THE SINGULARITY SPECTRUM OF THE FISH'S BOUNDARY

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Abstract. Let $\mathcal{M}(\mathbb{T}^1, T)$ be the convex set of Borel probability measures on the Circle $\mathbb{T}^1$ invariant under the action of the transformation $T : x \longmapsto 2x \mod (1)$. Its projection on the complex plane by the application $\mu \longmapsto \int e^{2i\pi x}\, d\mu(x)$ is a compact convex of the unit disc, symmetric with respect to the x-axis, called the "Fish" by T. Bousch [3]. Seeing the boundary of the upper half-Fish as a function, we focus on its local regularity. We show that its multifractal spectrum is concentrated at ∞, but that every pointwise regularity $\alpha \in [1, \infty]$ is realized in a uncountable dense set of points. The results rely on fine properties of Sturm measures.

1. Introduction

Multifractal analysis describes the fine local structure of functions or measures. On typical examples, the pointwise regularity exponent varies erratically from one point to another, and the level sets corresponding to a given regularity are usually fractal sets. The purpose of Multifractal analysis is to determine the Hausdorff dimension of these sets.

Interest in multifractal analysis came from fluid mechanics and also dynamical systems, see among many references [12, 14, 22]. Since then, multifractal analysis has developed in many contexts, for instance in Probability Theory [16, 1] (see [11, 2, 17] for other examples). In this article, we consider the example of a graph naturally appearing in an optimization problem in Ergodic Theory.

The notion of regularity we discuss in the sequel is the following. Given a real function $f \in L^\infty_{loc}$ on an open interval I and $x_0 \in I$, recall that f belongs to $\mathcal{C}^\alpha(x_0)$, for some $\alpha \geq 0$, if there exist a polynomial P of degree at most $\lfloor \alpha \rfloor$ and a constant $C > 0$ such that locally :

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \tag{1}$$

The local regularity of f at x_0 is measured by the *pointwise Hölder exponent* :

$$h_f(x_0) = \sup\{\alpha \geq 0 \mid f \in \mathcal{C}^\alpha(x_0)\}.$$

The relevant information is then provided by the *spectrum of singularities* d_f of f, which is the application :

$$d_f : s \in [0, \infty] \longmapsto \mathrm{Dim}_{\mathcal{H}}\{x_0 \in I \mid h_f(x_0) = s\},$$

where $\mathrm{Dim}_{\mathcal{H}}$ stands for the Hausdorff dimension. We adopt the convention that $\mathrm{Dim}_{\mathcal{H}}\emptyset = -\infty$.

We now detail the context of our example. Let $\mathbb{T}^1$ be the torus identified with $\mathbb{R}/\mathbb{Z}$ and equipped with the transformation $T(x) = 2x \mod (1)$. Introduce the convex set $\mathcal{M}(\mathbb{T}^1, T)$ of Borel probability measures on $\mathbb{T}^1$ invariant by T, endowed

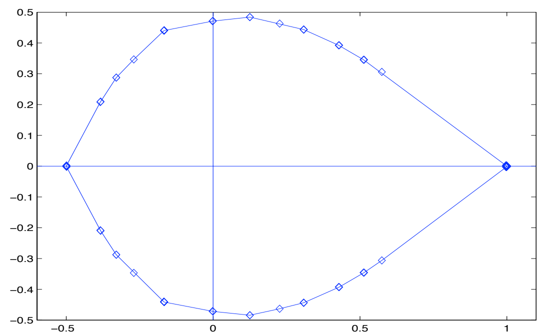

FIGURE 1. The Fish.

with the weak$*$ topology. The *Fish* is the compact convex subset of $\mathbb{C}$, image of the following linear map :

$$\begin{array}{ccc} \mathcal{M}(\mathbb{T}^1, T) & \longrightarrow & \mathbb{C} \\ \mu & \longmapsto & \int e^{2i\pi u}\, d\mu(u). \end{array}$$

The boundary of the Fish intersects the horizontal axis at the points $(-1/2, 0)$ and $(1, 0)$ and is symmetric with respect to this axis, since T commutes with the symmetry $x \longmapsto -x$ on $\mathbb{T}^1$. We shall then restrict our study to the upper half-Fish, whose boundary is denoted as a concave function $\mathcal{F}$:

$$\begin{array}{ccc} [-1/2, 1] & \longrightarrow & \mathbb{R}^+ \\ x & \longmapsto & \mathcal{F}(x). \end{array}$$

The goal of this paper is to determine the pointwise Hölder exponent and the singularity spectrum of the function $\mathcal{F}$.

As a preliminary remark, let us mention that, with respect to Fourier coefficients, there is only one Fish. Indeed, let $k \geq 2$ and consider the k^{th} Fourier coefficient of the elements of $\mathcal{M}(\mathbb{T}^1, T)$, that is the linear map :

$$\begin{array}{ccc} \mathcal{M}(\mathbb{T}^1, T) & \longrightarrow & \mathbb{C} \\ \mu & \longmapsto & \int e^{2i\pi k u}\, d\mu(u). \end{array}$$

Then the image of this map is also the Fish. This is a consequence of the fact that $\int e^{2i\pi k u}\, d\mu(u) = \int e^{2i\pi u}\, d(T_k\mu)(u)$, where $T_k x = kx \mod (1)$ on $\mathbb{T}^1$ and $T_k\mu$ is T-invariant. Reciprocally, fixing $\nu \in \mathcal{M}(\mathbb{T}^1, T)$, there always exists some μ invariant under T such that $T_k\mu = \nu$, for instance we can consider

$$\mu = \frac{1}{k} \sum_{0 \leq j \leq k-1} \nu(./k + j/k).$$

The Fish was introduced by Bousch [3] and Jenkinson [18], who considered the question of finding the maximizing measures for a degree one trigonometric polynomial $f_\omega : x \longmapsto \cos 2\pi(x - \omega)$, $\omega \in \mathbb{R}/\mathbb{Z}$. More generally, fixing some continuous $f : \mathbb{T}^1 \to \mathbb{R}$, the initial problematic is given by the variational problem :

$$\beta(f) = \sup \left\{ \int f\, d\mu \mid \mu \in \mathcal{M}(\mathbb{T}^1, T) \right\},$$

where one aims at describing the measures realizing the maximum. Such measures, which always exist as $\mathcal{M}(\mathbb{T}^1, T)$ is compact, are called *maximizing measures* for f. The link with the Fish is simply that if z_ω and z_x are the vectors of $\mathbb{R}^2$ with respective affixes $e^{2i\pi\omega}$ and $e^{2i\pi x}$, then $\langle z_\omega, z_x\rangle = \cos 2\pi(x-\omega)$. Therefore a maximizing measure μ of the function $x \longmapsto \cos 2\pi(x-\omega)$ is such that $\int e^{2i\pi u}\, d\mu(u)$ realizes the maximal orthogonal projection of the Fish on the line going through the origin and with angle $2\pi\omega$. We often adopt this point of view in the sequel.

The question of finding maximizing measures is closely related, via the Birkhoff Ergodic Theorem, to the difficult problem of studying the best pointwise growth of the ergodic sums $(f(x) + f(Tx) + \cdots + f(T^{n-1}x))_{n\geq 0}$ of a function f. General presentations of the topic can be found in Conze-Guivarc'h [10], Bousch-Mairesse [4] or Jenkinson [18], [19]. See also [6].

Back to the regularity of $\mathcal{F}$, concavity implies that the pointwise Hölder exponent is always larger or equal to 1 and that $\mathcal{F}$ is differentiable outside a at most countable subset. Here is our main result.

Theorem 1.1. *The singularity spectrum of* $\mathcal{F}$ *is*

$$d_{\mathcal{F}}(s) = \begin{cases} -\infty & \text{if } s \in [0,1), \\ 0 & \text{if } s \geq 1, \\ 1 & \text{if } s = +\infty. \end{cases}$$

More precisely :

(1) *For all x outside a zero Hausdorff-dimensional subset of $[-1/2, 1]$, there exist constants $C > 0$, $0 < \rho < 1$ and $0 < K \leq 1/2$ such that :*

$$\forall h,\ |\mathcal{F}(x+h) - \mathcal{F}(x) - h\mathcal{F}'(x)| \leq C\rho^{|h|^{-K}}. \tag{2}$$

In particular, $h_{\mathcal{F}}(x) = \infty$.

(2) *For every $1 \leq s < +\infty$, the level set $\{x : h_{\mathcal{F}}(x) = s\}$ is an uncountable and dense subset of $[-1/2, 1]$ of Hausdorff dimension 0.*

Hence, outside a set of Hausdorff dimension 0 $\mathcal{F}$ is locally exponentially close to its tangent (and in particular $\mathcal{F}$ is differentiable at these points). Hence, although "the Fish has no edges" [3] (more precisely, it is strictly convex), its boundary is very flat. To some extent, this also confirms a remark made by Bousch that the Fish is well approximable by polygons with few edges. We also mention that our proof implies that the constants C, ρ, K depend on x and cannot be fixed locally in $[-1/2, 1]$.

