COMBINING MULTIFRACTAL ADDITIVE AND MULTIPLICATIVE CHAOS

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ABSTRACT. In this work, we study the new class of multifractal measures, which combines additive and multiplicative chaos, defined by

$$\nu_{\gamma,\sigma} = \sum_{j\geq 1} \frac{b^{-j\gamma}}{j^2} \sum_{0\leq\, k\, \leq b^j -1} \mu([kb^{-j},(k+1)b^{-j}))^\sigma \delta_{kb^{-j}} \quad (\gamma \geq 0,\ \sigma\geq 1),$$

where μ is any positive Borel measure on $[0,1]$ and b is an integer ≥ 2.
The singularities analysis of the measures $\nu_{\gamma,\sigma}$ involves new results on the mass distribution of μ when μ describes large classes of multifractal measures. These results generalize ubiquity theorems associated with the Lebesgue measure.

Under suitable assumptions on μ, the multifractal spectrum of $\nu_{\gamma,\sigma}$ is linear on $[0, h_{\gamma,\sigma}]$ for some critical value $h_{\gamma,\sigma}$, and then it is strictly concave on the right of $h_{\gamma,\sigma}$, and deduced from the one of μ by an affine transformation. This untypical shape is the result of the combination between Dirac masses and atomless multifractal measures. These measures satisfy multifractal formalisms. These measures open interesting perspectives in modeling discontinuous phenomena.

1. INTRODUCTION

The multifractal nature of functions or measures possessing jump discontinuities has been investigated in several situations [30, 50, 51, 31, 22, 23]. The purpose of this article is the construction and the multifractal analysis of a new class of measures defined by infinite sums of Dirac masses. The study of these measures gives rise to yet unknown multifractal behaviors. Moreover, this class illustrates most of the multifractal behaviors one can expect from discontinuous measures which satisfy some multifractal formalism. This is important for the purpose of modeling discontinuous phenomena which are known to exhibit multifractal behaviors, for example in geophysics, telecommunications or finance [28, 38, 41].

The local regularity of a function or a measure μ at a point x is usually described by an Hölder exponent $h_\mu(x)$. Our work draws its interest from positive Borel measures, and in this case the Hölder exponent is defined by

$$h_\mu(x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ stands for the closed ball of radius r centered at x.

The multifractal analysis of μ consists in computing the size of the level sets of this Hölder exponent h, $E_h^\mu = \{x : h_\mu(x) = h\}$. More precisely, one often tries to find the Hausdorff multifractal spectrum d_μ of the measure μ defined by

$$h \mapsto d_\mu(h) = \dim(E_h^\mu),$$

where $\dim E$ stands for the Hausdorff dimension of the set E (with $\dim\emptyset = -\infty$).

Multifractal analysis started in the context of the study of fully developed turbulence with the following heuristics: In [26], Frisch and Parisi proposed a connexion, via a Legendre transform, between the Hausdorff multifractal spectrum of the energy dissipation measure μ and a kind of free energy function associated with μ. In the recent past years, a substantial amount of work has been devoted to compute the multifractal spectra of several classes of functions and measures [27, 16, 49, 15, 29, 20, 44, 1, 43, 37, 5, 7, 25]. These studies confirmed this connexion, which is now known as multifractal formalism. We make the definition of this formalism precise in a short moment.

Among the measures which multifractal analysis has been performed, two families can be distinguished by the typical shape of their spectrum.

Some measures, the construction of which is based on an additive scheme, exhibit linear increasing spectrum (see Figure 1): There exists $\beta \in (0,1]$ such that $d_\mu(h) = \beta h$ for $0 \leq h \leq 1/\beta$. Lévy subordinators [31] and the sums of Dirac masses of [22] belong to this class. These measures are a form of additive chaos. In these specific cases, the Hölder exponent at each point x is closely connected to the approximation rate of x by jump points as well as to the masses carried by these points. In this framework, the notion of "ubiquity" of some "resonant" sets [2, 18, 19] is accountable for the linear shape of the multifractal spectrum.

Atomless measures with a construction involving a multiplicative scheme usually have a strictly concave spectrum, including a decreasing part (see Figure 1). Multinomial measures, quasi-Bernoulli measures, Mandelbrot cascades and their extensions, as well as the recent compound Poisson cascades, are examples of such multiplicative chaos measures [14, 40, 33, 17, 7, 8]. These measures typically have a multifractal spectrum with the well-known $\cap$-shape, reflecting the validity of a multifractal formalism. This results from the Large Deviations theory (or from a similar argument) applied to the elements of a family of auxiliary "Gibbs" measures $\{\mu_h\}_{h\geq 0}$ such that each μ_h is carried by the level set E^μ_h.

It is natural to try to mix these two distinct construction schemes. In this article, we put forward the following scheme, where the jump points are the b-adic points. The heterogeneity in the distribution of the masses assigned to these points is created with the use of an auxiliary measure μ. More precisely, if μ is a positive Borel measure on $[0,1]$, let us consider the measure $\nu_{\gamma,\sigma}$ defined with the help of two parameters $\gamma \geq 0$ and $\sigma \geq 1$ by

$$\nu_{\gamma,\sigma} = \sum_{j\geq 1} \frac{1}{j^2} \sum_{0\leq k\leq b^j-1} b^{-j\gamma} \mu([kb^{-j},(k+1)b^{-j}))^\sigma \delta_{kb^{-j}}. \tag{1.1}$$

The factor j^{-2} makes the series converge when $\sigma = 1$ and $\gamma = 0$. In fact $\{j^{-2}\}_{j\geq 1}$ could be replaced by any decreasing positive sequence $\{a_j\}_{j\geq 1}$ such that $\sum_{j\geq 1} a_j < +\infty$ and $|\log a_j| = o(j)$.

This class of measures proves to have a fruitful structure, and it provides new important examples of measures that fulfill a multifractal formalism. Moreover, the measures $\nu_{\gamma,\sigma}$ have their natural counterparts in terms of discontinuous function series and wavelet series (see [12, 10]).

Let us mention that sets other than b-adic numbers could have been chosen for the location of the Dirac masses. Similar constructions will be performed in further works, using the rational numbers or some random families of points, as well as suitable associated weights. But the construction we deal with in this paper is key to understand the main ideas that rule the mixing between additive and multiplicative chaos. Our choice to work in the one-dimensional case is also motivated by this sake of comprehensibility.

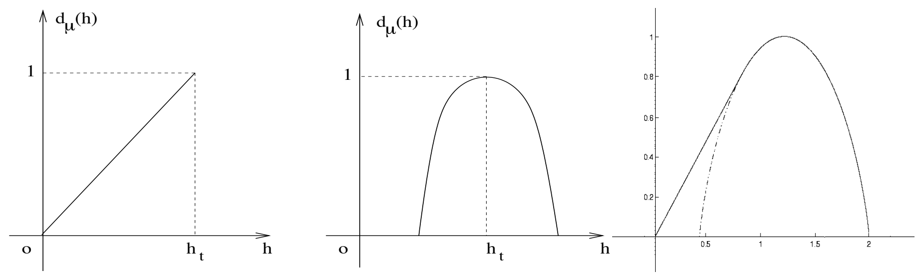

FIGURE 1. Typical multifractal spectrum of **Left:** a measure μ built on an additive scheme, **Middle:** on a multiplicative scheme, **Right:** a measure $\nu_{0,1}$ built where μ is a binomial measure. Here h_t is the Lebesgue-almost sure exponent.

In order to fully understand the next results, let us now come back to the notion of multifractal formalism. A multifractal formalism for measures is a formula which relates the multifractal spectrum d_μ to the Legendre transform of a scaling function associated with μ (see [14, 45] for complete mathematical foundations). A possible definition for the scaling function [14] is

$$\tau_\mu : q \in \mathbb{R} \mapsto \tau_\mu(q) = \liminf_{j\to+\infty} -\frac{1}{j}\log_b \sum_{0\le k\le b^j-1} \mu\big([kb^{-j},(k+1)b^{-j})\big)^q, \tag{1.2}$$

with the convention $0^q = 0\ \forall q$. In this paper, the multifractal formalism is said to hold for μ at exponent h when the multifractal spectrum coincides with the Legendre transform of the scaling function at h, i.e. when $\dim E_h^\mu = d_\mu(h) = \tau_\mu^*(h) := \inf_{q\in\mathbb{R}}(qh - \tau_\mu(h))$. This formalism combines some level sets considered in [45] and the scaling function of [14], and is satisfied by the classes of measures mentioned above. In terms of this formalism, a linear part in the spectrum corresponds to a non-differentiable point for τ_μ. Moreover, if $q_c = \inf\{q:\ \tau_\mu(q) = 0\}$, a linear spectrum starting at $(0,0)$ is equivalent to the fact that $\tau_\mu'(q_c^-) > 0$ and $\tau_\mu'(q_c^+) = 0$. Eventually, the spectrum exhibits a concave part on the right side of $\tau_\mu'(q_c^-)$ as soon as τ_μ is not linear when $q < q_c$.

We expose the properties and the multifractal structure of $\nu_{\gamma,\sigma}$ in two steps. It is convenient to begin with the basic construction $\nu = \nu_{0,1}$, and then to look at the influence of the parameters (γ,σ).

In order to state our results, three technical conditions, that we detail along the paper, are required: Condition **C1** ensures that the μ-mass of the b-adic intervals do not converge to 0 too fast as the intervals lengths converge to 0. **C2(**h**)** is related to the notion of ubiquity (detailed below) for some sets simultaneously related to the distribution of the measure μ and to the approximation rate by the b-adic numbers. **C3(**h**)** is comparable with the validity of the multifractal formalism of μ at h, but is slight stronger. Moreover **C2(**h**)** implies **C3(**h**)**.

Theorem 1.1. *Let μ be a positive Borel measure such that $\mathrm{supp}(\mu) = [0,1]$, and assume that* **C1** *holds for μ. Let $\nu = \nu_{0,1}$ be the measure given by formula (1.1). Let $q_c = \inf\{q \in \mathbb{R} : \tau_\mu(q) = 0\}$, and $h_c = \tau_\mu'(q_c^-)$.*
1. If $h_c > 0$, for every $h \in [0,h_c]$ one has $d_\nu(h) \le q_c h$.
If **C2(**h_c**)** *holds, for every $h \in [0,h_c]$ one has $d_\nu(h) = q_c h$, and the multifractal formalism holds at h.*

2. Let $h \geq h_c$. Then $d_\nu(h) \leq \tau^*_\mu(h)$ if $\tau^*_\mu(h) \geq 0$, and $E^\nu_h = \emptyset$ if $\tau^*_\mu(h) < 0$.
If **C3(**h**)** *holds, then $d_\mu(h) = d_\nu(h) = \tau^*_\mu(h) = \tau^*_\nu(h)$, and the multifractal formalism holds at h.*

Theorem 1.1 applies to the statistically self-similar measures μ ontained as limits of multiplicative processes described above. Examples of such measures are detailed in Section 3.2. Moreover, Theorem 1.1 applies to the measure ν itself: the process can be iterated, the spectrum being unchanged.

Theorem 1.1 shows that $\tau_\nu(q) \leq \tau_\mu(q)$ if $q \leq q_c$ and $\tau^*_\mu(\tau'_\mu(q^+)) \geq 0$, and that $\tau_\nu(q) = 0$ if $q > q_c$. There is equality everywhere when **C3(**$\tau'_\mu(q^+)$**)** holds for a dense countable set of q's such that $\tau^*_\mu(\tau'_\mu(q^+)) \geq 0$ and $\tau'_\mu(q^+) \geq h_c$. When $h_c > 0$, in the thermodynamical frame, the non differentiability of τ_ν at q_c corresponds to a phase transition (see [52, 25] for discussions on this phenomenon).

The following remark is key. Under the assumptions of Theorem 1.1 and when $h_c = \tau'_\mu(q_c^-) > 0$, multifractal formalisms that focus on level sets such as $\widetilde{E}^\nu_h = \{x : \lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} = h\}$ (defined using a limit rather than a $\liminf$) do not hold for ν at h when $0 < h < h_c$. This was noticed in [3] where the authors consider the measure $\nu_{\gamma,1}$ in the case where μ is the Lebesgue measure. The same difficulty is encountered in [51] with some self-similar sums of Dirac masses, which are close to our class $\nu_{0,1}$ when μ is multinomial. But [51] concludes to the failure of the multifractal formalism since the sets $\widetilde{E}^\nu_h$ were considered.

This phenomenon pleads for the choice of the sets E^ν_h defined using a $\liminf$, because no information is lost: These sets always form a partition of $[0,1]$. This choice incited us to investigate in detail the repartition of the mass of μ when μ differs from the Lebesgue measure, leading to new significant results of ubiquity.

The validity of Theorem 1.1 depends on the following theorem, which gives a lower bound of the dimension of sets that are linked to μ and to some approximation rate δ. Let ψ be a continuous positive function with $\psi(0) = 0$. Then $\mathcal{Q}_\psi(I)$ is said to hold for an interval I if $|I|^{h+\psi(|I|)} \leq \mu(I) \leq |I|^{h-\psi(|I|)}$.

Theorem 1.2. *Let μ be a positive Borel measure such that $\mathrm{supp}(\mu) = [0,1]$, and $h > 0$. For every $\delta > 1$, for every continuous positive function ψ with $\psi(0) = 0$ and for every positive sequence $\widetilde{\varepsilon} = \{\varepsilon_j\}_{j\geq 1}$ converging to 0, let us define*

$$S_{\delta,\widetilde{\varepsilon},\psi}(h) = \bigcap_{n\geq 1} \bigcup_{j\geq n} \bigcup_{\substack{k\in\{0,\dots,b^j-1\}: \\ \mathcal{Q}_\psi([kb^{-j},(k+1)b^{-j}])\text{ holds}}} [kb^{-j}, kb^{-j} + b^{-j(\delta-\varepsilon_j)}]. \tag{1.3}$$

Suppose that **C2(**h**)** *holds. There exists a function ψ such that for every $\delta > 1$, one can find a positive sequence $\widetilde{\varepsilon}$ converging to 0 and a positive Borel measure m_δ on $[0,1]$ with the following properties: $m_\delta(S_{\delta,\widetilde{\varepsilon},\psi}(h)) > 0$, and for every Borel set $E \subset [0,1]$ with $\dim E < \frac{\tau^*_\mu(h)}{\delta}$, $m_\delta(E) = 0$. Thus, $\dim S_{\delta,\widetilde{\varepsilon},\psi}(h) \geq \frac{\tau^*_\mu(h)}{\delta}$.*

Let us recall that if $x \in \mathbb{R}$, and $\delta \geq 1$, x is said to be δ-approximated if there exist an infinite number of b-adic numbers kb^{-j} such that $|kb^{-j} - x| \leq b^{-j\delta}$. With each x is associated its approximation rate

$$\delta_x = \sup\{\delta \geq 1 : x \text{ is } \delta\text{-approximated}\}. \tag{1.4}$$

One always has $\delta_x \geq 1$, and it is shown in [20, 32] for example that the set $\{x \in \mathbb{R} : \delta_x = \delta\}$ has a Hausdorff dimension equal to $\frac{1}{\delta}$.

Theorem 1.2 allows the computation of the Hausdorff dimension of the set of points that are infinitely often close at rate δ to b-adic numbers kb^{-j} that verify $\mu([kb^{-j},(k+1)b^{-j})) \sim b^{-jh}$. In fact Theorem 1.2 appears to be the consequence of a stronger result, Theorem 3.2, that we establish in Section 3.

