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A Time Domain Characterization of 2-Microlocal Spaces

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ABSTRACT. In [10], new functional spaces, denoted $K_{x_0}^{s,s'}$, were introduced. These spaces characterize the fine local regularity of functions, much in the spirit of 2-microlocal spaces $C_{x_0}^{s,s'}$. In contrast with $C_{x_0}^{s,s'}$ spaces, however, $K_{x_0}^{s,s'}$ spaces are defined through simple estimations on the pointwise values of the functions. In this work, we generalize the definition of $K_{x_0}^{s,s'}$ spaces and prove the equality $C_{x_0}^{s,s'} = K_{x_0}^{s,s'}$ for $s + s' > 0$, $s > 0$.

Using this result, we propose an algorithm able to estimate a part of the 2-microlocal frontier. Experiments on sampled data show that reasonable accuracy is achieved even for "difficult" functions such as continuous but nowhere differentiable ones. As a by-product, robust estimators of both the pointwise and the local exponents are obtained.

1. Introduction

The investigation of the local regularity properties of functions has proved to be useful in many domains, such as PDE theory [2], signal processing [6], or analysis of turbulence [5].

Various notions of local regularity exist. The easiest and most natural one is the *pointwise Hölder* exponent. Although it is powerful and useful in many applications, this exponent does not fully characterize the behavior of a function at a given point: As is shown by the example of the *chirp* function, it does not take into account oscillatory behaviors around a point. Other regularity exponents are thus needed for a complete description of the regularity of a function around a point. Several choices have been proposed in this view, e.g., the *local Hölder* exponent ([6] and [14]), or the *chirp* exponents introduced by S. Jaffard and Y. Meyer in [9] or [12].

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All these exponents fit in a more general frame, the so-called *2-microlocal analysis,* introduced in [2] by J.M. Bony. The 2-microlocal spaces $C_{x_0}^{s,s'}$ were first defined through a Littlewood–Paley analysis, and have later been characterized by a wavelet analysis [8]. The price to pay for this deeper approach is a greater complexity in the definitions and the estimations of the parameters s and s'.

In [10], new functional spaces were introduced. These spaces, denoted by $K_{x_0}^{s,s'}$, were defined for $0 < s + s' < 1$, $s < 1$, through simple conditions on the pointwise values of a real function f. This is sometimes an advantage, from a numerical point of view, over a definition by a Littlewood–Paley or wavelet analysis. Indeed, instead of losing information by integrating (and thus by *smoothing*) sampled data, every single point value is used. The main goals of this article are to extend the definition of $K_{x_0}^{s,s'}$ spaces to the case $s + s' \geq 0$, $s' \leq 0$, and to prove the equality $C_{x_0}^{s,s'} = K_{x_0}^{s,s'}$ when $s + s' \notin \mathbb{N}$ and $s' < 0$. Thus $K_{x_0}^{s,s'}$ spaces provide a time domain characterization of 2-microlocal spaces $C_{x_0}^{s,s'}$ when the above conditions on s and s' are fulfilled. These restrictions correspond to the case of real-valued functions. Since the definition of $K_{x_0}^{s,s'}$ spaces use pointwise values, one cannot hope to reach distributions (recall that $C_{x_0}^{s,s'}$ with $s + s' \leq 0$ is a functional space that contains distributions).

After some recalls, we deal in Section 2 with $K_{x_0}^{s,s'}$ spaces, and their basic properties. Section 3 contains the proof of the equality between $C_{x_0}^{s,s'}$ and $K_{x_0}^{s,s'}$ when $0 < s + s' < 1$ and $s < 1$. The proof in the general case, which is more technical, may be found in [13].

We propose in Section 4 an algorithm which uses this new characterization to estimate the 2-microlocal frontier of a continuous function. More precisely, given a signal f and a point x_0, it estimates the frontier of the domain of all exponents (s, s') such that $f \in K_{x_0}^{s,s'}$ (or equivalently such that $f \in C_{x_0}^{s,s'}$). Using this estimation of the frontier, the usual exponents are recovered. The results of this algorithm on different functions are presented in Section 5.

2. The Functional Spaces $K_{x_0}^{s,s'}$

2.1 Classical Regularity Exponents

Before introducing the $K_{x_0}^{s,s'}$ spaces, we recall some usual notions of regularity.

Definition 1. Let $x_0 \in \mathbb{R}$, s a positive real number with $s \notin \mathbb{N}$, and f a function: $\mathbb{R} \to \mathbb{R}$. The function f belongs to $C_{x_0}^{s}$ if there exist a constant C and a polynomial P of degree smaller than $[s]$ such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^s . \tag{2.1}$$

The *pointwise Hölder exponent* of f at x_0 is defined by $\alpha_p(x_0) = \sup\{s : f \in C_{x_0}^{s}\}$. Let us denote by $B(x_0, \rho)$ the open ball centered at x_0 of radius $\rho > 0$.

Definition 2. Let $x_0 \in \mathbb{R}$, $0 < s < 1$, and f a function: $\mathbb{R} \to \mathbb{R}$. f belongs to $C_l^s(B(x_0, \rho))$ if there exists a constant C such that, for all x, y in $B(x_0, \rho)$,

$$|f(x) - f(y)| \leq C|x - y|^s . \tag{2.2}$$

If $m < s < m+1$ ($m \in \mathbb{N}$), then $f \in C_l^s(B(x_0, \rho))$ means that there exists a constant C such that, for all x, y in $B(x_0, \rho)$, $|\partial^m f(x) - \partial^m f(y)| \leq C|x-y|^{s-m}$.

Set now $\alpha_l(x_0, B(x_0, \rho)) = \sup\{s : f \in C_l^s(B(x_0, \rho))\}$. The *local Hölder exponent* [6] of f at x_0, denoted by $\alpha_l(x_0)$, is defined by

$$\alpha_l(x_0) = \lim_{\rho \to 0} \alpha_l(x_0, B(x_0, \rho)) . \tag{2.3}$$

This exponent is well defined, since $\alpha_l(x_0, B(x_0, \rho))$ is clearly a non-increasing function of ρ. Note finally that one always has $\alpha_l(x_0) \leq \alpha_p(x_0)$.

These definitions lead to different exponents, as shown by the two following examples:
- the *cusp* function $f(x) = |x|^\alpha$: In this case, both Hölder exponents at 0 are equal to α.
- the *chirp* function $f(x) = |x|^\alpha \sin(\frac{1}{|x|^\beta})$: Here the pointwise exponent at 0 is α, while the local exponent at 0 is $\frac{\alpha}{1+\beta}$.

This difference is a consequence of the fact that the local Hölder exponent takes into account the whole local behavior of the function f around x_0, while the pointwise does not. In particular, the infinitely fast oscillations around x_0 in the case of the chirp are considered as a special behavior by $\alpha_l(x_0)$, while they are ignored by $\alpha_p(x_0)$. More details on the relations between α_l and α_p may be found in [14].

Unfortunately the knowledge of those two exponents does not fully describe the local regularity of a given function f. Obviously, there exist functions which have the same pointwise and local exponents at x_0, although they have a different behavior at this point. For example take $f_1(x) = |x|^{0.5} \sin(\frac{1}{|x|^1})$ and $f_2(x) = |x|^{0.8} \sin(\frac{1}{|x|^{2.2}}) + |x|^{0.5} \sin(\frac{1}{|x|^{0.25}})$. The reader can check that both functions have $\alpha_p = 0.5$ and $\alpha_l = 0.25$ at 0, while their behaviors around 0 are different. A deeper approach and a more complete description of regularity are thus needed to go further.

2.2 2-Microlocal Spaces: Definition and Properties

2-microlocal spaces $C_{x_0}^{s,s'}$ were introduced by J.M. Bony in [2] in the PDE's frame, through a Littlewood–Paley analysis. Let us quickly recall the definition of $C_{x_0}^{s,s'}$.

Let ϕ be a function that belongs to the Schwartz space $\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall(\gamma, \delta) \in \mathbb{N}^2, \ \sup_x |x^\gamma \partial^\delta f(x)| < \infty\}$, such that the Fourier transform of ϕ satisfies

$$\hat{\phi}(\xi) = 1 \quad \text{if} \quad |\xi| \leq 1/2 ,$$
$$\hat{\phi}(\xi) = 0 \quad \text{if} \quad |\xi| \geq 1 .$$

One defines $\phi_j(x) = 2^j \phi(2^j x)$, and $\psi_j = \phi_{j+1} - \phi_j$. Let f be a tempered distribution, thus belonging to the space $\mathcal{S}'(\mathbb{R})$ defined by

$$\mathcal{S}'(\mathbb{R}) = \left\{ f : \exists C, \ \exists q \in \mathbb{N}, \ \ \forall g \in \mathcal{S}(\mathbb{R}), \ |\langle f, g \rangle| \leq C \pi_q(g) \right\} , \tag{2.4}$$

(where $\pi_q(g) = \sup\{(1+|x|)^q \, |\partial^\delta g(x)| : |\delta| \leq q, x \in \mathbb{R}\}$). The Littlewood–Paley analysis of f is the set of distributions $\{S_0 f, \Delta_j f\}_{j \geq 0}$, where

$$S_0 f = \phi * f \ \text{ and } \ \Delta_j f = \psi_j * f . \tag{2.5}$$

One has the fundamental decomposition (see [12] for details)

$$f = S_0 f + \sum_{j=0}^{+\infty} \Delta_j f . \tag{2.6}$$

We are now able to define the 2-microlocal spaces.