In order to compute the pointwise Hölder exponents and the singularity spectrum of $\mathcal{F}$, we use a natural parametrization of the Fish's boundary, given by Bousch [3].

Let $(\nu_t)_{t\in\mathbb{R}/\mathbb{Z}}$ be the family of Sturm measures on $\mathbb{T}^1$, where ν_t is the Sturm measure with rotation number t. Precise definitions are given in the next section. Theorem A and Corollary 2 of [3] imply that any map $x \longmapsto \cos 2\pi(x-\omega)$ admits a unique maximizing measure which is a Sturm measure, and then that a bijective and bicontinuous parametrization of the boundary of the upper half-Fish is :

$$\begin{array}{ccl} [0,1/2] & \longrightarrow & \mathbb{C} = \mathbb{R}^2 \\ t & \longmapsto & \displaystyle\int e^{2i\pi u}\, d\nu_t(u) =: I(t) = (x(t), y(t)). \end{array}$$

In the sequel the notations $t \longmapsto I(t) = (x(t), y(t))$ are reserved to the above parametrization. Remark that when t increases from 0 to 1/2, the graph of $\mathcal{F}$ is described from the right side to the left side. Moreover, $\nu_0 = \delta_0$ and $\nu_{1/2} = 1/2(\delta_{1/3} + \delta_{2/3})$, explaining the extremal values.

It is shown by Bousch [3] that $\mathcal{F}$ admits an angular point (i.e. a point with two distinct semi-tangents) at $x(t)$ if and only if $t \in [0, 1/2] \cap \mathbb{Q}$. Moreover, the points $(-1/2, 0)$ and $(1, 0)$ are also angular points of the Fish. Consequently, using the symmetry of the Fish with respect to the x-axis, this implies that:

$$-\infty < \mathcal{F}'_-(1) < \mathcal{F}'_+(-1/2) < +\infty. \tag{3}$$

We shall use these informations in the sequel.

The set of angular points of $\mathcal{F}$ is countable and dense in $[-1/2, 1]$. At such a point $x(p/q)$ (with $p \wedge q = 1$), the Hölder exponent of $\mathcal{F}$ is equal to 1 (if larger, $\mathcal{F}$ would be differentiable at this point).

Let us now precise the angular defect at each point $x(p/q)$:

Proposition 1.1. *There is a constant $C > 0$ such that for any $p/q \in (0, 1/2)$, with $p \wedge q = 1$:*

$$\frac{1}{C} \cdot q2^{-q} \leq \mathcal{F}'_-(x(p/q)) - \mathcal{F}'_+(x(p/q)) \leq C \cdot q2^{-q}. \tag{4}$$

Proposition 1.1 is related to questions raised by Hunt, Ott and Jenkinson in [15, 18], see Corollary 3.1.

We next deal with the non-angular points of $\mathcal{F}$, i.e. the real numbers $x(t)$ with $t \in [0, 1/2] \backslash \mathbb{Q}$, and where $\mathcal{F}$ is differentiable. As a preliminary step, we study the regularity of the maps $t \longmapsto I(t)$ and $t \longmapsto x(t)$.

Theorem 1.2. *Let $c_0 = 2\sum_{n\geq 1} n \sin(\pi 2^{-n-1}) = 6.077491...$ and $c_1 = \sum_{n\geq 1} n(1 - \cos(\pi 2^{-n})) = 1.925255....$*

(1) *The map $t \longmapsto I(t)$ is c_0-Lipschitz, differentiable at $t \in [0, 1/2] \backslash \mathbb{Q}$ and left and right-differentiable but not differentiable at $t \in (0, 1/2) \cap \mathbb{Q}$. Also $I \longmapsto I'(t)$ is continuous when restricted to $[0, 1/2] \backslash \mathbb{Q}$.*

(2) *The map $t \longmapsto x(t)$ is a decreasing bi-Lipschitz homeomorphism from $[0, 1/2]$ onto $[-1/2, 1]$, verifying :*

$$\text{for every } t \in [0, 1/2] \backslash \mathbb{Q}, \quad x'(t) \in \frac{1}{\sqrt{1 + (\mathcal{F}'(x(t)))^2}} [-c_0, -c_1].$$

If $t \in [0, 1/2] \backslash \mathbb{Q}$, denote by $(p_n/q_n)_{n\geq 0}$ its sequence of convergents. The regularity of $\mathcal{F}$ at $x(t)$ is then read on the Diophantine properties of t.

Theorem 1.3. *Let $t \in [0, 1/2] \backslash \mathbb{Q}$, with convergents $(p_n/q_n)_{n\geq 0}$. Introduce :*

$$M(t) = \liminf_{n \to +\infty} \frac{q_n}{\log_2 q_{n+1}}.$$

(1) *We always have $h_{\mathcal{F}}(x(t)) \geq 1 + M(t)$, and the following relation holds :*

$$1 + M(t) = \sup\{\alpha \geq 0 \mid \exists C > 0, \ \forall h, \ \mathcal{F}(x(t) + h) - \mathcal{F}(x(t)) - h\mathcal{F}'(x(t))| \leq C \ |h|^\alpha\}.$$

(2) *We have $h_{\mathcal{F}}(x(t)) = 1 + M(t)$ in the following situations :*

(a) *$M(t) = +\infty$.*

(b) $M(t) \in \mathbb{R}^+ \backslash \{2m+1 \mid m \geq 0\}$.

(c) $M(t) \in \{2m+1 \mid m \geq 0\}$ *and* $\sup_n \left\{ q_n^{M(t)+1} \cdot 2^{-q_n} \cdot q_{n+1}^{M(t)} \right\} = \infty$.

Fix $M_0 > 0$ (resp. $M_0 = 0$). If $t \in [0, 1/2] \backslash \mathbb{Q}$ is highly Liouville in the sense that

(5) $$q_{n+1} \sim (2^{1/M_0})^{q_n} \quad \text{(resp. } q_{n+1} \sim 2^{q_n^2}\text{)},$$

then $h_{\mathcal{F}}(x(t)) = 1 + M_0$. Since it is known that the set of Liouville numbers satisfying (5) for a given $M_0 > 0$ (resp. $M_0 = 0$) is an uncountable dense subset of $[-1/2, 1]$, item (2) of Theorem 1.1 is deduced from this remark.

2. On Sturm measures

We sum up the informations on the family of Sturm measures that are used in the sequel. Details can be found in Morse-Hedlund [21], Bullet-Sentenac [9] and Bousch [3]. The classical notion of rotation number for homeomorphisms of the Circle is introduced in Katok-Hasselblatt [20]. Proofs in the below discontinuous context are given in [7] and [5].

Definition 2.1. *For $0 \leq \theta < 2$, the closed semi-circle $[\theta/2, \theta/2 + 1/2] \subset \mathbb{T}^1$ supports one and only one Borel T-invariant probability measure. Such a measure is ergodic and is called a Sturm measure.*

Distinct semi-circles may support the same Sturm measure, so a parametrization of these measures by the family of semi-circles is not intrinsic. In order to get a proper parametrization, we need the notion of rotation number of a Sturm measure.

First, a natural way of constructing the Sturm measure with support in $[\theta/2, \theta/2 + 1/2]$ is to introduce the transformations $\eta_{\theta,+}$ and $\eta_{\theta,-}$ of $\mathbb{T}^1$ verifying $T \circ \eta_{\theta,\pm} = Id$ and defined by :

$$\eta_{\theta,\pm}(x) = \frac{1}{2}(x + \epsilon_\theta(x)), \ \forall x \neq \theta \mod (1),$$

where $\epsilon_\theta(x) \in \{0, 1\}$ is chosen so that $\eta_{\theta,\pm}(x) \in (\theta/2, \theta/2 + 1/2)$. Complete the definition by setting $\eta_{\theta,+}(\theta) = \theta/2$ and $\eta_{\theta,-}(\theta) = \theta/2 + 1/2$. The graphs of $\eta_{\theta,\pm}$ are plotted in Figure 2.

Notice that $\eta_{\theta,+}$ is right-continuous, whereas $\eta_{\theta,-}$ is left-continuous. Concretely, $\eta_{\theta,+}$ acts on $\mathbb{T}^1$ as follows : the Circle $\mathbb{T}^1$ is cut into an interval at θ, is linearly contracted by a 1/2 and then rotated to the semi-circle $[\theta/2, \theta/2+1/2)$. For $\eta_{\theta,-}$, the image interval is this time $(\theta/2, \theta/2 + 1/2]$. The transformations $\eta_{\theta,\pm}$ are examples of quasi-contracting maps. More on this topic can be found in [5, 8, 13].