Theorem 1.2 and 3.2 apply to measures μ for which there exists a control of the "speed of renewal" of the level sets of the Hölder exponents (see properties **(3)** and **(4)** in Section 3). The measures mentioned above as illustrations of Theorem 1.1 are typical examples of such measures.

Theorems 1.2 and 3.2 are referred to as "measure-conditioned ubiquity". They yield a generalization of the notion of ubiquity (see [19]), in the sense that they involve an ubiquity property (i.e. an omnipresence) of sets of points that must satisfy some property. Here we work with b-adic points in $[0,1]$ and the property is related to the behavior of $\mu\big([kb^{-j},(k+1)b^{-j})\big)$. The "usual" ubiquity theorems [18, 19, 32] can be understood in some sense as Theorem 3.2 applied to $\mu=\lambda$ (the Lebesgue measure) or more generally to a monofractal measure μ (which corresponds in reality to an empty condition satisfied by all points). The property of the $\limsup$-sets $S_{\delta,\widetilde{\varepsilon},\psi}(h)$ to be non-empty is thus remarkable, and strongly depends on the measure μ considered.

Let us now consider the measures $\nu_{\gamma,\sigma}$ defined by (1.1), where $\gamma\geq 0$, $\sigma\geq 1$.

Theorem 1.1' *Let μ be a positive Borel measure such that $\mathrm{supp}(\mu)=[0,1]$, and assume that* **C1** *holds for μ. Let $\gamma\geq 0$ and $\sigma\geq 1$. Let $q_{\gamma,\sigma}=\inf\{q\in\mathbb{R}:\tau_\mu(\sigma q)+\gamma q=0\}$, and $h_{\gamma,\sigma}=\sigma\tau_\mu'(\sigma q_{\gamma,\sigma}^-)+\gamma$. One has $q_{\gamma,\sigma}\in(0,1]$ and $0\leq h_{\gamma,\sigma}\leq q_{\gamma,\sigma}^{-1}$.*

1. If $h_{\gamma,\sigma}>0$, for every $h\in[0,h_{\gamma,\sigma}]$, $d_{\nu_{\gamma,\sigma}}(h)\leq q_{\gamma,\sigma}h$. If **C2**$(\frac{h_{\gamma,\sigma}-\gamma}{\sigma})$ *holds, for every $h\in[0,h_{\gamma,\sigma}]$, $d_{\nu_{\gamma,\sigma}}(h)=q_{\gamma,\sigma}h$, and the multifractal formalism holds at h.*

2. Let $h\geq h_{\gamma,\sigma}$. Then $d_{\nu_{\gamma,\sigma}}(h)\leq\tau_\mu^*\big(\frac{h-\gamma}{\sigma}\big)$ if $\tau_\mu^*\big(\frac{h-\gamma}{\sigma}\big)\geq 0$, and $E_h^{\nu_{\gamma,\sigma}}=\emptyset$ if $\tau_\mu^*\big(\frac{h-\gamma}{\sigma}\big)<0$. If **C3**$(\frac{h-\gamma}{\sigma})$ *holds, then $d_{\nu_{\gamma,\sigma}}(h)=\tau_\mu^*\big(\frac{h-\gamma}{\sigma}\big)$, and the multifractal formalism holds at h.*

An attentive reading of the arguments developed while studying the measures $\nu=\nu_{0,1}$ yield the proof of this theorem. This is left to the reader.

The spectrum $d_{\nu_{\gamma,\sigma}}$ has in fact the same shape as the one of d_ν (i.e. composed of two parts), but γ and σ allow us to "play" with the slope of the linear part and the shape of the (strictly) concave part.

The novelty of the measures $\nu_{\gamma,\sigma}$ and the possibility to reach measures which illustrate all possible pairs $0<q_c\leq 1$, $0\leq h_c\leq q_c^{-1}$ make this class valuable. Until now, the case $q_c<1$ and $h_c>0$ was obtained only when $q_ch_c=1$ by Lévy subordinators, and in the particular case where μ is the Lebesgue measure [3].

The cases $q_c=1$, $0<h_c\leq 1$ are reached for example by using multinomial measures in Theorem 1.1'. The introduction of the parameters γ and σ allows us to reach all the possibilities $q_c<1$ and $h_c>0$.

The case $h_c=0$ is particularly remarkable. In this case condition **C2**(h_c) of Theorem 1.1 is useless. When **C3**(h) is satisfied by μ for every h such that $\tau_\mu^*(h)>0$, d_μ has the classical $\cap$-shape, and it begins at $(0,0)$. To our knowledge, this kind of behavior appears only in the case $q_c=1$ in [42, 50, 51, 5]. The construction of $\nu_{0,\sigma}$ with such measures illustrates the cases $q_c<1$ and $h_c=0$.

Section 2 recalls the definitions of Hölder exponents and of the multifractal formalism adapted to our construction. Conditions **C1** and **C3**(h) are given.

Section 3 holds the definition of Condition **C2**(h) and the proof of Theorem 3.2, which implies Theorem 1.2. Section 4 contains the proof of Theorem 1.1. Section 3.2 indicates classes of measures μ that fulfill conditions **C1-3**, and thus yield explicit examples of measures ν. Some observations, especially concerning the validity of multifractal formalism for $\nu_{\gamma,\sigma}$, are gathered in Section 5.

2. General settings

Fix b an integer greater than 2. For $j \geq 1$ and $k \in [0, \ldots, b^j - 1]$, one sets $I_{j,k} = [kb^{-j}, (k+1)b^{-j})$. $I^+_{j,k}$ and $I^-_{j,k}$ denote the intervals $I_{j,k} + b^{-j}$ and $I_{j,k} - b^{-j}$.

If $x \in (0,1)$, $\forall j \geq 1$ $I_j(x)$ denotes the b-adic interval of length b^{-j} that contains x. Then define $I_j^+(x) = I_j(x) + b^{-j}$ and $I_j^-(x) = I_j(x) - b^{-j}$. For each $j \geq 1$, $k_{j,x}$ is the unique integer such that $I_j(x) = [k_{j,x}b^{-j}, (k_{j,x}+1)b^{-j})$.

A b-adic number kb^{-j} is said to be *irreducible* if the fraction $\frac{k}{b^j}$ is irreducible. Eventually, for the rest of the paper, we adopt the convention $\log(0) = -\infty$.

2.1. **Local regularity of measures.**

Definition 2.1. Let ν be a positive Borel measure, and $x_0 \in [0,1]$. One sets

$$h_\nu(x_0) = \liminf_{r \to 0+} \frac{\log \nu(B(x_0, r))}{\log |B(x_0, r)|} = \liminf_{j \to +\infty} \frac{\log \nu(B(x_0, b^{-j}))}{\log |B(x_0, b^{-j})|}. \tag{2.1}$$

$|B|$ indicates the diameter of the set B. If we refer back to Definition 1.1 of $\nu = \nu_{0,1}$, the behavior of μ on the b-adic grid is fundamental to control the local regularity of ν. We thus also focus on exponents of μ based on the b-adic grid.

Definition 2.2. Let μ be a positive Borel measure on $[0,1]$. For $x_0 \in (0,1)$, the lower and upper Hölder exponents of μ at x_0 are respectively defined by

$$\underline{\alpha}_\mu(x_0) = \liminf_{j \to +\infty} \frac{\log \mu(I_j(x_0))}{\log |I_j(x_0)|} \quad \text{and} \quad \overline{\alpha}_\mu(x_0) = \limsup_{j \to +\infty} \frac{\log \mu(I_j(x_0))}{\log |I_j(x_0)|}$$

When $\underline{\alpha}_\mu(x_0) = \overline{\alpha}_\mu(x_0)$, their common value is denoted $\alpha_\mu(x_0)$ and called the Hölder exponent of μ at x_0.

The left and right lower and upper Hölder exponents of μ at x_0 are defined by

$$\underline{\alpha}^-_\mu(x_0) = \liminf_{j \to +\infty} \frac{\log \mu(I_j^-(x_0))}{\log |I_j^-(x_0)|} \quad \text{and} \quad \underline{\alpha}^+_\mu(x_0) = \liminf_{j \to +\infty} \frac{\log \mu(I_j^+(x_0))}{\log |I_j^+(x_0)|}$$
$$\text{and} \quad \overline{\alpha}^-_\mu(x_0) = \limsup_{j \to +\infty} \frac{\log \mu(I_j^-(x_0))}{\log |I_j^-(x_0)|} \quad \text{and} \quad \overline{\alpha}^+_\mu(x_0) = \limsup_{j \to +\infty} \frac{\log \mu(I_j^+(x_0))}{\log |I_j^+(x_0)|}.$$

Similarly, when they coincide, $\alpha^-_\mu(x_0)$ and $\alpha^+_\mu(x_0)$ denote their common value.

The reader can check that $h_\mu(x) = \min(\underline{\alpha}^-_\mu(x), \underline{\alpha}_\mu(x), \underline{\alpha}^+_\mu(x))$.

Definition 2.3. For every positive Borel measure μ on $[0,1]$ and for every $\alpha \geq 0$, let $E^\mu_\alpha = \{x : h_\mu(x) = \alpha\}$ and $\widetilde{E}^\mu_\alpha = \big\{x : \alpha_\mu(x) = \alpha^+_\mu(x) = \alpha^-_\mu(x) = \alpha\big\}$.

The mapping $d_\mu : \ \alpha \geq 0 \mapsto \dim(E^\mu_\alpha)$ is called the multifractal spectrum of μ. One also sets $\tilde{d}_\mu(\alpha) = \dim \widetilde{E}^\mu_\alpha$.

2.2. **Legendre and Large Deviation spectrum, multifractal formalism.** The Legendre transform of a function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is defined by $\varphi^* : h \mapsto \inf_{p\in\mathbb{R}} (ph - \varphi(p))$.

Let μ be a positive Borel measure on $[0,1]$. The function τ_μ defined by (1.2) is known to be concave, non-decreasing, and the mapping $h \mapsto \tau_\mu^*(h)$ is referred to as the Legendre spectrum of μ.

Definition 2.4. Let μ be a positive Borel measure on $\mathbb{R}$. Let us define, $\forall \alpha \geq 0$, $\eta > 0$ and $j \geq 1$, $N_{j,\eta}(\alpha) = \#\big\{k : \frac{\log \mu(I_{j,k})}{\log b^{-j}} \in [\alpha-\eta, \alpha+\eta]\big\}$, and $d^g_\eta(\alpha) = \limsup_{j\to+\infty} j^{-1} \log_b N_{j,\eta}(\alpha)$. The large deviation spectrum of μ is the mapping $d^g_\mu :\ \alpha \mapsto \lim_{\eta\to 0^+} d^g_\eta(\alpha)$.

The Legendre and large deviation spectra are useful in multifractal analysis. They are more tractable than d_μ, and they yield upper bounds of d_μ. Remark that the maximum of $\alpha \mapsto \tau_\mu^*(\alpha)$ is always reached at $\tau_\mu'(0^+)$.

Proposition 2.5. *1. Let $\alpha \geq 0$. One has $\tilde{d}_\mu(\alpha) \leq d^g_\mu(\alpha) \leq \tau^*_\mu(\alpha)$ and $\tilde{d}_\mu(\alpha) \leq d_\mu(\alpha) \leq \tau^*_\mu(\alpha)$. If $\tau^*_\mu(\alpha) < 0$ then $E^\mu_\alpha = \emptyset$.*
2. If $\alpha \in [0, \tau'_\mu(0^+)]$ then $\dim \bigcup_{\alpha' \leq \alpha} E^\mu_{\alpha'} \leq \tau^*_\mu(\alpha)$.
3. If $\alpha \geq \tau'_\mu(0^+)$ then $\dim \bigcup_{\alpha' \geq \alpha} E^\mu_{\alpha'} \leq \tau^*_\mu(\alpha)$.

This is deduced from Theorem 1 of [14], Proposition 2.5 of [45], Theorem 1 of [39], Lemma 4.2 of [6] and the fact that $\widetilde{E}^\mu_\alpha \subset E^\mu_\alpha \cap \{x : \alpha_\mu(x) = \alpha\}$.

Definition 2.6. A positive Borel measure μ on $[0,1]$ is said to obey the multifractal formalism at $\alpha \geq 0$ if $d_\mu(\alpha) = \dim(E^\mu_\alpha) = \tau^*_\mu(\alpha)$.

The following lemma follows from standard arguments. It gives a heuristic interpretation of the large deviation spectrum and is used in Section 4.3.

Lemma 2.7. *Let μ be a positive Borel measure on $[0,1]$. For every $0 \leq \beta \leq \alpha$, for every $\varepsilon > 0$ and $\eta > 0$, there exists a scale J such that $j \geq J$ implies*

$$\frac{\log\left(\#\left\{k : b^{-j\alpha} \leq \mu(I_{j,k}) \leq b^{-j\beta}\right\}\right)}{\log b^j} \leq \sup_{\max(\beta-\eta,0)\leq\alpha'\leq\alpha+\eta} d^g_\mu(\alpha') + \varepsilon.$$

2.3. **Conditions C1, C2(h) and C3(h).**

Definition 2.8. Let μ be a positive Borel measure with $\mathrm{supp}(\mu) = [0,1]$.
- **Condition C1:** There exists a constant B such that $\forall j$, $\forall k = 0,..,b^j-1$, $\mu(I_{j,k}) \geq b^{-Bj}$.

- **Condition C2(h):** see Definition 3.1 in next Section 3.

- **Condition C3(h):** There exists a positive Borel measure m_h on $[0,1]$ such that $m_h(\widetilde{E}^\mu_h) > 0$ and for every Borel set $E \subset [0,1]$ such that $\dim E < \tau^*_\mu(h)$, one has $m_h(E) = 0$.

3. Conditioned ubiquity

3.1. **Main result.** Let us detail the assumptions that make Theorem 3.2 below work. The measure ν is built on the b-adic numbers, but the analysis of the initial measure μ may be naturally done using another base c. This is the case for instance for multinomial measures built in basis c, or for the c-adic Mandelbrot random multiplicative cascades. We shall thus deal with the two bases simultaneously. When working in a base b, $I^b_{j,k}$ denotes the interval $[kb^{-j}, (k+1)b^{-j})$, $I^b_j(x)$ the b-adic interval at scale j that contains x, and $I^b_j(x) = [k^b_{j,x} b^{-j}, (k^b_{j,x}+1)b^{-j}]$.

Assume that the measure μ, whose support is $[0,1]$, is given, as well as two exponents $\alpha > 0$ and $\beta > 0$. Our assumptions are as follows.

H(α,β): **(1)** There exist two continuous non-decreasing functions φ and ψ defined on $\mathbb{R}_+$ such that:
- $\varphi(0) = \psi(0) = 0$, $r \mapsto r^{-\varphi(r)}$ and $r \mapsto r^{-\psi(r)}$ are non-increasing near 0^+, and $\lim_{r\to 0^+} r^{-\varphi(r)} = +\infty$.
- $\forall \varepsilon > 0$, $r \mapsto r^{\varepsilon-\varphi(r)}$ is non-decreasing near 0 (which implies that $r \mapsto r^{\frac{\beta}{\delta}-\gamma\varphi(r)}$ is non-decreasing near 0 for $\beta, \gamma, \delta > 0$).
- The next properties **(2)**, **(3)** and **(4)** hold.

(2) There exist a constant M (depending on b and c) and a measure m whose support is $[0,1]$ such that

$$m\text{-a.e, } \exists\, n,\ \forall\, j \geq n,\ m\big(I_j^c(x)\big) \leq |I_j^c(x)|^{\beta-\varphi(c^{-j})}. \tag{3.1}$$

$$m\text{-a.e, } \exists\, n,\ \forall\, j \geq n,\ \mathcal{P}_M(I_{j,k}^c) \text{ holds for } |k - k_{j,x}^c| \leq 2b^2 c. \tag{3.2}$$

where $\mathcal{P}_M(I)$ is said to hold for an interval I when

$$M^{-1}|I|^{\alpha+\psi(|I|)} \leq \mu\big(I\big) \leq M|I|^{\alpha-\psi(|I|)}. \tag{3.3}$$

Notice that $\beta \leq 1$ since we work in $\mathbb{R}$.