Definition 3. Let $x_0 \in \mathbb{R}$ and (s, s') two real numbers. A distribution $f \in \mathcal{S}'(\mathbb{R})$ is said to belong to $C_{x_0}^{s,s'}$ if there exists a constant C such that

$$\begin{aligned} |S_0 f(x)| &\leq C(1+|x-x_0|)^{-s'} , \\ |\Delta_j f(x)| &\leq C 2^{-js} \left(1+2^j|x-x_0|\right)^{-s'} . \end{aligned}$$

Definition 3 provides us with a generalization of the notion of regularity at a point x_0, as we shall see later.

In the following we focus on the local properties of functions. The above definition is not adapted to our study, because it takes into account the behavior of f at infinity. We will thus use a local version of 2-microlocal spaces, defined as follows [12]:

Definition 4. Let V be an open neighborhood of x_0 and $f \in \mathcal{D}'(V)$ a distribution on V. We say that f belongs to $C_{x_0}^{s,s'}$ *locally* if there exists a smaller neighborhood $V_0 \subset V$ of x_0 and a distribution $g \in C_{x_0}^{s,s'}$ (globally) such that $f = g$ on V_0.

By convention, from now on, $C_{x_0}^{s,s'}$ will denote *local* 2-microlocal spaces.

A useful characterization of $C_{x_0}^{s,s'}$ is given by the wavelet coefficients of f [8]. Indeed, let us take a function ψ in the Schwartz class $\mathcal{S}(\mathbb{R})$, such that $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)\}_{(j,k)\in\mathbb{R}^2}$ forms an orthonormal basis of $L^2(\mathbb{R})$ (see for example [11] for a construction of such a basis). The wavelet coefficients of f are defined by (note that we do not use an L^2 normalization factor for the wavelet coefficients, and take 2^j instead of $2^{j/2}$, for convenience in the following proofs)

$$d_{j,k} = \int f(x) 2^j \psi\left(2^j x - k\right) dx . \tag{2.7}$$

Moreover, if Ψ is an admissible analyzing wavelet (as defined in [11]), the continuous wavelet transform is defined by

$$W_f(a,b) = \frac{1}{a}\int f(x)\Psi\left(\frac{x-b}{a}\right) dx . \tag{2.8}$$

Theorem 1 ([9]).

Let s, s' be two real numbers. Let us assume that

- *ψ and Ψ have N vanishing moments, with $N > \max(s, s+s')$*
- *ψ and Ψ are in $\mathcal{S}(\mathbb{R})$ (the precise constraints on the regularity of the wavelets are given in [12]).*

Then, the three following conditions are equivalent

1. $f \in C_{x_0}^{s,s'}$,
2. $\forall j, k$ *such that* $|x_0 - k2^{-j}| \leq 1$, $|d_{j,k}| \leq C 2^{-js}(1+|k-2^j x_0|)^{-s'}$,
3. $\forall a > 0, |b - x_0| < 1,\quad |W_f(a,b)| \leq C a^s (1 + \frac{|b-x_0|}{a})^{-s'}$.

There also exists a wavelet characterization of the pointwise Hölder exponent, due to S. Jaffard [9]:

Theorem 2.

Assume that $f \in C^s_{x_0}$. If $|k2^{-j} - x_0| \le 1/2$, then

$$\left|d_{j,k}\right| \le C2^{-sj}\left(1 + 2^j \left|k2^{-j} - x_0\right|\right)^s . \tag{2.9}$$

Conversely, if (2.9) *holds for all (j,k)'s such that $|k2^{-j} - x_0| \le 2^{-j/(\log j)^2}$, and if $f \in C^{\log}$, then there exist a constant C and a polynomial P of degree at most $[s]$ such that*

$$|f(x) - P(x - x_0)| \le C|x - x_0|^s (\log(|x - x_0|))^2 . \tag{2.10}$$

$C^{\log}$ is the class of functions f whose wavelet coefficients verify $|d_{j,k}| \le C2^{-\frac{j}{\log j}}$. This regularity condition is stronger than uniform continuity, but does not imply a uniform Hölder continuity.

2.3 Definition of $K^{s,s'}_{x_0}$ Spaces

$K^{s,s'}_{x_0}$ spaces were defined in [10] for nowhere differentiable functions. We extend here this definition to a wider range of exponents

Definition 5. Let $x_0 \in \mathbb{R}$, and s, s' be two real numbers satisfying $s' \le 0$ and $s + s' \ge 0$ (and thus $s \ge 0$).

Let $m = [s + s']$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to belong to $K^{s,s'}_{x_0}$ if there exist $0 < \delta < 1/4$, a polynomial P of degree smaller than $[s] - m$, and a constant C, such that

$$\left|\frac{\partial^m f(x) - P(x)}{|x - x_0|^{[s]-m}} - \frac{\partial^m f(y) - P(y)}{|y - x_0|^{[s]-m}}\right| \le C|x - y|^{s+s'-m}(|x - y| + |x - x_0|)^{-s'-[s]+m} \tag{2.11}$$

for all x, y such that $0 < |x - x_0| < \delta$, $0 < |y - x_0| < \delta$.

Let us make a few remarks on this definition.

- If $s + s' < 1$ and $s < 1$ (i.e., $m = [s] = 0$), then the original definition of [10] is recovered

$$|f(x) - f(y)| \le C|x - y|^{s+s'}(|x - y| + |x - x_0|)^{-s'} . \tag{2.12}$$

- If $m < s + s' < m + 1$ and $s < m + 1$ (i.e., $[s] = m$), one obtains a simpler formulation of the definition

$$\left|\partial^m f(x) - \partial^m f(y)\right| \le C|x - y|^{s-m+s'}(|x - y| + |x - x_0|)^{-s'} . \tag{2.13}$$

- The right term in the above inequality seems to be asymmetric, but it is not. Remarking that $(|x - y| + |x - x_0|) \le 2(|x - y| + |y - x_0|)$, this right term of (2.11) can be re-written as one of the two following expressions (the last one being symmetric in x and y)

$$|x - y|^{s-m+s'}(|x - y| + |y - x_0|)^{-s'-[s]+m} ,$$
$$|x - y|^{s-m+s'}\left((|x - y| + |x - x_0|)(|x - y| + |y - x_0|)\right)^{(-s'-[s]+m)/2} .$$

- In the following, we will most of the time avoid the critical cases $s + s' \in \mathbb{N}$ and $s \in \mathbb{N}$. Indeed, they would require the use of Zygmund spaces instead of the usual homogeneous Hölder spaces $C^\alpha(\mathbb{R})$.

- If $g_m(x)$ denotes the function $\frac{\partial^m f(x) - P(x)}{|x-x_0|^{[s]-m}}$, one easily sees that g_m is at least continuous at each point, especially at x_0. Indeed, if $-s' - [s] + m > 0$, one writes

$$|g_m(x) - g_m(y)| \leq C|x - y|^{s+s'-m}$$

with $0 < s + s' - m < 1$. Thus $g_m \in C^{s+s'-m}$ around x_0. On the other hand, if $-s' - [s] + m \leq 0$, one has

$$|g_m(x) - g_m(y)| \leq C|x - y|^{s-[s]} \left(\frac{|x - y|}{|x - y| + |x - x_0|} \right)^{s'+[s]-m} \leq |x - y|^{s-[s]}$$

with $0 < s - [s] < 1$, thus $g_m \in C^{s-[s]}$ around x_0.
It is thus possible to take $y = x_0$ in (2.11) or in (2.12), and also to consider the real number $g_m(x_0)$.

- The left hand-side of (2.11) and the exponents in use may seem complex. The necessity of the different terms is however easily understood: One tries to reduce the study of f to the one of a new function derived from f that will belong to some $K_{x_0}^{t,t'}$ with $0 \leq t + t' < 1$ and $0 \leq t < 1$.
Roughly speaking, the subset of $\mathbb{R}^2$ $\{(s, s') : s + s' \geq 0,\ s' \leq 0 \text{ and } s > 0\}$ is partitioned into tiles of same size, and the problem is translated to the "initial" tile $\{(s, s') : 0 < s + s' < 1 \text{ and } s < 1\}$ (see Figure 1). For example, if a function f

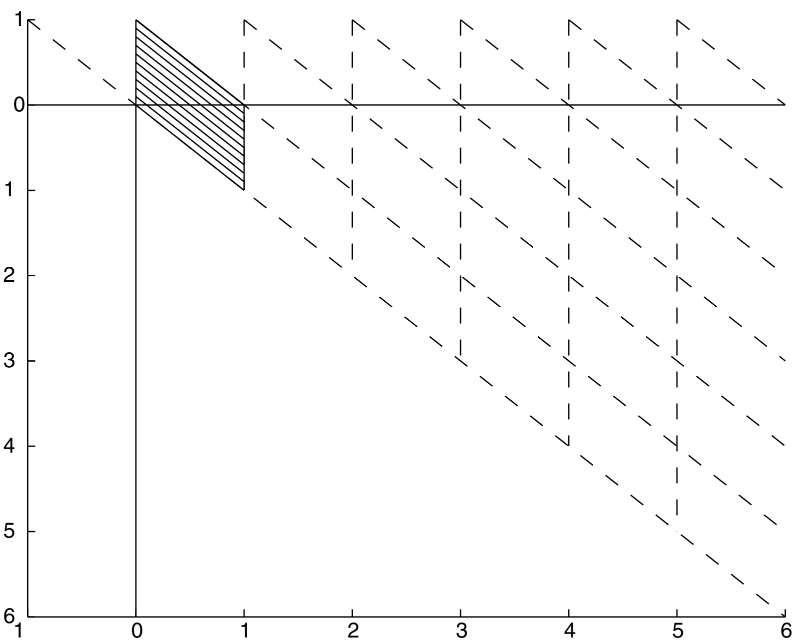

FIGURE 1 Paving of the half plane $s > 0$.

belongs to $K_{x_0}^{s,s'}$ with $m \leq s + s' < m + 1$, f admits around x_0 a derivative of order m. The formula simply states that $f \in K_{x_0}^{s,s'}$ means that $\partial^m f \in K_{x_0}^{s-m,s'}$.

