_{j,+} f(\mathbf{x}) = \partial_{j,-} f(\mathbf{x})$, hence $\partial_j f$ is continuous on $(0,1)^d$. □

However, the next lemma shows that typical functions f in $\mathcal{CC}^d$ are not differentiable on $[0,1]^d$. The problem comes from the boundary of the domain.

Proposition 9. *There is a dense G_δ set $\mathcal{G}_\infty$ in $\mathcal{G}$ such that every $f \in \mathcal{G}_\infty$ satisfies the following: for every $j \in \{1, ..., d\}$, for every $x_i \in [0,1]$ with $i \in \{1, ..., d\} \setminus \{j\}$, one has*

$$\partial_{j,+} f(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_d) = -\infty \tag{16}$$

and

$$\partial_{j,-} f(x_1, ..., x_{j-1}, 1, x_{j+1}, ..., x_d) = +\infty. \tag{17}$$

Moreover,

(18)
$$h_f(x_1,...,x_{j-1},0,x_{j+1},...,x_d)=0=h_f(x_1,...,x_{j-1},1,x_{j+1},...,x_d).$$

Proof. We are going to show that for a fixed j, there is a dense G_δ set $\mathcal{G}^{0,j}$ in $\mathcal{CC}^d$ such that if $f\in\mathcal{G}^{0,j}$ then (16) and the first equality in (18) holds.

Without loss of generality, one considers $j=1$. As in the proof of Theorem 8, one chooses a sequence of C^∞ functions $(f_m)_{m\geq 1}$. We also select integer constants $M_m\geq 1$ such that $|\partial_1 f_m|\leq M_m$ on $[0,1]^d$.

For every integer $l\geq 1$, let us introduce the mapping φ_l defined as

$$\varphi_l(x_1,...,x_d)=\begin{cases}-l^{l-1}x_1 & \text{if } 0\leq x_1\leq l^{-l},\\ -l^{-1} & \text{if } l^{-l}\leq x_1\leq 1.\end{cases}$$

Clearly, φ_l is convex and for any fixed n the functions $f_m+\varphi_{n\cdot m\cdot M_m}$, $m=1,...$ are dense in $\mathcal{CC}^d$. Set

$$\mathcal{G}^{0,1}_n=\bigcup_{m=1}^{\infty}B_{\|\cdot\|}(f_m+\varphi_{n\cdot m\cdot M_m},(n\cdot m\cdot M_m)^{-n\cdot m\cdot M_m}),$$

and

$$\mathcal{G}^{0,1}=\bigcap_{n=1}^{\infty}\mathcal{G}^{0,1}_n.$$

We prove that if $f\in\mathcal{G}^{0,1}$, then (18) and hence (16) hold. By definition, there exists an infinite sequence of integers $(m_n)_{n\geq 1}$ such that

$$f\in B_{\|\cdot\|}\big(f_{m_n}+\varphi_{n\cdot m_n\cdot M_{m_n}},(n\cdot m_n\cdot M_{m_n})^{-n\cdot m_n\cdot M_{m_n}}\big).$$

For ease of notation, set $l_n=n\cdot m_n\cdot M_{m_n}$, that is

$$f\in B_{\|\cdot\|}(f_{m_n}+\varphi_{l_n},l_n^{-l_n}).$$

Consider $\mathbf{x}\in[0,1]^d$, with $x_1=0$. Then for every $n\geq 5$,

$$\begin{aligned}
& f(0,x_2,...,x_d)-f(l_n^{-l_n},x_2,...,x_d)\\
\geq\ & (f_{m_n}+\varphi_{l_n})(0,x_2,...,x_d))-(f_{m_n}+\varphi_{l_n})(l_n^{-l_n},x_2,...,x_d))-2\cdot l_n^{-l_n}\\
\geq\ & -M_{m_n}l_n^{-l_n}+l_n^{-1}-2\cdot l_n^{-l_n}\\
=\ & l_n^{-1}\Big(1-\frac{M_{m_n}}{l_n}l_n^{-l_n+2}-\frac{2}{l_n}l_n^{-l_n+1}\Big)\\
>\ & \frac{1}{2l_n},
\end{aligned}$$

where we have used the boundedness of ∂f_{m_n} by M_{m_n} and the fact that $l_n \gg M_{m_n}$. Hence, for any $\alpha > 0$ we have

$$\frac{f(0, x_2, ..., x_d) - f(l_n^{-l_n}, x_2, ..., x_d)}{l_n^{-l_n \alpha}} > \frac{1}{2} l_n^{l_n \alpha - 1},$$

which tends to infinity when n goes to infinity. Hence (16) holds true, and one also deduces that $h_f(0, x_2, ..., x_d) = 0$.

Similar arguments yield G_δ sets $\mathcal{G}^{0,j}$ for $j > 1$ and $\mathcal{G}^{1,j}$ for $j = 1, ..., d$ such that the functions f in these sets satisfy the corresponding equalities in (16-18).