(3) (Self-similarity property of m) For every closed c-adic subinterval I of $[0,1]$, f_I standing for the affine increasing mapping from I onto $[0,1]$, there exists a measure m^I on I, equivalent to the restriction of m to I, such that the measure $m^I \circ f_I^{-1}$ satisfies (3.1), and with the same exponent β.

For every $n \geq 1$, for every closed c-adic interval I of $[0,1]$, let

$$E_n^I = \Big\{ x \in I :\ \forall\, j \geq n + \log_c(|I|^{-1}),\ m^I\big(I_j^c(x)\big) \leq \Big(\frac{|I_j^c(x)|}{|I|}\Big)^{\beta - \varphi\left(\frac{c^{-j}}{|I|}\right)} \Big\}.$$

The sets E_n^I form a non-decreasing sequence and by (3.1) $\bigcup_{n\geq 1} E_n^I$ is of full m^I-measure. Let us define

$$n_I = \inf\big\{ n \geq 1 :\ m^I(E_n^I) \geq \|m^I\|/2 \big\}.$$

For $x \in [0,1)$ and $j \geq 0$, let $\mathcal{I}_j(b,x)$ the set of b-adic intervals of maximal length included in $[k_{j,x}c^{-j}, (k_{j,x}+\frac{1}{2})c^{-j}]$. Then if $L = [kb^{-j}, (k+1)b^{-j}) \in \mathcal{I}_j(b,x)$, for $\delta > 1$ let $\mathcal{L}^\delta$ be the set of c-adic intervals of maximal length included in $[kb^{-j}, kb^{-j} + b^{-j\delta}]$. Finally we define

$$\mathcal{I}_j^\delta(x) = \bigcup_{L \in \mathcal{I}_j(b,x)} \mathcal{L}^\delta. \tag{3.4}$$

(4) (Control of the speed of renewal n_I and of the mass $\|m^I\|$) There exists a dense subset $\mathcal{D}$ of $(1,\infty)$ such that for every $\delta \in \mathcal{D}$, the property $\mathcal{P}(\delta)$ holds, where $\mathcal{P}(\delta)$ is: for m-almost every $x \in (0,1)$, for every j large enough, there exists $I \in \mathcal{I}_j^\delta(x)$ such that

$$n_I \leq \log_c\big(|I|^{-1}\big)\varphi(|I|) \ \text{ and } \ |I|^{\varphi(|I|)} \leq \|m^I\|. \tag{3.5}$$

Definition 3.1. Let μ be a positive Borel measure supported by $[0,1]$. **C2(h)** is said to hold for μ if **H(h, $\tau_\mu^*(h)$)** holds.

The assumptions on φ and ψ in **(1)** are purely technical, but non restrictive in practice. Assumption **(2)** allows to control m-almost everywhere the local behaviors of the analyzed measure μ and of the analyzing measure m. Assumption **(3)** emphasizes a self-similarity property of the analyzing measure m. Eventually, assumption **(4)** is a control (in some c-adic intervals I) of $\|m^I \circ f_I^{-1}\|$ and of the speed of renewal of the control (3.1) for the measure $m^I \circ f_I^{-1}$.

Assumptions **(3)** and **(4)** supply the monofractality property of the measures m used in [19, 32]. For these monofractal measures, there exist $\beta > 0$, $C > 0$ and $r_0 > 0$ such that $\forall\, x \in \mathrm{supp}(m), \forall\, 0 < r \leq r_0, C^{-1} r^\beta \leq \mu\big(B(x,r)\big) \leq C r^\beta$.

Theorem 3.2. *Let μ be a positive Borel measure such that $\mathrm{supp}(\mu) = [0,1]$. For $\delta, M' \geq 1$, $\alpha > 0$ and $\widetilde{\varepsilon} = \{\varepsilon_j\}_{j\geq 1}$ a non-negative sequence let*

$$S_{\delta,\widetilde{\varepsilon},M'}(\alpha) = \bigcap_{n\geq 1} \bigcup_{j\geq n} \bigcup_{k\in\{0,\dots,b^j-1\}:\mathcal{P}_{M'}(I^b_{j,k}) \text{ holds}} [kb^{-j}, kb^{-j} + b^{-j(\delta-\varepsilon_j)}].$$

Let $\alpha, \beta > 0$ and suppose that **H**(α,β) *holds. There exists $M' \geq 1$ such that for every $\delta > 1$, one can find a non-increasing sequence $\widetilde{\varepsilon}$ converging to 0 and a positive Borel measure m_δ on $[0,1]$ such that $m_\delta(S_{\delta,\widetilde{\varepsilon},M'}(\alpha)) > 0$, and for every $x \in S_{\delta,\widetilde{\varepsilon},M'}(\alpha)$, one has*

$$\limsup_{r\to 0^+} \frac{m_\delta\big(B(x,r)\big)}{r^{\frac{\beta}{\delta} - 5\varphi(r)}} < \infty. \tag{3.6}$$

Moreover, if $\delta \in \mathcal{D}$ then $\widetilde{\varepsilon}$ can be taken equal to $\{0\}_{n\geq 1}$.

Corollary 3.3. *If* **H**(α,β) *holds, there exists $M' \geq 1$ such that for every $\delta > 1$, one can find a sequence $\widetilde{\varepsilon}$ such that $\mathcal{H}^f(S_{\delta,\widetilde{\varepsilon},M'}(\alpha)) > 0$, where $\mathcal{H}^f$ is the generalized Hausdorff dimension $\mathcal{H}^f$ associated with the dimension (or* gauge*) function $f : r \mapsto r^{\frac{\beta}{\delta} - 5\varphi(r)}$.*

The mass distribution principle [20] *implies that $\dim S_{\delta,\widetilde{\varepsilon},M'}(\alpha) \geq \frac{\beta}{\delta}$, and for every Borel set E such that $\dim E < \frac{\beta}{\delta}$, $m_\delta(E) = 0$.*

Theorem 1.2 is thus a consequence of the above corollary (the condition $\mathcal{Q}_\psi$ is equivalent to the condition $\mathcal{P}_M$ up to a small correction of the function ψ).

The following property is used repeatedly in the sequel. Due to the assumption on φ and ψ, there exists a constant $C > 0$ such that

$$\text{for every } 0 < r \leq s \leq 1,\ s^{-\varphi(s)} \leq C r^{-\varphi(r)} \text{ and } s^{-\psi(s)} \leq C r^{-\psi(r)}. \tag{3.7}$$

Moreover, all along the proof, each time it occurs, C denotes a positive constant which depends only on α, β, δ, φ and ψ.

Before starting the proof, let us establish the following lemma.

Lemma 3.4. *Let $N \in \mathbb{N}$, and $x \in (0,1)$ such that $\mathcal{P}_M(I^c_{j,k})$ holds for $k \in \{k^c_{j,x} - 2b^2c, \dots, k^c_{j,x} + 2b^2c\}$ for every $j \geq N$. Then there exists a constant M' (depending only on b, c and μ) such that for every $j \geq N$ and every b-adic interval I of maximal length contained in $[k^c_{j,x}c^{-j}, (k^c_{j,x} + \frac{1}{2})c^{-j}]$, $\mathcal{P}_{M'}(I)$ holds.*

Proof. Let us fix $j \geq N$ and I a b-adic interval of maximal length contained in $[k^c_{j,x}c^{-j}, (k^c_{j,x} + \frac{1}{2})c^{-j}]$. One has $|I| \geq \frac{c^{-j}}{2b^2}$. Consequently, since $I \subset I^c_j(x)$ and since both $\mathcal{P}_M(I^c_j(x))$ and (3.7) hold, there exists $M' \geq 1$ depending only on b, c and μ such that $\mu(I) \leq M'|I|^{\alpha - \psi(|I|)}$.

Conversely, I contains at least one c-adic interval J of generation $j' = j + [\log_c(2b^2)] + 1$ which is distant from $I^c_{j'}(x)$ by at most $2b^2c \cdot c^{-j'}$. By our assumption this implies that $\mathcal{P}_M(J)$ holds so $\mu(I) \geq M|J|^{\alpha+\psi(|J|)} \geq M|J|^{\alpha+\psi(|I|)}$ if $|I|$ is small enough (ψ is non-increasing near 0). Since $\frac{|I|}{|J|}$ is bounded, there exists $M' \geq 1$ which depends only on b, c and μ such that $\mu(I) \geq M'^{-1}|I|^{\alpha+\psi(|I|)}$. □

Proof. Let $\delta > 1$ and let $\{\delta_n\}_{n\geq 0} \in \mathcal{D}^{\mathbb{N}}$ be a non-decreasing sequence converging to δ. To each δ_n can be applied $\mathcal{P}(\delta_n)$. Let $M' \geq 1$ be the constant computed in last Lemma 3.4. We shall construct step by step a generalized Cantor set K_δ in $S_{\delta,\widetilde{\varepsilon},M'}(\alpha)$, and simultaneously the measure m_δ on K_δ and the sequence $\widetilde{\varepsilon}$.

In the sequel, the closure of an interval $I^c_{j,k}$ is also denoted $I^c_{j,k}$.

- First step: The first generation of intervals involved in the construction of K_δ is taken as follows. Let us focus on δ_1.

Let $L_0 = [0,1]$. By assumptions **(2)** and $\mathcal{P}(\delta_1)$, there exist a subset $\widetilde{E}^{L_0}$ of $E^{L_0}_{n_{L_0}}$ of m-measure larger than $\frac{1}{4}\|m\|$ and an integer $n'_{L_0} \geq n_{L_0}$ such that for every $x \in \widetilde{E}^{L_0}$, for every $j \geq n'_{L_0}$, there exists $I \in \mathcal{I}^{\delta_1}_j(x)$ such that (3.5) holds and simultaneously

$$\forall\, j \geq n'_{L_0},\ \mathcal{P}_M(I^c_{j,k}) \text{ holds for } k \in \{k^c_{j,x} - 2b^2c, \ldots, k^c_{j,x} + 2b^2c\}. \tag{3.8}$$

The set $\widetilde{E}^{L_0}$ possesses a Cantor-like structure:

$$\widetilde{E}^{L_0} = \bigcap_{j\geq n'_{L_0}} \ \bigcup_{k:\exists x\in\widetilde{E}^{L_0},\, I^c_{j,k}=I^c_j(x)} I^c_{j,k}. \tag{3.9}$$

For $j \geq n'_{L_0}$, let us define $\widetilde{G}_1(j) = \left\{I^c_{j,k} :\ \exists\, x \in \widetilde{E}^{L_0},\ I^c_{j,k} = I^c_j(x)\right\}$.

Let $I^c_{j,k}$ be a c-adic interval in $\widetilde{G}_1(j)$, $x \in \widetilde{E}_{L_0} \cap I^c_{j,k}$ and a c-adic interval $I \in \mathcal{I}^{\delta_1}_j(x)$ such that (3.5) holds. Let $I^b_{J,K} \in \mathcal{I}_j(b,x)$ such that $I \subset [Kb^{-J}, Kb^{-J\delta_1}] \subset I^b_{J,K}$. By Lemma 3.4, $\mathcal{P}_{M'}(I^b_{J,K})$ holds.

One remarks that by construction one ensured the existence of a constant C (depending on b and c only) such that $\forall I \in \mathcal{I}^{\delta_1}_j(x)$, $C^{-1}|I| \leq |I^c_{j,k}|^{\delta_1} \leq C|I|$.

With every c-adic interval $I^c_{j,k}$ in $\widetilde{G}_1(j)$ is associated another (closed) c-adic interval $I = I^c_{j',k'}$. Eventually (this is the key property to belong to $S_{\delta,\widetilde{\varepsilon},M'}(\alpha)$) $I^c_{j',k'} \subset [Kb^{-J}, Kb^{-J} + b^{-J\delta_1}]$ for some Kb^{-J} such that $\mathcal{P}_{M'}(I^b_{J,K})$ holds.

We denote $I^c_{j',k'} = \underline{I^c_{j,k}}$. Conversely, if a c-adic interval I can be written $\underline{J}$ for some larger c-adic interval J, one writes $J = \overline{I}$. These small intervals $\underline{I^c_{j,k}}$, for some choice of j, will be the first generation of c-adic intervals used to construct K_δ. Let us define $G_1(j) = \left\{\underline{I^c_{j,k}} :\ \ I^c_{j,k} \in \widetilde{G}_1(j)\right\}$. Notice that if I and I' are two distinct elements of $G_1(j)$, the distance between I and I' is at least $|\overline{I}|/2$.

On the algebra generated by the elements of $G_1(j)$, a probability measure m_δ is defined by

$$m_\delta(I) = \frac{m(\overline{I})}{\sum_{I^c_{j,k}\in\widetilde{G}_1(j)} m(I^c_{j,k})}.$$

By the assumption made on the measure m and (3.7), one has

$$m(\overline{I}) \leq |\overline{I}|^{\beta-\varphi(|\overline{I}|)} \leq C|I|^{\frac{\beta}{\delta_1}}|\overline{I}|^{-\varphi(|\overline{I}|)} \leq C|I|^{\frac{\beta}{\delta_1}}|I|^{-\varphi(|I|)}.$$

Moreover, using the Cantor-like structure (3.9), $\sum_{I^c_{j,k}\in\widetilde{G}_1(j)} m(I^c_{j,k}) \geq m(\widetilde{E}^{L_0}) \geq \|m\|/4$. As a consequence,

$$\forall\, I \in G_1(j),\ m_\delta(I) \leq 4\|m\|^{-1} C |I|^{-\varphi(|I|)} |I|^{\frac{\beta}{\delta_1}}.$$

By **(1)**, j_1 can be chosen large enough so that $\forall\, I \in G_1(j_1),\ 4\|m\|^{-1}C \leq |I|^{-\varphi(|I|)}$. We choose the c-adic elements of the first generation of the construction of K_δ as being those of $G_1 := G_1(j_1)$. By construction,

$$\forall\, I \in G_1,\ m_\delta(I) \leq |I|^{\frac{\beta}{\delta_1}-2\varphi(|I|)}. \tag{3.10}$$

- Second step: We construct the second generation of intervals. Consider δ_2. For every $L \in G_1$, using assumptions **(3)** and **(4)**, one can find a subset $\widetilde{E}^L$ of $E^L_{n^L}$ such that $m^L\big(\widetilde{E}^L\big) \geq \frac{1}{4}\|m^L\|$ and an integer $n'_L \geq n_L$ such that $\forall\, x \in \widetilde{E}^L$, for every $j \geq n'_L + \log_c\big(|L|^{-1}\big)$, there exists $I \in \mathcal{I}^{\delta_2}_j(I^c_j(x))$ such that (3.5) holds and (as in (3.8))

$$\forall\, j \geq n'_L + \log_c\big(|L|^{-1}\big),\ \mathcal{P}_M(I^c_{j,k}) \text{ holds for } |k - k^c_{j,x}| \leq 2b^2c. \tag{3.11}$$

One has $\widetilde{E}^L = \bigcap_{j\geq n'_L+\log_c\left(|L|^{-1}\right)} \bigcup_{k:\exists x\in\widetilde{E}^L, I^c_{j,k}=I^c_j(x)} I^c_{j,k}$, and one can define for every $j \geq n'_L + \log_c\big(|L|^{-1}\big)$ the set $\widetilde{G}^L_2(j) = \big\{I^c_{j,k} :\ \exists\, x \in \widetilde{E}^L,\ I^c_{j,k} = I^c_j(x)\big\}$.