Then, if $f \in K_{x_0}^{s,s'}$ with $0 < s+s' < 1$ with $s > 1$, f is replaced in formula (2.12) by $\frac{f(x)-P(x)}{|x-x_0|^{[s]}}$, and this new function will belong to $K_{x_0}^{s-[s],s'+[s]}$.
These thoughts can be summarized by saying that the operator

$$f \to \frac{\partial^m f(x) - P(x)}{|x-x_0|^{[s]-m}}$$

(with the appropriate polynomial P) maps $K_{x_0}^{s,s'}$ into $K_{x_0}^{s-[s],s'+[s]-m}$.

2.4 First Properties

We stress several interesting properties of these spaces $K_{x_0}^{s,s'}$. Propositions 1 to 4 are just extensions of the corresponding ones in [10] to the general case.

The first proposition is an embedding property between the spaces $K_{x_0}^{s,s'}$.

Proposition 1.

Let $x_0 \in \mathbb{R}$, and s, s', t, t' be four real numbers such that $t \leq s$ and $t + t' \leq s + s'$. Then $K_{x_0}^{s,s'} \subset K_{x_0}^{t,t'}$.

The proof of Proposition 1 is split into two simpler lemmas.

Lemma 1.

Let $x_0 \in \mathbb{R}$, and s, s', t, t' be four real numbers such that $t \leq s$ and $t + t' = s + s'$. Then $K_{x_0}^{s,s'} \subset K_{x_0}^{t,t'}$.

Proof. Let us treat the case $s + s' < 1$ (i.e., $m = 0$), the general case is then deduced by replacing f by $\partial^m f$ and s by $s - m$.

If $[t] = [s]$, the result is obvious. Let us treat the case $[t] = [s] - 1$, the general result will then easily follow by iteration.

Let us assume that $f \in K_{x_0}^{s,s'}$, $x_0 = 0$ and that $|y| \leq |x|$, without loss of generality. One also assumes that $x > 0$, by replacing $z \to f(z)$ by $z \to f(-z)$. There exists a polynomial P such that (2.11) holds. One is now looking for a polynomial P_t that satisfies

$$\left| \frac{f(x) - P_t(x)}{|x|^{[s]-1}} - \frac{f(y) - P_t(y)}{|y|^{[s]-1}} \right| \leq C|x-y|^{t+t'}(|x-y| + |x|)^{-t'-[t]} . \qquad (2.14)$$

Let us denote by P_t the polynomial of degree $[t] = [s] - 1$ with the same coefficients as P up to degree $[t]$. To simplify the notations, let us define $g(x) = \frac{f(x)-P(x)}{|x|^{[s]}}$ and $g_t(x) = \frac{f(x)-P_t(x)}{|x|^{[s]-1}}$. Then,

$$\forall x, \quad g_t(x) = |x|\, g(x) + \frac{P(x) - P_t(x)}{|x|^{[s]-1}} .$$

Now,

$$|g_t(x) - g_t(y)| \leq \left| \left(|x| g(x) + \frac{P(x) - P_t(x)}{|x|^{[s]-1}} \right) - \left(|y| g(y) + \frac{P(y) - P_t(y)}{|y|^{[s]-1}} \right) \right| .$$

By construction, $P(x) - P_t(x)$ is a polynomial with only one non-zero coefficient. Thus

$P(x) - P_t(x) = ax^{[s]}$, and

$$\begin{aligned} |g_t(x) - g_t(y)| &\leq ||x|g(x) - |y|g(y)| + a\left|\frac{x^{[s]}}{|x|^{[s]-1}} - \frac{y^{[s]}}{|y|^{[s]-1}}\right| \\ &\leq ||x|(g(x) - g(0)) - |y|(g(y) - g(0))| + (|a| + |g(0)|)|x - y| \,. \end{aligned}$$

A useful remark is that

$$\begin{aligned} |x - y| &= |x - y|^{t+t'}|x - y|^{-t'+1-t} \\ &\leq |x - y|^{t+t'}(|x - y| + |x|)^{-t'+1-t} \\ &\leq |x - y|^{t+t'}(|x - y| + |x|)^{-t'-[t]} \,, \end{aligned}$$

(note that $1 - t > -[t]$). A direct upper bound for g is obtained by taking $y = 0$ in (2.11)

$$\forall x, \quad |g(x) - g(0)| \leq C|x|^{s-[s]} \,. \tag{2.15}$$

Applying (2.15) and (2.11), the last term $(T) = ||x|(g(x) - g(0)) - |y|(g(y) - g(0))|$ is treated as follows

$$\begin{aligned} (T) &\leq ||x| - |y||g(x) - g(0)| + ||y|(g(x) - g(0)) - |y|(g(y) - g(0))| \\ &\leq |x - y||g(x) - g(0)| + |y||g(x) - g(y)| \\ &\leq C|x - y||x|^{s-[s]} + |y||x - y|^{s+s'}(|x| + |x - y|)^{-s'-[s]} \\ &\leq C|x - y|^{t+t'}|x - y|^{1-(t+t')}|x|^{s-[s]} \\ &\quad + |y||x - y|^{s+s'}(|x| + |x - y|)^{-s'-[s]} \,. \end{aligned}$$

Since $s - [s]$ and $1 - (t + t')$ are positive, one upper-bounds $|x|^{s-[s]}$, $|x - y|^{1-(t+t')}$ and $|y|$, respectively by $(|x| + |x - y|)^{s-[s]}$, $(|x| + |x - y|)^{1-(t+t')}$ and $(|x| + |x - y|)$. This gives, using $[t] = [s] - 1$,

$$\begin{aligned} (T) &\leq C|x - y|^{t+t'}(|x| + |x - y|)^{-t'-[t]+s-t+2} \\ &\quad + |x - y|^{s+s'}(|x| + |x - y|)^{-s'-[t]+2}. \end{aligned}$$

Finally, since $t + t' = s + s'$, one has $-s' - [t] + 2 = -t' - [t] + 2 + s - t$. One concludes, using $2 + s - t > 0$, that

$$|g_t(x) - g_t(y)| \leq C|x - y|^{t+t'}(|x| + |x - y|)^{-t'-[t]}$$

which gives the required result. □

Lemma 2.

Let $x_0 \in \mathbb{R}$, and s, s', t' be three real numbers such that $t' \leq s'$. Then $K_{x_0}^{s,s'} \subset K_{x_0}^{s,t'}$.

Proof. If $[s+s'] = [s+t']$, the result is obvious. Assume that $m = [s+s'] = [s+t']+1$, the general result will then easily follow. As usual now, we will assume without loss of generality that $x_0 = 0$ and $|y| \leq x$. By assumption, (2.11) holds, and one wants to prove

$$\begin{aligned} &\left|\frac{\partial^{m-1} f(x) - P(x)}{|x|^{[s]-(m-1)}} - \frac{\partial^{m-1} f(y) - P(y)}{|y|^{[s]-(m-1)}}\right| \\ &\quad \leq C|x - y|^{s-(m-1)+t'}(|x - y| + |x|)^{-t'-[s]+(m-1)} \,. \end{aligned}$$

Taking $y = 0$ in (2.11) yields

$$\left|\partial^m f(x) - P(x)\right| \le C|x|^{s-m}$$

for a certain polynomial P of degree at most $[s] - m$. Integrating first this last inequality between 0 and x, and then between y and x, one obtains

$$\left|\partial^{m-1} f(x) - P_{t'}(x)\right| \le C|x|^{s-m+1} \tag{2.16}$$

and

$$\left|\left(\partial^{m-1} f(x) - P_{t'}(x)\right) - \left(\partial^{m-1} f(y) - P_{t'}(y)\right)\right| \le C|x-y||x|^{s-m} \tag{2.17}$$

for a polynomial $P_{t'}$ of degree at most $[s]-m+1$ [the same one for both (2.16) and (2.17)]. One first writes

$$\left|\frac{\partial^{m-1} f(x) - P_{t'}(x)}{|x|^{[s]-m+1}} - \frac{\partial^{m-1} f(y) - P_{t'}(y)}{|y|^{[s]-m+1}}\right| \le (I) + (II)\ ,$$

where

$$(I) = \left|\frac{\partial^{m-1} f(x) - P_{t'}(x)}{|x|^{[s]-m+1}} - \frac{\partial^{m-1} f(y) - P_{t'}(y)}{|x|^{[s]-m+1}}\right|$$

and

$$(II) = \left|\frac{\partial^{m-1} f(y) - P_{t'}(y)}{|x|^{[s]-m+1}} - \frac{\partial^{m-1} f(y) - P_{t'}(y)}{|y|^{[s]-m+1}}\right|\ .$$