It is an observation (see [5], Lemma 3.2) that a Borel measure μ on $\mathbb{T}^1$ is T-invariant and with support in $[\theta/2, \theta/2 + 1/2]$ if and only if it is invariant under either $\eta_{\theta,+}$ or $\eta_{\theta,-}$. The maps $\eta_{\theta,+}$ and $\eta_{\theta,-}$ are order-preserving transformations of the Circle $\mathbb{T}^1$: this means that, given any three points on $\mathbb{T}^1$, the images of these three points by $\eta_{\theta,+}$ or $\eta_{\theta,-}$ are ordered as the initial points with respect to cyclic order on $\mathbb{T}^1$.

Therefore, $\eta_{\theta,+}$ and $\eta_{\theta,-}$ admit a rotation number, the same one for both, written as $t \in \mathbb{R}/\mathbb{Z}$. Recall that any order-preserving transformation $\chi : \mathbb{T}^1 \to \mathbb{T}^1$ has a

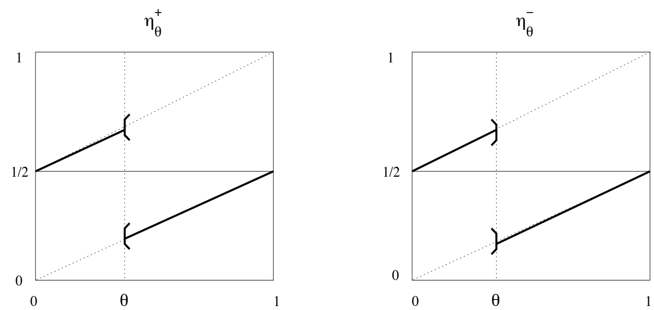

FIGURE 2. Graphs of $\eta_{\theta,+}$ and $\eta_{\theta,-}$.

rotation number τ_χ. This quantity measures the average speed of rotation under iterations and is defined as :

$$\tau_\chi = \lim_{n\to+\infty} \frac{1}{n}(\widetilde{\chi})^n(x) \bmod (1),$$

where $\widetilde{\chi} : \mathbb{R} \to \mathbb{R}$ is any order-preserving lift of χ, commuting with integer translations. This limit is independent on x.

Reciprocally, it can be shown that for any $t \in \mathbb{R}/\mathbb{Z}$, there is a closed interval of parameters θ for which the applications $\eta_{\theta,+}$ and $\eta_{\theta,-}$ have rotation number t. Moreover, all these applications correspond to a unique Sturm measure. This measure will be written as ν_t in the sequel. Let us detail the relations between the rational character of t, the corresponding parameters θ and the support of ν_t in $\mathbb{T}^1$:

(1) When $t \in [0,1)\backslash\mathbb{Q}$, there is a unique $\theta_t \in [0,1)$ such that $\mathrm{supp}(\nu_t) \subset [\theta_t/2, \theta_t/2 + 1/2]$. In this case ν_t is diffusive and its support is a minimal and uniquely ergodic Cantor set.

(2) Suppose now that $t = p/q \in [0,1)$, $p \wedge q = 1$.

   (a) There is a closed interval $[\theta^-_{p/q}, \theta^+_{p/q}]$ such that :

$$\mathrm{supp}(\nu_{p/q}) \subset [\theta/2, \theta/2 + 1/2], \text{ iff } \theta \in [\theta^-_{p/q}, \theta^+_{p/q}].$$

   Moreover, we have :

$$\theta^+_{p/q} - \theta^-_{p/q} = 1/(2^q - 1). \tag{6}$$

   The points $\theta^+_{p/q}$ and $\theta^-_{p/q}$ are q-periodic and lay in the same T-orbit. In this case, $\nu_{p/q}$ is the T-invariant periodic measure supported by this orbit. Mention that $\theta^+_{p/q}/2$ is periodic under T, whereas $\theta^+_{p/q}/2 + 1/2$ is not.
   Symmetrically, $\theta^-_{p/q}/2 + 1/2$ is periodic under T, whereas $\theta^-_{p/q}/2$ is not. In order to unify the proofs, we set $\theta^+_t = \theta^-_t = \theta_t$, when $t \in [0,1/2]\backslash\mathbb{Q}$.

   (b) If $0 \le p/q < p'/q' < 1$ are adjacent rationals, in the sense that $p'q - pq' = 1$, we will use the information that :

$$\theta^-_{p'/q'} - \theta^+_{p/q} = (2^q - 1)^{-1}(2^{q'} - 1)^{-1}, \tag{7}$$

given in the proof of Lemma 2 of Bullett-Sentenac [9]. It is a consequence of the following relations : $T^q(\theta^-_{p'/q'}) = \theta^+_{p'/q'}$ and $T^{q'}(\theta^+_{p/q}) = \theta^-_{p/q}$.

(3) Another property is that $\cup_{p/q\in[0,1),p\wedge q=1}[\theta^-_{p/q},\theta^+_{p/q}]$ has full measure in $[0,1)$. As a corollary, the mapping $\theta \longmapsto t$ is a non-decreasing Devil staircase, i.e. a non-constant continuous map from $[0,1)$ onto $[0,1)$ which is locally constant on a set of full Lebesgue-measure.

We finally develop the connexions between the maps $\eta_{\theta,\pm}$ and the rational character of the rotation number t of the Sturm measure ν_t. For any $\gamma\in\mathbb{T}^1$, introduce first the open semi-circle $U_\gamma = (\gamma/2+1/2,\gamma/2)\subset\mathbb{T}^1$ complementary to $[\gamma/2,\gamma/2+1/2]$. We sum up some results contained in Lemma 3.2 and Proposition 4.8 of [5] :

Proposition 2.1.

(1) *The sets $(\eta^n_{\theta,+}(U_\theta))_{n\geq 0}$ are all disjoint and their union has full Lebesgue measure in $\mathbb{T}^1$.*

(2) *If $t\in[0,1]\backslash\mathbb{Q}$, then the sets $(\eta^n_{\theta_t,+}(U_{\theta_t}))_{n\geq 0}$ are intervals. Moreover, each one can be written as $\eta^n_{\theta_t,+}(U_{\theta_t}) = (\eta^{n+1}_{\theta_t,-}(\theta_t),\eta^{n+1}_{\theta_t,+}(\theta_t))$, and has length 2^{-n-1}.*

(3) *Let $q\geq 1$. A real number θ is in the closure of $\eta^{q-1}_{\theta,+}(U_\theta)$ if and only if there exists $0\leq p<q$ with $p\wedge q=1$ such that the rotation number of $\eta_{\theta,+}$ is p/q. This property is equivalent to saying that $\theta\in\cup_{0\leq p<q:\,p\wedge q=1}[\theta^-_{p/q},\theta^+_{p/q}]$. In this case, the sets $\eta^{q-1}_{\theta,+}(U_\theta)$ are not always intervals. More precisely,*

$$
\begin{cases} 0\leq n\leq q-1 \;:\; (\eta_{\theta,+})^n(U_\theta) = \big((\eta_{\theta,-})^{n+1}(\theta),(\eta_{\theta,+})^{n+1}(\theta)\big) \\ \quad\;\; n\geq q \;:\; (\eta_{\theta,+})^n(U_\theta) = ((\eta_{\theta,-})^{n+1}(\theta),(\eta_{\theta,-})^{n+1-q}(\theta)] \\ \qquad\qquad\qquad\qquad\qquad\; \cup((\eta_{\theta,+})^{n+1-q}(\theta),(\eta_{\theta,+})^{n+1}(\theta)). \end{cases} \tag{8}
$$

In the extremal cases :

$$
\eta^q_{\theta^+_{p/q},+}(\theta^+_{p/q}) = \theta^+_{p/q} \quad \text{and} \quad \eta^q_{\theta^-_{p/q},-}(\theta^-_{p/q}) = \theta^-_{p/q}.
$$

Finally, for every $\theta\in(\theta^-_{p/q},\theta^+_{p/q})$ and $n\geq 1$, we have $\eta^n_{\theta,-}(\theta) = 2^{-n}\theta + Z_n$, for some fixed real number Z_n, which depends only on p/q.

The following result will be used several times :

Proposition 2.2.
Let $t\in[0,1)$ and fix $n\geq 1$. Then $\eta^n_{\theta,-}(\theta)$ tends to $\eta^n_{\theta^+_t,-}(\theta^+_t)$, as $\theta\to\theta^+_t$ with $\theta>\theta^+_t$. If $t\in(0,1]$, then $\eta^n_{\theta,+}(\theta)$ tends to $\eta^n_{\theta^-_t,+}(\theta^-_t)$, as $\theta\to\theta^-_t$ with $\theta<\theta^-_t$.

Proof. Consider the first situation (the second one being identical) and remark that θ^+_t is the only discontinuity of $\eta_{\theta^+_t,-}$.

If $\theta^+_t\neq\eta^k_{\theta^+_t,-}(\theta^+_t)$, for every $1\leq k\leq n-1$, then the result is obvious since $\theta\mapsto\eta^n_{\theta,-}(\theta)$ is continuous around θ^+_t.