Then, another set $G^L_2(j)$ of closed c-adic intervals is obtained from $\widetilde{G}^L_2(j)$ by the same procedure as $G_1(j)$ is constructed from $\widetilde{G}_1(j)$ in the first step. Thus, with every c-adic interval $I^c_{j,k}$ in $\widetilde{G}^L_2(j)$ is now associated a b-adic interval $[Kb^{-J}, (K+1)b^{-J}]$ and another closed c-adic interval $I^c_{j',k'}$ with the following properties:
- their lengths satisfy $C^{-1}|I^c_{j',k'}| \leq |I^c_{j,k}|^{\delta_2} \leq C|I^c_{j',k'}|$,
- $I^c_{j',k'} \subset [Kb^{-J}, Kb^{-J} + b^{-J\delta_2}]$ and $\mathcal{P}_{M'}(I^b_{J,K})$ holds.

Here again, one writes $I^c_{j',k'} = \underline{I^c_{j,k}}$ and $I^c_{j,k} = \overline{I^c_{j',k'}}$.

Let us define $G^L_2(j) = \big\{\underline{I^c_{j,k}} :\ I^c_{j,k} \in \widetilde{G}^L_2(j)\big\}$. On the algebra generated by the elements I of $G^L_2(j)$, an extension of the restriction to the interval I of the measure m_δ is defined by

$$m_\delta(I) = \frac{m^L(\overline{I})}{\sum_{I^c_{j,k}\in\widetilde{G}^L_2(j)} m^L(I^c_{j,k})}\, m_\delta(L).$$

By the assumption made on the measure m^I, one shows that

$$m^L(\overline{I}) \leq \Big(\frac{|\overline{I}|}{|L|}\Big)^{\beta-\varphi\left(\frac{|\overline{I}|}{|L|}\right)} \leq C|I|^{\frac{\beta}{\delta_2}}|L|^{-\beta}\Big(\frac{|\overline{I}|}{|L|}\Big)^{-\varphi\left(\frac{|\overline{I}|}{|L|}\right)} \leq C|I|^{\frac{\beta}{\delta_2}}|L|^{-\beta}|I|^{-\varphi(|I|)},$$

where (3.7) has been used. Moreover $\sum_{I^c_{j,k}\in\widetilde{G}^L_2(j)} m^L(I^c_{j,k}) \geq m^L(\widetilde{E}^L) \geq \|m^L\|/4$. Consequently, using (3.10) to upper bound $m_\delta(L)$, one obtains

$$m_\delta(I) \leq m_\delta(L)\frac{4}{\|m^L\|}C|I|^{\frac{\beta}{\delta_2}}|L|^{-\beta}|I|^{-\varphi(|I|)} \leq \frac{4C|L|^{\frac{\beta}{\delta_1}-\beta-2\varphi(|L|)}}{\|m^L\|}|I|^{\frac{\beta}{\delta_2}-\varphi(|I|)}.$$

One can choose $j_2(L)$ large enough so that for every integer $j \geq j_2(L)$, for every c-adic interval I in $G^L_2(j)$, $4\|m^L\|^{-1}C|L|^{\frac{\beta}{\delta_1}-\beta-2\varphi(|L|)} \leq |I|^{-\varphi(|I|)}$.

Then, taking $j_2 = \max\big\{j_2(L) :\ L \in G_1\big\}$, and defining $G_2 = \bigcup_{L\in G_1} G^L_2(j_2)$, this yields an extension of m_δ to the algebra generated by the elements of $G_1 \bigcup G_2$ and such that for every $I \in G_1 \bigcup G_2$, since $\delta_2 \geq \delta_1$,

$$m_\delta(I) \leq |I|^{\frac{\beta}{\delta_2} - 2\varphi(|I|)}. \tag{3.12}$$

- Third step: We end the induction. Assume that the first n^{th} generations of intervals $G_1, \dots, G_n$ are found for some integer $n \geq 2$. Assume also that a probability measure m_δ on the algebra generated by $\bigcup_{1 \leq p \leq n} G_p$ is defined and that the following properties hold (the fact that this holds for $n = 2$ comes from the two previous steps):

(i) the elements of G_p are closed c-adic intervals and pairwise disjoint. With each $I \in G_p$ is associated an interval $\overline{I}$ such that the $\overline{I}$'s, $I \in G_p$, are pairwise distinct c-adic intervals of the same generation, with $C^{-1}|\overline{I}|^{\delta_p} \leq |I| \leq C|\overline{I}|^{\delta_p}$ for some universal constant C. If I and I' are two distinct elements of G_p, the distance between I and I' is at least $|\overline{I}|/2$.

(ii) For every $2 \leq p \leq n$, each element I of G_p is a subinterval of an element L of G_{p-1}. Moreover, $\overline{I} \subset L$, $\log_c \left(|\overline{I}|^{-1}\right) \geq n_L + \log_c \left(|L|^{-1}\right)$ and $\overline{I} \cap E^L_{n_L} \neq \emptyset$.

(iii) For every $1 \leq p \leq n$ and $I \in G_p$, there is a b-adic interval $I^b_{j,k} = [kb^{-j}, (k+1)b^{-j})$ such that $I \subset [kb^{-j}, kb^{-j} + b^{-j\delta_p}] \subset \overline{I}$ and $\mathcal{P}_{M'}(I^b_{j,k})$ holds.

(iv) For every $I \in \bigcup_{1 \leq p \leq n} G_p$, $m_\delta(I) \leq |I|^{\frac{\beta}{\delta_p} - 2\varphi(|I|)} \leq |I|^{\frac{\beta}{\delta} - 2\varphi(|I|)}$.

(v) For every $1 \leq p \leq n-1$, $L \in G_p$, and $I \in G_{p+1}$ such that $I \subset L$,

$$m_\delta(I) \leq 4\|m^L\|^{-1} m_\delta(L) m^L(\overline{I}).$$

The construction of a generation G_{n+1} of c-adic intervals and an extension of m_δ to the algebra generated by the elements of $\bigcup_{1 \leq p \leq n+1} G_p$ such that properties **(i)** to **(v)** hold for $n+1$ instead of n is done in the same way as when $n = 1$.

For every $n \geq 1$, let $J_n = \sup\{J : \exists I \in G_n,\ \exists K,\ I \subset [Kb^{-J}, Kb^{-J} + b^{-J\delta_n}] \subset \overline{I}$ and $\mathcal{P}_{M'}(I^b_{J,K})$ holds$\}$ and $J_0 = 1$. Then for every $n \geq 1$, for every $j \in [J_{n-1}+1, J_n]$, one sets $\varepsilon_j = \delta - \delta_n$.

By induction, and due to the separation property **(i)**, we obtain a sequence $(G_n)_{n \geq 1}$ and a probability measure m_δ on $\sigma\big(I : I \in \bigcup_{n \geq 1} G_n\big)$ such that properties **(i)** to **(v)** hold for every $n \geq 2$. Let us define $K_\delta = \bigcap_{n \geq 1} \bigcup_{I \in G_n} I$. By construction, $m_\delta(K_\delta) = 1$ and because of property **(iii)** $K_\delta \subset S_{\delta, \widetilde{\varepsilon}, M'}(\alpha)$. Eventually, the measure m_δ is extended to $\mathcal{B}([0,1])$ in the usual way: $m_\delta(B) := m_\delta(B \cap K_\delta)$ for every $B \in \mathcal{B}([0,1])$.

- Last step: Proof of (3.6). If $I \in G_n$, we set $g(I) = n$ (the generation of the interval I). Let us fix I an open subinterval of $[0,1]$ of length smaller than the lengths of the elements of G_1, and assume that $I \cap K_\delta \neq \emptyset$. Let L be the element of largest diameter in $\bigcup_{n \geq 1} G_n$ such that I intersects at least two elements of $G_{g(L)+1}$ included in L. This implies that I does not intersect any other element of $G_{g(L)}$, and as a consequence $m_\delta(I) \leq m_\delta(L)$. We distinguish three cases:

• If $|I| \geq |L|$, one has

$$m_\delta(I) \leq m_\delta(L) \leq |L|^{\frac{\beta}{\delta} - 2\varphi(|L|)} \leq C|I|^{\frac{\beta}{\delta} - 2\varphi(|I|)}. \tag{3.13}$$

• If $|I| \leq c^{-n_L - 1}|L|$, let $L_1, \dots, L_d$ be the elements of $G_{g(L)+1}$ which intersect I. They are all sons of L. Property **(v)** above yields

$$m_\delta(I) = \sum_{i=1}^{d} m_\delta(I \cap L_i) \leq m_\delta(L) \frac{4}{\|m^L\|} \sum_{i=1}^{d} m^L(\overline{L}_i).$$

Let n be the unique integer such that $c^{-n} \leq |I| < c^{-n+1}$. Recall

$$E^L_{n_L} = \bigcap_{j \geq n_L + \log_c(|L|^{-1})} \ \bigcup_{k: I^c_{j,k} \cap E^L_{n_L} \neq \emptyset} I^c_{j,k}. \tag{3.14}$$

Due to property **(i)**, $d \geq 2$ implies $|I| \geq |\overline{L}_i|/2$. Hence the scale of the intervals $\overline{L}_i$ (which equals $-\log_c |\overline{L}_i|$) is larger than $n-1$. Combining this with **(ii)** and (3.14), one can write that $\bigcup_{i=1}^d \overline{L}_i \subset \bigcup_{k: I \cap I^c_{n-1,k} \cap E^L_{n_L} \neq \emptyset} I^c_{n-1,k}$. There are at most 2 terms in the previous union. Since $|I| \leq c^{-n_L-1}|L|$, one has $n-1 \geq n_L + \log_c\left(|L|^{-1}\right)$. Thus for each of the intervals included in $\bigcup_{k: I \cap I^c_{n-1,k} \cap E^L_{n_L} \neq \emptyset} I^c_{n-1,k}$, $m^L(I^c_{n-1,k}) \leq \left(\frac{|I^c_{n-1,k}|}{|L|}\right)^{\beta - \varphi\left(\frac{|I^c_{n-1,k}|}{|L|}\right)} \leq C\left(\frac{|I|}{|L|}\right)^{\beta}\left(\frac{|I|}{|L|}\right)^{-\varphi\left(\frac{|I|}{|L|}\right)}$, where C depends only on β. This yields

$$\begin{aligned} m_\delta(I) &\leq m_\delta(L)\frac{4}{\|m^L\|}\sum_{i=1}^d m^L(\overline{L}_i) \leq m_\delta(L)\frac{4}{\|m^L\|}2C\left(\frac{|I|}{|L|}\right)^{\beta}\left(\frac{|I|}{|L|}\right)^{-\varphi\left(\frac{|I|}{|L|}\right)} \\ &\leq m_\delta(L)\frac{C}{\|m^L\|}\left(\frac{|I|}{|L|}\right)^{\beta}|I|^{-\varphi(I)} \end{aligned}$$

We then use consecutively two facts. First by **(iv)**, $m_\delta(L) \leq |L|^{\frac{\beta}{\delta}}|L|^{-2\varphi(|L|)} \leq C|L|^{\frac{\beta}{\delta}}|I|^{-2\varphi(|I|)}$. This implies $m_\delta(I) \leq \dfrac{C}{\|m^L\|}|I|^{\frac{\beta}{\delta}}|I|^{-3\varphi(|I|)}\left(\dfrac{|I|}{|L|}\right)^{\beta(1-1/\delta)}$, which is smaller than $c\|m^L\|^{-1}|I|^{\frac{\beta}{\delta}}|I|^{-3\varphi(|I|)}$ since $r \mapsto r^{\beta(1-1/\delta)}$ is bounded near 0. Then **(4)** upper bounds $\|m^L\|^{-1}$ and

$$m_\delta(I) \leq C|L|^{-\varphi(|L|)}|I|^{\frac{\beta}{\delta}}|I|^{-3\varphi(|I|)} \leq C|I|^{\frac{\beta}{\delta}}|I|^{-4\varphi(|I|)}. \tag{3.15}$$

• $c^{-n_L-1}|L| < |I| \leq |L|$: one needs at most c^{n_L+2} contiguous intervals of length $c^{-n_L-1}|L|$ to cover I. For these intervals, the estimate (3.15) can be used. Thus for $|I|$ small enough, and using again assumption **(4)**,

$$\begin{aligned} m_\delta(I) &\leq Cc^{n_L+2}\left(c^{-n_L-1}|L|\right)^{\frac{\beta}{\delta}}\left(c^{-n_L-1}|L|\right)^{-4\varphi(c^{-n_L-1}|L|)} \\ &\leq Cc^{n_L}|I|^{\frac{\beta}{\delta}}|I|^{-4\varphi(|I|)} \leq C|L|^{-\varphi(|L|)}|I|^{\frac{\beta}{\delta}}|I|^{-4\varphi(|I|)} \leq C|I|^{\frac{\beta}{\delta}}|I|^{-5\varphi(|I|)}. \end{aligned}$$

The constant $C > 0$ does not depend on the interval I.

Remembering (3.13) and (3.15), and using assumption **(1)**, one gets that for every non-trivial subinterval L of $[0,1]$, $m_\delta(L) \leq C|L|^{\frac{\beta}{\delta}}|L|^{-5\varphi(|L|)}$. □

3.2. **Examples of measures μ that satisfy C1, C2 and C3.** We are going to describe four classes of statistically self-similar measures. For all these measures, property **C1** follows easily from their study in the papers mentioned below.

3.2.1. *Deterministic Gibbs measures.* Let μ be a Gibbs measure associated with an Hölder potential ϕ in the dynamical system $([0,1),T)$, where $T(x) = cx$ mod 1 with c an integer ≥ 2 (see [46]). The multifractal analysis of μ is performed for instance in [14, 47, 24]. In this case the function τ_μ is analytic, and the fact that **C3(**h**)** holds for all h of the form $\tau'_\mu(q)$, $q \in \mathbb{R}$, is an easy consequence of the works mentioned above.

The fact that **C2(**$\tau'_\mu(q)$**)** holds for all $q \in \mathbb{R}$ is also simple in this case. Let $q \in \mathbb{R}$. To see that **H(**$\tau'_\mu(q)$**,** $\tau^*_\mu(\tau'_\mu(q))$**)** holds, choose the analyzing measure m to be the Gibbs measure associated with the potential $q\phi$ (instead of ϕ for μ). The law of the iterated logarithm applied to the Birkhoff sums associated with ϕ with respect to m (see Chapter 7 of [48]) show that property **(2)** holds with $\varphi(t)=\psi(t)=C\big(\frac{\log\log|\log t|}{|\log(t)|}\big)^{1/2}$ for some $C>0$. Also, if one defines the probability measure μ^I by $m^I\circ f_I^{-1}=m$, it is obvious that **(3)** and **(4)**, and the speed of renewal n_I does not depend on I.