Dividing (2.17) by $|x|^{[s]-m+1}$, (I) is bounded by $C|x-y||x|^{s-[s]-1}$. On the other hand, using (2.16), one has

$$\begin{aligned}
(II) &\le \left|\partial^{m-1} f(y) - P_{t'}(y)\right| \left|\frac{1}{|x|^{[s]-m+1}} - \frac{1}{|y|^{[s]-m+1}}\right| \\
&\le \left|\partial^{m-1} f(y) - P_{t'}(y)\right| \frac{|x|^{[s]-m+1} - |y|^{[s]-m+1}}{|xy|^{[s]-m+1}} \\
&\le \left|\partial^{m-1} f(y) - P_{t'}(y)\right| \frac{|x-y||x|^{[s]-m}}{|xy|^{[s]-m+1}} \\
&\le C|y|^{s-m+1} \frac{|x-y||x|^{-1}}{|y|^{[s]-m+1}} \le C\frac{|y|^{s-[s]}}{|x|}|x-y| \\
&\le C\left(\frac{|y|}{|x|}\right)^{s-[s]} |x|^{s-[s]-1}|x-y| \le C|x-y||x|^{s-[s]-1}\ ,
\end{aligned}$$

since $\frac{|y|}{|x|}$ is bounded by 1.

The same kind of manipulations of exponents (but easier) as at the end of the previous proposition can be performed. Remarking that $|x|^{-s'} \le C(|x-y| + |x|)^{-s'}$ and using $0 < s + t' - (m-1) < 1$, it is easily verified that

$$\begin{aligned}
|x-y||x|^{s-[s]-1} &= |x-y|^{s+t'-(m-1)}|x-y|^{1-(s+t'-(m-1))}|x|^{s-[s]-1} \\
&\le |x-y|^{s+t'-(m-1)}(|x-y| + |x|)^{1-(s+t'-(m-1))+s-[s]-1} \\
&\le |x-y|^{s+t'-(m-1)}(|x-y| + |x|)^{-t'+(m-1)-[s]}
\end{aligned}$$

which gives the result. □

Combining Lemmas 1 and 2, Proposition 1 is proved.

We now compare these $K_{x_0}^{s,s'}$ with the classical pointwise Hölder spaces.

Proposition 2.

Let $x_0 \in \mathbb{R}$, and s be a real number such that $s > 0$, $s \notin \mathbb{N}$. Then $C_{x_0}^s = K_{x_0}^{s,-s}$.

Proof. Let f be a function in $C_{x_0}^s$. One writes, using the approximating polynomial P found in (2.1),

$$\begin{aligned}\left|\frac{f(x)-P(x)}{|x-x_0|^{[s]}} - \frac{f(y)-P(y)}{|y-x_0|^{[s]}}\right| &\leq \left|\frac{f(x)-P(x)}{|x-x_0|^{[s]}}\right| + \left|\frac{f(y)-P(y)}{|y-x_0|^{[s]}}\right| \\ &\leq C\left(|x-x_0|^{s-[s]} + |y-x_0|^{s-[s]}\right) \\ &\leq C(|x-y|+|x-x_0|)^{s-[s]} .\end{aligned}$$

This proves $f \in K_{x_0}^{s,-s}$.

On the other hand, let f be a function in $K_{x_0}^{s,-s}$. Taking $y = x_0$ in (2.11) leads to (remember that $g_m(x)$ is defined by $\frac{\partial^m f(x)-P(x)}{|x-x_0|^{[s]-m}}$)

$$|g_m(x) - g_m(0)| = \left|\frac{\partial^m f(x)-P(x)}{|x-x_0|^{[s]-m}} - g_m(0)\right| \leq |x-x_0|^{s-[s]} ,$$

thus $|\partial^m f(x) - P_1(x)| \leq C|x-x_0|^{s-m}$, where P_1 is a polynomial of order smaller than $[s]-m$. The last inequality can be reformulated in

$$\left|\partial^m (f-P_2)(x)\right| \leq C|x-x_0|^{s-m} ,$$

where P_2 is a polynomial of degree less than $[s]$. This reads $\partial^m f \in C_{x_0}^{s-m}$, which implies $f \in C_{x_0}^s$. □

It is easily verified that $\forall s' \leq 0$ with $s+s' > 0$, $K_{x_0}^{s,s'} \subset C_{x_0}^s$, thus, for all $s' \leq 0$, $K_{x_0}^{s,s'} \subset K_{x_0}^{s,-s}$. As soon as a function belongs to $K_{x_0}^{s,s'}$ for some s', it automatically belongs to $C_{x_0}^s$. That also means, by reciprocity, that a function f whose pointwise Hölder exponent is $s > 0$ cannot belong to any $K_{x_0}^{t,t'}$, for $t > s$, whatever t' is. This is an important property of $K_{x_0}^{s,s'}$ spaces.

2.5 Domain of Admissible Exponents

Definition 6. Let f be a function: $\mathbb{R} \to \mathbb{R}$. $E(f, x_0)$ denotes the set in the half plane $\{(s,s') : \; s+s' \geq 0, \; s' \leq 0 \text{ and } s > 0\}$ of all couples (s,s') such that $f \in K_{x_0}^{s,s'}$.

Let us notice that this set is convex by Proposition 1. Moreover, it cannot intersect the open half-space defined by $\{(s,s') : s > \alpha_p(f)\}$, because of the last remark in the previous section. Finally, if Γ denotes the boundary of $E(f, x_0)$, it is easily shown that Γ is, again by Lemmas 1 and 2, the graph of a function $s = \gamma(s')$. The function γ is concave, decreasing, with slope greater than -1. We shall call Γ the K-frontier.

The following proposition links $K_{x_0}^{s,s'}$ spaces with the local Hölder exponent defined in the first section.

Proposition 3.

Let $x_0 \in \mathbb{R}$*, and* $f : \mathbb{R} \to \mathbb{R}$ *such that* $\gamma(0) > 0$*. The local Hölder exponent corresponds to the intersection of the* K*-frontier with the* x*-axis, i.e.,* $\alpha_l = \gamma(0)$*.*

Proof. From the definition of $K_{x_0}^{s,0}$, one obtains, for $|x - x_0| < \delta$, $|y - x_0| < \delta$, (remark that $s' = 0$ implies $[s] = m$)

$$\left|\partial^m f(x) - \partial^m f(y)\right| \leq C|x - y|^{s-m} .$$

On the one hand, if $s < \gamma(0)$, then $f \in C_l^s(B(x_0, \delta))$, for any $\delta < 1/4$. Then, taking the limit when $\delta \to 0$, one has $\alpha_l \geq s$ for any $s < \gamma(0)$. Eventually, one concludes $\alpha_l \geq \gamma(0)$.

On the other hand, if $\alpha_l > \gamma(0)$, then there exists s such that $\gamma(0) < s < \alpha_l$ and $\delta > 0$ such that, for $|x - x_0| < \delta$, $|y - x_0| < \delta$,

$$\left|\partial^m f(x) - \partial^m f(y)\right| \leq C|x - y|^{s-m} .$$

Thus $f \in K_{x_0}^{s,0}$, and $\gamma(0) \leq s$, which is absurd. □

Proposition 4.

Let $x_0 \in \mathbb{R}$*, and* $f : \mathbb{R} \to \mathbb{R}$ *such that* $\gamma(0) > 0$*. The pointwise Hölder exponent of* f *is the unique* $\alpha > 0$ *where the* K*-frontier intersects the second diagonal, i.e.,* α_p *is the unique* α *with* $\alpha = \gamma(-\alpha)$*.*

Proof. The existence and the unicity of this intersection α is clear, since $\gamma(0) > 0$ and since $|\gamma'(s')| < 1$ for $s' > 0$.

First, this intersection is located to the left of the pointwise exponent of f at x_0 since f cannot belong to $K_{x_0}^{s,s'}$ for $s > \alpha_p$. Thus $\alpha \leq \alpha_p$.

On the other hand, if $\alpha < \alpha_p$, then there exists s such that $\alpha < s < \alpha_p$. By definition of α_p, $f \in C_{x_0}^s$, and by Proposition 2, $f \in K_{x_0}^{s,-s}$, which is in contradiction with the unicity of α. □

Propositions 3 and 4 show that, as soon as f has a minimal local regularity (i.e., $\gamma(0) > 0$), one can read the pointwise and local Hölder exponents from the K-frontier.

To end up with this section, let us notice that all the above propositions are only consequences of simple manipulations of the several exponents s, s', and $s + s'$. In fact the definitions of $K_{x_0}^{s,s'}$ spaces combine two notions of regularity: The *global* regularity around x_0 and the *pointwise* regularity at x_0. They provide us with a deep understanding of the behavior of the considered function f around x_0.

3. Relation with 2-Microlocal Spaces

The main result of the article is the following theorem, which identifies in the most interesting cases the 2-microlocal spaces $C_{x_0}^{s,s'}$ with our spaces $K_{x_0}^{s,s'}$. The previous Lemma 1 to Proposition 4, are in fact consequences of the following Theorem 3, since the corresponding properties have been proved to hold for $C_{x_0}^{s,s'}$ spaces. However, we detailed them to show how easier Propositions 3 and 4 were to prove in our frame than in the 2-microlocal frame.