Finally, the set

$$\mathcal{G}_\infty = \bigcap_{j=1}^{d} \mathcal{G}^{0,j} \cap \mathcal{G}^{1,j}$$

satisfies the conditions of Proposition 9. □

We conclude this section by proving that typical convex functions have only pointwise exponents less than 2.

Proposition 10. *There exists a G_δ-set $\mathcal{G}^2$ such that for every $f \in \mathcal{G}^2$, $E_f(h) = \emptyset$ for every $h > 2$.*

Before proving Proposition 10, we introduce some perturbation functions already used in Section 4 of [4].

Definition 3. For every $l \in \mathbb{N}$, the function $\gamma_l : [0, 1] \to [0, 1]$ is defined as follows:

- γ_l is continuous,
- For every integer $j = 0, ..., 2^{l^2} - 1$, if $x_1 \in [j 2^{-l^2}, (j + 1) 2^{-l^2} - 2^{-l^4}]$, one sets $\gamma_l(x_1) = j 2^{-l^2 - l}$. So γ_l is constant on these intervals,
- $\gamma_l(1) = 2^{-l}$,
- For every integer $j = 0, ..., 2^{l^2} - 1$, the mapping γ_l is affine on the intervals $[(j + 1) 2^{-l^2} - 2^{-l^4}, (j + 1) 2^{-l^2}]$,

These functions γ_l are continuous, ranging from 0 to 2^{-l}, and are strictly increasing only on the 2^{l^2} many very small, uniformly distributed, intervals of length 2^{-l^4}.

Proof. We start by selecting a set of C^∞ functions $\{f_m : m = 1, 2, ...\}$ which is dense in $\mathcal{CC}^d$. We choose an integer $M_{m,3} \geq 1$ such that the second and third partial derivatives $\partial_1^3 f_m$ with respect to the first variable x_1 of f_m satisfy $|\partial_1^2 f_m| + |\partial_1^3 f_m| \leq M_{m,3}$.

Observe that the x_1-partial derivatives $\partial_1 f_m$ of these functions are monotone in the first variable.

Then, one introduces the auxiliary functions, depending only on the first variable: for $l \geq 0$,

$$\overline{\gamma}_l(x_1, x_2, ..., x_d) = \gamma_l(x_1). \tag{19}$$

The perturbation functions are defined for $l \geq 0$ by

$$\overline{f}_l(\mathbf{x}) = \overline{f}_l(x_1, x_2, ..., x_d) = \int_0^{x_1} \overline{\gamma}_l(t, x_2, ..., x_d)dt = \int_0^{x_1} \gamma_l(t)dt. \tag{20}$$

Next we apply Lemma 3 to the functions $f_m + \overline{f}_{m+M_{m,3}+n}$ with $\varepsilon = \varepsilon_{m,n} := 2^{-(m+M_{m,3}+n)^8}$ and $j = 1$. There exists a constant, denoted by $\varrho_{m,n} > 0$, such that if $f \in B_{\|\cdot\|}(f_m + \overline{f}_{m+M_{m,3}+n}, \varrho_{m,n})$, then for any $x_1 \in [\varepsilon_{m,n}, 1 - \varepsilon_{m,n}]$ and $x_j \in [0, 1]$ for $j = 2, ..., d$, one has

$$|\partial_{1,\pm} f(\mathbf{x}) - \partial_1(f_m + \overline{f}_{m+M_{m,3}+n})(\mathbf{x})| < \varepsilon_{m,n}. \tag{21}$$

Without limiting generality one can assume that $\varrho_{m,n} \leq \varepsilon_{m,n}$.

Set

$$\mathcal{R}_n = \bigcup_{m=1}^{\infty} B_{\|\cdot\|}(f_m + \overline{f}_{m+M_{m,3}+n}, \varrho_{m,n}).$$

It is not difficult to see that $\mathcal{R}_n$ is open and dense in $\mathcal{CC}^d$. Suppose that $\mathcal{G}_\infty$ is the dense G_δ set from Proposition 9. Since $\mathcal{G}_\infty$ is a subset of $\mathcal{G}$ from Proposition 8 all $f \in \mathcal{G}_\infty$ are continuously differentiable on $(0, 1)^d$. Moreover, the Hölder exponent is zero of these functions on the boundary of $[0, 1]^d$.

Finally, set

$$\mathcal{G}^2 = \mathcal{G}_\infty \bigcap \left(\bigcap_{n=1}^{\infty} \mathcal{R}_n \right).$$

By construction, there exists a sequence of integers $(m_n)_{n\geq 1}$ such that $f \in B_{\|\cdot\|}(f_{m_n} + \overline{f}_{m_n+M_{m_n,3}+n}, \varrho_{n,m_n})$ for every n.