3.2.2. *Random Gibbs measures.* We consider the following particular class. We fix a potential φ as above, and a sequence $\omega=(\omega_n)_{n\geq 0}$ of independent random phases uniformly distributed in $[0,1]$. If $j\geq 1$ one denotes by $\omega^{(j)}$ the sequence $(\omega_n)_{n\geq j}$. For $n\geq 1$ and $x\in[0,1]$, let $S_n(\varphi,\omega)(x)=\sum_{k=0}^{n-1}\varphi(T^kx+\omega_k)$. It follows from the thermodynamic formalism for random transformations (see [35]) that, with probability one, the sequence of measures

$$\mu_j^{\varphi,\omega}(dx)=\frac{\exp\big(S_j(\varphi,\omega)(x)\big)}{\int_{[0,1]}\exp\big(S_j(\varphi,\omega)(u)\big)\,du}\,dx$$

converges weakly to a Gibbs measure μ. The fact that **C3(**h**)** holds for every h of the form $\tau_\mu'(q)$, almost surely, is a consequence of [36]. The stronger property "**C3(**h**)** holds almost surely for all h of the form $\tau_\mu'(q)$" is established in [9]. The fact that, with probability one, **H(**$\tau_\mu'(q)$**,** $\tau_\mu^*(\tau_\mu'(q))$**)** holds for all $q\in\mathbb{R}$ is established in [11]. Given ω in the probability space such that $\mu(\omega)$ is defined, for $q\in\mathbb{R}$ one takes m as a weak limit of a subsequence of the sequence $(\mu_j^{q\varphi,\omega})$. In the same way, for $j\geq 1$, one defines $m^{(j)}$ as a weak limit of a subsequence of $(\mu^{q\varphi,\omega^{(j)}})$. Then, if I is a c-adic interval of generation j, the measure m^I is defined so that $m^I\circ f_I^{-1}=m^{(j)}$. One gets **(2)**, **(3)** and **(4)** with $\psi(t)=\varphi(t)=|\log_b(t)|^{-\frac18}\big(\log|\log_b(t)|\big)^{\frac12+\eta}$ for some $\eta>0$. Moreover, since all the measure m^I only depends on the generation of I, [11] shows that the control (3.5) holds for all I of sufficiently large generation.

3.2.3. *Canonical cascades measures.* These measures are studied in particular in [40, 34, 29, 43, 5, 6, 11]. Let W be a positive random variable with expectation equal to 1, and let $(W_J)_{J\in\mathcal{I}}$ be a sequence of independent copies of W indexed by the set $\mathcal{I}$ of c-adic subintervals of $[0,1)$. The canonical cascade measure μ is the almost sure weak limit of the sequence of measures μ_j defined on $[0,1]$ by

$$\mu_j(dx)=\prod_{c^{-j}\leq|J|\leq c^{-1},\,x\in J}W_J\,dx.$$

Let $\widetilde{\tau}:\ q\in\mathbb{R}\mapsto q-1-\log_c\mathbb{E}(W^q)$. The condition $\widetilde{\tau}'(1^-)>0$ is necessary and sufficient to ensure that, with probability one, μ is non-degenerate, that is non equal to zero [34]. We assume $\widetilde{\tau}'(1^-)>0$ and then define J, the interior of the interval $\{q\in\mathbb{R}:\widetilde{\tau}'(q)q-\widetilde{\tau}(q)>0\}$. We assume that J is a neighborhood of $[0,1]$. It is proved in the works mentioned above that, with probability one, $\widetilde{\tau}$ and τ_μ coincide on the closure of J, and also that **C3(**h**)** holds for all h of the form $\tau_\mu'(q)$, $q\in J$.

The following fact is established in [11]: For every $q\in J$, with probability one, **H(**$\tau_\mu'(q)$**,** $\tau_\mu^*(\tau_\mu'(q))$**)** holds. Also, with probability one, **H(**$\tau_\mu'(q)$**,** $\tau_\mu^*(\tau_\mu'(q))$**)** holds for almost-every $q\in J$ (with respect to the Lebesgue measure).

For $q\in J$, the analyzing measure m is obtained as μ but with the weights $W_{q,J}=W_J^q/\mathbb{E}(W^q)$ instead of the W_J's, and the measure $m_I\circ f_I^{-1}$ is the measure obtained as m, but with the weights $W_{q,J}^I:=W_{q,f_I^{-1}(J)}$. Moreover, $\psi(t)=|\log(t)|^{-\frac12}\big(\log|\log(t)|\big)^{\frac12+\eta}$ and $\varphi(t)=\big(\log|\log(t)|\big)^{-\kappa}$ for some $\eta,\kappa>0$.

Contrary to the case of random Gibbs measures, the measures m^I are pairwise distinct. This reflects a higher degree of randomness in the construction (j i.i.d random phases are needed to construct $\mu_j^{\varphi,\omega}$ while b^j independent copies of W enter in the definition of μ_j) and makes impossible to get uniformly over the c-adic intervals of sufficiently large generation the control (3.5) with a suitable function φ.

3.2.4. *Coumpound Poisson cascades.* Theses meaures were recently introduced in [7]. Their construction is as follows (we do not too much enter the details here). Let $\rho > 0$ and let Λ be the measure on the strip $\mathbb{R} \times (0, 1]$ given by its density $\Lambda(dtd\lambda) = \rho\lambda^{-2}dtd\lambda$. Let S be a Poisson point process with intensity Λ, and with each $M = (t_M, \lambda_M) \in S$, associate a positive integrable random variable W_M in such a way that the W_M's are i.i.d, and also indendent of S. Then for $(t, \varepsilon) \in [0, 1]\times(0, 1]$ define $\mathcal{C}_\varepsilon(t) = \{(s, \lambda) \in \mathbb{R}\times[0, 1] : \varepsilon \le \lambda < 1, t-\lambda/2 < s \le t+\lambda/2\}$. The coumpound Poisson cascade measure μ on $[0, 1]$ is the almost sure weak limit, as $\varepsilon \to 0$, of the measure-valued martingale

$$\mu_\varepsilon(dt) = \varepsilon^{\rho(\mathbb{E}(W)-1)} \prod_{M\in S\cap\mathcal{C}_\varepsilon(t)} W_M.$$

Let $\widetilde{\tau}(q) = -1 + q\big(1 + \rho(\mathbb{E}(W) - 1)\big) - \rho(\mathbb{E}(W^q) - 1)$. It is shown in [11] that under the same assumptions on $\widetilde{\tau}$ as for canonical cascades measures, one has formally the same consclusions on the validity of **C2(**h**)** and **C3(**h**)**.

Extensions of this class are proposed in [4], as well as in [8], which gives for instance the following alternative to μ: φ being chosen as for Gibbs measures,

$$\mu_\varepsilon(dt) = \varepsilon^{\rho\left(\int_{[0,1]} \exp(\varphi(t)\, dt - 1\right)} \prod_{M\in S\cap\mathcal{C}_\varepsilon(t)} \exp\left(\varphi\big(\lambda_M^{-1}(t - t_M - \lambda_M/2)\big)\right).$$

There is no doubt that the same properties also hold for these measures.

4. Proof of Theorem 1.1

The points concerning the multifractal formalism are postponed to Section 5.
Before stating some results, remark that for every b-adic interval $I_{j,k}$, one has

$$\nu(I_{j,k}) \ge \nu(\{kb^{-j}\}) \ge j^{-2}\mu(I_{j,k}). \tag{4.1}$$

For some constant C independent of j, k, and μ, one has when kb^{-j} is irreducible,

$$\nu(I_{j,k}) = \sum_{j'\ge j} \frac{1}{j'^2} \sum_{k'b^{-j'}\in I_{j,k}} \mu(I_{j',k'}) = \sum_{j'\ge j} \frac{1}{j'^2}\mu(I_{j,k}) \le C\frac{1}{j}\mu(I_{j,k}), \tag{4.2}$$

4.1. **First properties of** ν**.** Remember the definition (1.4) of δ_x. Obviously, if x is a b-adic number kb^{-j} and if Kb^{-J} is its irreducible representation, either $h_\nu(x) = 0$ if $\mu(I_{J,K}) > 0$, or $h_\nu(x) = +\infty$ if $\mu(I_{J,K}) = 0$.

Lemma 4.1. *Assume* **C1** *for* μ*. If* $x \in \operatorname{supp}(\mu)$ *and* $\delta_x = +\infty$*,* $h_\nu(x) = 0$*.*

Proof. By Definition 2.8 of **C1**, there exists a constant B such that **C1** holds.

Let $M > B$. Since $\delta_x = +\infty$, there exists an infinite number of b-adic numbers kb^{-j} with $j \ge J$ such that $|kb^{-j} - x| \le b^{-jM}$. Let $k_0b^{-j_0}$ be such a b-adic number. Let $J_0 = [Mj_0] - 2$ and let K_0 be such that $K_0b^{-J_0} = k_0b^{-j_0}$. Since $|k_0b^{-j_0} - x| \le b^{-j_0M}$, one has $|K_0b^{-J_0} - x| \le b^{-(J_0+1)}$, and thus $K_0b^{-J_0} \in B(x, b^{-J_0})$. Using (4.1), for some constant C depending on B and M one has

$$\nu(B(x, b^{-J_0})) \ge \nu(\{k_0b^{-j_0}\}) \ge j_0^{-2}\mu(I_{j_0,k_0}) \ge j_0^{-2}b^{-Bj_0} \ge CJ_0^{-2}b^{-\frac{B}{M}J_0}.$$

There exists an infinite number of integers J_0 such that last inequality holds, thus $h_\nu(x) \le B/M$. This remains true for any $M > B$, thus $h_\nu(x) = 0$. □

Proposition 4.2. *Let $x \in E^\mu_\alpha$ for some $\alpha \geq 0$, and assume that its approximation rate by the b-adic numbers δ_x is finite. Then $\frac{\alpha}{\delta_x} \leq h_\nu(x) \leq \alpha$.*

Proof. Let $\varepsilon > 0$. Let us first upper bound $h_\nu(x)$. By definition of α, there exists an infinite number of integers j_0 such that $\max\big(\mu(I^-_{j_0}(x)), \mu(I_{j_0}(x)), \mu(I^+_{j_0}(x))\big) \geq b^{-j_0(\alpha+\varepsilon)}$. Let j_0 be such an integer, and let us then find a lower bound of $\nu(B(x, b^{-(j_0-1)}))$. It is obvious that $I^-_{j_0}(x) \cup I_{j_0}(x) \cup I^+_{j_0}(x) \subset B(x, b^{-(j_0-1)})$. Thus using (4.1), one gets

$$\begin{aligned}\nu(B(x, b^{-(j_0-1)})) &\geq \max(\nu(I^-_{j_0}(x)), \nu(I_{j_0}(x)), \nu(I^+_{j_0}(x)))\\ &\geq j_0^{-2}\max(\mu(I^-_{j_0}(x)), \mu(I_{j_0}(x)), \mu(I^+_{j_0}(x))) \geq j_0^{-2} b^{-j_0(\alpha+\varepsilon)}.\end{aligned}$$

This implies $h_\nu(x) = \liminf_{j\to+\infty} \frac{\log \nu(B(x,b^{-(j-1)}))}{\log |B(x,b^{-(j-1)})|} \leq \alpha + \varepsilon$. This remains true for every $\varepsilon > 0$, hence the result.

Let us move to the lower bound. By definition of δ_x, there exists J' such that $j \geq J'$ implies $\forall k$, $|kb^{-j} - x| \geq b^{-j(\delta_x+\varepsilon)}$. Moreover, $x \in E^\mu_\alpha$, thus there exists a scale J'' such that $j \geq J''$ implies $\max(\mu(I^-_j(x)), \mu(I_j(x)), \mu(I^+_j(x))) \leq b^{-j(\alpha-\varepsilon)}$.

One sets $J = \max([2(\delta_x+1)J'], [2(\delta_x+1)J''])$.

Let $j_0 \geq J$, and consider $B(x, b^{-j_0})$. For every $j \geq j_0 + 1$, one has

$$\sum_{kb^{-j} \in B(x,b^{-j_0})} \mu(I_{j,k}) \leq \mu(I^-_{j_0}(x)) + \mu(I_{j_0}(x)) + \mu(I^+_{j_0}(x)) \leq 3\, b^{-j_0(\alpha-\varepsilon)}.$$

since $B(x, b^{-j_0}) \subset I^-_{j_0}(x) \cup I_{j_0}(x) \cup I^+_{j_0}(x)$. This yields

$$\begin{aligned}\nu(B(x, b^{-j_0})) &= \nu\Big(\Big\{\frac{k_{j_0,x}}{b^{j_0}}\Big\}\Big) + \nu\Big(\Big\{\frac{k_{j_0,x}+1}{b^{j_0}}\Big\}\Big) + \sum_{j\geq j_0+1} \frac{1}{j^2} \sum_{kb^{-j}\in B(x,b^{-j_0})} \mu(I_{j,k})\\ &\leq \nu\Big(\Big\{\frac{k_{j_0,x}}{b^{j_0}}\Big\}\Big) + \nu\Big(\Big\{\frac{k_{j_0,x}+1}{b^{j_0}}\Big\}\Big) + \sum_{j\geq j_0+1} \frac{1}{j^2} 3\, b^{-j_0(\alpha-\varepsilon)}.\end{aligned}$$

Thus for any $x \in E^\mu_\alpha$ and for j_0 large enough, one has

$$\nu(B(x, b^{-j_0})) \leq \nu(\{k_{j_0,x} b^{-j_0}\}) + \nu(\{(k_{j_0,x}+1) b^{-j_0}\}) + C j_0^{-1} b^{-j_0(\alpha-\varepsilon)}. \tag{4.3}$$

This inequality will later be of a great importance. We distinguish three cases:

- if $k_{j_0,x}$ is a multiple of b: $k_{j_0,x} b^{-j_0}$ can be written as an irreducible fraction $K_0 b^{-J_0}$ with $J_0 < j_0$. Since $|K_0 b^{-J_0} - x| \leq b^{-j_0} \leq b^{-(J_0+1)}$, $K_0 b^{-J_0}$ is the b-adic number that is the closest to x at scale J_0.

But J has been chosen large enough so that the reduced scale J_0 is greater than J'. Hence one gets that $|K_0 b^{-J_0} - x| \geq b^{-J_0(\delta_x+\varepsilon)}$.

Thus $b^{-J_0(\delta_x+\varepsilon)} \leq |K_0 b^{-J_0} - x| \leq b^{-j_0}$, which implies $j_0 \leq J_0(\delta_x+\varepsilon)$. Moreover, since $J_0 \geq J \geq J''$, one obtains $\mu(I_{J_0,K_0}) \leq b^{-J_0(\alpha-\varepsilon)}$. One can now upper bound $\nu(\{k_{j_0,x} b^{-j_0}\})$. Indeed, for some constant C_{δ_x} that depends on δ_x,

$$\nu(\{k_{j_0,x} b^{-j_0}\}) \leq \sum_{j\geq J_0} j^{-2} \mu(I_{J_0,K_0}) \leq C J_0^{-1} b^{-J_0(\alpha-\varepsilon)} \leq C_{\delta_x} j_0^{-1} b^{-j_0 \frac{\alpha-\varepsilon}{\delta_x+\varepsilon}}.$$

- if $k_{j_0,x} + 1$ is a multiple of b: the same arguments apply also here, and $\nu(\{(k_{j_0,x} + 1) b^{-j_0}\}) \leq C_{\delta_x} j_0^{-1} b^{-j_0 \frac{\alpha-\varepsilon}{\delta_x+\varepsilon}}$.

- if $k_{j_0,x}$ (or $k_{j_0,x}$) is not a multiple of b: then by (4.2) one has $\nu(\{k_{j_0,x} b^{-j_0}\}) \leq C j_0^{-1} b^{-j_0(\alpha-\varepsilon)}$ (or $\nu(\{(k_{j_0,x}+1) b^{-j_0}\}) \leq C j_0^{-1} b^{-j_0(\alpha-\varepsilon)}$.