3.1 Main Result

Theorem 3.
Let $x_0 \in \mathbb{R}$, and s, s' be two real numbers such that $s + s' > 0$, $s + s' \notin \mathbb{N}$, and $s' < 0$. Then

$$\left(f \in C_{x_0}^{s,s'}\right) \Leftrightarrow \left(f \in K_{x_0}^{s,s'}\right) . \tag{3.1}$$

Let us say first a few words about the constraints on s and s'. As shown before, the condition $s + s' > 0$ implies a minimal global regularity for the function in a neighborhood of x_0, and the existence of a sort of Taylor expansion of f at x_0.

Theorem 3 does not contain the critical case $s + s' = 0$. Indeed, $C_{x_0}^{s,-s}$ contains distributions [8], which obviously do not belong to any $K_{x_0}^{s,-s}$ spaces. $K_{x_0}^{s,-s}$ spaces are thus strictly included in $C_{x_0}^{s,-s}$ spaces.

Theorem 3 can be compared with Theorem 2: Theorem 2 assumes a minimal global regularity for the considered function f (namely $f \in C^{\log}$) to estimate quantities of the type $|f(x) - P(x - x_0)|$. Theorem 3 provides an equivalence and allows to estimate differences of the type $|f(x) - f(y)|$, for any couple of points (x, y) in a neighborhood of x_0. This gain of accuracy is due to the fact that we fully use the assumption of local regularity (i.e., $s + s' > 0$).

A consequence of Theorem 3 is that the K-frontier and the 2-microlocal frontier coincide in the domain $s + s' > 0$, $s' < 0$.

3.2 Proof in a Simple Case

We shall prove Theorem 3 in the case $0 < s + s' < 1$, $s < 1$. In addition, we assume that the analysis is done using an orthonormal basis of compactly supported wavelets with at least 2 vanishing moments (see for example [4] for the existence and the construction of such a wavelet). This is slightly different and easier than the general case. This restriction is of great interest for practical purposes, as we shall see later. The proof of Theorem 3 in the general case is given in [13].

Proof. Without loss of generality, we assume that $x_0 = 0$, and that $s' < 0$, $0 < s < 1$ and $0 < s + s' < 1$. The definition of $K_{x_0}^{s,s'}$ spaces takes here a nice form, i.e.,

$$|f(x) - f(y)| \leq C|x - y|^{s+s'}(|x - y| + |x - x_0|)^{-s'} . \tag{3.2}$$

The important case of functions which are continuous but nowhere differentiable is contained in this frame.

For each couple (j, k), denote by $S_{j,k}$ the support of $\psi_{j,k}$, the translated-dilated version of ψ. Namely, one has $\psi_{j,k}(x) = \psi(2^j x - k)$ and $S_{j,k} = [(k-K)2^{-j}, (k+K)2^{-j}]$, where $2K + 1$ is the length of the support of ψ. The corresponding wavelet coefficient is

$$d_{j,k} = 2^j \int f(x)\psi_{j,k}(x)\, dx .$$

To prove the result, we will use the characterization of the $C_{x_0}^{s,s'}$ spaces by wavelet coefficients, recalled in Theorem 1.

1. $K_0^{s,s'} \subset C_0^{s,s'}$**.**

Assume that $f \in K_0^{s,s'}$. Then (3.2) holds. We want to study the wavelet coefficients $d_{j,k}$.

Using the first vanishing moment of the wavelet, one writes

$$\begin{aligned} |d_{j,k}| &= \left| \int f(x) 2^j \psi_{j,k}(x)\, dx \right| = 2^j \left| \int_{S_{j,k}} \left(f(x) - f\left(k2^{-j}\right)\right) \psi_{j,k}(x)\, dx \right| \\ &= 2^j \int_{S_{j,k}} \left| x - k2^{-j} \right|^{s+s'} \left(\left| x - k2^{-j} \right| + \left| k2^{-j} \right| \right)^{-s'} |\psi_{j,k}(x)|\, dx \,. \end{aligned}$$

On the interval $S_{j,k}$, $|x - k2^{-j}|$ is bounded by $K2^{-j}$, where K does not depend on x, y, j or k. Then

$$\left| x - k2^{-j} \right| + \left| k2^{-j} \right| \leq C2^{-j}(1 + |k|) \,,$$

thus $(|x - k2^{-j}| + |x|)^{-s'} \leq C2^{js'}(1 + |k|)^{-s'}$ Moreover, on $S_{j,k}$, one also has $|x - k2^{-j}|^{s+s'} \leq C2^{-j(s+s')}$ on $S_{j,k}$. One deduces that

$$|d_{j,k}| \leq C2^{js'}(1 + |k|)^{-s'} 2^{-j(s+s')} \int_{S_{j,k}} 2^j |\psi_{j,k}(x)|\, dx \leq C2^{-js}(1 + |k|)^{-s'} \,,$$

since $\int 2^j |\psi_{j,k}(x)|\, dx$ is a constant. Thus, f indeed belongs to $C_0^{s,s'}$.

2. $C_0^{s,s'} \subset K_0^{s,s'}$**.**

We suppose that the wavelet coefficients of f verify $|d_{j,k}| \leq C2^{-js}(1 + |k|)^{-s'}$ ($f \in C_0^{s,s'}$). We aim to show that f satisfies (3.2).

Since $s + s' > 0$, $C_0^{s,s'} \subset C^{s+s'}$ around 0, and we are allowed to use the reconstruction formula

$$f(x) = \sum_j \sum_k d_{j,k} \psi_{j,k}(x) \,. \tag{3.3}$$

As explained before, it is enough to treat the case $|y| \leq x$. We have to study the difference

$$|f(x) - f(y)| = \left| \sum_j \sum_k d_{j,k} (\psi_{j,k}(x) - \psi_{j,k}(y)) \right| \,. \tag{3.4}$$

Denote by j_0 be the integer such that

$$2^{-j_0-1} \leq |x - y| < 2^{-j_0} \,. \tag{3.5}$$

The difference (3.4) can be split into three different expressions

$$\begin{aligned} |f(x) - f(y)| \leq & \sum_{j \leq j_0 - 1} \sum_k |d_{j,k}| |\psi_{j,k}(x) - \psi_{j,k}(y)| && (I) \\ & + \sum_{j \geq j_0} \sum_k |d_{j,k}| |\psi_{j,k}(x)| && (II) \\ & + \sum_{j \geq j_0} \sum_k |d_{j,k}| |\psi_{j,k}(y)| && (III) \,. \end{aligned}$$

Let us study first the term (II).

$$\sum_{j\geq j_0}\sum_k |d_{j,k}||\psi_{j,k}(x)| \leq C\sum_{j\geq j_0}\sum_k 2^{-js}(1+|k|)^{-s'}|\psi_{j,k}(x)| \ .$$

The crucial fact here is the following: If x and j are fixed, only a fixed number of $\psi_{j,k}(x)$ are different from 0, namely $2K+1$, i.e., the length of the support of ψ. This corresponds to the couples of indices (j,k) such that $|x-k2^{-j}|\leq K2^{-j}$, i.e., $(1+|k|)\sim(1+2^j|x|)$.

Then, using that the dilated-translated $\psi_{j,k}$'s are bounded by the same constant M (M and K are independent of s, s', j and k), one has

$$\sum_{j\geq j_0}\sum_k |d_{j,k}||\psi_{j,k}(x)| \leq C\sum_{j\geq j_0}(2K+1)M2^{-js}\left(1+2^j|x|\right)^{-s'} \ .$$

If $j\geq j_0$, $2^j|x|>1/2$, thus $2^j|x|<2^j|x|+1\leq 4.2^j|x|$. One thus has

$$\begin{aligned}\sum_{j\geq j_0}\sum_k |d_{j,k}||\psi_{j,k}(x)| &\leq C\sum_{j\geq j_0}2^{-j(s+s')}|x|^{-s'}\leq C2^{-j_0(s+s')}|x|^{-s'}\\ &\leq C|x-y|^{s+s'}|x|^{-s'}\end{aligned}$$

since (3.5) holds. Then, using that $|x-y|\leq 2|x|$, the last inequality gives

$$\sum_{j\geq j_0}\sum_k |d_{j,k}||\psi_{j,k}(x)| \leq C|x-y|^{s+s'}(|x-y|+|x|)^{-s'} \ ,$$

which is the correct bound.

The third term is bounded by the same method as described above.