Suppose then that this is not true and that there exists $k\leq n-1$ such that $\theta^+_t = \eta^k_{\theta^+_t,-}(\theta^+_t)$, whereas $\theta^+_t\neq\eta^l_{\theta^+_t,-}(\theta^+_t)$, for $1\leq l<k$. By item *3.* of Proposition 2.1, t is an irreducible fraction of the form r/k. We use the following property:

when $\theta \in \left[\theta^-_{r/k}, \theta^+_{r/k}\right]$, we have $\theta \in \left[\eta^k_{\theta,-}(\theta), \eta^k_{\theta,+}(\theta)\right]$. Moreover, $\theta = \eta^k_{\theta,-}(\theta)$ if and only if $\theta = \theta^-_{r/k}$ (and similarly $\theta = \eta^k_{\theta,+}(\theta)$ if and only if $\theta = \theta^+_{r/k}$).

Consequently, the condition $\theta^+_{r/k} = \eta^k_{\theta^+_{r/k},-}(\theta^+_{r/k})$ implies that $\theta^+_{r/k} = \theta^-_{r/k}$, which is impossible. □

3. Preliminary results

As a first remark, $\mathcal{F}$ is obtained by integrating twice a sum of Dirac masses.

Lemma 3.1. *In $(-1/2, 1)$, the second derivative $\mathcal{F}''$ of $\mathcal{F}$ in the sense of Distributions is a sum of Dirac masses at the angular points of $\mathcal{F}$:*

$$\mathcal{F}'' = \sum_{t\in(0,1/2)\cap\mathbb{Q}} \left[\mathcal{F}'_+(x(t)) - \mathcal{F}'_-(x(t))\right] \delta_{x(t)}.$$

Proof. Corollary 1 of [3], the maximizing measure of $x \longmapsto \cos 2\pi(x-\omega)$ is periodic for λ-almost all $\omega \in [0, 1/2]$. A reformulation is that for λ-almost all $\omega \in [0, 1/2]$, the maximal orthogonal projection of the Fish on the straight line going through 0 and with angle $2\pi\omega$ is realized by an angular point. Thus for every $-1/2 < a < b < 1$:

$$\begin{aligned}\sum_{x(p/q)\in(a,b)} &\arctan\left(\mathcal{F}'_-(x(p/q))\right) - \arctan\left(\mathcal{F}'_+(x(p/q))\right)\\ &= \arctan\left(\mathcal{F}'_+(a)\right) - \arctan\left(\mathcal{F}'_-(b)\right),\end{aligned}$$

which can be rewritten as :

$$\sum_{x(p/q)\in(a,b)} \int_{\mathcal{F}'_+(x(p/q))}^{\mathcal{F}'_-(x(p/q))} \frac{1}{1+u^2}\, du = \int_{\mathcal{F}'_-(b)}^{\mathcal{F}'_+(a)} \frac{1}{1+u^2}\, du$$

and equivalently :

$$\cup_{x(p/q)\in(a,b)} \left[\mathcal{F}'_+(x(p/q)), \mathcal{F}'_-(x(p/q))\right] = \left[\mathcal{F}'_-(b), \mathcal{F}'_+(a)\right],\ \lambda - \text{a.s.},$$

where the union is disjoint. Consequently :

$$\sum_{x(p/q)\in(a,b)} \mathcal{F}'_-(x(p/q)) - \mathcal{F}'_+(x(p/q)) = \mathcal{F}'_+(a) - \mathcal{F}'_-(b),$$

which implies the lemma. □

We now consider Proposition 1.1, which asserts that the angular defect of $\mathcal{F}$ at each angular point $x(p/q)$, with $p \wedge q = 1$, has exact order $q2^{-q}$. The proof is based on the next Definition and Lemma.

Definition 3.1. *As in [3], let us introduce, for $0 \le \gamma < 2$, the exit time $E_\gamma : x \in \mathbb{T}^1 \longmapsto \mathbb{N} \cup \{\infty\}$ of the semi-circle $[\gamma/2, \gamma/2 + 1/2]$ under iterations of T. In other words, $E_\gamma(x)$ is the smallest $n \ge 0$ for which $T^n(x) \notin [\gamma/2, \gamma/2 + 1/2]$. This map E_γ belongs to $L^1(\mathbb{T}^1)$. We denote, for every $\gamma \in [0, 2)$ the quantity*

$$J(\gamma) = \int e^{2i\pi u} E_\gamma(u)\, du. \tag{9}$$

By Lemma p. 505 of [3], $\gamma \in [0, 2) \longmapsto E_\gamma \in \mathrm{Ł}^1(\mathbb{T})$ is a continuous map. Consequently, the map $\gamma \in [0, 2) \longmapsto J(\gamma)$ is also continuous. We shall invoke this essential property several times hereafter.

Lemma 3.2.

(1) *For* $t \in [0,1/2)$ *:*

$$\lim_{p/q\to t,\ t<p/q,\ p\wedge q=1} \frac{2^q}{q}\cdot\left(J(\theta^+_{p/q})-J(\theta^-_{p/q})\right)=\xi^+_t, \tag{10}$$

where $\xi^+_t=\sum_{n\geq 1}2^{-n}\,e^{2i\pi\eta^n_{\theta^+_t,-}(\theta^+_t)}\left(1-e^{2i\pi 2^{-n}}\right)$.

(2) *For* $t \in (0,1/2]$ *:*

$$\lim_{p/q\to t,\ t>p/q,\ p\wedge q=1} \frac{2^q}{q}\cdot\left(J(\theta^+_{p/q})-J(\theta^-_{p/q})\right)=\xi^-_t, \tag{11}$$

where $\xi^-_t=\sum_{n\geq 1}2^{-n}\,e^{2i\pi\eta^n_{\theta^-_t,+}(\theta^-_t)}\left(e^{-2i\pi 2^{-n}}-1\right)$.

Proof. Fixing $0\leq\gamma<1$, we first rewrite $J(\gamma)$. It is readily checked that for $u\in\mathbb{T}^1$, $E_\gamma(u)$ is the integer $n\geq 0$ such that $u\in\eta^n_{\gamma,+}(U_\gamma)$, quantity defined $\lambda-a.s.$ Consequently :

$$J(\gamma)=\sum_{n\geq 1} n\int_{\eta^n_{\gamma,+}(U_\gamma)} e^{2i\pi u}\,du. \tag{12}$$

Let $p/q\in(0,1/2)$ with $p\wedge q=1$ and fix γ and γ' such that $\theta^-_{p/q}<\gamma<\gamma'<\theta^+_{p/q}$. Using (8) and (12), we write $J(\gamma')-J(\gamma)=A+B+C$, where :

$$\begin{aligned}
A &= \sum_{n=1}^{q-1} n\left(\int_{\eta^{n+1}_{\gamma',-}(\gamma')}^{\eta^{n+1}_{\gamma',+}(\gamma')} e^{2i\pi u}\,du-\int_{\eta^{n+1}_{\gamma,-}(\gamma)}^{\eta^{n+1}_{\gamma,+}(\gamma)} e^{2i\pi u}\,du\right)\\
B &= \sum_{n\geq 0}(n+q)\left(\int_{\eta^{n+q+1}_{\gamma',-}(\gamma')}^{\eta^{n+1}_{\gamma',-}(\gamma')} e^{2i\pi u}\,du-\int_{\eta^{n+q+1}_{\gamma,-}(\gamma)}^{\eta^{n+1}_{\gamma,-}(\gamma)} e^{2i\pi u}\,du\right)\\
C &= \sum_{n\geq 0}(n+q)\left(\int_{\eta^{n+1}_{\gamma',+}(\gamma')}^{\eta^{n+q+1}_{\gamma',+}(\gamma')} e^{2i\pi u}\,du-\int_{\eta^{n+1}_{\gamma,+}(\gamma)}^{\eta^{n+q+1}_{\gamma,+}(\gamma)} e^{2i\pi u}\,du\right).
\end{aligned}$$

Consider first A. We use two informations:

- For $\gamma\in(\theta^-_{p/q},\theta^+_{p/q})$ and $n\geq 1$, $\eta^n_{\gamma,-}(\gamma)=2^{-n}\gamma+Z_n$, by Proposition 2.1.
- For $\gamma\in[\theta^-_{p/q},\theta^+_{p/q}]$ and $1\leq n\leq q-1$, $\eta^{n+1}_{\gamma,+}(\gamma)-\eta^{n+1}_{\gamma,-}(\gamma)=2^{-n-1}$.