Eventually, $\nu(B(x,b^{-j_0})) \leq 2C_{\delta_x} j_0^{-1} b^{-j_0 \frac{\alpha-\varepsilon}{\delta_x+\varepsilon}} + C j_0^{-1} b^{-j_0(\alpha-\varepsilon)} \leq C j_0^{-1} b^{-j_0 \frac{\alpha-\varepsilon}{\delta_x+\varepsilon}}$. As a consequence, $h_\nu(x) \geq \frac{\alpha-\varepsilon}{\delta_x+\varepsilon}$, and this is true $\forall \varepsilon > 0$, hence the result. □

4.2. **Decomposition of E_h^ν.** The following sets are needed.

Definition 4.3. Let μ be a positive Borel measure, and $\alpha \geq 0$, $\delta \geq 1$ be two real numbers. Let $\varepsilon > 0$. For every point x, the property $\mathcal{L}(\alpha,\delta,\varepsilon)$ is said to hold at x if there exist $\eta \leq \varepsilon$, and an infinite number of b-adic numbers kb^{-j} that verify

$$b^{-j(\alpha+\eta)} \leq \mu([kb^{-j},(k+1)b^{-j})) \leq b^{-j(\alpha-\eta)} \text{ and } |kb^{-j} - x| \leq 2\, b^{-j\delta}. \tag{4.4}$$

Let now $h \geq 0$. The set F_h is defined by

$$F_h = \left\{ x : \begin{array}{c} \forall \varepsilon > 0,\ \exists\, \alpha \geq 0,\ \delta \geq 1 \text{ such that} \\ \frac{\alpha}{\delta} \leq h+\varepsilon \text{ and } \mathcal{L}(\alpha,\delta,\varepsilon) \text{ holds at } x \end{array} \right\} \tag{4.5}$$

It is obvious to verify that for any $0 \leq h \leq h'$, $F_h \subset F_{h'}$.

Proposition 4.4. *Let $h > 0$. One has $E_h^\nu = F_h \setminus \bigcup_{h'<h} F_{h'}$.*

Before proving Proposition 4.4, we first study the sets F_h.

Lemma 4.5. *If $x \in F_h$ for some $h \geq 0$, then $h_\nu(x) \leq h$.*

Proof. Let $\varepsilon > 0$, and (α,δ) such that $\frac{\alpha}{\delta} \leq h+\varepsilon$ and $\mathcal{L}(\alpha,\delta,\varepsilon)$ holds at x. For some $\eta < \varepsilon$, denote by $k_n b^{-j_n}$ an infinite sequence of b-adic numbers such that

$$b^{-j_n(\alpha+\eta)} \leq \mu([k_n b^{-j_n},(k_n+1)b^{-j_n})) \leq b^{-j_n(\alpha-\eta)} \text{ and } |k_n b^{-j_n} - x| \leq 2\, b^{-j_n\delta}.$$

Since $\nu(B(x,2b^{-j_n\delta})) \geq \frac{1}{j_n^2}\mu([k_n b^{-j_n},(k_n+1)b^{-j_n}))$, one gets $\frac{\log \nu(B(x,2b^{-j_n\delta}))}{\log 2b^{-j_n\delta}} \leq \frac{-\log j_n^2}{\log 2b^{-j_n\delta}} + \frac{j_n(\alpha+\eta)}{j_n\delta - \log 2}$. The right term tends to $\frac{\alpha+\eta}{\delta}$ when $j_n \to +\infty$, hence $\forall \varepsilon > 0$, $h_\nu(x) \leq \frac{\alpha+\eta}{\delta} \leq h+2\varepsilon$. □

The following proposition is important to prove Proposition 4.4 and also to find the upper bound in the next section.

Proposition 4.6. *Let $h > 0$ and $x \in E_h^\nu$. Assume* **C1** *holds for μ. Then $x \in F_h$.*

Proof. Let $\varepsilon > 0$, and $x \in E_h^\nu$. We want to show that there exists a couple (α,δ) such that $\frac{\alpha}{\delta} \leq h+\varepsilon$ and $\mathcal{L}(\alpha,\delta,\varepsilon)$ holds at x. Let $\alpha_x > 0$ the unique exponent such that $x \in E_{\alpha_x}^\mu$ (remember that by Proposition 4.2, $\alpha_x = 0 \Rightarrow h_\nu(x) = 0$).

1. $\delta_x = 1$: by Proposition 4.2, one has $h = \alpha_x$. One can take $\delta = 1$, $\alpha = h+\varepsilon$. Indeed, if $x \in E_\mu^h$, there exists an infinite number of intervals $I \in \{I_j^-(x), I_j(x), I_j^+(x)\}$ such that $b^{-j(h+\varepsilon)} \leq \mu(I) \leq b^{-j(h-\varepsilon)}$. Such intervals I satisfy (4.4).

2. $\delta_x > 1$ and $h = \alpha_x$: the arguments of item **1.** apply with $\delta = 1$ and $\alpha = \alpha_x + \varepsilon$.

3. $\delta_x > 1$ and $h < \alpha_x$: we assume that ε is small enough so that $h+\varepsilon < \alpha_x - \varepsilon$. Remark that if b-adic numbers that satisfy (4.4) exist, then $k = k_{j,x}$ or $k = k_{j,x}+1$.

By definition of δ_x, there exists a scale J such that $j \geq J$ implies $\forall k$, $|kb^{-j} - x| \geq b^{-j(\delta_x+\frac{\varepsilon}{3})}$, and since $x \in E_{\alpha_x}^\mu$, one can similarly impose J large enough so that for every $j \geq J$, $\max(\mu(I_j^-(x)), \mu(I_j(x)), \mu(I_j^+(x))) \leq b^{-j(\alpha_x-\frac{\varepsilon}{3})}$.

Since $x \in E_h^\nu$, there exists an infinite number of integers $j_n \geq J$ such that $\nu(B(x,b^{-j_n})) \geq b^{-j_n(h+\frac{\varepsilon}{3})}$. Consider one of these j_n. Since $h+\frac{\varepsilon}{3} < \alpha_x - \frac{\varepsilon}{3}$, (4.3) yields for j_n large enough and for some constant C depending on x, h and α_x

$$C^{-1} b^{-j_n(h+\frac{\varepsilon}{3})} \leq \nu(\{k_{j_n,x} b^{-j_n}\}) + \nu(\{(k_{j_n,x}+1) b^{-j_n}\}) \tag{4.6}$$

Remark that one of $k_{j_n,x}$ and $k_{j_n,x}+1$ must be a multiple of b. Indeed, otherwise we would have by (4.2) $\nu(\{k_{j_n,x}b^{-j_n}\})+\nu(\{(k_{j_n,x}+1)b^{-j_n}\}) \le \frac{2}{j_n}\mu(I_{j_n}(x)) \le \frac{2}{j_n}b^{-j_n(\alpha_x-\frac{\varepsilon}{3})}$. Thus if ε is small enough so that $\alpha-\varepsilon>h+\varepsilon$, this is impossible.

- If $k_{j_n,x}$ is a multiple of b: then $(k_{j_n,x}+1)b^{-j_n}$ is irreducible, and by (4.2)

$$\nu(\{(k_{j_n,x}+1)b^{-j_n}\}) \le Cj_n^{-1}\mu(I^+_{j_n}(x)) \le Cj_n^{-1}b^{-j_n(\alpha_x-\frac{\varepsilon}{3})}$$

Thus (4.6) rewrites for j_n large enough $\nu(\{k_{j_n,x}b^{-j_n}\}) \ge \frac{1}{C}b^{-j_n(h+\frac{\varepsilon}{3})}$.

Let us write $k_{j_n,x}b^{-j_n}=K_nb^{-J_n}$, where K_n is not a multiple of b. By construction $|K_nb^{-J_n}-x|=2b^{-J_n\delta_n}$ where $\delta_n\ge 1$. Moreover, by (4.2),

$$J_n^{-2}\mu(I_{J_n,K_n}) \ge \nu(\{k_{j_n,x}b^{-j_n}\}) \ge C^{-1}b^{-j_n(h+\frac{\varepsilon}{3})}=C^{-1}b^{-J_n\delta_n(h+\frac{\varepsilon}{3})}$$

Thus for j_n large enough, $\mu(I_{J_n,K_n})=b^{-J_n\alpha_n}$ where $\alpha_n\le\delta_n(h+2\frac{\varepsilon}{3})$.

Eventually, for the b-adic number $K_nb^{-J_n}$ and its corresponding interval I_{J_n,K_n}, (4.4) is satisfied with the couple (α_n,δ_n). Remark that $\delta_n\in[1,\delta_x+\frac{\varepsilon}{3}]$ (because $J_n\ge J$) , and that $\frac{\alpha_n}{\delta_n}\le h+2\frac{\varepsilon}{3}<h+\varepsilon$.

- If $k_{j_n,x}+1$ is a multiple of b: the same arguments as above also apply here.

Since **C1** is satisfied, by Definition 2.8 there exists B such that for every j and k, $\mu(I_{j,k})\ge b^{-Bj}$. One can thus extract an infinite subsequence of b-adic numbers $K_nb^{-J_n}$ that verify (4.4) with (α_n,δ_n) ranging in the square $S=[\alpha_x-\frac{\varepsilon}{3},B]\times[1,\delta_x+\frac{\varepsilon}{3}]$ and satisfying $\frac{\alpha_n}{\delta_n}\le h+2\frac{\varepsilon}{3}$.

One can extract from $(\alpha_n,\delta_n)_n$ a subsequence $(\alpha_{\phi(n)},\delta_{\phi(n)})$ converging to some value (α_0,δ_0), that also satisfies $\frac{\alpha_0}{\delta_0}\le h+2\frac{\varepsilon}{3}$. Now choose η small enough such that $\frac{\alpha_0+\eta}{\max(1,\delta_0-\eta)}\le h+\varepsilon$, define $\delta_0'=\max(1,\delta_0-\eta)$ and consider the square $S_\eta=[\alpha_0-\eta,\alpha_0+\eta]\times[\delta_0',\delta_0+\eta]$. There exists a scale N such that $n\ge N$ implies $(\alpha_{\phi(n)},\delta_{\phi(n)})\in S_\eta$. By construction, for every $n\ge N$, one has $b^{-J_{\phi(n)}(\alpha_0-\eta)}\ge\mu(I_{J_{\phi(n)},K_{\phi(n)}})\ge b^{-J_{\phi(n)}(\alpha_0+\eta)}$ and $|K_{\phi(n)}b^{-J_{\phi(n)}}-x|=b^{-J_{\phi(n)}\delta_{\phi(n)}}\le b^{-J_{\phi(n)}\delta_0'}$. Hence $\mathcal{L}(\alpha_0,\delta_0',\varepsilon)$ holds at x. □

Proof. (of Proposition 4.4) Last Proposition shows that $E_h^\nu\subset F_h$ (one also has $E_h^\nu\subset\bigcap_{h'>h}F_{h'}$). Moreover, applying Lemma 4.5 to $h'<h$ yields $E_h^\nu\cap F_{h'}=\emptyset$. Hence $E_h^\nu\subset F_h\setminus\bigcup_{h'<h}F_{h'}$.

Conversely, let $x\in F_h\setminus\bigcup_{h'<h}F_{h'}$. $x\in F_h$ implies by Lemma 4.5 $h_\nu(x)\le h$. But if $h_\nu(x)<h$, $x\in\bigcup_{h'<h}F_{h'}$ by Proposition 4.6. Hence $h_\nu(x)=h$, and $x\in E_h^\nu$. □

Let us finish this section by saying that in the Definition 4.3 of $\mathcal{L}(\alpha,\delta,\varepsilon)$ one can impose the choice of only irreducible b-adic numbers. Then the characterizations we proved, and the next results, remain the same.

4.3. **Upper bound of the multifractal spectrum.** Let us first mention that item 1. of Proposition 2.5 combined with Proposition 4.2 yields that, as soon as $\tau_\mu^*(h)<0$ and $h\ge\tau_\mu'(0^+)$, $E_h^\nu=\emptyset$. We focus now on the exponents h such that $\tau_\mu^*(h)\ge 0$.

Proposition 4.7. *If $h\ge\tau_\mu'(0^+)$, $\dim(E_h^\nu)\le\tau_\mu^*(h)$.*

Proof. Let $h>\tau_\mu'(0^+)$, and $x\in E_h^\nu$. Let α be the unique exponent such that $x\in E_\alpha^\mu$. By Proposition 4.2, $h=h_\nu(x)\le\alpha$, hence $x\in\bigcup_{\alpha'\ge h}E_{\alpha'}^\mu$. Finally, by Proposition 2.5, $\dim E_h^\nu\le\dim\bigcup_{\alpha'\ge h}E_{\alpha'}^\mu\le\tau_\mu^*(h)$. □

To prove the upper bound when $h\le\tau_\mu'(0^+)$, one uses the next technical lemma.

Lemma 4.8. *Assume* **C1** *holds for* μ. *Let* $f : \mathbb{R} \mapsto \mathbb{R}$ *be a positive strictly increasing continuous function such that* $\lim_{+\infty} f(x) = +\infty$. *Let us define*

$$G_h(f) = \left\{ x : \begin{array}{c} \forall \varepsilon > 0, \, \exists\, \alpha \geq 0, \, \delta \geq 1 \text{ such that} \\ \frac{f(\alpha)}{\delta} \leq h + \varepsilon \text{ and } \mathcal{L}(\alpha, \delta, \varepsilon) \text{ holds at } x \end{array} \right\}.$$

Then $\dim G_h(f) \leq h \sup_{\alpha : f(\alpha) \geq h} \frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{f(\alpha)}$.

Proof. Let $\varepsilon > 0$, and for every $i \in \mathbb{N}$, let $\delta_i = 1 + i\frac{\varepsilon}{2h}$, and α_i be such that $f(\alpha_i) = \delta_i(h + 2\varepsilon)$. Remark that δ_i and α_i have been chosen so that for $\varepsilon > 0$ small enough, for every $\delta \in [\delta_i, \delta_{i+1}]$, one has

$$f(\alpha_i) = \delta_i(h + 2\varepsilon) \geq \delta(h + \varepsilon). \tag{4.7}$$

Thus let $\varepsilon > 0$ such that (4.7) holds, and let us define the sets T_{α_i,δ_i} by

$$T_{\alpha_i,\delta_i} = \bigcap_{J \geq 0} \bigcup_{j \geq J} \bigcup_{k : \mu(I_{j,k}) \geq b^{-j\alpha_i}} [kb^{-j} - 2\, b^{-j\delta_i}, kb^{-j} + 2\, b^{-j\delta_i}]. \tag{4.8}$$

Any point of T_{α_i,δ_i} is infinitely many often close at rate δ_i to a b-adic number kb^{-j} that verifies $\mu(I_{j,k}) \geq b^{-j\alpha_i}$. By definition of $G_h(f)$, every $x \in G_h(f)$ belongs to T_{α_i,δ_i} with i the unique integer such that $\delta \in [\delta_i, \delta_{i+1})$. One thus gets the inclusion $G_h(f) \subset \bigcup_{i \in \mathbb{N}} T_{\alpha_i,\delta_i}$.