For simplification, we set $l_n = m_n + M_{m_n,3} + n$, $\rho_n := \rho_{m_n,n}$ and $\varepsilon_n := \varepsilon_{m_n,n}$, so that for every $n \geq 1$, $\rho_n \leq \varepsilon_n = 2^{-(l_n)^8}$, $f \in B_{\|\cdot\|}(f_{m_n} + \overline{f}_{l_n}, \varrho_n)$, and for any $x_1 \in [\varepsilon_n, 1 - \varepsilon_n]$ and $x_j \in [0, 1]$ for $j = 2, ..., d$

$$|\partial_{1,\pm} f(\mathbf{x}) - \partial_1(f_{m_n} + \overline{f}_{l_n})(\mathbf{x})| < \varepsilon_n. \tag{22}$$

Proceeding towards a contradiction, suppose that there exists $\mathbf{x} = (x_1, x_2..., x_d) \in [0, 1]^d$ where $h_f(\mathbf{x}) > 2$. Since the Hölder exponent of f is zero on the boundary of $[0, 1]^d$, necessarily $\mathbf{x} \in (0, 1)^d$.

Since $h_f(\mathbf{x}) > 2$, one can find $\varepsilon > 0$, $D_2, C_{\mathbf{x}} \in \mathbb{R}$ such that for every small h,

$$|f(\mathbf{x} + h\mathbf{e}_1) - f(\mathbf{x}) - \partial_1 f(\mathbf{x})h - D_2 h^2| \leq C_{\mathbf{x}} |h|^{2+\varepsilon}. \tag{23}$$

Without limiting generality we can suppose that $\varepsilon < 1/2$ holds as well.

Consider the unique integer $j_n \in \mathbb{N}$ such that $x_1 \in [j_n 2^{-l_n^2}, (j_n + 1)2^{-l_n^2})$. Next we consider two cases depending on whether $x_1 \in [j_n 2^{-l_n^2}, (j_n + 1)2^{-l_n^2} - 2^{-l_n^4}]$, or $x_1 \in [(j_n+1)2^{-l_n^2} - 2^{-l_n^4}, (j_n+1)2^{-l_n^2}]$.

Case 1. Assume that $x_1 \in [j_n 2^{-l_n^2}, (j_n + 1)2^{-l_n^2} - 2^{-l_n^4}]$ for infinitely many integers $n \geq 1$.

We set $\mathbf{h}_n = h_n \mathbf{e}_1$ with $|\mathbf{h}_n| = |h_n| = 2^{-l_n^2}/4$, such that the first coordinates of $\mathbf{x}$ and $\mathbf{x} + \mathbf{h}_n$ both belong to $[j_n 2^{-l_n^2}, (j_n + 1)2^{-l_n^2} - 2^{-l_n^4}]$.

Combining (22), (23) and the fact that $f \in B_{\|\cdot\|}(f_{m_n} + \overline{f}_{l_n}, \varrho_n)$, we obtain

$$\begin{aligned}
&\Big| \Big(f_{m_n}(\mathbf{x}+\mathbf{h}_n) + \overline{f}_{l_n}(\mathbf{x}+\mathbf{h}_n) \Big) - \Big(f_{m_n}(\mathbf{x}) + \overline{f}_{l_n}(\mathbf{x}) \Big) \\
&-h_n(\partial_1 f_{m_n} + \partial_1 \overline{f}_{l_n})(\mathbf{x}) - D_2 h_n^2 \ \Big| \\
&\leq C_{\mathbf{x}} |h_n|^{2+\varepsilon} + 2\varrho_n + \varepsilon_n |h_n| \\
&\leq C_{\mathbf{x}} |h_n|^{2+\varepsilon} + 2 \cdot 2^{-(l_n)^8} + 2^{-(l_n)^8} |h_n| \\
&\leq (C_{\mathbf{x}} + 1)|h_n|^{2+\varepsilon}, \qquad (24)
\end{aligned}$$

where the last inequality holds since $\rho_n \leq \varepsilon_n \leq 2^{-(l_n)^8} \ll |h_n|^3$ for large n.

By using the Taylor polynomial estimate of the C^∞ function f_{m_n}, one deduces that

$$\Big| f_{m_n}(\mathbf{x}+\mathbf{h}_n) - f_{m_n}(\mathbf{x}) - \partial_1 f_{m_n}(\mathbf{x}) h_n - \frac{\partial_1^2 f_{m_n}(\mathbf{x})}{2!} h_n^2 \Big| \leq \frac{M_{m_n,3}}{3!} |h_n|^3. \qquad (25)$$

Since by its definition $\partial_1 \overline{f}_{l_n}(\mathbf{y}) = \gamma_{l_n}(x_1)$ (i.e. it is constant) for any $\mathbf{y}$ on the line segment connecting $\mathbf{x}$ and $\mathbf{x} + \mathbf{h}_n$, we also have

$$\overline{f}_{l_n}(\mathbf{x}+\mathbf{h}_n) - \overline{f}_{l_n}(\mathbf{x}) - \partial_1 \overline{f}_{l_n}(\mathbf{x}) h_n = 0. \qquad (26)$$

Using (24), (25) and (26) we infer

$$\begin{aligned}
&\Big| \frac{\partial_1^2 f_{m_n}(\mathbf{x})}{2!} h_n^2 - D_2 h_n^2 \Big| \qquad (27) \\
&\leq (C_{\mathbf{x}} + 1)|h_n|^{2+\varepsilon} + \frac{M_{m_n,3}}{3!} |h_n|^3 < (C_{\mathbf{x}} + 2)|h_n|^{2+\varepsilon}
\end{aligned}$$

where, using the fact that $\varepsilon < 1/2$, the last inequality holds if n is sufficiently large since $M_{n,3} \leq l_n \leq 2^{l_n} \leq |h_n|^{-1/2}$ for n large.