Hence

$$\begin{aligned}
A &= \sum_{n=1}^{q-1} n\left(\int_{2^{-n-1}\gamma'+Z_{n+1}-2^{-n-1}}^{2^{-n-1}\gamma'+Z_{n+1}} e^{2i\pi u}\,du-\int_{2^{-n-1}\gamma+Z_{n+1}-2^{-n-1}}^{2^{-n-1}\gamma+Z_{n+1}} e^{2i\pi u}\,du\right)\\
&= \sum_{n=1}^{q-1} ne^{2i\pi\eta^{n+1}_{\gamma,+}(\gamma)}\left(e^{2i\pi 2^{-n-1}(\gamma'-\gamma)}-1\right)\int_{-2^{-n-1}}^{0} e^{2i\pi u}\,du.
\end{aligned}$$

Since $|e^{iu}-1|\leq|u|$ and $\gamma'-\gamma<\theta^+_{p/q}-\theta^-_{p/q}=1/(2^q-1)$, we deduce the upper-bound :

$$|A|\leq\frac{\pi}{2(2^q-1)}\sum_{n\geq 1} n4^{-n}. \tag{13}$$

We now show that B and C are much greater than $|A|$. Using the same informations as above, we get :

$$B = \frac{1}{2i\pi}\sum_{n\geq 0}(n+q)\cdot\Big[e^{2i\pi\eta^{n+1}_{\gamma,-}(\gamma)}\left(e^{2i\pi 2^{-n-1}(\gamma'-\gamma)}-1\right) - e^{2i\pi\eta^{n+q+1}_{\gamma,-}(\gamma)}\left(e^{2i\pi 2^{-n-q}(\gamma'-\gamma)}-1\right)\Big].$$

Let then $t\in[0,1/2)$ and suppose that $p/q\to t$, with $p/q>t$, $p\wedge q=1$. We fix $n\geq 0$. By Proposition 2.2, $\eta^{n+1}_{\gamma,-}(\gamma)\to\eta^{n+1}_{\theta_t^+,-}(\theta_t^+)$, when $q\to+\infty$. Moreover, still for $q\to+\infty$, the quantity $e^{2i\pi 2^{-n-1}(\gamma'-\gamma)}-1$ is equivalent to $2i\pi 2^{-n-1}(\gamma'-\gamma)$ and the last term in the right-hand side above is negligeable.

Therefore, uniformly in $\theta^-_{p/q}<\gamma<\gamma'<\theta^+_{p/q}$:

$$\frac{B}{q(\gamma'-\gamma)}\longrightarrow\sum_{n\geq 0}2^{-n-1}\cdot e^{2i\pi\eta^{n+1}_{\theta_t^+,-}(\theta_t^+)},\ \text{as } p/q\to t,\ t<p/q,\ p\wedge q=1. \tag{14}$$

In a similar way, uniformly in $\theta^-_{p/q}<\gamma<\gamma'<\theta^+_{p/q}$:

$$\frac{C}{q(\gamma'-\gamma)}\longrightarrow-\sum_{n\geq 0}2^{-n-1}\cdot e^{2i\pi\left(\eta^{n+1}_{\theta_t^+,-}(\theta_t^+)+2^{-n-1}\right)},\ \text{as } p/q\to t,\ t<p/q,\ p\wedge q=1. \tag{15}$$

Since the convergences (13), (14) and (15) are uniform, and using the continuity of $J(\gamma)$ at $\gamma=\theta^-_{p/q}$ and $\theta^+_{p/q}$, (13), (14) and (15) still hold with $\gamma=\theta^-_{p/q}$ and $\gamma'=\theta^+_{p/q}$. The final result simply follows from (6), which yields $\theta^+_{p/q}-\theta^-_{p/q}=1/(2^q-1)$.

The second item is shown in the same way. □

We now move to Proposition 1.1.

Proof. For every irreducible fraction p/q, let $[\omega_{p/q,-},\omega_{p/q,+}]$ be the set of ω such that the maximizing measure of $x\longmapsto\cos 2\pi(x-\omega)$ is $\nu_{p/q}$. Then :

$$\omega_{p/q,+}-\omega_{p/q,-}=\arctan\left(\mathcal{F}'_-(x(p/q))\right)-\arctan(\mathcal{F}'_+(x(p/q))).$$

Remembering (3), we have $-\infty<\mathcal{F}'_-(1)\leq\mathcal{F}'_+(x)<\mathcal{F}'_-(x)\leq\mathcal{F}'_+(-1/2)<+\infty$. This implies that for some universal constant $C'>0$:

$$C'^{-1}(\omega_{p/q,+}-\omega_{p/q,-})\leq\mathcal{F}'_-(x(p/q))-\mathcal{F}'_+(x(p/q))\leq C'(\omega_{p/q,+}-\omega_{p/q,-}).$$

We thus focus on $\omega_{p/q,+}-\omega_{p/q,-}$ instead of $\mathcal{F}'_-(x(p/q))-\mathcal{F}'_+(x(p/q))$.

From Bousch [3], for any ω, the maximizing measure of $x\longmapsto\cos 2\pi(x-\omega)$ has support in $[\gamma,\gamma+1/2]$ and the parameter ω is uniquely determined in terms of γ by the two conditions (Proposition p. 503 and Remark p. 506 in [3]) :

$$e^{2i\pi(\omega+1/4)}\perp J(\gamma)\quad\text{and}\quad|\gamma+1/4-\omega|\leq 0.111, \tag{16}$$

where the first expression is a shorthand notation for orthogonality of the corresponding vectors of the plane. It is also known that $J(\gamma)\neq 0$.

Theorem B of [3] indicates that the correspondence $\gamma\longmapsto\omega$ is a homeomorphism with a modulus of continuity of the form $Kx\log(1/x)$. If $p/q\in(0,1/2)$, $p\wedge q=1$, then we get $\omega_{p/q,+}-\omega_{p/q,-}\leq K\ (\theta^+_{p/q}-\theta^-_{p/q})\log(\theta^+_{p/q}-\theta^-_{p/q})\leq C\ q2^{-q}$, via (6), for some universal constant $C>0$. This gives the right-hand side inequality in (4).

We now prove the other direction. Suppose that an infinite sequence of distinct rationals $(p_n/q_n)_{n\geq 0}$ in $(0,1/2)$, with $p_n\wedge q_n=1$, is such that $\frac{2^{q_n}}{q_n}\Big(\mathcal{F}'_-(x(p_n/q_n))-\mathcal{F}'_+(x(p_n/q_n))\Big)\to 0$ as $n\to+\infty$. By the remarks above, this is equivalent to :

$$\frac{2^{q_n}}{q_n}\cdot\left(\omega_{p_n/q_n,+}-\omega_{p_n/q_n,-}\right)\to 0,\text{ as } n\to+\infty. \tag{17}$$

Up to extraction, we suppose that $p_n/q_n\to t\in[0,1/2)$, $t<p_n/q_n$, $p_n\wedge q_n=1$. In this case we shall use (10). The other case $p_n/q_n\to t\in(0,1/2]$, $t>p_n/q_n$, $p_n\wedge q_n=1$ is treated similarly, using (11). By (16), we have

$$\begin{cases} e^{2i\pi(\omega_{p_n/q_n,-}+1/4)}\perp J(\theta^-_{p_n/q_n})\\ e^{2i\pi(\omega_{p_n/q_n,+}+1/4)}\perp J(\theta^+_{p_n/q_n}),\end{cases}$$

hence the angular variation between $J(\theta^+_{p_n/q_n})$ and $J(\theta^-_{p_n/q_n})$ equals $\omega_{p_n/q_n,+}-\omega_{p_n/q_n,-}$. But by (10), and provided that $\xi^+_t\neq 0$, $J(\theta^+_{p_n/q_n})-J(\theta^-_{p_n/q_n})$ is of order $q_n2^{-q_n}$. As both $J(\theta^+_{p_n/q_n})$ and $J(\theta^-_{p_n/q_n})$ tend to $J(\theta^+_t)\neq 0$ (since J is a continuous map as noticed above), it is enough to prove that $\xi^+_t\neq 0$ and that ξ^+_t and $J(\theta^+_t)$ do not have the same directions, so that (10) and (17) will be in contradiction.

We develop a numerical argument, combined with an estimate from [3]. Set :

$$u_n=e^{2i\pi\eta^n_{\theta^+_t,-}(\theta^+_t)}\left(1-e^{2i\pi2^{-n}}\right),\ n\geq 1.$$

With these notations, $\xi^+_t=\sum_{n\geq 1}2^{-n}u_n$. Remark that $|u_1|=2$ and $|u_2|=\sqrt{2}$. We introduce the straight line $\mathcal{D}_t$ supported by $e^{2i\pi(\theta^+_t/2)}$ and denote by $proj$ the orthogonal projection on $\mathcal{D}_t$ and by $proj_\perp$ the orthogonal projection on $(\mathcal{D}_t)^\perp$. One checks that :

$$\begin{aligned} 1-proj(u_2)/4-\sum_{n\geq 3}2^{-n}\cdot 2\sin(2\pi 2^{-n-1}) &\leq |proj(\xi^+_t)|\\ proj_\perp(u_2)/4+\sum_{n\geq 3}2^{-n}\cdot 2\sin(2\pi 2^{-n-1}) &\geq |proj_\perp(\xi^+_t)|.\end{aligned}$$

We next show that $|proj(\xi^+_t)|>|proj_\perp(\xi^+_t)|$ (which implies that $\xi^+_t\neq 0$). Since $\sum_{n\geq 3}2^{-n}\ 2\sin(2\pi2^{-n-1})\leq\pi/24$, this amounts to proving that $1-proj(u_2)/4>proj_\perp(u_2)/4+\pi/12$. Since $\alpha=proj(u_2)$ and $proj_\perp(u_2)$ are bounded by $|u_2|=\sqrt{2}$, the proof reduces to $1>\sqrt{2}/2+\pi/12=0.968...$, which is true.