It is time to use Lemma 2.7 to upper bound the dimension of a set $T_{\alpha,\delta}$. Indeed, let $\alpha > 0$, $\delta \geq 1$ and $\varepsilon' < \varepsilon$. By Lemma 2.7 applied to $\eta = \varepsilon'/2$ and $\varepsilon = \varepsilon'$, one gets that for j large enough (one also uses that $d_\mu^g(\alpha)$ is always smaller than $\tau_\mu^*(\alpha)$, see Proposition 2.5)

$$\frac{\log\left(\#\left\{k : \mu(I_{j,k}) \geq b^{-j\alpha}\right\}\right)}{\log b^j} \leq \sup_{\alpha' \leq \alpha + \varepsilon'/2} d_\mu^g(\alpha') + \varepsilon' \leq \sup_{\alpha' \leq \alpha + \varepsilon'/2} \tau_\mu^*(\alpha') + \varepsilon'.$$

We denote $\tau_\mu^*(\alpha') + \varepsilon'$ by $\tau_{\alpha,\varepsilon'}$. Let us upper bound the Hausdorff dimension of $T_{\alpha,\delta}$. Let $d > \frac{\tau_{\alpha,\varepsilon'}}{\delta}$. This set $T_{\alpha,\delta}$ is covered by $\bigcup_{j \geq J} \bigcup_{k : \mu(I_{j,k}) \geq b^{-j\alpha}} [kb^{-j} - b^{-j\delta}, kb^{-j} + b^{-j\delta}]$, and

$$\sum_{j \geq J} \sum_{k : \mu(I_{j,k}) \geq b^{-j\alpha}} |[kb^{-j} - b^{-j\delta}, kb^{-j} + b^{-j\delta}]|^d \leq C \sum_{j \geq J} b^{j\tau_{\alpha,\varepsilon'}} b^{-jd\delta} \leq C b^{-J(\tau_{\alpha,\varepsilon'} - d\delta)},$$

where C is a constant that does not depend on d or J. This double sum goes to zero when $J \to +\infty$, and the d-dimensional Hausdorff measure of $T_{\alpha,\delta}$ is finite for every $d > \frac{\tau_{\alpha,\varepsilon'}}{\delta}$. Thus the Hausdorff dimension of $T_{\alpha,\delta}$ is less than $\frac{\tau_{\alpha,\varepsilon'}}{\delta}$. This remains true for any $\varepsilon' > 0$, so, using the continuity of τ_μ^*, one gets $\dim T_{\alpha,\delta} \leq \frac{\inf_{\varepsilon'} \tau_{\alpha,\varepsilon'}}{\delta} = \frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{\delta}$. The inclusion $G_h(f) \subset \bigcup_{i \in \mathbb{N}} T_{\alpha_i,\delta_i}$ implies

$$\begin{aligned} \dim G_h(f) &\leq \sup_{i \in \mathbb{N}} (\dim T_{\alpha_i,\delta_i}) \leq \sup_{i \in \mathbb{N}} \frac{\sup_{\alpha' \leq \alpha_i} \tau_\mu^*(\alpha')}{\delta_i} \\ &\leq (h + 2\varepsilon) \sup_{i \in \mathbb{N}} \frac{\sup_{\alpha' \leq \alpha_i} \tau_\mu^*(\alpha')}{f(\alpha_i)} \leq (h + 2\varepsilon) \sup_{\alpha : f(\alpha) \geq h} \frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{f(\alpha)}, \end{aligned}$$

where the range of α's is $f(\alpha) \geq h$ since $f(\alpha_i)$ is by definition always greater than h. Letting ε go to zero yields the conclusion. □

Proposition 4.9. *Assume* **C1**. *If* $h \in (0, \tau_\mu'(0^+))$, $\dim(E_h^\nu) \leq h \sup_{u \geq h} \frac{\tau_\mu^*(u)}{u}$.

Proof. Let $h > 0$, and $x \in E_h^\nu$. Since $x \in E_h^\nu$, by Proposition 4.6 $x \in F_h$, which corresponds in view of Lemma 4.8 to $G_h(f)$ with the function f being the identity $f(x) = x$. Hence by Lemma 4.8, $\dim E_h^\nu \leq h \sup_{\alpha \geq h} \frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{\alpha}$.

Let us now simplify this formula. Remember that $q_c = \inf\{q : \tau_\mu(q) = 0\}$. Since τ_μ^* is concave, the function $\alpha \mapsto \frac{\tau_\mu^*(\alpha)}{\alpha}$ is concave, and reaches its maximum at $h_c = \tau_\mu'(q_c^-)$, with $q_c = \frac{\tau_\mu^*(h_c)}{h_c}$. Hence $\forall \alpha$, $\frac{\tau_\mu^*(\alpha)}{\alpha} \leq \frac{\tau_\mu^*(h_c)}{h_c}$. Moreover, when $\alpha \geq \tau_\mu'(0^+)$, $\frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{\alpha} \leq \frac{\tau_\mu^*(\tau_\mu'(0^+))}{\alpha} \leq \frac{\tau_\mu^*(\tau_\mu'(0^+))}{\tau_\mu'(0^+)} \leq \frac{\tau_\mu^*(h_c)}{h_c} = q_c$.

Two cases can thus be distinguished

- $h_c < h < \tau_\mu'(0^+)$: if $h \leq \alpha \leq \tau_\mu'(0^+)$, $\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha') = \tau_\mu^*(\alpha)$, so $\frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{\alpha} = \frac{\tau_\mu^*(\alpha)}{\alpha}$. If $\alpha \geq \tau_\mu'(0^+)$, $\frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{\alpha} \leq \frac{\tau_\mu^*(\tau_\mu'(0^+))}{\tau_\mu'(0^+)}$. Hence one gets $\dim E_h^\nu \leq h \sup_{\tau_\mu'(0^+) \geq \alpha \geq h} \frac{\tau_\mu^*(\alpha)}{\alpha} = h \sup_{\alpha \geq h} \frac{\tau_\mu^*(\alpha)}{\alpha}$.
- $0 < h \leq h_c$: if $\alpha \geq h_c$, the same arguments as above still work. If $h \leq \alpha < h_c$, $\frac{\sup_{\alpha' \leq \alpha} \tau_\mu^*(\alpha')}{\alpha} \leq \frac{\tau_\mu^*(\alpha)}{\alpha} \leq \frac{\tau_\mu^*(h_c)}{h_c} = q_c$. Thus $\dim E_h^\nu \leq h q_c = h \sup_{\alpha \geq h} \frac{\tau_\mu^*(\alpha)}{\alpha}$. □

Let now verify that the upper bound $h \sup_{u \geq h} \frac{\tau_\mu^*(u)}{u}$ coincides with the one announced in Theorem 1.1. When $h \leq h_c$, $\sup_{u \geq h} \frac{\tau_\mu^*(u)}{u} = q_c$, and the upper bound becomes $\dim E_h^\nu \leq q_c h$. When $h \geq h_c$, $\sup_{u \geq h} \frac{\tau_\mu^*(u)}{u} = \frac{\tau_\mu^*(h)}{h}$ (the mapping $\alpha \mapsto \frac{\tau_\mu^*(\alpha)}{\alpha}$ is non-increasing when $\alpha \geq h_c$), hence $\dim E_h^\nu \leq h \frac{\tau_\mu^*(h)}{h} = \tau_\mu^*(h)$.

A simple adaptation of the last proof yields the following corollary

Corollary 4.10. *Let $h \in [0, \tau_\mu'(0^+)]$, and F_h be the set (4.5). Then $\dim F_h \leq q_c h$.*

4.4. **Lower bound of the multifractal spectrum.** Applying Theorem 1.2 gives a lower bound of the dimension of the set of points x that are infinitely often well-approached by b-adic numbers kb^{-j} that verify $\mu(I_j(x)) \sim b^{-j\alpha}$. For every j, k and δ, one denotes $I_{j,k}^{(\delta)} = [kb^{-j}, kb^{-j} + b^{-j\delta}]$.

Proposition 4.11. *Let μ satisfying* **C2**(h_c)*. Then $\forall \delta \geq 1$, $\dim E_{\frac{h_c}{\delta}}^\nu \geq \frac{\tau_\mu^*(h_c)}{\delta}$.*

Proof. Let $\delta > 1$ and $d = \frac{\tau_\mu^*(h_c)}{\delta}$. For any positive sequence $\widetilde{\varepsilon} = \{\varepsilon_j\}_{j \geq 1}$ converging to 0, let

$$S_{\delta,\widetilde{\varepsilon},\psi}(h_c) = \bigcap_{n \geq 1} \bigcup_{j \geq n} \bigcup_{\substack{k \in \{0,\dots,b^j-1\}: \\ |I_{j,k}|^{h_c + \psi(b^{-j})} \leq \mu(I_{j,k}) \leq |I_{j,k}|^{h_c - \psi(b^{-j})}}} I_{j,k}^{(\delta - \varepsilon_j)}, \tag{4.9}$$

We apply Theorem 1.2. There exist a sequence $\widetilde{\varepsilon}$, a function ψ (converging to 0 at 0^+) and a measure m_δ such that $m_\delta(S_{\delta,\widetilde{\varepsilon},\psi}(h_c)) > 0$ and for every Borel set E with $\dim E < d$, $m_\delta(E) = 0$. Recall also that $E_h^\nu = F_h \setminus \bigcup_{h' < h} F_{h'} = F_h \setminus \bigcup_{i \geq [h^{-1}]+1} F_{h - \frac{1}{i}}$, the second equality due to the monotonicity of the sets $\{F_{h'}\}$ when $h \leq \tau_\mu'(0^+)$. Using Corollary 4.10, for every $i \geq [h^{-1}] + 1$, $\dim F_{h-\frac{1}{i}} < q_c h$. This implies, by Theorem 1.2, that $m_\delta(\bigcup_{i \geq [h^{-1}]+1} F_{h-\frac{1}{i}}) = 0$.

One easily verifies that $S_{\delta,\widetilde{\varepsilon},\psi}(h_c) \subset F_h$, since every point of $S_{\delta,\widetilde{\varepsilon},\psi}(h_c)$ satisfies $\mathcal{L}(h_c, \delta, \varepsilon)$ for every $\varepsilon > 0$. This implies that $m_\delta(E_h^\nu) \geq m_\delta(S_{\delta,\widetilde{\varepsilon},\psi}(h_c)) > 0$, and thus that $\dim E_h^\nu \geq d$.

If $\delta = 1$, since **C2**(h_c) implies **C3**(h_c), see the proof of Proposition 4.12. □

Proposition 4.12. *Let μ be a positive Borel measure supported by $[0,1]$, and let us assume that* **C3**(h) *holds for some $h \geq h_c$. Then $d_\nu(h) = \dim E_h^\nu \geq \tau_\mu^*(h)$.*

Proof. Consider $\widetilde{E}_h^\mu$, the measure m_h provided by **C3**(h) and $\varepsilon > 0$. One has

$$\widetilde{E}_h^\mu \subset \bigcap_{J\geq 1} \bigcup_{j \geq J} \bigcup_{k\in\{0,\dots,b^j-1\}:\, b^{-j(h+\varepsilon)} \leq \mu(I_{j,k}) \leq b^{-j(h-\varepsilon)}} [kb^{-j},(k+1)b^{-j}).$$

Using Lemma 2.7 applied with $\eta = \varepsilon$, one gets that

$$\#\left\{k : b^{-j(h+\varepsilon)} \leq \mu(I_{j,k}) \leq b^{-j(h-\varepsilon)}\right\} \leq b^{j(\sup_{\max(\beta-\varepsilon,0)\leq\alpha'\leq\alpha+\varepsilon} d_\mu^g(\alpha')+\varepsilon)}.$$

But $\forall \alpha' \in [\max(\beta-\varepsilon,0),\alpha+\varepsilon]$, $d_\mu^g(\alpha') \leq \tau_\mu^*(\alpha')$. Let us denote by $\tau_{h,\varepsilon}$ the quantity $\sup_{\max(\beta-\varepsilon,0)\leq\alpha'\leq\alpha+\varepsilon} \tau_\mu^*(\alpha')+\varepsilon$.

Let $\delta > 1$. Let us estimate the dimension of $\widetilde{E}_{h,\delta}^\mu = \{x \in \widetilde{E}_h^\mu : \delta_x = \delta\}$. The same lines of computations as in Lemma 4.8 show that, for every J,

$$\bigcup_{\delta'>\delta} \widetilde{E}_{h,\delta'}^\mu \subset \bigcup_{j\geq J} \bigcup_{\substack{k\in\{0,\dots,b^j-1\}:\\ b^{-j(h+\varepsilon)}\leq\mu(I_{j,k})\leq b^{-j(h-\varepsilon)}}} [kb^{-j}-b^{-j(\delta-\varepsilon)}, kb^{-j}+b^{-j(\delta-\varepsilon)}].$$

Using this covering, one deduces that $\dim \bigcup_{\delta'>\delta} \widetilde{E}_{h,\delta'}^\mu \leq \frac{\tau_{h,\varepsilon}}{\delta-\varepsilon}$. This is true $\forall\varepsilon>0$, hence using the continuity of τ_μ^* on its support, $\dim \bigcup_{\delta'>\delta} \widetilde{E}_{h,\delta'}^\mu \leq \frac{\tau_\mu^*(h)}{\delta}$.

Now, let $\widetilde{E}_{h,1}^\mu = \widetilde{E}_h^\mu \setminus \bigcup_{i\geq 2}\bigcup_{\delta'>1+i^{-1}} \widetilde{E}_{h,\delta'}^\mu$. For $i \geq 2$, $\dim \bigcup_{\delta'>1+i^{-1}} \widetilde{E}_{h,\delta'}^\mu < \tau_\mu^*(h)$, and thus $m_h(\bigcup_{\delta'>1+i^{-1}} \widetilde{E}_{h,\delta'}^\mu) = 0$. Hence $m_h(\widetilde{E}_{h,1}^\mu) = m_h(\widetilde{E}_h)$, which is > 0 by **C3**(h). But the points x belonging to $\widetilde{E}_{h,1}^\mu$ all have their δ_x equal to 1. Then, by Proposition 4.2, $h_\nu(x) = h$. Hence $\widetilde{E}_{h,1}^\mu \subset E_h^\nu$. This yields $m_h(E_h^\nu) > 0$ and $\dim E_h^\nu \geq \tau_\mu^*(h)$. □

5. Additional properties derived from Theorems 1.1 and 3.2

Let us begin with two remarks:

- Replacing the initial measure μ by the sum of Dirac masses ν does not change anything from the point of view of the bad-approximated points. Indeed, the real numbers x with $\delta_x = 1$ verify $h_\mu(x) = h_\nu(x)$.

- The reader can check that the *upper* multifractal spectrum defined by $\overline{d}_\nu(h) = \dim\{x : \limsup_{r\to 0+} \frac{\log \nu(B(x,r))}{\log |B(x,r)|} = h\}$ is equal to the one of μ (when **C3**(h) holds). Although the $\limsup$ at a given point x may be different for the two measures μ and ν, the upper spectrum is thus left unchanged.

5.1. Hausdorff dimension of some specific sets.

Let us define for $\alpha > 0$, $\delta \geq 1$

$$E_{\alpha,\delta}^\mu = \left\{ x : \begin{array}{c} \text{There exist an infinite sequence of } b\text{-adic numbers} \\ k_n b^{-j_n} \text{ that verify } |k_n b^{-j_n} - x| \leq b^{-j_n\delta} \\ \text{and } \liminf_{n\to+\infty} \frac{\log \mu(I_{j_n,k_n})}{\log |I_{j_n,k_n}|} = \alpha \end{array} \right\} \tag{5.1}$$

Corollary 5.1. *For any $\alpha \geq 0$, $\delta \geq 1$, if* **H**(α,$\tau_\mu^*(\alpha)$) *holds, $\dim E_{\alpha,\delta}^\mu = \tau_\mu^*(\alpha)/\delta$.*

5.2. **Multifractal formalisms for $\nu_{\gamma,\sigma}$.** We set $\alpha_{\max} = \sup\{\alpha : \tau_\mu^*(\alpha) > 0\}$.