We now move to the first term, which is a little bit more delicate to study. We will use the derivative of the wavelet ψ. Indeed, one remarks that

$$|\psi_{j,k}(x)-\psi_{j,k}(y)|\leq |x-y|\sup_{z\in[y,x]}\left|\psi'_{j,k}(z)\right| \ .$$

But we know that $\psi'_{j,k}(x)=(\psi(2^jx-k))'=2^j\psi'(2^jx-k)$, which is uniformly bounded by $C2^j$. Thus one has the property

$$|\psi_{j,k}(x)-\psi_{j,k}(y)|\leq C|x-y|2^j \ . \tag{3.6}$$

The sum in k contains only a fixed number $2K+1$ of non-zero terms, and for these k's, by the same arguments as before, $(1+|k|)^{-s'}\sim(1+2^j|x|)^{-s'}$. Using (3.6), one writes

$$\begin{aligned}(I) &\leq C\sum_{j\leq j_0-1}\sum_k |d_{j,k}|2^j|x-y|\leq C|x-y|\sum_{j\leq j_0-1}(2K+1)2^{-js}(1+|k|)^{-s'}\\ &\leq C|x-y|\sum_{j\leq j_0-1}2^{j(1-s)}\left(1+2^j|x|\right)^{-s'} \ .\end{aligned}$$

Moreover, $(1+2^j|x|)^{-s'}\leq C1+(2^j|x|)^{-s'}$, and

$$\begin{aligned}(I) &\leq C|x-y|2^{j_0(1-s)}+2^{j_0(1-(s+s'))}|x|^{-s'}\\ &\leq C|x-y|\left(|x-y|^{s-1}+|x-y|^{s+s'-1}|x|^{-s'}\right)\\ &\leq C|x-y|^{s+s'}\left(|x-y|^{-s'}+|x|^{-s'}\right)\leq C|x-y|^{s+s'}(|x-y|+|x|)^{-s'} \ .\end{aligned}$$

The key at this point was that $1-(s+s')$ is strictly positive by construction. One now upper-bounds $|x|^{-s'}$ by $(|x-y|+|x|)^{-s'}$.

This ends the proof of Theorem 3. □

3.3 Applications

We give here two applications of Theorem 3. We first exhibit some classes of functions that will belong to $C_{x_0}^{s,s'}$. Second we give a decomposition of any function of $C_{x_0}^{s,s'}$ into "simpler" functions. These propositions were already proved in a more general frame in [12], but our approach shows how easier they are to prove with the help of the $K_{x_0}^{s,s'}$ characterization.

Proposition 5.

Let us assume that $s' < 0$ and $s+s' > 0$. Let $\{U_j(x)\}_{j\in\mathbb{N}}$ be a sequence of functions that satisfy, for every $|\alpha| \le N$,

$$\left|\partial^\alpha U_j(x)\right| \le C(1+|x|)^{-s'} . \tag{3.7}$$

Then the function f defined by

$$f(x) = \sum_{j=0}^{+\infty} 2^{-sj} U_j\left(2^j(x-x_0)\right) \tag{3.8}$$

belongs to $C_{x_0}^{s,s'}$.

Proof. We give here the proof of this proposition in the case where $s+s' < 1$, $s < 1$, the general case only needs an easy adaptation of the following.

With f defined by (3.8), let us study the differences $|f(x)-f(y)|$. Let j_0 be the integer such that $2^{-j_0} \le |x-y| \le 2^{-j_0-1}$. Then,

$$\begin{aligned}
|f(x)-f(y)| &= \left|\sum_{j=0}^{+\infty} 2^{-sj}\left(U_j\left(2^j(x-x_0)\right) - U_j\left(2^j(y-x_0)\right)\right)\right| \\
&\le \sum_{j=0}^{j_0} 2^{-sj}\left|U_j\left(2^j(x-x_0)\right) - U_j\left(2^j(y-x_0)\right)\right| \qquad (I) \\
&\quad + \sum_{j=j_0+1}^{+\infty} 2^{-sj}\left|U_j\left(2^j(x-x_0)\right) - U_j\left(2^j(y-x_0)\right)\right| \qquad (II) .
\end{aligned}$$

When $j \ge j_0+1$, by (3.7), one obtains

$$\left|U_j\left(2^j(x-x_0)\right) - U_j\left(2^j(y-x_0)\right)\right| \le 2\left|U_j\left(2^j(x-x_0)\right)\right| \le C\left(1+2^j|x-x_0|\right)^{-s'},$$

and thus

$$\begin{aligned}
(II) &\le C\sum_{j=j_0+1}^{+\infty} 2^{-sj}\left(1+2^j|x-x_0|\right)^{-s'} \le C\sum_{j=j_0+1}^{+\infty} 2^{-sj} + 2^{-(s+s')j}|x-x_0|^{-s'} \\
&\le C2^{-sj_0} + 2^{-(s+s')j_0}|x-x_0|^{-s'} \le C|x-y|^s ,
\end{aligned}$$

since $2^{-j_0} \sim |x - x_0|$.

Consider now the other term (I).

$$
\begin{aligned}
(I) &\leq C \sum_{j=0}^{j_0} 2^{-sj} \left| 2^j(x-x_0) - 2^j(y-x_0) \right| \sup_{z\in[2^j(x-x_0),2^j(y-x_0)]} \left| \partial^1 U_j(z) \right| \\
&\leq C \sum_{j=0}^{j_0} 2^{-sj} 2^j |x-y| \max\left(\left(1+2^j|x-x_0|\right)^{-s'}, \left(1+2^j|y-x_0|\right)^{-s'} \right) \\
&\leq C 2^{(1-s)j_0} \left(1+2^{j_0}|x-x_0|\right)^{-s'} |x-y| \\
&\leq C|x-y|^s \left(1+\frac{|x-x_0|}{|x-y|}\right)^{-s'} \leq C|x-y|^{s+s'}(|x-y|+|x-x_0|)^{-s'} ,
\end{aligned}
$$

where one has assumed that $(1+2^j|x-x_0|)^{-s'} \geq (1+2^j|y-x_0|)^{-s'}$. □

The following proposition gives a decomposition of any function $f \in K_{x_0}^{s,s'}$ into two terms of different behaviors, the first one being regular and the second one containing the "oscillatory" behavior of f around x_0. It has already been proved in a more general case by Y. Meyer in [12].

Proposition 6.

Let s, s' be two real numbers such that $s + s' > 0$, $s' < 0$, and $x_0 \in \mathbb{R}$. Then the following propositions are equivalent

- $f \in K_{x_0}^{s,s'}$
- *there exist a constant $\delta > 0$, a polynomial P of degree smaller than $[s]$, and a function h which satisfies $h \in C^{s+s'}([x_0-\delta, x_0+\delta])$ and $|h(x)| \leq C|x-x_0|^{s+s'}$, such that*

$$f(x) = P(x) + |x-x_0|^{-s'} h(x) . \tag{3.9}$$

Proof. We treat the case $0 < s + s' < 1$ and $x_0 = 0$.

Assume $f \in K_0^{s,s'}$, and $|y| \leq x$. There exists a polynomial P of degree smaller than s such that (2.11) holds. Let us define the functions g and h by

$$g(x) = \frac{f(x) - P(x)}{|x|^{[s]}}, \quad h(x) = \frac{f(x) - P(x) - g(0)x^{[s]}}{|x|^{-s'}} .$$

One knows that, for all x, y close enough to 0,

$$|g(x) - g(y)| \leq C|x-y|^{s+s'}(|x-y|+|x|)^{-s'-[s]} ,$$

and that $h(x) = |x|^{s'+[s]}(g(x) - g(0))$. One always has $-1 < s' + [s] < 1$. If $s' + [s] = 0$, the result is obvious, thus we restrict the study to $s' + [s] \neq 0$.

First note that if $y = 0$,

$$|h(x) - h(y)| = |h(x)| = |x|^{s'+[s]}|g(x) - g(0)| \leq C|x|^{s'+[s]}|x|^{s-[s]} = C|x|^{s+s'} .$$

Now one can assume that $x \neq 0$ and $y \neq 0$, and thus, denoting $\tilde{g}(x) = g(x) - g(0)$,

$$|h(x) - h(y)| \leq \left| |x|^{s'+[s]}\tilde{g}(x) - |y|^{s'+[s]}\tilde{g}(y) \right| .$$

Using that $|\tilde{g}(x)| \leq C|x|^{s-[s]}$ for all x close enough to 0 and (2.11), one obtains

$$\begin{aligned} |h(x)-h(y)| &\leq |x|^{s'+[s]}\,|\tilde{g}(x)-\tilde{g}(y)| + \left||x|^{s'+[s]}\tilde{g}(y) - |y|^{s'+[s]}\tilde{g}(y)\right| \\ &\leq |x|^{s'+[s]}\,|\tilde{g}(x)-\tilde{g}(y)| + |\tilde{g}(y)|\left||x|^{s'+[s]} - |y|^{s'+[s]}\right| \\ &\leq C\Big(|x|^{s'+[s]}|x-y|^{s+s'}\,(|x-y|+|x|)^{-s'-[s]} \\ &\quad + |g(y)|\left||x|^{s'+[s]} - |y|^{s'+[s]}\right|\ . \end{aligned}$$

Then we make the same kind of manipulations as before. For the first term, one uses $|x| \leq |x-y| + |x| \leq 3|x|$ to get

$$\begin{aligned} |x-y|^{s+s'}|x|^{s'+[s]}(|x-y|+|x|)^{-s'-[s]} &\leq C|x-y|^{s+s'}|x|^{s'+[s]}|x|^{-s'-[s]} \\ &\leq C|x-y|^{s+s'}\ . \end{aligned}$$

The second term is more delicate to study:

- if $|x-y| \leq |y|$, then

$$\begin{aligned} |\tilde{g}(y)|\left||x|^{s'+[s]} - |y|^{s'+[s]}\right| &\leq C|y|^{s-[s]}|x-y||y|^{s'+[s]-1} \\ &\leq C|x-y|^{s+s'}\left(\frac{|x-y|}{|y|}\right)^{1-(s+s')} \leq C|x-y|^{s+s'}. \end{aligned}$$

- if $|y| \leq |x-y|$ (i.e., $y < x/2$), two cases must be separated
  - if $-1 < s'+[s] < 0$, $||x|^{s'+[s]} - |y|^{s'+[s]}| \leq C|y|^{s'+[s]}$, and

$$|\tilde{g}(y)|\left||x|^{s'+[s]} - |y|^{s'+[s]}\right| \leq C|y|^{s-[s]}|y|^{s'+[s]} \leq C|y|^{s+s'} \leq C|x-y|^{s+s'}.$$

  - if $0 < s'+[s] < 1$, $||x|^{s'+[s]} - |y|^{s'+[s]}| \leq C|x-y|^{s'+[s]}$ (indeed, $t \to t^{s'+[s]}$ is concave), and

$$\begin{aligned} |\tilde{g}(y)|\left||x|^{s'+[s]} - |y|^{s'+[s]}\right| &\leq C|y|^{s-[s]}|x-y|^{s'+[s]} \\ &\leq C|x-y|^{s-[s]}|x-y|^{s'+[s]} \leq C|x-y|^{s+s'}. \end{aligned}$$

The function h belongs to $C^{s+s'}([x_0-\delta, x_0+\delta])$.