Consequently, as claimed above, $|proj(\xi^+_t)|>|proj_\perp(\xi^+_t)|$. This yields :

$$Arg(\xi^+_t)\in[\theta^+_t/2-0.125,\theta^+_t/2+0.125]\mod (1/2).$$

Let $\omega_{t,+}$ be such that the maximizing measure of $x\longmapsto\cos 2\pi(x-\omega_{t,+})$ has its support contained in $[\theta^+_t/2,\theta^+_t/2+1/2]$. If ξ^+_t and $J(\theta^+_t)$ have the same direction, then $\omega_{t,+}=Arg(\xi^+_t)\mod(1/2)$. Thus $\omega_{t,+}\in[\theta^+_t/2-0.125,\theta^+_t/2+0.125]\mod(1/2)$. But by (16), we get $\omega_{t,+}\in[\theta^+_t/2+0.139,\theta^+_t/2+0.361]$. The two conditions on $\omega_{t,+}$ are not compatible, hence the contradiction. □

For p/q with $p\wedge q=1$, still denote by $[\omega_{p/q,-},\omega_{p/q,+}]$ the interval of ω such that the maximizing measure of $x\longmapsto\cos 2\pi(x-\omega)$ is $\nu_{p/q}$. As a by-product of the proof, we obtain :

Corollary 3.1. *There is a constant $C > 0$ such that for $p/q \in (0,1/2)$ with $p \wedge q = 1$:*

$$\frac{1}{C} \cdot q2^{-q} \leq \omega_{p/q,+} - \omega_{p/q,-} \leq C \cdot q2^{-q}.$$

Moreover, if $p/q \to t$ with $p \wedge q = 1$ and $p/q < t \in (0,1/2]$ (or $p/q > t \in [0,1/2)$), then $(2^q/q)(\omega_{p/q,+} - \omega_{p/q,-})$ converges to a real number in $(0,+\infty)$.

Corollary 3.1 proves a conjecture by Hunt and Ott, enounced in a weaker form in [15]. This problem was also mentioned by Jenkinson at the end of [18].

We now turn to another preliminary study, concerning the analysis of the regularity of $t \longmapsto I(t)$ and $t \longmapsto x(t)$. The next proposition gives Theorem 1.2.

Proposition 3.1.

(1) *Let $t \in [0,1/2)$. Then the map $u \longmapsto I(u)$ is right-differentiable at t and :*

$$I'_+(t) = \sum_{n \geq 1} n \; e^{2i\pi \eta^{n+1}_{\theta_t^+,-}(\theta_t^+)} \left(e^{2i\pi 2^{-n-1}} - 1 \right).$$

(2) *Let $t \in (0,1/2]$. Then $u \longmapsto I(u)$ is left-differentiable at t and :*

$$I'_-(t) = \sum_{n \geq 1} n \; e^{2i\pi \eta^{n+1}_{\theta_t^-,+}(\theta_t^-)} \left(1 - e^{-2i\pi 2^{-n-1}} \right).$$

(3) *Let $t \in [0,1/2] \backslash \mathbb{Q}$. Then $u \longmapsto I(u)$ is differentiable at t and :*

$$I'(t) = \sum_{n \geq 1} n \; \left(e^{2i\pi \eta^{n+1}_{\theta_t,+}(\theta_t)} - e^{2i\pi \eta^{n+1}_{\theta_t,-}(\theta_t)} \right). \tag{18}$$

(4) *The application $u \longmapsto I(u)$ is not differentiable at $t \in (0,1/2) \cap \mathbb{Q}$ and $t \longmapsto I'(t)$ is continuous in restriction to $[0,1/2] \backslash \mathbb{Q}$.*

(5) *Let c_0 and c_1 as in Theorem 1.2. Then $t \longmapsto I(t)$ is c_0-Lipschitz. Moreover, $t \longmapsto x(t)$ is a decreasing bi-Lipschitz homeomorphism from $[0,1/2]$ on $[-1/2,1]$, with :*

$$x'(t) \in -\frac{1}{\sqrt{1 + (\mathcal{F}'(x(t)))^2}} [c_0, c_1], \; t \in [0,1/2] \backslash \mathbb{Q}.$$

Proof. We prove item *1.* Consider adjacent rationals $0 \leq p/q < p'/q' < 1/2$, with $p'q - pq' = 1$ (therefore $p/q - p'/q' = 1/(qq')$). For a function f defined on $\mathbb{T}^1$ and $n \geq 0$, introduce the ergodic sum $S_n f(x) = \sum_{k=0}^{n-1} f(T^k x)$. Recall that $\nu_{p/q}$ is the T-invariant Sturm measure supported by the orbit of $\theta^+_{p/q}$ and $\theta^-_{p/q}$. We can write :

$$\begin{aligned} I(p/q) &= \int e^{2i\pi u} \, d\nu_{p/q}(u) = \frac{1}{q'} \int \left(S_{q'} e^{2i\pi \, \cdot} \right)(u) \, d\nu_{p/q}(u) \\ &= \frac{1}{qq'} \sum_{0 \leq n < q, 0 \leq m < q'} e^{2i\pi 2^{n+m} \theta^+_{p/q}}. \end{aligned}$$

Similarly :

$$I(p'/q') = \frac{1}{qq'} \sum_{0 \leq n < q, \, 0 \leq m < q'} e^{2i\pi 2^{n+m} \theta^-_{p'/q'}}.$$

Combining the previous two equalities together with (7), we deduce that :

$$\begin{aligned} & qq'\big(I(p/q) - I(p'/q')\big) \\ = & \sum_{0\le n<q,\, 0\le m<q'} e^{2i\pi 2^{n+m}\theta^+_{p/q}}\left(1 - e^{2i\pi \frac{2^{n+m-q-q'}}{(1-2^{-q})(1-2^{-q'})}}\right) \\ = & \sum_{1\le n\le q,\, 1\le m\le q'} e^{2i\pi 2^{q+q'-n-m}\theta^+_{p/q}}\left(1 - e^{2i\pi \frac{2^{-n-m}}{(1-2^{-q})(1-2^{-q'})}}\right). \end{aligned} \tag{19}$$

Let now $t \in [0, 1/2)$. We shall show that if two adjacent rational numbers $p/q < p'/q'$ verify $t < p/q < p'/q'$ and if they both tend to t, then $qq'(I(p/q) - I(p'/q'))$ converges to a real number.

First, the modulus of the generic expression in (19) rewrites

$$\left|e^{2i\pi 2^{q+q'-n-m}\theta^+_{p/q}}\left(1 - e^{2i\pi \frac{2^{-n-m}}{(1-2^{-q})(1-2^{-q'})}}\right)\right| = 2\left|\sin\left(\pi \frac{2^{-n-m}}{(1-2^{-q})(1-2^{-q'})}\right)\right|.$$

Since $q, q' \ge 1$, this term is bounded by $8\pi 2^{-n-m}$.

Second, the term between brackets in (19) tends to $(1-e^{2i\pi 2^{-n-m}})$, as $\min\{q, q'\}$ goes to $+\infty$.

Third, as recalled in the section of Sturm measures, $2^{q+q'}\theta^+_{p/q} = \theta^-_{p'/q'} \mod (1)$, which yields $2^{q+q'-n-m}\theta^+_{p/q} = \eta^{n+m}_{\theta^-_{p'/q'},-}(\theta^-_{p'/q'}) \mod (1)$. Then, by Proposition 2.2, for fixed n and m, we get that $\eta^{n+m}_{\theta^-_{p'/q'},-}(\theta^-_{p'/q'}) \to \eta^{n+m}_{\theta^+_t,-}(\theta^+_t)$.