Now take $\overline{\mathbf{h}}_n = 8|h_n|\mathbf{e}_1$. Then (24) and (25) used with $\overline{\mathbf{h}}_n$ instead of $\mathbf{h}_n$ for sufficiently large n yield

$$\begin{aligned}
&\left|\overline{f}_{l_n}(\mathbf{x}+\overline{\mathbf{h}}_n) - \overline{f}_{l_n}(\mathbf{x}) - \partial_1\overline{f}_{l_n}(\mathbf{x})|\overline{\mathbf{h}}_n| + \frac{\partial_1^2 f_{m_n}(\mathbf{x})}{2!}|\overline{\mathbf{h}}_n|^2 - D_2|\overline{\mathbf{h}}_n|^2\right| \\
&\leq \frac{M_{m_n,3}}{3!}|\overline{\mathbf{h}}_n|^3 + (C_{\mathbf{x}}+1)|\overline{\mathbf{h}}_n|^{2+\varepsilon} \\
&= |\overline{\mathbf{h}}_n|^{2+\varepsilon}(\frac{M_{m_n,3}}{3!}|\overline{\mathbf{h}}_n|^{1-\varepsilon} + C_{\mathbf{x}}+1) \\
&\leq |\overline{\mathbf{h}}_n|^{2+\varepsilon}(C_{\mathbf{x}}+2). \qquad (28)
\end{aligned}$$

Now for large n, it follows from (27) and $|h_n| < |\overline{\mathbf{h}}_n|$ that

$$\left|\overline{f}_{l_n}(\mathbf{x}+\overline{\mathbf{h}}_n) - \overline{f}_{l_n}(\mathbf{x}) - \partial_1\overline{f}_{l_n}(\mathbf{x})|\overline{\mathbf{h}}_n|\right| \leq |\overline{\mathbf{h}}_n|^{2+\varepsilon}(2C_{\mathbf{x}}+4). \qquad (29)$$

Next we obtain a contradiction by using a lower estimate of the left-hand side of (29). By convexity of $\overline{f}_{l_n}$ we have $\partial_1\overline{f}_{l_n}(\mathbf{y}) \geq \partial_1\overline{f}_{l_n}(\mathbf{x})$ when $\mathbf{y}$ is on the line segment connecting $\mathbf{x}$ and $\mathbf{x}+\overline{\mathbf{h}}_n$. Even more, this interval contains a subinterval of length larger than $|\overline{\mathbf{h}}_n|/8 = |h_n|$ where

$$\partial_1\overline{f}_{l_n}(\mathbf{y}) = \overline{\gamma}_{l_n}(\mathbf{y}) = \partial_1\overline{f}_{l_n}(\mathbf{x}) + 2^{-l_n^2-l_n} = \gamma_{l_n}(x_1) + 2^{-l_n^2-l_n}.$$

Thus,

$$\overline{f}_{l_n}(\mathbf{x}+\overline{\mathbf{h}}_n) - \overline{f}_{l_n}(\mathbf{x}) \geq \partial_1\overline{f}_{l_n}(\mathbf{x})|\overline{\mathbf{h}}_n| + \frac{|\overline{\mathbf{h}}_n|}{8}\cdot 2^{-l_n^2-l_n}.$$

By (29), one should have

$$\frac{|\overline{\mathbf{h}}_n|}{8}\cdot 2^{-l_n^2-l_n} \leq |\overline{\mathbf{h}}_n|^{2+\varepsilon}(2C_{\mathbf{x}}+4).$$

Since $8|h_n| = |\overline{\mathbf{h}}_n| = 2\cdot 2^{-l_n^2}$ we would obtain that

$$2^{-l_n^2-l_n} \leq (2\cdot 2^{-l_n^2})^{1+\varepsilon}(2C_{\mathbf{x}}+4),$$

a contradiction when n is large.

Case 2. Suppose that $x_1 \in [(j_n+1)2^{-l_n^2} - 2^{-l_n^4}, (j_n+1)2^{-l_n^2}]$ for infinitely many $n \geq 1$.

We set $\mathbf{h}_n = h_n\mathbf{e}_1$ with $|\mathbf{h}_n| = |h_n| = 2^{-l_n^4}/2$, such that the first coordinates of $\mathbf{x}$ and $\mathbf{x}+\mathbf{h}_n$ both belong to $[(j_n+1)2^{-l_n^2} - 2^{-l_n^4}, (j_n+1)2^{-l_n^2}]$.