Let us assume now that f satisfies (3.9) for a certain polynomial P and a function h, but does not satisfy (2.11).

Since (2.11) is not verified, one can find two sequences of real numbers $\{x_n\}_n$ and $\{y_n\}_n$, such that, for all n,

$$|g(x_n) - g(y_n)| \geq n|x_n - y_n|^{s+s'}(|x_n - y_n| + |x_n|)^{-s'-[s]}\ . \tag{3.10}$$

Since all the properties are local, around x_0, one can extract from these sequences two subsequences (still denoted by x_n and y_n) that will satisfy $\lim_n x_n = X$ and $\lim_n y_n = Y$, and

$$|g(x_n) - g(y_n)| \geq C_n|x_n - y_n|^{s+s'}(|x_n - y_n| + |x_n|)^{-s'-[s]}\ , \tag{3.11}$$

with $C_n \to +\infty$ when $n \to +\infty$.

First case. $X \neq Y$, $X \neq 0$ and $Y \neq 0$. Since g is continuous, using (3.11) and the fact that $|x_n - y_n| \to_n |X - Y|$ and $|x_n| \to_n |X|$, one obtains that $\forall n$ large enough, $|g(X) - g(Y)| \geq n|X - Y|^{s+s'}(|X - Y| + |X|)^{-s'-[s]}$, which is absurd.

Second case. $X = 0$ and $Y \neq 0$. This case is treated similarly as the preceding one.

Third case. $X = Y = 0$. One can assume that, for all n, $|x_n| \geq |y_n|$.

By definition one has, for all x, $|g(x)| \leq C|x|^{s-[s]}$. Thus, for all n, $|g(x_n) - g(y_n)| \leq C(|x_n|^{s-[s]} + |y_n|^{s-[s]}) \leq 2C|x_n|^{s-[s]}$. On the other hand, using (3.11), one has

$$\begin{aligned} 2C|x_n|^{s-[s]} &\geq |g(x_n) - g(y_n)| \geq C_n|x_n - y_n|^{s+s'}(|x_n - y_n| + |x_n|)^{-s'-[s]} \\ &\geq C_n|x_n - y_n|^{s+s'}|x_n|^{-s'-[s]} . \end{aligned}$$

The lower-bound $(|x_n - y_n| + |x_n|)^{-s'-[s]}$ by $|x_n|^{-s'-[s]}$ holds even when $-s' - [s] < 0$, still because $|x_n| \sim (|x_n - y_n| + |x_n|)$ for all n. One thus obtains $|x_n| \geq C_n^{\frac{1}{s+s'}}|x_n - y_n|$, which can be rewritten as

$$|x_n - y_n| = o(|x_n|) . \tag{3.12}$$

(3.12) says that the couples of points where the inequality (3.9) may fail must satisfy some strong properties: Both converge to 0, and the differences $|x_n - y_n|$ are small while the differences $|g(x_n) - g(y_n)|$ stay large. Intuitively it corresponds to the case of strong oscillations around 0.

Let us show that this is impossible. One would have

$$|y_n|^{-s'-[s]} = |x_n|^{-s'-[s]} + \big(-s'\big)|x_n - y_n||z_n|^{-s'-[s]-1} ,$$

where z_n is a real number between x_n and y_n. Then,

$$\begin{aligned} |g(x_n) - g(y_n)| &= \Big||x_n|^{-s'-[s]}h(x_n) - |y_n|^{-s'-[s]}h(y_n)\Big| \\ &\leq |x_n|^{-s'-[s]}|h(x_n) - h(y_n)| \\ &\quad + C|h(y_n)||x_n - y_n||z_n|^{-s'-[s]-1} \\ &\leq C|x_n|^{-s'-[s]}|x_n - y_n|^{s+s'} \\ &\quad + C|y_n|^{s+s'}|x_n - y_n||z_n|^{-s'-[s]-1} \end{aligned}$$

since $h \in C^{s+s'}(\mathbb{R})$.

The first term in the last inequality is bounded by $|x_n - y_n|^{s+s'}(|x_n - y_n| + |x_n|)^{-s'-[s]}$. Let us deal with the last term. Using that $|z_n| \sim |x_n| \sim |y_n|$, one verifies that

$$\begin{aligned} |y_n|^{s+s'}|x_n - y_n||z_n|^{-s'-[s]-1} &\leq C|y_n|^{s+s'}|x_n - y_n||y_n|^{-s'-[s]-1} \\ &\leq C|x_n - y_n||y_n|^{s-[s]-1} \\ &\leq C|x_n - y_n|^{s+s'}|x_n - y_n|^{1-(s+s')}|y_n|^{s-[s]-1} \\ &\leq C|x_n - y_n|^{s+s'}(|x_n - y_n| + |y_n|)^{-s'-[s]} . \end{aligned}$$

This eventually gives

$$|g(x_n) - g(y_n)| \leq C|x_n - y_n|^{s+s'}(|x_n - y_n| + |y_n|)^{-s'-[s]} ,$$

in contradiction with (3.11). □

The main interest of the last proofs is to show that it is possible to check all the properties of functions belonging to $C_{x_0}^{s,s'}$ spaces (with $s + s' \geq 0$) using only elementary arguments and a time domain analysis.

4. Algorithms

4.1 Background Ideas

There are three major justifications for the use of $K_{x_0}^{s,s'}$ spaces for the characterization of regularity in practical applications.

- $K_{x_0}^{s,s'}$ spaces give a rather rich description of the regularity structure.
- The computation of both exponents (s, s') is performed using directly the values of the function. One does not lose information by integrating or smoothing the data. Moreover, one can extract from the frontier the usual information, i.e., the Hölder exponents.
- it may seem harder to estimate a frontier of a domain in $\mathbb{R}^2$ than only one regularity exponent. But this is not the case. The main reason is that we are using more information: In fact, using (2.11) we extract the whole information available in the data. Combined with the fact that a frontier must satisfy a number of constraints such that its general aspect is known, this leads to reliable estimation procedures.

The formula defining the spaces $K_{x_0}^{s,s'}$ for $s > 1$ involves a polynomial which approximates the data (a kind of Taylor expansion), which is accessible only with the help of finite differences. This makes harder the implementation of an algorithm in these situations. We then focus on the simpler case $s < 1$ and $s + s' < 1$, where we have already seen that there is no polynomial in the definition of $K_{x_0}^{s,s'}$. An algorithm for estimating this part of the frontier has been proposed in [10]. We describe here another approach.

4.2 Implementation

We want to estimate the $K_{x_0}^{s,s'}$-frontier of a function f at a point x_0. We start with formula (3.2)

$$|f(x) - f(y)| \leq C|x - y|^{s+s'}(|x - y| + |x - x_0|)^{-s'} .$$

We assume that we have at our disposal the discrete values $\{f_i\}_{i=1,\dots,N}$ of a function f at the points $\{x_i\}_{i=1,\dots,N}$ (note that we do not need to assume that the $\{x_i\}_{i=1,\dots,N}$ are equidistant). If $f \in K_{x_0}^{s,s'}$, $\forall i, j$,

$$|f_i - f_j| \leq C|x_i - x_j|^{s+s'}(|x_i - x_j| + |x_i - x_0|)^{-s'} . \tag{4.1}$$

Define $x_{i,j}$ by $x_{i,j} = \log(|x_i - x_j|)$ and $y_{i,j,s'}$ by

$$y_{i,j,s'} = \log(|f_i - f_j|) + s' \log\left(1 + \frac{|x_i - x_0|}{|x_i - x_j|}\right) .$$

Then, (4.1) reads $\forall\, \lambda = (i, j)$, $y_{\lambda,s'} \leq Cx_\lambda$.

Now fix an exponent s'. In order to obtain the other exponent s as a function of s', it suffices to make a regression on the maxima of the set of couples (x_λ, y_λ) (where $\lambda \in [1, \dots, N]^2$) to find the corresponding exponent $s = \Gamma(s')$.