Finally, when $t < p/q < p'/q'$, $p'q - pq' = 1$, $p'/q' \to t$:

$$\begin{aligned} qq'\big(I(p/q) - I(p'/q')\big) \longrightarrow -Z^+_t \;&=\; -\sum_{n\ge1, m\ge1} e^{2i\pi\eta^{n+m}_{\theta^+_t,-}(\theta^+_t)}\left(e^{2i\pi 2^{-n-m}} - 1\right) \\ &=\; -\sum_{n\ge1} n\; e^{2i\pi\eta^{n+1}_{\theta^+_t,-}(\theta^+_t)}\left(e^{2i\pi 2^{-n-1}} - 1\right). \end{aligned} \tag{20}$$

Let us fix $h > 0$. The Farey construction of the rational numbers gives the existence of an increasing bi-infinite sequence $(p_s/q_s)_{s\in\mathbb{Z}}$ of irreducible fractions checking :

$$\begin{cases} t < p_s/q_s < t + h, \text{ for } s \in \mathbb{Z}, \\ p_s/q_s \to t, \text{ as } s \to -\infty, \\ p_s/q_s \to t + h, \text{ as } s \to +\infty, \\ p_{s+1}q_s - p_s q_{s+1} = 1, \text{ for } s \in \mathbb{Z}. \end{cases}$$

By construction :

$$\sum_{s\in\mathbb{Z}} \frac{1}{q_s q_{s+1}} = h. \tag{21}$$

We now decompose the increment $I(t) - I(t + h)$ as :

$$I(t) - I(t + h) = \sum_{s\in\mathbb{Z}} I(p_s/q_s) - I(p_{s+1}/q_{s+1})$$

Observe that $\min_{s\in\mathbb{Z}} q_s \longrightarrow \infty$ as $h \to 0$. Consequently, combining (20) with (21), the previous uniform calculus easily implies that $(I(t) - I(t + h))/h - Z^+_t \longrightarrow 0$ when $h \to 0^+$. Hence $u \longmapsto I(u)$ is right-differentiable at t with right-derivative Z^+_t.

The proof of item *2.* is similar.

Consider items *3.* and *4.* Let $t \in [0,1/2]\backslash\mathbb{Q}$. Then $\theta_t^- = \theta_t^+ = \theta_t$ and $\eta^n_{\theta_t,+}(\theta_t) - \eta^n_{\theta_t,-}(\theta_t) = 2^{-n}$, for every $n \geq 2$. Thus $Z_t^- = Z_t^+$ and $u \longmapsto I(u)$ is differentiable at t. The continuity of $u \longmapsto I'(u)$ when restricted to $[0,1/2]\backslash\mathbb{Q}$ is a consequence of the remark that for fixed $n \geq 2$, the maps $u \longmapsto \eta^n_{\theta_u,+}(\theta_u)$ and $u \longmapsto \eta^n_{\theta_u,-}(\theta_u)$ are continuous at t.

Assume now that $t = r/s \in [0,1/2] \cap \mathbb{Q}$, $r \wedge s = 1$. Since $(x(r/s), \mathcal{F}(x(r/s)))$ is an angular point for $\mathcal{F}$, $Z^+_{r/s} = Z^-_{r/s} = 0$ is a necessary condition for I to be differentiable at r/s. Hence, in order to show that $u \longmapsto I(u)$ is not differentiable at r/s, it is enough to show that $Z^+_{r/s} \neq 0$.

To see this, remark that for $n \geq 2$, the interval $[\eta^n_{\theta^+_{r/s},-}(\theta^+_{r/s}), \eta^n_{\theta^+_{r/s},-}(\theta^+_{r/s}) + 2^{-n}]$ is contained in $[\theta^+_{r/s}/2, \theta^+_{r/s}/2 + 1/2]$. Indeed, as already mentioned in Section 2 on Sturm measures, for $2 \leq n \leq s$ this interval is $[\eta^n_{\theta^+_{r/s},-}(\theta^+_{r/s}), \eta^n_{\theta^+_{r/s},+}(\theta^+_{r/s})]$ and is disjoint from the interval corresponding to $n = 1$, which is $[\theta^+_{r/s}/2+1/2, \theta^+_{r/s}/2]$. For $n > s$, clearly $\eta^n_{\theta^+_{r/s},-}(\theta^+_{r/s}) \in [\theta^+_{r/s}/2, \theta^+_{r/s}/2+1/2]$, together with $\mathrm{dist}(\eta^n_{\theta^+_{r/s},-}(\theta^+_{r/s}), \theta^+_{r/s}/2+1/2) \geq 2^{-s-1}$. This proves the claim.

Recall now (20). By the above remark, each term $n\, e^{2i\pi\eta^{n+1}_{\theta_t^+,-}(\theta_t^+)} \left(e^{2i\pi 2^{-n-1}} - 1\right)$, with $t = r/s$, has a strictly negative orthogonal projection on the line D spanned by $e^{2i\pi(\theta^+_{r/s}/2)}$. Consequently, $Z^+_{r/s}$ itself has a strictly negative projection on D, and $Z^+_{r/s}$ is necessarily non-zero. This concludes item *4.*

Consider *5.* We first prove that $t \longmapsto I(t)$ is a Lipschitz map. Via (19) and the remark following (19), for any adjacent rationals $p/q < p'/q'$, one has $|I(p/q) - I(p'/q')| \leq 8\pi|p/q - p'/q'|$. The Farey construction of the rational numbers implies that this relation is valid between any neighbor points on an arbitrary thin net of $[0,1/2]$. This shows that $t \longmapsto I(t)$ is 8π-Lipschitz. Consequently, $t \longmapsto I(t)$ is absolutely continuous and then the integral of its derivative.

In order to get the result, it is thus enough to prove that c_0 is an upper-bound for $|I'(t)|$, for $t \in [0,1/2]\backslash\mathbb{Q}$. This easily follows from (18), which yields (by using again $\eta^{n+1}_{\theta_t,+}(\theta_t) - \eta^{n+1}_{\theta_t,-}(\theta_t) = 2^{-n-1}$) that :

$$|I'(t)| \leq \sum_{n\geq 1} 2n \sin(2\pi 2^{-n-2}) = c_0 = 6.077491...$$

It remains us to study the regularity of $t \mapsto x(t)$. For $t \in [0,1/2]\backslash\mathbb{Q}$, we have $x'(t) = Re(I'(t))$. A first remark is that $|I'(t)|$ is uniformly strictly positive, by a projection argument very similar to the one just above. Indeed for every $n \geq 2$, the interval $\left[\eta^n_{\theta_t,-}(\theta_t), \eta^n_{\theta_t,+}(\theta_t)\right]$ is $[\eta^n_{\theta_t,-}(\theta_t), \eta^n_{\theta_t,-}(\theta_t) + 2^{-n}]$ and is contained in the semi-circle $[\theta_t/2, \theta_t/2 + 1/2]$. Then, projecting orthogonally the vector with affix $I'(t)$ on the line spanned by $e^{2i\pi(\theta_t/2)}$ gives :

$$|I'(t)| \geq \sum_{n\geq 1} n(1 - \cos(\pi 2^{-n})) = c_1 = 1.925255... > 0.$$

Since $|x'(t)| = \cos(\arctan(\mathcal{F}'(x(t))))|I'(t)|$ and $x'(t) \leq 0$, the proof is complete. $\square$

4. Proof of the main Theorems

4.1. **Proof of Theorem 1.1.** Fix $t \in [0, 1/2]\backslash\mathbb{Q}$ and recall that $\mathcal{F}$ is differentiable at $x(t)$. Let $h \in \mathbb{R}$ and introduce the difference for h small enough

$$\Delta_t(h) = \mathcal{F}(x(t)) - \mathcal{F}(x(t+h)) + \mathcal{F}'(x(t)) \cdot (x(t) - x(t+h)).$$

Remark that $\Delta_t(h) \geq 0$, as $\mathcal{F}$ is concave. By Lemma 3.1, $\Delta_t(h)$, which is a second-order difference, depends only on the angular variation of $\mathcal{F}$ at the irreducible fractions p/q lying in $[t, t+h]$. This implies :

$$\Delta_t(h) \quad \leq \sum_{p/q \in (t,t+h), p\wedge q=1} |x(p/q) - x(t+h)| \left(\mathcal{F}'_-(x(p/q) - \mathcal{F}'_+(x(p/q))\right).$$

Then, using Proposition 1.1 ($C > 0$ is the constant appearing in (4)), we get :

$$\Delta_t(h) \quad \leq \quad C\ |x(t) - x(t+h)| \sum_{p/q \in (t,t+h), p\wedge q=1} q2^{-q}. \tag{22}$$

Introduce the convergents $(p_k/q_k)_{k\geq 0}$ of t. From Jarnik's Theorem (see Falconer [11]), for every $t \notin A$, where $A \subset [0, 1/2]$ is a set of Hausdorff dimension 0, there exist an integer $N_t \geq 1$ and a constant $C_t > 0$ such that $q_{k+1} \leq C_t\ (q_k)^{N_t}$, for all k. We focus on such a $t \notin A$.