Since ρ_n and ε_n are still much smaller than $|h_n|$, equations (24) and (25) still hold, but now (26) is replaced by

$$\overline{f}_{l_n}(\mathbf{x}+\mathbf{h}_n) - \overline{f}_{l_n}(\mathbf{x}) - \partial_1\overline{f}_{l_n}(\mathbf{x})h_n = \frac{\partial_1^2\overline{f}_{l_n}(\mathbf{x})}{2!}h_n^2 = 2^{l_n^4-l_n-l_n^2}\frac{h_n^2}{2}. \qquad (30)$$

Using the same arguments as before but with (30), we deduce that

$$\Big|\frac{\partial_1^2 f_{m_n}(\mathbf{x})}{2!}h_n^2 - D_2 h_n^2 + 2^{l_n^4 - l_n - l_n^2}\frac{h_n^2}{2}\Big| < (C_{\mathbf{x}}+2)|h_n|^{2+\varepsilon}.$$

This last inequality becomes impossible when n becomes large, since $|\frac{\partial_1^2 f_{m_n}(\mathbf{x})}{2}| \leq M_{m_n,3} \leq l_n \leq 2^{l_n^3}$. Hence a contradiction.

□

4. The upper estimate

We start with the upper bound for the Hausdorff dimensions of the sets $E_f^{\leq}(h)$.

Proposition 11. *If $1 < h \leq 2$ and $f \in \mathcal{CC}^d$, then $\dim E_f^{\leq}(h) \leq d+h-2$.*

Proof. It is sufficient to treat the case $h \in (1,2)$.

Assume that $\dim E_f^{\leq}(h) > d+h-2$.

For every $\mathbf{x} \in E_f^{\leq}(h)$, by Lemma 7, there exists a cone of direction C_{i_x}, where $i_x \in \{1,...,d\}$ such that for every $\mathbf{z} \in C_{i_x}$, the restriction of f to the straight line passing through $\mathbf{x}$ parallel to $\mathbf{z}$ has exponent less than h.

Let us call E_i the set of elements of $E_f^{\leq}(h)$ satisfying this property with $i_x = i \in \{1,...,N\}$. Obviously, $E_f^{\leq}(h) = \bigcup_{i=1}^N E_i$, so there exists at least one $i \in \{1,...,N\}$ such that $\dim E_i > d+h-2$.

Let us recall the following special case of Marstrand's slicing theorem, Theorem 10.10 in Chapter 10 of [7]. Recall that S_d is the unit sphere in $\mathbb{R}^d$.

Theorem 12. *Let $E \subset [0,1]^d$ be a Borel set with Hausdorff dimension $\alpha \in (d-1,d)$. Then for almost every $\mathbf{z} \in S_d$ (in the sense of $(d-1)$-dimensional "surface" measure), there exists a set $E_{\mathbf{z}}$ of positive $(d-1)$-dimensional Hausdorff measure in the hyperplane orthogonal to $\mathbf{z}$ such that for every $\mathbf{x} \in E_{\mathbf{z}}$, $\dim E \cap (\mathbf{x}+\mathbb{R}\mathbf{z}) = \alpha-(d-1)$.*

Each C_i has non-empty interior in the subspace topology of S_d, hence it is of positive $d-1$-dimensional measure. Applying Theorem 12 to E_i, one can find $\mathbf{z} \in C_i$ and $\overline{\mathbf{x}} \in [0,1]^d$ such that if $\mathcal{D} = (\overline{\mathbf{x}}+\mathbb{R}\mathbf{z})$, then $\dim E_i \cap \mathcal{D} \geq d+h-2-(d-1) = h-1$.

Let us call g the restriction of f to $\mathcal{D}$. Then g is still a convex function of one variable.

By definition of E_i, every $\mathbf{x} \in \mathcal{D} \cap E_i$ satisfies $h_g(\mathbf{x}) \leq h$.

Next, applying Lemma 6 to g, we deduce that $\min(h_{g'_+}(\mathbf{x}), h_{g'_-}(\mathbf{x})) \leq h-1$, for every $\mathbf{x} \in \mathcal{D} \cap E_i$.

Hence, at least one of the two sets $E^{\leq}_{g'_+}(h-1)$ and $E^{\leq}_{g'_-}(h-1)$ has Hausdorff dimension strictly greater than $h-1$.