Proposition 5.2. *Let μ be a positive Borel measure on $[0,1]$, and let $\gamma \geq 0$ and $\sigma \geq 1$. Let $q_{\gamma,\sigma} = \inf\{q \in \mathbb{R} : \tau_\mu(\sigma q) + \gamma q = 0\}$, and $h_{\gamma,\sigma} = \sigma\tau_\mu'(\sigma q_{\gamma,\sigma}^-) + \gamma$.*

Assume that **C2(**$\frac{h_{\gamma,\sigma}-\gamma}{\sigma}$**)** *holds, and that* **C3(**$\frac{h-\gamma}{\sigma}$**)** *holds for every exponent $h \in [\frac{h_{\gamma,\sigma}-\gamma}{\sigma}, \alpha_{\max})$. The measure $\nu_{\gamma,\sigma}$ (1.1) satisfies the multifractal formalism at every h such that $\tau_\mu^*(h) > 0$.*

Proof. We give the proof in the case of ν, i.e. when $\gamma = 0$ and $\sigma = 1$.
Let us compute the scaling function τ_ν of ν (see definition (1.2) of τ).

- if $q < 0$ **:** For every (j,k), by (4.1) $\nu(I_{j,k}) \geq j^{-2}\mu(I_{j,k})$, which shows $\sum_{k=0}^{b^j-1} \nu^q(I_{j,k}) \leq j^{-2q}\sum_{k=0}^{b^j-1} \mu^q(I_{j,k})$. Hence $\tau_\nu(q) \geq \tau_\mu(q)$. Moreover, when **C3(**$h$**)** holds at $h \in [\tau_\mu'(0^+), \alpha_{\max})$, $\tau_\mu^*(h) = d_\nu(h) \leq \tau_\nu^*(h)$. If this holds on a dense set of exponents $h \in [\tau_\mu'(0^+), \alpha_{\max})$, by inverse Legendre transform one gets $\tau_\nu(q) \leq \tau_\mu(q)$ for every $q < 0$ with $\tau_\mu'(q^+) \leq \alpha_{\max}$. The equality follows for these q's.

- if $0 < q < q_c$ **:** Let $J \geq 2$. One has $\sum_{K=0}^{b^J-1} \nu^q(I_{J,K}) = \sum_{Kb^{-J}\text{irreducible}} \nu^q(I_{J,K}) + \sum_{K\text{multiple of } b} \nu^q(I_{J,K})$. When Kb^{-J} is irreducible, one uses (4.2) to get $\nu^q(I_{J,K}) \leq C^q\frac{1}{J^q}\mu^q(I_{J,K})$. When K is a multiple of b, let kb^{-j} be its unique irreducible representation ($0 \leq j \leq J-1$). As already noticed before, in this case

$$\begin{aligned}\nu(I_{J,K}) &= \nu(\{kb^{-j}\}) + \sum_{j'\geq J+1}\frac{1}{j'^2}\sum_{k':I_{j',k'}\subset I_{J,K}}\mu(I_{j',k'})\\ &\leq \frac{C}{j^2}\mu(I_{j,k}) + \sum_{j'\geq J+1}\frac{C}{j'^2}\mu(I_{J,K}) \leq \frac{1}{j^2}\mu(I_{j,k}) + C\frac{1}{J}\mu(I_{J,K}).\end{aligned}$$

Since $q < 1$, one gets $\nu^q(I_{J,K}) \leq C^q\Big(\frac{1}{j^2}\mu(I_{j,k}) + \frac{1}{J}\mu(I_{J,K})\Big)^q \leq C^q\ \Big(\frac{1}{j^{2q}}\mu^q(I_{j,k}) + \frac{1}{J^q}\mu^q(I_{J,K})\Big)$. The term $\frac{1}{j^{2q}}\mu^q(I_{j,k})$ is bounded by $\frac{1}{j^{2q}}\Big|\sum_{K':I_{J,K'}\subset I_{j,k}}\mu(I_{J,K'})\Big|^q \leq \frac{1}{j^q}\sum_{K':I_{J,K'}\subset I_{j,k}}\mu^q(I_{J,K'})$. This results yields

$$\sum_{0\leq K<b^J}\nu^q(I_{J,K}) \leq \frac{C^q}{J^q}\sum_{0\leq K<b^J}\mu^q(I_{J,K}) + C^q\sum_{\substack{K\text{ multiple of } b,\text{ and } kb^{-j}\\ \text{its irreducible representation}}}\ \sum_{K':I_{J,K'}\subset I_{j,k}}\frac{\mu^q(I_{J,K'})}{j^q}.$$

Each irreducible b-adic number kb^{-j} with $0 \leq j \leq J-1$ appears one time in the above double sum. Conversely, for a given integer $K \in \{0,\ldots,b^J-1\}$ and for each scale j, there exists only one irreducible b-adic number kb^{-j} such that $I_{J,K} \subset I_{j,k}$. Hence, the double sum can be bounded by $\sum_{j=1}^{J-1}\frac{1}{j^q}\sum_{K=0}^{b^J-1}\mu^q(I_{J,K})$, and eventually by $J^{1-q}\sum_{K=0}^{b^J-1}\mu^q(I_{J,K})$. Then $\sum_{K=0}^{b^J-1}\nu^q(I_{J,K})$ is bounded by

$$C^qJ^{-q}\sum_{0\leq K<b^J}\mu^q(I_{J,K}) + C^qJ^{1-q}\sum_{0\leq K<b^J}\mu^q(I_{J,K}) \leq CJ^{1-q}\sum_{0\leq K<b^J}\mu^q(I_{J,K}),$$

where C denotes some constant independent of μ and J. Computing the liminf of $\frac{1}{-J}\log_b\sum_{K=0}^{b^J-1}\nu^q(I_{J,K})$ when J goes to infinity shows that $\tau_\nu(q) \geq \tau_\mu(q)$.

On the other hand, when **C3(**h**)** holds on a dense set of values of $h \in [h_c, \tau_\mu'(0^+)]$, at these exponents one has $\tau_\nu^*(h) \geq d_\nu(h) = \tau_\mu^*(h)$, which yields by inverse Legendre transform $\tau_\nu(q) \leq \tau_\mu(q)$ for every $q \in [0, q_c]$. Hence the equality holds.

- if $q \geq q_c$ **:** Let us distinguish two cases.

If $q_c = 1$, then Theorem 1.1 yields $d_\nu(h) = h$ for $h \in [0, \tau'_\mu(1)]$. Hence $\tau^*_\nu(h) \geq h$ for $h \in [0, \tau'_\mu(1)]$, but one always has $\tau^*_\nu(h) \leq h$, hence $\tau^*_\nu(h) = h$, which gives by inverse Legendre transform $\tau_\nu(q) = 0$ for $q \geq q_c = 1$.
If $q_c < 1$, $\tau_\nu = \tau_\mu$ when $q \in (-\infty, q_c)$, hence $\tau_\nu(q_c) = \tau_\mu(q_c) = 0$. Since $\tau_\nu(1) = 0$, the concavity of τ_ν forces $\tau_\nu(q) = 0$ for $q \geq q_c$.

Theorem 1.1 and the above identification of τ_ν show that under our assumptions, $d_\nu(h) = \tau^*_\nu(h)$ for every $h \in [0, \alpha_{\max})$. □

Finally, it can also be verified using [6] that under the assumptions of Proposition 5.2, the multifractal formalisms defined in [14], [13] and [45] are verified if one uses the level sets E^μ_h. The formalisms do not hold if the sets $\widetilde{E}^\mu_h$ are considered.

REFERENCES

[1] Arbeiter, M., Patzschke, N., Random self-similar multifractals, Math. Nachr. **181**, 5–42 (1996)

[2] Arnol'd, V.I., Geometrical Methods in the Theory of Ordinary Differential Equations, translated by J. Szücs, Springer-Verlag, New York, 1983

[3] Aversa, V., Bandt,C., The multifractal spectrum of discrete measures, Acta Uni. Caro.- Math. et Phys. **31**(21), 5–8 (1990)

[4] Bacry, E., Muzy, J.-F., Log-infinitely divisible multifractal processes, Commun. Math. Phys. **236**, 449–475 (2003)

[5] Barral, J., Continuity of the multifractal spectrum of a random statistically self-similar measures, J. Theor. Probab. **13**, 1027–1060 (2000)

[6] Barral, J, Ben Nasr, F., Peyrière, J., Comparing Multifractal Formalisms: the neighboring condition, Asian J. Math. **7**, 149–166 (2003)

[7] Barral, J., Mandelbrot, B., Multifractal products of cylindrical pulses, Probab. Theory Relat. Fields **124**(3), 409–430 (2002)

[8] Barral, J., Mandelbrot, B., Random multiplicative multifractal measures, Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot, Proc. Symp. Pure Math., AMS, Providence, RI, 2004

[9] Barral, J., Coppens, M.-O., Mandelbrot, B.B., Multiperiodic multifractal martingale measures, J. Math. Pures Appl. (9) **82**, 1555–1589 (2003)

[10] Barral, J., Seuret, S., From multifractal measures to multifractal wavelet series, Preprint (2002)

[11] Barral, J., Seuret, S., Speed of renewal of level sets for statistically self-similar measures, Preprint (2004)

[12] Barral, J., Seuret, S., Function series with multifractal variations, preprint (2004)

[13] Ben Nasr, F., Analyse multifractale de mesures, C. R. Acad. Sci. Paris Série I **319**, 807–810 (1994)

[14] Brown, G., Michon, G., Peyrière, J., On the multifractal analysis of measures, J. Stat. Phys. **66**(3-4), 775–790 (1992)

[15] Cawley, R., Mauldin, R.D., Multifractal decompositions of Moran fractals, Adv. Math. **92**, 196–236 (1992)

[16] Collet, P., Lebowitz, J.L., Porzio, A., The dimension spectrum of some dynamical systems, J. Stat. Phys. **47**, 609–644 (1987)

[17] Collet, P., Koukiou, F., Large deviations for multiplicative chaos, Commun. Math. Phys. **147**, 329–342 (1992)

[18] Dodson, M.M., Rynne, B.P., Vickers, J.A.G., Diophantine approximation and a lower bound for Hausdorff dimension, Mathematika **37**, 59–73 (1990)

[19] Dodson, M.M., Melián, M.V., Pestane, D., Vélani, S.L., Patterson measure and Ubiquity, Ann. Acad. Sci. Fenn. Ser. A I Math. **20**(1), 37–60 (1995)

[20] Falconer, K.J., The multifractal spectrum of statistically self-similar measures, J. Theor. Prob. **7**, 681–702 (1994)

[21] Falconer, K.J., Fractal Geometry: Mathematical Foundations and Applications, John Wiley, New York, 1990

[22] Falconer, K.J., Representation of families of sets by measures, dimension spectra and Diophantine approximation, Math. Proc. Camb. Phil. Soc. **128**, 111–121 (2000)

[23] Falconer, K.J., One-sided multifractal analysis and points of non-differentiability of devil's staircases, Math. Proc. Cambridge Philos. Soc. **136**, 167–174 (2004)

[24] Fan, A.H., Multifractal analysis of infinite products, J. Stat. Phys. **86**(5/6), 1313–1336 (1997)

[25] Feng, D.-J., Olivier, E., Multifractal analysis of weak Gibbs measures and phase transition—application to some Bernoulli convolutions. Ergodic Theory Dynam. Systems **23**, 1751–1784 (2003)

[26] Frisch, U., Parisi, G., Fully developped turbulence and intermittency, Proc. International Summer school Phys., Enrico Fermi, 84-88, North Holland, 1985
[27] Halsey, T.C., Jensen, M.H., Kadanoff, L.P., Procaccia, I., Shraiman B.I., Fractal measures and their singularities: The characterization of strange sets, Phys. Rev. A **33**(2), 1141–1151 (1986)
[28] Harte, D., Multifractals: Theory and Applications, Chapman & Hall, 2001
[29] Holley, R., Waymire, E.C., Multifractal dimensions and scaling exponents for strongly bounded random cascades, Ann. Appl. Probab. **2**, 819–845 (1992)
[30] Jaffard, S., Old friends revisited: the multifractal nature of some classical functions, J. Fourier Anal. Appl. **3**(1), 1–22 (1997)
[31] Jaffard, S., The multifractal nature of Lévy processes, Probab. Theory Relat. Fields **114**(2), 207–227 (1999)
[32] S. Jaffard, S., On lacunary wavelet series, Ann. of Appl. Prob. **10**(1), 313-329 (2000)
[33] Kahane, J.-P., Sur le chaos multiplicatif, Ann. Sci. Math. Québec **9**, 105–150 (1985)
[34] Kahane, J.-P., Peyrière, J., Sur certaines martingales de Benoît Mandelbrot, Adv. Math **22**, 131–145 (1976)
[35] Khanin, K., Kifer, Y., Thermodynamic formalism for random transformations and statistical mechanics. Sinaĭ's Moscow Seminar on Dynamical Systems, 107–140, Amer. Math. Soc. Transl. Ser. 2, 171, Amer. Math. Soc., Providence, RI (1996)
[36] Kifer, Y., Fractals via random iterated function systems and random geometric constructions, Fractal geometry and stochastics (Finsterbergen, 1994), 145–164, Progr. Probab. **37**, Birkhäuser, Basel (1995)
[37] Ledrappier, F., Porzio, A., On the multifractal analysis of Bernoulli convolutions. I. Large-deviation results, II. Dimensions, J. Stat. Phys. **82**, 367–420 (1996)
[38] Lévy Véhel, J., Riedi, R.H., TCP traffic is multifractal: a numerical study, INRIA research report, RR-3129 (1997)
[39] Lévy Véhel, J., Vojak, R., Multifractal analysis of Choquet capacities, Adv. Appl. Math. **20**, 1–43 (1998)
[40] Mandelbrot, B., Intermittent turbulence in self-similar cascades: divergence of hight moments and dimension of the carrier, J. Fluid. Mech. **62**, 331–358 (1974)
[41] Mandelbrot, B., Fractals and Scaling in finance (Discontinuity, Concentration, Risk), Springer, 1997
[42] Mandelbrot, B., Evertsz, C.J., Hayakawa, Y., Exactly self-similar left-sided multifractal measures, Phys. Rev. A **42**(8), 4528–4536 (1990)
[43] Molchan, G.M., Scaling exponents and multifractal dimensions for independent random cascades, Commun. Math. Phys. **179**, 681–702 (1996)
[44] Olsen,L., Random Geometrically Graph Directed Self-similar Multifractals, Pitman Res. Notes Math. Ser., **307**, 1994
[45] Olsen, L., A multifractal formalism, Adv. Math. **116**, 92–195 (1995)
[46] Parry, W., Policott, M., Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics, Société Mathématique de France, Astérisque, **187–188** (1990)
[47] Pesin, Y., Weiss, H., The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples, Chaos, **7(1)**, 89–106 (1997)
[48] Philipp, W., Stout, W., Almost sure invariance principles for partial sums of weakly dependent random variables, Mem. Amer. Math. Soc. **2**, 161, 1975
[49] Rand, D.A., The singularity spectrum $f(\alpha)$ for cookie-cutters, Ergod. Th. & Dynam. Sys. **9**, 527–541 (1989)
[50] Riedi, R.H., Mandelbrot, B., Inverse measures, the inversion formula, and discontinuous multifractals, Adv. Appl. Math. **18**, 50–58 (1997)
[51] Riedi, R.H., Mandelbrot, B., Exceptions to the Multifractal Formalism for Discontinuous Measures, Math. Proc. Cambr. Phil. Soc. **123**, 133–157 (1998)
[52] Zinsmeister, M., Thermodynamic formalism and holomorphic dynamical systems. SMF/AMS Texts and Monographs, 2. AMS, Providence, RI (2000); SMF, Paris (1996)