The practical implementation proceeds as follows:

- Choose a set of n discretized values of s', $\{s'_1, s'_2, \dots, s'_n\}$, ranging typically in $[-1, 0.5]$.
- For each s'_i, compute the corresponding y_{λ,s'_i} and x_λ.
- Find the largest y_{λ,s'_i}, when λ belongs to $\Lambda_{x_0} = \{(k,l)/ \ |x_0 - x_k| < 1/4 \text{ and } |x_0 - x_l| < 1/4\}$. We obtain the y_{μ,s'_i}, where μ belongs to a subset of Λ_{x_0}.
- Perform a linear regression on the set of couples $(x_{\mu,s'}, y_{\mu,s'})$.
- The slope of the straight line obtained by regression is the estimation of the exponent s corresponding to the s'.

A set of n samples in the frontier of exponents $s = \Gamma(s')$ is then obtained. By a simple method of convexification, it can be modified into a convex set of samples. Applying this method we obtain an approximation of the frontier, which satisfies its basic theoretical properties: It is convex, non-decreasing, with a derivative with modulus less than 1.

A last but important remark is the following: Since the simple version of the $K^{s,s'}_{x_0}$ spaces (i.e., without polynomials) is considered, the formula we use can only be applied in the triangle $0 < s < 1$, $s' < 0$ and $0 < s + s' < 1$. This implies that, for a function f whose pointwise Hölder exponent is $\alpha < 1$, the algorithm can not detect any regularity larger than $s = \alpha$, even when $s + s' < 0$. This equivalently means that the frontier the algorithm tries to estimate can not intersect the half-plane $\{(s, s')/ \ s > \alpha\}$. This provides us with a sharp localization of the pointwise Hölder exponent.

5. Numerical Results

We present the results of the algorithm implemented in different cases. We first treat the case of an isolated singularity, with three examples: A cusp singularity, a chirp singularity, and a sum of two chirps at the same point. Then the more complicated cases of functions which are everywhere continuous, but nowhere differentiable are considered: We deal with the Weierstrass function, the fractional Brownian motion, and the generalized Weierstrass function.

In all the figures, we plot the frontier found by the algorithm, and compare it with the theoretical one. We also plot the straight lines $s + s' = 0$, and $s + s' = 1$, which bound the validity of the results (indeed, remember that the formula we are using is only valid for $0 < s + s' < 1$ and $s' < 0$). All the results were obtained using functions sampled on 1024 points.

5.1 A Cusp

The function considered here is $x \to |x|^{0.25}$. Since there is no oscillation phenomenon, the local and the pointwise Hölder exponents are both equal to $\alpha = 0.25$. The theoretical frontier is a vertical line.

The estimation found for the common value of the two regularity exponents is 0.252, which is extremely precise (see Figure 2).

5.2 A Chirp

The function we study here is the chirp function, $|x|^{0.6} \sin(\frac{1}{|x|^{1.4}})$. The theoretical Hölder exponents are $\alpha_p = 0.6$, and $\alpha_l = \frac{0.6}{1+1.4} = 0.25$.

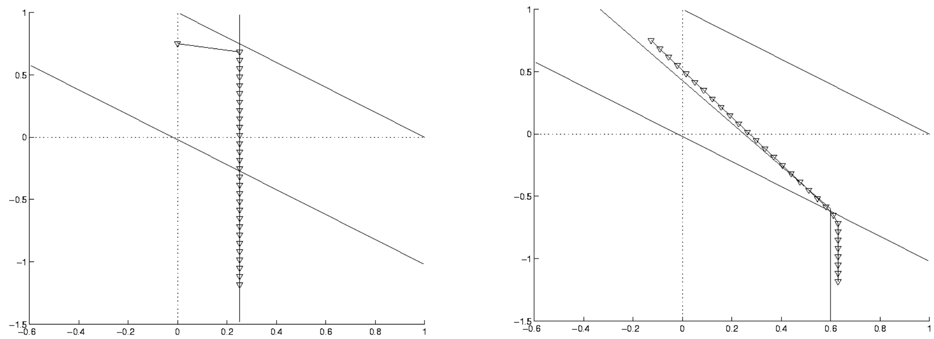

FIGURE 2 **Left:** Estimation of the frontier of the cusp $|x|^{0.25}$ at 0. The estimation is plotted with triangles. **Right:** Estimation of the frontier of the chirp $|x|^{0.6} \sin(\frac{1}{|x|^{1.4}})$ at 0.

The frontier computed by the algorithm (Figure 2) yields the estimations 0.27 and 0.62 for, respectively, the local and the pointwise exponent of f. One notices one more time the precision of the results. The whole frontier is also estimated with good accuracy.

5.3 A Sum of Two Chirps

The case of the sum of two chirps located at the same point is delicate. Indeed, it is very hard to distinguish the two behaviors, since there are two types of oscillations (at different frequencies).

We see that, in Figure 3, the two behaviors are identified, since, for example, the theoretical pointwise and local exponents, (respectively 0.5 and 0.25) are found with a good precision. The phase transition between the two chirps is not well estimated, but, away from it, the estimation of the frontier is accurate.

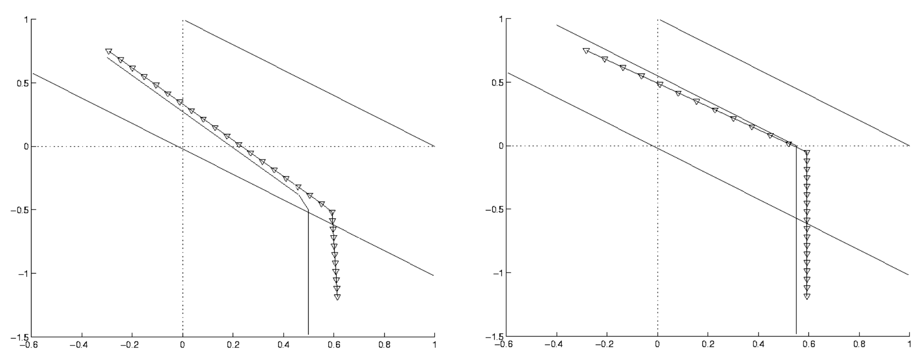

FIGURE 3 **Left:** Estimation of the frontier of the function $|x|^{0.8} \sin(\frac{1}{|x|^3}) + |x|^{0.5} \sin(\frac{1}{|x|^{0.25}})$ at 0. **Right:** Estimation of the frontier of the Weierstrass function with $\lambda = 3.23$ and $H = 0.55$ at an arbitrary point.

5.4 A Weierstrass Function

The Weierstrass function, defined by

$$W_H(x) = \sum_{j=0}^{+\infty} \lambda^{-jH} \sin\left(\lambda^j x\right) ,$$

where $\lambda \geq 2$ and $0 < H < 1$, belongs to a more complicated type of functions. Indeed, it is well known that W_H is everywhere continuous, nowhere differentiable, and that it has a Hölder exponent equal to H at all points (see for example the original article [15] and [7]).

Remark that, in Figure 3, the knee of the frontier located around the axis $s' = 0$ is found by the algorithm.

5.5 An fBm

A path of a Fractional Brownian Motion is another way of obtaining a signal for which the pointwise Hölder function is controlled, and is almost surely everywhere equal to a given exponent H (see [1] for example).

We have tested the algorithm on an fBm with $H = 0.7$ (see Figure 4).

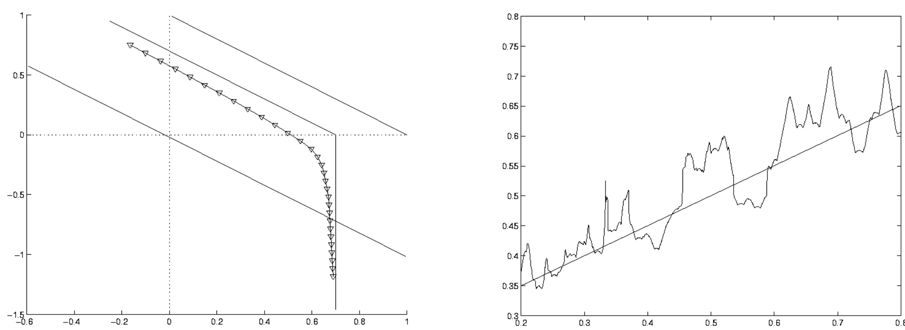

FIGURE 4 **Left:** Estimation of the frontier at an arbitrary point of an fBm with $H = 0.7$. The knee in the frontier (at $s' = 0$) is still recovered by the algorithm, but it is smoother in the estimation than in the theoretical frontier. **Right:** Estimation of the pointwise Hölder exponents of a generalized Weierstrass function satisfying $\alpha_p(t) = t/2 + 1/4$, from $t = 0.2$ to $t = 0.8$. The straight line is the theoretical pointwise Hölder function.

5.6 A Generalized Weierstrass Function

The generalized Weierstrass function [3] is defined by

$$W_H(x) = \sum_{j=0}^{+\infty} \lambda^{-jH(x)} \sin\left(\lambda^j x\right) ,$$

where $\lambda \geq 2$ and $x \to H(x)$ is a C^1-function ranging in $(0, 1)$. It verifies, for every x,

$\alpha_l(x) = \alpha_p(x) = H(x)$. The difference with the classical Weierstrass functions is that one can now prescribe time varying pointwise exponents.

We have run the algorithm on a function generated with $H(x) = x/2 + 1/4$. The estimated pointwise exponents are plotted on Figure 4 for x ranging in $[0.2, 0.8]$.

The algorithm has been implemented in **FracLab**, a software toolbox available at: `http://www-rocq.inria.fr/fractales/`

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