But this is impossible, since both functions g'_+ and g'_- are monotone, and for such functions, by (4), the Hausdorff dimension of $E^{\leq}_{g'_+}(h-1)$ and $E^{\leq}_{g'_-}(h-1)$ is necessarily less than $h-1 \in [0,1]$. Hence a contradiction, and the conclusion that $\dim E^{\leq}_f(h) \leq d+h-2$. □

Proposition 13. *If* $0 \leq h \leq 1$*,* $f \in \mathcal{CC}^d$*, then* $\dim E^{\leq}_f(h) \leq d-1$*.*

Proof. The proof is immediate: if $f \in \mathcal{CC}^d$, the pointwise exponent of f at any $\mathbf{x} \in (0,1)^d$ is necessarily larger or equal than 1. The remaining points are located on the boundary, whose dimension is $d-1$. And for every $h \in [0,1]$, it is easy to build examples of convex functions such that $h_f(\mathbf{x}) = h$ for every $\mathbf{x}$ satisfying $x_1 = 0$, so the upper bound $d-1$ for the Hausdorff dimension of $E^{\leq}_f(h)$ is optimal. □

5. The lower estimate

Using Lemma 4 it is rather easy to "integrate" the result about functions in $\mathcal{M}^1 = \mathcal{M}$ to obtain the one-dimensional result.

Theorem 14. *There is a dense* G_δ *set* $\mathcal{G}$ *in* $\mathcal{CC}^1$ *such that for any* $f \in \mathcal{G}$*, and for any* $1 \leq h \leq 2$*,* $\dim E_f(h) = h-1$*.*

Proof. Suppose $0 < \delta_0 < 1/2$ is fixed.

Recalling the result on typical monotone continuous functions, there exists a G_δ set of functions $\mathcal{G}^{\mathcal{M},\delta_0}$ in the set $\mathcal{M}^{1,\delta_0} := \{f : [\delta_0, 1-\delta_0] \to \mathbb{R},\ f \text{ monotone}\}$ such that every $f \in \mathcal{G}^{\mathcal{M},\delta_0}$ satisfies (3).

Let us write $\mathcal{G}^{\mathcal{M},\delta_0} = \bigcap_{n\geq 1} \mathcal{G}^{\mathcal{M},\delta_0}_n$, where $\mathcal{G}^{\mathcal{M},\delta_0}_n$ is a dense open set in $\mathcal{M}^{1,\delta_0}$.

Let us choose a dense sequence $(g_{n,m})_{m=1}^{\infty}$ in $\mathcal{G}^{\mathcal{M},\delta_0}_n$.

By taking antiderivatives of the elements of this sequence and by a suitable definition on the intervals $[0,\delta_0) \cup (1-\delta_0, 1]$, one can choose a sequence of convex functions $(f_{n,m,k})_{m,k=1}^{+\infty}$ which is dense in $\mathcal{CC}^1$ such that for every $m,k \geq 1$, $f'_{n,m,k}(x) = g_{n,m}(x)$ for $x \in [\delta_0, 1-\delta_0]$. These functions $f_{n,m,k}$ are continuously differentiable on $[\delta_0, 1-\delta_0]$.

Now, choose $\varepsilon_{n,m} > 0$ such that

$$B_{\|\cdot\|}(g_{n,m}, \varepsilon_{n,m}) \subset \mathcal{G}^{\mathcal{M},\delta_0}_n,$$

where the ball $B_{\|\cdot\|}$ is taken in the set $\mathcal{M}^{1,\delta_0}$ using the L^∞-norm.

Lemma 4 gives the existence of $\varrho_{n,m,k} > 0$ such that for every $f \in B_{\|\cdot\|}(f_{n,m,k}, \varrho_{n,m,k}) \subset \mathcal{CC}^1$, the inequality

$$|f'_\pm(x) - g_{n,m}(x)| < \varepsilon_{n,m}$$

holds for all $x \in [\delta_0, 1-\delta_0]$.

Let us now introduce

$$\mathcal{G}_n = \bigcup_{n,k} B_{\|\cdot\|}(f_{n,m,k}, \varrho_{n,m,k}).$$

By construction, $\mathcal{G}_n$ is dense in $\mathcal{CC}^1$.

Proposition 8 yields the existence of a dense G_δ set $\mathcal{D}$ in $\mathcal{CC}^1$ consisting of functions f continuously differentiable on $(0,1)$.

We finally set

$$\mathcal{G} = \mathcal{D} \bigcap \left(\bigcap_{n=1}^{\infty} \mathcal{G}_n \right).$$

Clearly, $\mathcal{G}$ is a dense G_δ set in $\mathcal{CC}^1$.

Suppose that $f \in \mathcal{G}$, and set $g = f'$ on $(0,1)$ and $g_0 = g|_{[\delta_0, 1-\delta_0]}$.

Then $g_0 \in \bigcap_{n=1}^{\infty} \mathcal{G}_n^{\mathcal{M},\delta_0}$ and equation (3) implies that for any $1 \le h \le 2$, $\dim E_{g_0}(h-1) = h-1$.

But Lemma 5 yields, $E_{g_0}(h-1) = E_{f|_{[\delta_0,1-\delta_0]}}(h)$, hence $\dim E_f(h) = h-1$ for any $1 \le h \le 2$.

Since δ_0 can be chosen arbitrarily small by taking a sequence of δ_0's tending to zero, we can conclude the proof of the theorem. □

Next we turn to the higher dimensional case.