Fixing $h > 0$, let then k be such that $t < p_{2k+1}/q_{2k+1} \leq t + h < p_{2k-1}/q_{2k-1}$. By definition of the convergents and k, for some universal constant $C_0 > 0$:

$$\sum_{p/q \in (t,t+h), p\wedge q=1} q2^{-q} \leq C_0 \cdot q_{2k+1} \cdot 2^{-q_{2k+1}}. \tag{23}$$

By Proposition 3.1, we can choose $C' > 0$ such that $t \longmapsto x^{-1}(t)$ is C'-Lipschitz. Then, remembering that $t \rightarrow x(t)$ is a decreasing map, we obtain that $x(t) - x(t+h) \geq x(t) - x(p_{2k+1}/q_{2k+1})$, implying :

$$x(t) - x(t+h) \geq C'|t - p_{2k+1}/q_{2k+1}| \quad \geq \quad \frac{C'}{2 \cdot q_{2k+1} q_{2k+2}} \tag{24}$$

$$\geq \quad \left(\frac{C'}{2C_t}\right) \frac{1}{(q_{2k+1})^{1+N_t}}. \tag{25}$$

Set next $C'' = C'/(2C_t)$. We inject relations (23) and (25) in (22), using the remark that the map $u \longmapsto u2^{-u}$ is decreasing for $u \geq 1/\log 2$:

$$\begin{aligned}
\Delta_t(h) \quad &\leq \quad CC_0\ |x(t) - x(t+h)| \left(\frac{C''}{x(t) - x(t+h)}\right)^{1/(1+N_t)} \ 2^{-\left(\frac{C''}{x(t)-x(t+h)}\right)^{1/(1+N_t)}} \\
&\leq \quad CC_0 (C'')^{1/(1+N_t)}\ |x(t) - x(t+h)|^{N_t/(1+N_t)}\ 2^{-\left(\frac{C''}{x(t)-x(t+h)}\right)^{1/(1+N_t)}} \\
&\leq \quad C'''\ 2^{-\left(\frac{C''}{x(t)-x(t+h)}\right)^{1/(1+N_t)}},
\end{aligned}$$

which exactly implies (2) (C''' depends on t). The case $h < 0$ is treated similarly.

As $t \longmapsto x(t)$ is Lipschitz (by Theorem (1.2)), the image of A by $t \longmapsto x(t)$ also has zero Hausdorff-dimension. Hence, outside a set of Hausdorff-dimension zero, (2) holds true. This completes the proof of item (1) of Theorem 1.1.

Recall that item (2) of Theorem 1.1. follows from Theorem 1.3 and the remarks made in the introduction.

4.2. **Proof of Theorem 1.3.** Fix $t \in [0,1/2]\backslash\mathbb{Q}$ and consider the convergents $(p_n/q_n)_{n\geq 1}$ associated with t. Define :

$$E(t) = \{M \geq 0 \mid \sup_n \{q_n^{M+1} \cdot 2^{-q_n} \cdot q_{n+1}^M\} < +\infty\} \tag{26}$$

and set $M(t) = \sup E(t)$. Remark that this definition of $M(t)$ coincides with the one given in the introduction : $M(t) = \liminf_{n\to+\infty} q_n/\log_2 q_{n+1}$.

Let $h > 0$ and k be such that $t < p_{2k+1}/q_{2k+1} \leq t+h < p_{2k-1}/q_{2k-1}$. Again, the case $h < 0$ is similar. Let now $M \in E(t)$. From (24), we obtain that

$$|x(t) - x(t+h)|^M \geq \left(\frac{C'}{2q_{2k+1}q_{2k+2}}\right)^M.$$

It follows then from (23) that :

$$\sum_{p/q\in(t,t+h),p\wedge q=1} q2^{-q} \leq C_0 \frac{2^M}{(C')^M} \left[(q_{2k+1})^{1+M}(q_{2k+2})^M \ 2^{-q_{2k+1}}\right] (x(t)-x(t+h))^M.$$

Since $M \in E(t)$, the quantity between brackets is bounded in k, thus (22) clearly implies that for some constant C_M (depending on t and M),

$$\Delta_t(h) \leq C_M |x(t) - x(t+h)|^{1+M}.$$

Hence, $h_{\mathcal{F}}(x(t)) \geq 1 + M$, for every $M \in E(t)$ and finally $h_{\mathcal{F}}(x(t)) \geq 1 + M(t)$. This obviously implies that $h_{\mathcal{F}}(x(t)) = +\infty$ when $M(t) = +\infty$ (this yields item (2a) of Theorem 1.3).

In order to prove items (2b) and (2c) of Theorem 1.3, we now study the converse inequality. Suppose that $M(t) < +\infty$. Notice that it is very classical to see that the set of points satisfying $M(t) < +\infty$ has Hausdorff dimension 0. Assume that $\mathcal{F}$ is $\mathcal{C}^{1+M}(x(t))$, for $M > M(t)$. Let k and then $h > 0$ (chosen later) be such that :

$$t < p_{2k+1}/q_{2k+1} \leq t + h.$$

Consider again the difference $\Delta_t(h)$. The same argument as in the proof of Theorem 1.1 also holds true here. Hence, using successively Lemma 3.1, Proposition 1.1 and Proposition 3.1, we obtain :

$$\begin{aligned}\Delta_t(h) &\geq \left(\mathcal{F}'_-(x(p_{2k+1}/q_{2k+1}) - \mathcal{F}'_+(x(p_{2k+1}/q_{2k+1}))\right) \ |x(p_{2k+1}/q_{2k+1}) - x(t+h)| \\ &\geq \frac{C'}{C} \ q_{2k+1}2^{-q_{2k+1}} \ |p_{2k+1}/q_{2k+1} - (t+h)|. \end{aligned} \tag{27}$$

By hypothesis, for some constant $\widetilde{C}$ and a polynomial P of degree at most $1+\lfloor M\rfloor$:

$$|\Delta_t(h) - P(x(t) - x(t+h))| \leq \widetilde{C} \ |x(t) - x(t+h)|^{1+M}. \tag{28}$$

The previous study gives $|\Delta_t(h)| \leq C_\varepsilon \ |x(t) - x(t+h)|^{1+M(t)-\varepsilon}$, for any $\varepsilon > 0$. Consequently, the polynomial P has the form $P(X) = \sum_{m=1+\lceil M(t)\rceil}^{1+\lfloor M\rfloor} \alpha_m X^m$. Let M' be the first integer in the interval $[\lceil M(t)\rceil, M]$ such that $\alpha_{1+M'} \neq 0$. We deduce that for another constant C, we have $\Delta_t(h) \leq C \ |x(t)-x(t+h)|^{1+\min\{M,M'\}}$. Now, using that $u \longmapsto x(u)$ is bi-Lipschitz, we finally obtain for some $\widetilde{C}$:

$$\Delta_t(h) \leq \widetilde{C} \ |h|^{1+\min\{M,M'\}}. \tag{29}$$

Combining (27) with (29), there exists a constant C_1, independent on h, such that :

$$q_{2k+1}2^{-q_{2k+1}} \ \frac{|p_{2k+1}/q_{2k+1} - (t+h)|}{h^{1+\min\{M,M'\}}} \leq C_1. \tag{30}$$

Choose $h = 2(p_{2k+1}/q_{2k+1} - t)$, so that $t + h - p_{2k+1}/q_{2k+1} = p_{2k+1}/q_{2k+1} - t$. Since $1/(2q_{2k+1}q_{2k+2}) \le |p_{2k+1}/q_{2k+1} - t| \le 1/(q_{2k+1}q_{2k+2})$, inequality (30) becomes, with another constant C_2 :

$$q_{2k+1}^{1+\min\{M,M'\}} 2^{-q_{2k+1}} q_{2k+2}^{\min\{M,M'\}} \le C_2. \tag{31}$$

This procedure gives the same result for the other half of indices, i.e.

$$q_{2k}^{1+\min\{M,M'\}} 2^{-q_{2k}} q_{2k+1}^{\min\{M,M'\}} \le C_2. \tag{32}$$

Recall that $M > M(t)$, and $M' \ge M(t)$. We distinguish some cases:

- If $\min(M, M') > M(t)$, then (31) and (32) contradict the definition (26) of $M(t)$. Hence $\mathcal{F} \notin \mathcal{C}^{1+M}(x(t))$, for any $M > M(t)$. This implies that $h_{\mathcal{F}}(x(t)) = 1 + M(t)$.
- If $\min(M, M') = M(t)$, then $M(t) = M' \in \mathbb{N}$. If $M(t) \notin E(t)$, again (31) and (32) contradict (26). If $M(t) \in E(t)$, since M' is the non-zero coefficient of P of smallest degree, (28) gives :

$$\Delta_t(h) \sim \alpha_{1+M(t)} \ (x(t) - x(t+h))^{1+M(t)}, \text{ as } h \to 0.$$

  - If $M(t)$ is even, then the last equivalence is impossible since $\Delta_t(h)$ is always positive. The proof of the theorem is now complete.
  - The remaining case ($M(t) \in E(t)$ and $M(t)$ odd) would require to push further the developments.

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