Theorem 15. *There is a dense G_δ set $\mathcal{G}$ in $\mathcal{CC}^d$ such that for any $f \in \mathcal{G}$ and $1 \le h \le 2$, one has $\dim E_f(h) \ge h+d-2$. In addition, $E_f(h) = \emptyset$ for $h > 2$, $E_f(h) \cap [0,1]^d = \emptyset$ for $0 < h < 1$, and $\partial([0,1]^d) = E_f(0)$.*

Proof. Now instead of the functions, we can "integrate" the proof used in Section 4 of [4]. The idea is again to reduce the problem to the one-dimensional case. We select one coordinate direction, for ease of notation the first, the x_1-axis. We use in our proof the perturbation functions which are constant in the directions of the coordinate axes x_j, $j = 2, ..., d$ already used in this paper, given in Definition 3.

Let us select a dense set of C^∞ functions $\{f_m : m = 1, 2, ...\}$ which is dense in $\mathcal{CC}^d$. The x_1-partial derivatives, $\partial_1 f_m$, of these functions will be denoted by g_m. The important feature of these functions g_m is the fact that they are monotone increasing in the x_1-variable. As our example at the beginning of the paper shows (see Remark 1), these functions are not necessarily monotone in the other variables, and this is why one cannot "integrate" simply the MISV genericity results.

Now, as in [4] and in the proof of Proposition 10, we use the functions $\overline{\gamma}_l$ and the perturbations $\overline{f}_l$ defined in (19) and (20).

Next, apply Lemma 3 to the functions $f_m + \overline{f}_{m+n}$ with $\varepsilon = \varepsilon_{n+m}$ and $j = 1$. There exists a constant, denoted by $\varrho_{m,n} > 0$, such that if $f \in B_{\|\cdot\|}(f_m + \overline{f}_{m+n}, \varrho_{m,n})$, then for any $x_1 \in [\varepsilon_{n+m}, 1-\varepsilon_{n+m}]$ and $x_j \in [0,1]$ for $j = 2, ..., d$, one has

$$|\partial_{1,\pm} f(\mathbf{x}) - \partial_1(f_m + \overline{f}_{m+n})(\mathbf{x})| < \varepsilon_{n+m},$$

that is,

$$|\partial_{1,\pm} f(\mathbf{x}) - (g_m + \overline{\gamma}_{m+n}(\mathbf{x}))| < \varepsilon_{n+m} \tag{31}$$

holds. Without limiting generality one can assume that $\varrho_{m,n} \to 0$ if n is fixed and $m \to \infty$.

Set

$$\mathcal{R}_n = \bigcup_{m=1}^{\infty} B_{\|\cdot\|}(f_m + \overline{f}_{m+n}, \varrho_{m,n}).$$

It is not difficult to see that $\mathcal{R}_n$ is open and dense in $\mathcal{CC}^d$. Denote by $\mathcal{D}$ a dense G_δ set in $\mathcal{CC}^d$, which consists of functions differentiable on $(0,1)^d$, and such that (according to Proposition 10) these functions also have nowhere a pointwise exponent strictly greater than 2.

Finally, set

$$\mathcal{G} = \mathcal{D} \bigcap \left(\bigcap_{n=1}^{\infty} \mathcal{R}_n \right).$$

Let $f \in \mathcal{G}$, and $1 \le h \le 2$. We are going to prove that $\dim E_f(h) \ge h + d - 2$, by reducing the argument to a situation already totally taken care of in [4].

By construction, there exists a sequence of integers $(m_n)_{n\ge 1}$ such that $f \in B_{\|\cdot\|}(f_{m_n} + \overline{f}_{m_n+n}, \varrho_{n,m_n})$ for every n.

Set $g = \partial_1 f$.

By (31), when $x_1 \in [\varepsilon_{n+m_n}, 1-\varepsilon_{n+m_n}]$ and $x_j \in [0,1]$ for $j = 2, ..., d$, one has for $\mathbf{x} = (x_1, x_2, ..., x_d)$

$$|g(\mathbf{x}) - (g_{m_n} + \overline{\gamma}_{m_n+n})(\mathbf{x})| < \varepsilon_{n+m_n}. \tag{32}$$

Given $0 < \delta_0 < 1/10$, choose an integer n_1 such that $\varepsilon_{n_1} < \delta_0/100$.

Select an increasing subsequence $(n_k)_{k\ge 1}$ such that the following conditions are fulfilled: put $l_k = m_{n_k} + n_k$. One assumes that

$$l_k > 2^k, ((l_k)^2 + l_k)k + 1 < (l_k)^4,\ 2^{-((l_{k-1})^2 + l_{k-1})(k-1)-1} > 100 \cdot 2^{-(l_k)^2}$$

and if $D_k = 2^{(l_k)^2} \cdot 2^{-((l_k)^2 + l_k)k-2} < 1$, then one also assumes that k is so large that

$$D_1 \cdots D_{k-1} > 2^{-l_k}.$$

These conditions are equations (27) and (28) in [4].

Denote by φ_k the restriction of $g_{m_{n_k}}$ onto $[\delta_0, 1-\delta_0] \times [0,1]^{d-1}$ and by $\widetilde{g}_{l_k}$ the restriction of $\overline{\gamma}_{m_{n_k}+n_k}$ onto $[\delta_0, 1-\delta_0] \times [0,1]^{d-1}$. For ease of notation for the restriction of g onto $[\delta_0, 1-\delta_0] \times [0,1]^{d-1}$ we will still use the notation g.

Then $g \in B_{\|\cdot\|}(\varphi_k + \widetilde{g}_{l_k}, \varepsilon_{l_k})$ for all $k = 1, ...$, where the ball is taken with respect to the supremum norm in the space of continuous functions monotone in the first variable.

Now, we are exactly in the context of our previous article [4], in which we proved the following sequence of propositions (cf Propositions 13-18 of [4]). We reproduce the definitions given in [4] and the associated propositions.

For every $h \in (1,2)$ and $k \geq 2$, let

$$F_{h-1,k} = \bigcup_{j=0}^{2^{(l_k)^2}-1} \left[(j+1)2^{-(l_k)^2} - 2^{\frac{(l_k)^2+l_k}{h-1}}, (j+1)2^{-(l_k)^2} - \frac{1}{2} 2^{\frac{(l_k)^2+l_k}{h-1}} \right].$$

For $h = 1$, set $F_{h-1,k} = F_{0,k}$ as

$$F_{0,k} = \bigcup_{j=0}^{2^{(l_k)^2}-1} \left[(j+1)2^{-(l_k)^2} - 2^{k((l_k)^2+l_k)}, (j+1)2^{-(l_k)^2} - \frac{1}{2} 2^{k((l_k)^2+l_k)} \right].$$

One has [4]:

Proposition 16. *For $h \in [1,2)$, let $k_h = \max(3, [1/h]+2)$ and*

$$F_{h-1} = \bigcap_{k \geq k_h} F_{h-1,k} \ \subset \ [\delta_0, 1-\delta_0].$$

For every $\mathbf{x} \in F_{h-1} \times [0,1]^{d-1}$, $h_g(\mathbf{x}) \leq h-1$.

In particular, $\dim E_g(0) = d-1$.

Proposition 17. *For $h \in [1,2)$, there exists a probability measure μ_{h-1} such that $\mu_{h-1}(F_{h-1} \times [0,1]^{d-1}) = 1$ and for every $\mathbf{x} \in F_{h-1} \times [0,1]^{d-1}$,*

$$\liminf_{r \to 0+} \frac{\log \mu_{h-1}\big(B(\mathbf{x}, r)\big)}{\log r} \geq d+h-2. \tag{33}$$

Hence, using the mass distribution principle, one deduces from (33) that $\dim(F_{h-1} \times [0,1]^{d-1}) \geq d+h-2$. From the last two propositions, one deduces that for every $h \in [1,2]$,

$$\dim E_g^{\leq}(h-1) \geq d+h-2.$$

Finally, Proposition 18 of [4] gives:

Proposition 18. *For every $h > 2$, $E_g(h-1) = \emptyset$.*

Finally, we use that the functions f have nowhere a pointwise Hölder exponent greater than 2 (by Proposition 10), and we finish the proof of Theorem 2.

If $\mathbf{x} \in F_{h-1} \times [0,1]^{d-1}$, then $h_g(\mathbf{x}) \leq h-1 \in [0,1]$. Necessarily, $h_g(\mathbf{x})+1 \leq h_f(\mathbf{x}) \leq 2$. By Lemma 5, the only possibility is $h_f(\mathbf{x}) = h_g(\mathbf{x})+1 \leq (h-1)+1 = h$. Hence $\dim E_f^{\leq}(h) = \dim E_g^{\leq}(h-1) \geq d+h-2$. Even more, the same argument as above (Proposition 16 combined with Proposition 11) gives $\mu_{h-1}(E_f^{\leq}(h)) > 0$.

Using the upper bound of Theorem 1, one knows that for every large integer $n \geq 1$, $\dim(E_f^{\leq}(h-1/n)) \leq h-1/n+d-2$, hence $\mu_{h-1}\big(E_f^{\leq}(h-1/n)\big) = 0$. Since

$$E_f(h) = E_f^{\leq}(h) \setminus \bigcap_{n\geq 1} E_f^{\leq}(h-1/n),$$

one deduces that $\mu_{h-1}(E_f(h)) > 0$, which implies that $\dim E_f(h) \geq d+h-2$. Since the converse inequality also holds true, one concludes that $\dim E_f(h) = d+h-2$. □

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Zoltán Buczolich, Department of Analysis, Eötvös Loránd University, Pázmány Péter Sétány 1/c, 1117 Budapest, Hungary, www.cs.elte.hu/∼buczo

E-mail address: buczo@cs.elte.hu

Stéphane Seuret, Université Paris-Est, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, F-94010, Créteil, France

E-mail address: seuret@u-pec.fr